DURFEE SQUARES, SYMMETRIC PARTITIONS AND BOUNDS ON KRONECKER COEFFICIENTS

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Abstract. We resolve two open problems on Kronecker coefficients $g(\lambda, \mu, \nu)$ of the symmetric group. First, we prove that for partitions λ, μ, ν with fixed Durfee square size, the Kronecker coefficients grow at most polynomially. Second, we show that the maximal Kronecker coefficients $g(\lambda, \lambda, \lambda)$ for self-conjugate partitions λ grow superexponentially. We also give applications to explicit special cases.

1. Introduction

1.1. **Foreword.** How do you approach a massive open problem with countless cases to consider? You start from the beginning, of course, trying to resolve either the most natural, the most interesting or the simplest yet out of reach special cases. For example, when looking at the billions and billions of stars contemplating the immense challenge of celestial cartography, you start with the *closest* (Alpha Centauri and Barnard's Star), the *brightest* (Sirius and Canopus), or the *most useful* (Polaris aka North Star), but not with the galaxy far, far away.

The same principle applies to the *Kronecker coefficients* $g(\lambda, \mu, \nu)$. Introduced by Murnaghan in 1938, they remain among the great mysteries of Algebraic Combinatorics. In part due to the fact that they lack a combinatorial interpretation, even the most basic questions present seemingly insurmountable challenges, while even the simplest examples are already hard to compute. Yet, this should not prevent us from pursuing both.

In our previous paper [PP20], we briefly surveyed the dispiriting state of art on Kronecker bounds, and identified two promising problems which are both interesting, simple looking, yet not immediately approachable with the tools previously used:

(1) give upper bounds for $g(\lambda, \mu, \nu)$, where λ, μ, ν have a *small Durfee square*,

(2) give lower bounds for the maximal $g(\lambda, \lambda, \lambda)$, where λ is *symmetric*: $\lambda = \lambda'$.

We largely resolve both problems, getting estimates up to a constant in the leading terms of the asymptotics. For the *small Durfee square problem* (1), we employ symmetric functions technology and obtain new estimates on the *Littlewood–Richardson coefficients* which are of independent interest. For the *fully symmetric Kronecker problem* (2), we use a combinatorial argument based on the *monotonicity property*. We then use this argument to derive the first nontrivial lower bounds in several explicit examples.

1.2. **Small Durfee square problem.** For a partition $\lambda \vdash n$, denote by $\ell(\lambda)$ the *length* of λ, i.e. the number of rows in the Young diagram λ. Denote by $d(\lambda)$ the *Durfee square size*, i.e. the size of the largest square which fits λ. Clearly, $d(\lambda) \leq \ell(\lambda)$.

The *Kronecker coefficients* $g(\lambda, \mu, \nu) \in \mathbb{N}$ are defined as the structure constants in the ring of characters of S_n:

$$\chi^\mu \cdot \chi^\nu = \sum_{\lambda \vdash n} g(\lambda, \mu, \nu) \, \chi^\lambda , \quad \text{where} \quad \mu, \, \nu \vdash n$$

and χ^α denotes the character of the irreducible representation (Specht module) indexed by the partition α. Note that $g(\lambda, \mu, \nu)$ are symmetric with respect to permutations of three partitions.

March 27, 2023.

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It is known that $g(\lambda,\mu,\nu) \le \min\{f^\lambda, f^\mu, f^\nu\}$, where $f^\alpha := \chi^\alpha(1)$ is the dimension of the Specht module, see [PPY19], but there are no other general bounds. On the other hand, for partitions with few rows, we have the following general upper bound:

Theorem 1.1 ([PP20]). *Let $\lambda,\mu,\nu \vdash n$, such that $\ell(\lambda)=\ell$, $\ell(\mu)=m$, and $\ell(\nu)=r$. Then:*

$$g(\lambda,\mu,\nu) \ \le \ \left(1+\frac{\ell m r}{n}\right)^{n}\left(1+\frac{n}{\ell m r}\right)^{\ell m r}.$$

In particular, we have:

Corollary 1.2 (see §2.5). *Let $\lambda,\mu,\nu \vdash n$, such that $\ell(\lambda),\ell(\mu),\ell(\nu) \le k$. Then:*

$$g(\lambda,\mu,\nu) \ \le \ n^{k^3}. \tag{1.1}$$

In other words, for partitions with fixed number of rows, the Kronecker coefficients are bounded polynomially.[1] Recently, we conjectured that the same holds for partitions with fixed Durfee square size.

Conjecture 1.3 ([PP20, Rem. 5.10]). *Fix $k\ge 1$ and let $\lambda,\mu,\nu \vdash n$, such that $d(\lambda), d(\mu), d(\nu) \le k$. Then $g(\lambda,\mu,\nu) \le n^c$ for some constant $c=c(k)>0$.*

The contingency arrays estimates we used in the proof of Theorem 1.1 are inapplicable in this case. Using symmetric functions techniques, here we prove the conjecture with an explicit constant $c(k)$.

Theorem 1.4. *Let $n,k\ge 1$, and let $\lambda,\mu,\nu \vdash n$, such that $d(\lambda), d(\mu), d(\nu) \le k$. Then:*

$$g(\lambda,\mu,\nu) \ \le \ \frac{1}{k^{8k^2}\,2^{8k^3}}\ n^{4k^3+13k^2+31k}. \tag{1.2}$$

Note that the upper bound (1.2) is slightly weaker than the upper bound in (1.1), but only by constant 4 is the leading term. In fact, Corollary 1.2 is crucial for the proof of Theorem 1.4.

To appreciate the power of the theorem, compare it with the previous bounds. For example, when $\lambda=(m+1,1^m)$ is a hook of size $n=2m+1$, so $d(\lambda)=1$, the dimension bound is exponential:

$$g(\lambda,\lambda,\lambda) \ \le \ f^\lambda \ = \ \tbinom{2m}{m} \ = \ \Theta\big(2^n/\sqrt{n}\big).$$

By contrast, Theorem 1.4 gives a polynomial upper bound: $g(\lambda,\lambda,\lambda) \le n^{40}$. In fact, it is known that $g(\lambda,\lambda,\lambda)=1$ in this special case, see e.g. [Rem89, Ros01].

Similarly, in [PP20, Prop. 5.9], we used contingency tables and an ad hoc orbit counting argument to give a weakly exponential upper bound for the case when ν is a hook and λ,μ are *double hooks*, i.e. $d(\lambda), d(\mu) \le 2$:

$$g(\lambda,\mu,\nu) \ \le \ n^{450}p(n)^{400} \ = \ e^{\Theta(\sqrt{n})},$$

where $\lambda,\mu,\nu\vdash n$ and $p(n)$ is the number of partitions of n. By contrast, Theorem 1.4 gives a polynomial upper bound: $g(\lambda,\mu,\nu) \le n^{146}/2^{96}$.[2]

In the opposite direction, let us show that the bound in (1.1) is tight up to the lower order terms in the following strong sense. Let

$$\mathrm{A}(n,k) \ := \ \max\big\{\, g(\lambda,\mu,\nu) \ : \ \lambda,\mu,\nu\vdash n \ \text{ and } \ \ell(\lambda),\ell(\mu),\ell(\nu)\le k \,\big\}.$$

Theorem 1.5. *For all $k\ge 1$, there is a constant $C_k>0$, such that*

$$\mathrm{A}(n,k) \ \ge \ C_k\, n^{k^3-3k^2-3k+3} \qquad \text{for all} \quad n\ge 1. \tag{1.3}$$

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[1] Let us mention [MT22] which appeared after the first version of this paper, and improves some of our constants from [PP17, PP20] in some special cases of partitions with bounded number of rows.

[2] The constant 146 is likely very far from optimal, and we make no effort to improve it as we are mostly interested in the asymptotic estimates.

In other words, the theorem says that for all n, there exist partitions $\lambda, \mu, \nu \vdash n$, such that $\ell(\lambda), \ell(\mu), \ell(\nu) \le k$ and we have $g(\lambda,\mu,\nu) \ge$ right-hand side of (1.3). In particular, the theorem implies that the upper bound (1.2) is tight up to a constant 4 in the leading term, cf. §6.6.

1.3. **Symmetric Kronecker problem.** [3] Let

$$\mathrm{K}(n) \,:=\, \max\left\{ g(\lambda,\mu,\nu) \ : \ \lambda,\mu,\nu \vdash n \right\}$$

denote the *maximal Kronecker coefficient*. It was shown by Stanley [Sta16, slide 44, item (d)], that[4]

$$\mathrm{K}(n) \,=\, \sqrt{n!}\, e^{-O(\sqrt{n})}. \tag{1.4}$$

Later,[5] it was shown in [PPY19] that the maximum can only occur when all three partitions have *Vershik–Kerov–Logan–Shepp shape*, and their limit curve is of course self-conjugate, see e.g. [Rom15]. It is thus natural to ask whether the following maximal Kronecker coefficients for the *symmetric* and *fully symmetric* problem have the same asymptotics as $\mathrm{K}(n)$:

$$\mathrm{K}^s(n) \,:=\, \max\left\{ g(\lambda,\lambda,\lambda) \ : \ \lambda \vdash n \right\}, \quad \text{and}$$
$$\mathrm{K}^{fs}(n) \,:=\, \max\left\{ g(\lambda,\lambda,\lambda) \ : \ \lambda \vdash n, \ \lambda = \lambda' \right\}.$$

Clearly, $\mathrm{K}^{fs}(n) \le \mathrm{K}^s(n) \le \mathrm{K}(n)$.

We showed in [PP20, §6.3], that $\mathrm{K}^s(n) = e^{\Omega(n^{2/3})}$ using an explicit construction and the asymptotics of plane partitions.[6] Although no nontrivial lower bound was known for $\mathrm{K}^{fs}(n)$, we (somewhat audaciously) stated:

Conjecture 1.6 ([PP20, Conj. 6.7])**.**

$$\log \mathrm{K}^{fs}(n) \,=\, \frac{1}{2}\, n \log n \,-\, O(n).$$

Here we give a surprisingly simple proof of a superexponential lower bound, resolving the problem up to a constant factor in the logarithm.

Theorem 1.7. *For all $\varepsilon > 0$, we have:*

$$\log \mathrm{K}^{fs}(n) \,\ge\, \frac{1}{(16\,+\,\varepsilon)}\, n \log n \,-\, O(n).$$

1.4. **Explicit constructions.** A key problem we identified in [PP20] is an explicit construction of partitions $\lambda, \mu, \nu \vdash n$ with *large* $g(\lambda,\mu,\nu)$. Here by *explicit construction* we mean an algorithm which outputs the triple (λ,μ,ν) in $\mathrm{poly}(n)$ time. As we mentioned above, in [PP20, §6.2] we gave an explicit construction of $\lambda \vdash n$, such that $\ell(\lambda) = \Theta(n^{1/3})$ and $g(\lambda,\lambda,\lambda) = e^{\Omega(n^{2/3})}$.

In the *symmetric case* of Kronecker coefficients $g(\lambda,\lambda,\lambda)$ with $\lambda = \lambda'$, until now very little was known. It was shown in [BB04], that $g(\lambda,\lambda,\lambda) \ge 1$ for all $\lambda = \lambda'$. In connection with the *Saxl conjecture*, the *staircase shape* $\rho_k = (k-1,\ldots,2,1) \vdash \binom{k}{2}$ is especially important, see [Ike15, LS17, PPV16]. Unfortunately, the best lower and upper bounds in this case remain

$$1 \,\le\, g\big(\rho_k\,,\rho_k\,,\rho_k\big) \,\le\, f^{\rho_k} \,=\, \sqrt{n!}\, e^{-O(n)}. \tag{1.5}$$

While we conjecture the asymptotics on the right is the correct estimate, we are nowhere close to proving this claim (cf. §6.8). However, we are able to obtain lower bounds for the other two shapes considered in [PPV16] in connection with the Saxl conjecture.

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[3] Here "symmetric" refers to the problem, not the coefficient. The *symmetric Kronecker coefficients* are defined in [BCI11], and not studied in this paper.

[4] Here and throughout the paper we use $f = g\,e^{-O(n^\alpha)}$ to mean that there is a universal constant $c > 0$ such that $f \ge g\,e^{-cn^\alpha}$ for all $n \ge 1$. The notation $f = g - O(n^\alpha)$ is defined analogously.

[5] Formula (1.4) is stated in [Sta16] in a different, slightly weaker form: $\log \mathrm{K}(n) \sim \frac{1}{2}\, n \log n$. However, the one-sentence proof Stanley gives in fact implies (1.4) as stated. We expound on the connection in [PPY19, §3.2].

[6] The statement of Theorem 1.3 in [PP20] has an error: we prove the lower bound for $\mathrm{K}^s(n)$, not $\mathrm{K}^{fs}(n)$ as claimed in the theorem.

Theorem 1.8. *Let* $r \geq 1$, $k = 2^{2r+1}$, $n = k^2$ *and let* $\delta_k := (k^k) \vdash n$ *be the square shape. Then:*

$$g\big(\delta_k, \delta_k, \delta_k\big) \geq e^{\Omega(n^{1/4})}.$$

Similarly, let $r \geq 1$, $k = 2^{2r} - 1$, $n = 3k^2 + 1$, *and let*

$$\tau_k := \big(3k-1, 3k-3, \ldots, k+3, (k+1)^2, (k-1)^2, \ldots, 2^2, 1^2\big) \vdash n$$

be the caret shape. Then:

$$g\big(\tau_k, \tau_k, \tau_k\big) \geq e^{\Omega(n^{1/4})}.$$

Although both lower bounds are rather weak compared to what we believe to be the correct asymptotics (see §6.7), these are the first nontrivial bounds we obtain in this case. Note that they are weaker than the bound $g(\lambda, \lambda, \lambda) \geq e^{\Omega(n^{2/3})}$ from our earlier explicit construction.

1.5. **Structure of the paper.** We start with a brief Section 2 with definitions and some background. In the next Section 3 we give estimates for the Kostka and Littlewood–Richardson coefficients. We then proceed to obtain bounds on the Kronecker coefficients and prove Theorems 1.4 and 1.5 in Section 4. We then prove Theorem 1.7 and give explicit constructions (Theorem 1.8) in Section 5. In Section 6, we conclude with final remarks, conjectures and open problems.

2. Definitions and basic results

We assume the reader is familiar with the notation and standard results in the literature, see e.g. [Mac95] and [Sta99, §7]. In this section we recall several useful basic results to help the reader navigate through the paper.

2.1. **Partitions.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a *partition* of *size* $n := |\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 1$. We write $\lambda \vdash n$ in this case. As in the introduction, let $\ell(\lambda) := \ell$ be the *length* of λ, and let $d(\lambda) = \max\{k \,:\, \lambda_k \geq k\}$ denote the *Durfee square size*.

Denote by $p(n)$ the number of partitions $\lambda \vdash n$. Let λ' denote the *conjugate partition* of λ. Let $\lambda + \mu$ denote the *sum of partitions*: $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$. Similarly, let $\lambda \cup \mu$ denotes the *union of partitions* defined as $\lambda \cup \mu := (\lambda' + \mu')'$. We write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i \geq 1$. The *skew shape* λ/μ is the difference of two straight shapes (Young diagrams).

We use $(a^b) = (a, \ldots, a)$, b times, to denote the *rectangular shape*, and $\rho_\ell = (\ell - 1, \ldots, 2, 1)$ denotes the *staircase shape*. Other special partitions include the *hooks shape* $(k, 1^{n-k})$ and the *two-row shape* $(n-k, k)$.

2.2. **Basic inequalities.** Throughout the paper we will use several basic inequalities allowing us to reach the polynomial bounds. By AM–GM we will refer to the Arithmetic vs Geometric mean inequality

$$\frac{\sum_{i=1}^m x_i}{m} \geq \left(\prod_{i=1}^m x_i\right)^{1/m}$$

for nonnegative real numbers x_i. In particular, applying this with $x_i = N + 1 - i$ gives

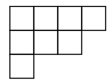
$$k!\binom{N}{k} = \prod_{i=1}^k (N+1-i) \leq \left(\frac{\sum_{I=1}^k N+1-i}{k}\right)^k = \left(N - \frac{k-1}{2}\right)^k \tag{2.1}$$

We will also use the fact that $\big(1 + \frac{x}{n}\big)^n$ is an increasing sequence in n with limit e^x.

We will also use the log-concavity of binomial coefficients, namely

$$\binom{x}{k}^2 \geq \binom{x-1}{k}\binom{x+1}{k} \geq \cdots \geq \binom{x-r}{k}\binom{x+r}{k}. \tag{2.2}$$

2.3. **Partition inequalities.** We draw partitions as Young diagrams in the English notation. For example, we use



for $\lambda = (4,3,1)$. We will use several bounds on the number of partitions which can be easily seen from this graphical representation.

The Young diagram of a partition can be determined by its boundary, which is a North-East monotone lattice path. If the partition has length k and at most n boxes, then, after removing its first column, it fits in an $k \times (n-k)$ rectangle and we can bound it by the number of lattice paths as

$$\#\{\lambda \vdash n \,:\, \ell(\lambda) = k\} \;\le\; \binom{n}{k}. \tag{2.3}$$

If $\lambda \vdash n$ and $\ell(\lambda) \le k$, then the number of partitions can be bounded by the (unsorted) number of weak compositions of n into k parts, given by $\binom{n+k-1}{k-1}$ and so

$$\#\{\lambda \vdash n \,:\, \ell(\lambda) \le k\} \;\le\; \binom{n+k-1}{k-1} \;=\; O(n^{k-1}), \tag{2.4}$$

where the last equality holds when k is fixed and $n \to \infty$.

For a partition $\gamma \subset \lambda$ with $\ell(\lambda) = k$ and $\lambda \vdash n$, then γ can be viewed as a lattice path inside the $k \times (n-k+1)$ rectangle bounding λ. This gives:

$$\#\{\gamma \,:\, \gamma \subseteq \lambda\} \;\le\; \binom{n-k+1}{k}.$$

For $\lambda \vdash n$ and $d(\lambda) \le k$, we have then:

$$\#\{\gamma \,:\, \gamma \subseteq \lambda\} \;\le\; (n/2)^{2k}. \tag{2.5}$$

This follows by considering γ as a NE lattice path from the lower left corner of λ at $(0, -\ell(\lambda))$, through a point on the diagonal $(i, -i)$ with $i \in [1, k]$, to $(\lambda_1, 0)$. For $\ell(\lambda) = \ell$, then $\lambda_1 \le n + 1 - \ell$, and the total lattice path count gives

$$\sum_{i=1}^{k} \binom{\ell}{i}\binom{n+1-\ell}{i} \;\le\; \sum_{i=1}^{k} \binom{\lfloor n/2 \rfloor}{i}\binom{\lceil n/2 \rceil}{i} \;\le\; \sum_{i=1}^{k} \frac{(n/2)^{2i}}{(i!)^2} \;\le\; (n/2)^{2k}.$$

Here the first inequality follows from (2.2), and the last inequality is easily seen by induction on k.

2.4. **Kronecker coefficients.** Recall an equivalent definition of *Kronecker coefficients*:

$$g(\lambda,\mu,\nu) \;=\; \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma).$$

From here it is easy to see both the *symmetry* and the *conjugation* properties:

$$g(\lambda,\mu,\nu) \;=\; g(\mu,\lambda,\nu) \;=\; g(\lambda,\nu,\mu) \;=\; \ldots \qquad \text{and} \qquad g(\lambda,\mu,\nu) \;=\; g(\lambda',\mu',\nu). \tag{2.6}$$

We will use the following lesser known *monotonicity property*, which is an extension of the *semigroup property*, see [CHM07].

Theorem 2.1 ([Man15])**.** *Suppose $\alpha,\beta,\gamma \vdash m$, such that $g(\alpha,\beta,\gamma) > 0$. Then for all $\lambda,\mu,\nu \vdash m$, we have $g(\lambda+\alpha,\mu+\beta,\nu+\gamma) \ge g(\lambda,\mu,\nu)$.*

2.5. **Dimension bound.** As in the introduction, denote $f^\lambda = \chi^\lambda(1)$, which is also the number of Standard Young Tableaux of shape λ. We make frequent use of the *dimension bound* $g(\lambda,\mu,\nu) \le f^\lambda \le \sqrt{n!}$, see e.g. [PPY19]. For example, we have:

Proof of Corollary 1.2. If $k^3 \ge n$, then the result follows from the dimension bound: $g(\lambda,\mu,\nu) \le n! \le n^{k^3}$. If $k^3 \le n$ and $k \ge 2$, it follows from (1.1), that

$$g(\lambda,\mu,\nu) \le \left(1+\tfrac{k^3}{n}\right)^n \left(1+\tfrac{n}{k^3}\right)^{k^3} \le n^{k^3}.$$

The last inequality is a consequence of the fact that if $2e \le a \le b$ then

$$\left(1+\frac{a}{b}\right)^b \left(1+\frac{b}{a}\right)^a \le e^a \left(1+\frac{b}{a}\right)^a \le b^a.$$

Finally, if $k=1$, then all three partitions are hooks, and the result follows from [Ros01]. $\square$

2.6. **Symmetric functions.** Recall the *homogenous symmetric functions* h_λ, *elementary symmetric functions* e_λ, *monomial symmetric functions* m_λ and *Schur functions* s_λ. The *Kostka numbers* can be defined as

$$h_\alpha = \sum_{\lambda \vdash n} K_{\lambda,\alpha}\, s_\lambda \quad \text{for all} \quad \alpha \vdash n. \tag{2.7}$$

More generally, for skew shapes λ/μ we have *skew Kostka numbers*

$$s_\mu \cdot h_\alpha = \sum_{\lambda \vdash |\mu|+|\alpha|} K_{\lambda/\mu,\alpha}\, s_\lambda\,.$$

The Kronecker coefficients can be equivalently defined as follows:

$$s_\lambda[xy] := \sum_{\mu,\nu \vdash n} g(\lambda,\mu,\nu)\, s_\mu(x)\, s_\nu(y) \quad \text{for all} \quad \lambda \vdash n, \tag{2.8}$$

where $[xy] := (x_1y_1, x_1y_2, \ldots, x_iy_j, \ldots)$ denote all pairwise products of variables. The last identity can be written as the *tripple Cauchy identity* given by

$$\sum_{\lambda,\mu,\nu} g(\lambda,\mu,\nu) s_\lambda(x) s_\mu(y) s_\nu(z) = \prod_{i,j,k} \frac{1}{1-x_iy_jz_k}. \tag{2.9}$$

We will also need *Littlewood's identity* [Lit58]:

$$s_\lambda * (s_\alpha s_\beta) = \sum_{\theta \vdash |\alpha|, \eta \vdash |\beta|} c^\lambda_{\theta\eta} \big(s_\alpha * s_\theta\big)\big(s_\beta * s_\eta\big) \tag{2.10}$$

where $c^\lambda_{\mu\nu}$ denote the *Littlewood–Richardson coefficients*, and "$*$" denotes the *Kronecker product of symmetric functions*:

$$s_\mu \cdot s_\nu = \sum_{\lambda \vdash |\mu|+|\nu|} c^\lambda_{\mu\nu}\, s_\lambda \quad \text{and} \quad s_\mu * s_\nu = \sum_{\lambda \vdash n} g(\lambda,\mu,\nu)\, s_\lambda\,, \quad \text{for all} \quad \mu,\nu \vdash n.$$

3. Bounds on Kostka numbers and Littlewood–Richardson coefficients

In this section, we obtain bounds on the Littlewood–Richardson coefficients in terms of Durfee square size of partitions. We begin with the *skew Kostka numbers*:

Lemma 3.1. *Let λ/μ be a skew shape, $|\lambda/\mu| = m$, and let $\alpha \vdash m$. Suppose $d(\lambda) \le k$ and $\ell(\alpha) \le r$. Then:*

$$K_{\lambda/\mu,\alpha} \le 2^{2r} \left(\frac{m}{r} + \frac{k}{2}\right)^{r(k-1)}.$$

Proof. Recall the *Pieri rule* for $s_\mu \cdot h_a$. We have:

$$K_{\lambda/\mu,\alpha} \ = \ \#\{\mu \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(r)} \ = \ \lambda\},$$

where the set has sequences of partitions $\lambda^{(1)},\ldots,\lambda^{(r)}$, such that $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip of size α_i, which can be zero if $\ell(\alpha) < r$. Such a strip can have at most k rows of total size k, which are below the diagonal in our English notation, and then at most k rows of total size at most α_i. The first number is bounded by 2^k and the second number is bounded by the number of weak compositions of α_i into k parts, i.e. $\binom{\alpha_i+k-1}{k-1}$.

Overall, we have:

$$K_{\lambda/\mu,\alpha} \ \le \ 2^{rk} \prod_{i=1}^{r} \binom{\alpha_i + k - 1}{k-1} \ \le \ \frac{2^{rk}}{((k-1)!)^r} \prod_{i=1}^{r} \left[\,(\alpha_i+1)\cdots(\alpha_i+k-1)\right].$$

Applying the AM–GM inequality as (2.1) to each product term at the end we get the bound $\left(\alpha_i + \frac{k}{2}\right)^r$. Another application of AM–GM gives

$$\prod_{i=1}^{r}\left(\alpha_i + \frac{k}{2}\right) \le \left(\frac{rk/2 + \sum_i \alpha_i}{r}\right)^r = \left(\frac{m}{r} + \frac{k}{2}\right)^{r(k-1)}$$

for the big product. Since $2^{k-2} \le (k-1)!$, the first factor is bounded by 2^{2r}. Putting it all together we obtain the result. $\square$

Lemma 3.2. *Let $\lambda \vdash n$, $\mu \vdash n-m$ and $\nu \vdash m$, such that $\ell(\lambda) \le k$. Then:*

$$c^{\lambda}_{\mu\nu} \ \le \ \left(\frac{2m}{k} \ + \ \frac{k+1}{3}\right)^{\binom{k}{2}}$$

Proof. Recall that $c^{\lambda}_{\mu\nu}$ is equal to the number of Littlewood–Richardson tableaux of shape λ/μ and type ν. These are characterized by having only 1's in the first row, only 1's and 2's in the second row, etc. Thus, we have:

$$c^{\lambda}_{\mu\nu} \ \le \ \tbinom{\lambda_2-\mu_2+1}{1}\tbinom{\lambda_3-\mu_3+2}{2}\cdots\tbinom{\lambda_k-\mu_k+k-1}{k-1}.$$

By the AM–GM applied to each product term in the numerators of the binomial coefficients, the right-hand side is bounded by

$$\frac{1}{\prod_{i=1}^{k-1} i!}\left(\frac{\sum_{i=2}^{k}(\lambda_i-\mu_i)(i-1) + \binom{k+1}{3}}{\binom{k}{2}}\right)^{\binom{k}{2}} \ \le \ \left(\frac{2m}{k} \ + \ \frac{k+1}{3}\right)^{\binom{k}{2}},$$

which completes the proof. $\square$

Remark 3.3. Lemma 3.2 is slightly sharper than the upper bound in Theorem 4.14 of [PPY19]. Both upper bounds are the same asymptotically when the length k is fixed and $n \to \infty$, and match the lower bound in the same theorem. The proof of the lemma is concise and very different from that in [PPY19], so we included it for completeness.

Lemma 3.4. *Let $\lambda \vdash n$ such that $d(\lambda) \le k$. Then for every $\mu,\nu \subseteq \lambda$ with $|\mu|+|\nu| = n$, we have:*

$$c^{\lambda}_{\mu\nu} \ \le \ \left(\frac{n}{k} \ + \ k\right)^{2k^2}.$$

Proof. Let $\nu = \alpha \cup \beta'$, where $\alpha = (\nu_1,\ldots,\nu_k)$ and $\beta' = (\nu_{k+1},\ldots)$. Since $d(\nu) \le d(\lambda) \le k$, we have $\ell(\beta) \le k$. Then we have:

$$c^{\lambda}_{\mu\nu} \ = \ \langle s_\lambda, s_\mu s_\nu\rangle \ \le \ \langle s_\lambda, s_\mu h_\alpha e_\beta\rangle.$$

By definition, we have

$$s_\mu\, h_\alpha \ = \ \sum_{\gamma} K_{\gamma/\mu,\alpha}\, s_\gamma$$

Multiplying by e_β, expanding in the Schur basis and extracting the s_λ term, we get:

$$\langle s_\lambda, s_\mu h_\alpha e_\beta\rangle \, = \, \sum_\gamma K_{\gamma/\mu,\alpha}\, K_{\lambda'/\gamma',\beta}\,.$$

Applying the bounds from Lemma 3.1 and using the fact that $\ell(\alpha), \ell(\beta) \le k$, we obtain

$$c^\lambda_{\mu\nu} \, \le \, \sum_{\gamma\vdash m} 2^{4k} \left(\frac{m}{k} \, + \, \frac{k}{2}\right)^{k(k-1)} \left(\frac{n-m}{k} \, + \, \frac{k}{2}\right)^{k(k-1)}, \quad \text{where} \quad m := |\alpha|.$$

Observe that the number of partitions $\gamma \subset \lambda$ is bounded above by $(n/2)^{2k}$ from inequality (2.5). Note also that, by the AM–GM inequality, we have:

$$\left(\frac{m}{k} \, + \, \frac{k}{2}\right)\left(\frac{n-m}{k} \, + \, \frac{k}{2}\right) \, \le \, \frac{1}{4}\left(\frac{n}{k} \, + \, k\right)^2.$$

Putting everything together, after some cancellations we obtain:

$$c^\lambda_{\mu\nu} \, \le \, (n/2)^{2k}\, \frac{2^{4k}}{4^k}\, \left(\frac{n}{k} + k\right)^{2k(k-1)} \, \le \, \left(\frac{n}{k} + k\right)^{2k^2},$$

as desired. □

4. Bounds on Kronecker coefficients

In this section we prove Theorem 1.4 and Theorem 1.5, main results of the paper.

4.1. **Upper bounds.** We approach the problem gradually, starting from the case where two partitions have bounded number of rows (Lemma 4.2), then just one (Lemma 4.3), and then in full generality.

Before we proceed with those, we need to add a Lemma complementing the results in [PP20].

Lemma 4.1. *Let $\lambda, \mu, \nu \vdash n$ and $\ell(\lambda) \le a$, $\ell(\mu) \le b$, $\ell(\nu) \le c$. Then $g(\lambda, \mu, \nu') \le 2^{abc}$.*

Proof. From the triple Cauchy identity (2.9), applying the involution ω on the symmetric functions in z we get that $\omega(s_\nu(z)) = s_{\nu'}(z)$ on the left-hand side, so

$$\sum_{\lambda,\mu,\nu} g(\lambda, \mu, \nu') s_\lambda(x) s_\mu(y) s_\nu(z) = \prod_{i,j,k} (1 + x_i y_j z_k).$$

Expanding both sides in monomials in x, y, z and taking the coefficient at $x^\lambda y^\mu z^\nu$ on both sides we get

$$g(\lambda, \mu, \nu') \le [x^\lambda y^\mu z^\nu] \prod_{i,j,k=1}^{a,b,c} (1 + x_i y_j z_k)$$

as only the variables $x_1, \ldots, x_a, y_1, \ldots, y_b, z_1, \ldots, z_c$ can appear. The right-hand side is the generating function of $a \times b \times c$ binary arrays B, where for each coordinate (i, j, k) we have a multiplicative weight $(x_i y_j z_k)^{B_{ijk}}$. Thus, the coefficient on the right-hand side above is bounded by the total number of such binary arrays, which is 2^{abc}. □

Lemma 4.2. *Let $\lambda, \mu, \nu \vdash n$, such that $\ell(\lambda), \ell(\mu) \le k$ and $d(\nu) \le k$. Then*

$$g(\lambda, \mu, \nu) \, \le \, 2^{k^3} n^{k^3+k^2+3k}.$$

Proof. Let $\nu = \alpha \cup \beta'$, where $\alpha = (\nu_1, \ldots, \nu_k)$ and $\beta' = (\nu_{k+1}, \ldots)$. Again, since $d(\nu) \le d(\lambda) = k$, we have $\ell(\beta) \le k$. Let $m = |\alpha|$, so $n - m = |\beta|$

Since $s_\alpha s_\beta = s_\nu + \cdots$ is a Schur positive sum containing s_ν, we have:

$$g(\lambda, \mu, \nu) \, = \, \langle s_\lambda * s_\nu, s_\nu\rangle \, \le \, \big\langle s_\lambda * (s_\alpha s_{\beta'}), s_\mu\big\rangle. \tag{4.1}$$

Applying Littlewood's identity (2.10), we get:

$$\big\langle s_\lambda * (s_\alpha s_{\beta'}), s_\mu\big\rangle \, = \, \sum_{\theta,\eta,\xi,\gamma} c^\lambda_{\theta\eta}\, c^\mu_{\gamma\xi}\, g(\theta, \alpha, \gamma)\, g(\eta, \beta', \xi)\,. \tag{4.2}$$

Since $c^{\mu}_{\gamma\xi} > 0$ only if $\gamma, \xi \subset \mu$, and similarly $c^{\lambda}_{\theta,\eta} > 0$ only if $\theta, \eta \subset \lambda$, it follows that all partitions in the right-hand side above have length at most k.

We now apply previous results to estimate the right-hand side of (4.2). By Corollary 1.2, we have:

$$g(\theta, \alpha, \gamma) \, \le \, m^{k^3}.$$

For the term $g(\eta, \beta', \xi)$, note that we have $\ell(\eta), \ell(\xi) \le k$. Furthermore, if $g(\eta, \beta', \xi) \ne 0$, then by [Reg80], we must have $\ell(\beta') \le k^2$. Since we also have $\ell(\beta) = \nu_{k+1} \le k$, we can apply Lemma 4.1 and see that $g(\eta, \beta', \xi) \le 2^{k^3}$.

Applying Lemma 3.2, we also have upper bounds for the Littlewood–Richardson coefficients involved. Indeed, denote by $r := \min\{n-m, m\}$. Then for the Littlewood–Richardson coefficients in (4.2), we have:

$$c^{\lambda}_{\theta\eta}\,,\, c^{\mu}_{\gamma\xi} \;\le\; \left(\frac{2r}{k} \,+\, \frac{k+1}{3}\right)^{\binom{k}{2}}.$$

Therefore, equations (4.1) and (4.2) give

$$g(\lambda, \mu, \nu) \;\le\; \sum_{\theta,\eta,\xi,\gamma} \left(\frac{2r}{k} \,+\, \frac{k+1}{3}\right)^{2\binom{k}{2}} 2^{k^3}\, m^{k^3}$$

The sum above is over $\theta, \eta \subset \lambda$ of sizes m, $n-m$, and over $\xi, \gamma \subset \mu$ of sizes m, $n-m$, respectively. The number of such pairs can be bound following (2.4) by $\binom{m+k-1}{k-1}\binom{n-m+k-1}{k-1} \le_{(2.2)} \binom{n/2+k-1}{k-1}^2$ will suffice. We conclude:

$$g(\lambda, \mu, \nu) \;\le\; \binom{n/2+k-1}{k-1}^4 \left(\frac{2r}{k} \,+\, \frac{k+1}{3}\right)^{2\binom{k}{2}} 2^{k^3} m^{k^3} \;\le\; 2^{k^3} n^{k^3+k^2+3k} \;\le\; C_k\, n^{(k+1)^3},$$

as desired. □

Lemma 4.3. *Let $\lambda, \mu, \nu \vdash n$, such that $d(\mu), d(\nu) \le k$ and $\ell(\lambda) \le k$. Then*

$$g(\lambda, \mu, \nu) \;\le\; \frac{1}{k^{2k^2}}\, n^{2k^3 + \frac{9}{2}k^2 + \frac{19}{2}k}.$$

Proof. As before, let $\mu = \alpha \cup \beta'$, where $\alpha = (\mu_1, \ldots, \mu_k)$ and $\beta = (\mu_{k+1}, \ldots)'$. Let $m = |\alpha|$, $n - m = |\beta|$, and let $r := \min\{m, n-m\}$. Since $s_\alpha s_\beta = s_\mu + \cdots$ is a Schur positive sum containing s_μ, we have:

$$g(\lambda, \mu, \nu) \,=\, \langle s_\lambda * s_\mu, s_\nu \rangle \;\le\; \big\langle s_\lambda * (s_\alpha\, s_{\beta'}), s_\nu \big\rangle. \tag{4.3}$$

Applying Littlewood's identity (2.10), we get

$$\big\langle s_\lambda * (s_\alpha\, s_{\beta'}), s_\nu \big\rangle \;=\; \sum_{\theta,\eta,\gamma,\xi} c^{\lambda}_{\theta\eta}\, c^{\nu}_{\gamma\xi}\, g(\theta, \alpha, \gamma)\, g(\eta, \beta', \xi). \tag{4.4}$$

We will bound the terms in the right-hand side of (4.4). For partitions θ, η such that $c^{\lambda}_{\theta,\eta} > 0$ we must have $\theta, \eta \subset \lambda$, and so $\ell(\theta), \ell(\eta) \le k$. By Lemma 3.2, we thus have:

$$c^{\lambda}_{\theta\eta} \;\le\; \left(\frac{2r}{k} + \frac{k+1}{3}\right)^{\binom{k}{2}}.$$

On the other hand, since we only select partitions γ, ξ, for which $c^{\nu}_{\gamma\xi} > 0$, then we must have $\gamma, \xi \subset \nu$, and so $d(\gamma), d(\xi) \le k$. By Lemma 3.4, we thus have:

$$c^{\nu}_{\gamma\xi} \;\le\; \left(\frac{n}{k} + k\right)^{2k^2}.$$

For the Kronecker coefficients in the summation, by Lemma 4.2, we have:

$$g(\theta, \alpha, \gamma) \;\le\; 2^{k^3} m^{k^3+k^2+3k}.$$

Similarly, we have:

$$g(\eta, \beta', \xi) \;=\; g(\eta, \beta, \xi') \;\le\; 2^{k^3} (n-m)^{k^3+k^2+3k}.$$

Now, the summation in the right-hand side of (4.4), we bound the number of pairs of partitions θ, η following inequalities (2.3) and (2.2) by $\binom{n/2+k-1}{k-1}^2$, and the number of partitions γ, ξ by $(n/2)^{2k}$ from (2.5).

Combining (4.3), (4.4) and the upper bounds above, we conclude:

$$\begin{aligned} g(\lambda,\mu,\nu) \ &\leq \ \binom{n/2+k-1}{k-1}^2 \left(\frac{n}{2}\right)^{4k} 2^{2k^3} \left(\frac{n}{k}+k\right)^{2k^2} \left(\frac{2r}{k}+\frac{k+1}{3}\right)^{\binom{k}{2}} m^{k^3+k^2+3k}(n-m)^{k^3+k^2+3k} \\ &\leq \ \frac{1}{k^{2k^2}}\, n^{2k^3+2k^2+6k+6k+2k^2+\binom{k}{2}} \ = \ \frac{1}{k^{2k^2}}\, n^{2k^3+\frac{9}{2}k^2+\frac{23}{2}k}, \end{aligned}$$

where the constant factors involving k are altogether bounded by k^{-2k^2}. □

Proof of Theorem 1.4. We use the same setup as in the proofs of Lemma 4.3, where $\mu = \alpha \cup \beta'$ and $m = |\alpha|$. We have:

$$g(\lambda,\mu,\nu) \ \leq \ \left\langle s_\lambda * (s_\alpha s_{\beta'}), s_\nu \right\rangle \ = \ \sum_{\theta,\eta,\gamma,\xi} c^\lambda_{\theta\eta}\, c^\nu_{\gamma\xi}\, g(\theta,\alpha,\gamma)\, g(\eta,\beta',\xi).$$

Again, we must have $d(\theta)$, $d(\eta)$, $d(\gamma)$, $d(\xi) \leq k$. Thus, we can apply the upper bounds on the Kronecker coefficients from Lemma 4.3, and on the Littlewood–Richardson coefficients from Lemma 3.4. Bounding the number of partitions $\theta, \eta, \gamma, \xi$ from (2.5) by $(n/2)^{2k}$, we obtain

$$\begin{aligned} g(\lambda,\mu,\nu) \ &\leq \ \sum_{\theta,\eta,\gamma,\xi} \left(\frac{n}{k}+k\right)^{4k^2} \frac{1}{k^{4k^2}}\, m^{2k^3+\frac{9}{2}k^2+\frac{23}{2}k}\,(n-m)^{2k^3+\frac{9}{2}k^2+\frac{23}{2}k} \\ &\leq \ \left(\frac{n}{2}\right)^{8k} \frac{1}{k^{4k^2}} \left(\frac{n}{k}+k\right)^{4k^2} m^{2k^3+\frac{9}{2}k^2+\frac{23}{2}k}(n-m)^{2k^3+\frac{9}{2}k^2+\frac{23}{2}k} \\ &\leq \ \frac{1}{k^{8k^2}} \left(\frac{n}{2}\right)^{8k} \left(\frac{n}{2}\right)^{4k^3+9k^2+23k} n^{4k^2} \ = \ \frac{1}{k^{8k^2}\,2^{8k^3}}\, n^{4k^3+13k^2+31k}, \end{aligned}$$

which completes the proof. □

4.2. **Proof of Theorem 1.5.** Let $n = ak$. Combining (2.7) and (2.8), we have the following identity:

$$h_{a^k}[xy] \ = \sum_{\lambda\vdash n,\ \ell(\lambda)\leq k} K_{\lambda,a^k}\, s_\lambda[xy] \ = \sum_{\lambda,\mu,\nu\vdash n} K_{\lambda,a^k}\, g(\lambda,\mu,\nu)\, s_\mu(x)\, s_\nu(y)\,.$$

Let $x = (x_1,\ldots,x_k)$ and $y = (y_1,\ldots,y_k)$, so all partitions in the above identity have lengths bounded by k. Compare the coefficients at $m_{a^k}(x)\cdot m_{a^k}(y)$ on both sides, where m_α are *monomial symmetric functions*. We then have:

$$[x_1^a\ldots x_k^a\, y_1^a\ldots y_k^a]\, h_{a^k}[xy] \ = \sum_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu)\, K_{\lambda,a^k}\, K_{\mu,a^k}\, K_{\nu,a^k}\,. \tag{4.5}$$

Consider the term on the left-hand side. We have that

$$h_a[xy] \ = \ \sum_M \prod_{i,j} (x_i y_j)^{M_{ij}} \ = \ \sum_M x^{row(M)}\, y^{col(M)}$$

is the generating function for *contingency tables* $M = (M_{ij})$ with respect to their row and column sums. Since $h_{a^k} = (h_a)^k$, we conclude that the coefficients at $x_1^a\cdots x_k^a\, y_1^a\cdots y_k^a$ are equal to the number of *3-dim contingency arrays* A with all *2-dim marginals* equal to a. We refer to [PP20] for precise definitions and further details.

Geometrically, these contingency arrays A are integer points in a three-way *transportation polytope* $T_k(m) \subset \mathbb{R}^{k^3}$ such that $\dim T_k(a) = (k^3-3k)$. By the Ehrhart theory for rational polytopes (see e.g. [BR07, §3.7]), the number of such points is given by a quasipolynomial in a of degree (k^3-3k). Thus there exists a constant $G_k > 0$ (see also §6.4), such that

$$\left[x_1^a\ldots x_k^a\, y_1^a\ldots y_k^a\right] h_{a^k}[xy] \ \geq \ G_k\, a^{k^3-3k}. \tag{4.6}$$

On the other hand, in (4.5) we have $K_{\lambda,a^k} \le a^{k^2-k}$ by Lemma 3.1, and a similar bound for the other Kostka numbers. We conclude:

$$\begin{aligned}\sum_{\lambda,\mu,\nu\in\mathcal{P}_k(n)} g(\lambda,\mu,\nu)\, K_{\lambda,a^k}\, K_{\mu,a^k}\, K_{\nu,a^k} \ &\le\ a^{3k^2-3k} \sum_{\lambda,\mu,\nu\in\mathcal{P}_k(n)} g(\lambda,\mu,\nu) \\ &\le\ \bigl|\mathcal{P}_k(n)\bigr|^3 a^{3k^2-3k} \max_{\lambda,\mu,\nu\in\mathcal{P}_k(n)} g(\lambda,\mu,\nu) \ \le\ a^{3k^2-3} \max_{\lambda,\mu,\nu\in\mathcal{P}_k(n)} g(\lambda,\mu,\nu),\end{aligned} \tag{4.7}$$

where $\mathcal{P}_k(n) = \bigl\{\lambda\vdash n \,:\, \ell(\lambda)\le k\bigr\}$, so that $|\mathcal{P}_k(n)| = O\bigl(n^{k-1}\bigr)$. Comparing the inequalities from (4.6) and (4.7), we obtain

$$\max_{\lambda,\mu,\nu\in\mathcal{P}_k(n)} g(\lambda,\mu,\nu) \ \ge\ G_k\, a^{k^3-3k-3k^2+3},$$

as desired. □

5. Kronecker bounds via the monotonicity property

5.1. **Bounds for the symmetric Kronecker problem.** For all $n,k\ge 1$, define

$$\begin{aligned}\mathrm{A}^s(n,k) \ &:=\ \max\bigl\{g(\lambda,\lambda,\lambda) \ :\ \lambda\vdash n,\ \ell(\lambda)\le k\bigr\}, \quad \text{and}\\ \mathrm{B}^{fs}(n,k) \ &:=\ \max\bigl\{g(\lambda,\lambda,\lambda) \ :\ \lambda\vdash n,\ \lambda=\lambda',\ d(\lambda)\le k\bigr\}.\end{aligned}$$

Clearly, $\mathrm{A}^s(n,k)\le \mathrm{K}^s(n)$ and $\mathrm{B}^{fs}(n,k)\le \mathrm{K}^{fs}(n)$.

Lemma 5.1. *For all $n\ge 1$, we have:*

$$\mathrm{K}^s(3n) \ \ge\ \mathrm{K}(n) \quad \textit{and} \quad \mathrm{A}^s(3n,k)\ge \mathrm{A}(n,k)\,. \tag{5.1}$$

Proof. Let $g(\alpha,\beta,\gamma)=\mathrm{K}(n)$, for some $\alpha,\beta,\gamma\vdash n$. Let $\lambda:=(\alpha+\beta+\gamma)\vdash 3n$. By the symmetry property (2.6) and monotonicity property (Theorem 2.1), we have:

$$\begin{aligned}\mathrm{K}^s(3n) \ &\ge\ g(\lambda,\lambda,\lambda) \ =\ g\bigl(\alpha+\beta+\gamma,\ \beta+\gamma+\alpha,\ \gamma+\alpha+\beta\bigr)\\ &\ge\ \max\bigl\{g(\alpha,\beta,\gamma),\ g(\beta+\gamma,\ \gamma+\alpha,\ \alpha+\beta)\bigr\}\\ &\ge\ \max\bigl\{g(\alpha,\beta,\gamma),\ g(\beta,\gamma,\alpha),\ g(\gamma,\alpha,\beta)\bigr\} \ =\ \mathrm{K}(n).\end{aligned}$$

This proves the first inequality in (5.1). The second inequality follows verbatim the argument above and the fact that $\ell(\lambda)=\max\{\ell(\alpha),\ell(\beta),\ell(\gamma)\}$. □

Corollary 5.2. *For all $k\ge 1$, there is a constant $C_k>0$, such that*

$$\mathrm{A}^s(n,k) \ \ge\ C_k\, n^{k^3-3k^2-3k+3} \qquad \textit{for all} \quad n\ge 1\,. \tag{5.2}$$

Proof. Combining Theorem 1.5 and Lemma 5.1, we obtain the result for $3\,|\,n$. For general n, note that $g(\lambda+1,\mu+1,\nu+1)\ge g(\lambda,\mu,\nu)$, again by the monotonicity property. Thus, we have $\mathrm{A}^s(n+1,k)\ge \mathrm{A}^s(n,k)$. This completes the proof. □

Lemma 5.3. *For all $n,k\ge 1$, we have:*

$$\mathrm{B}^{fs}(4n+k^2,k) \ \ge\ \mathrm{A}^s(n,k)\,. \tag{5.3}$$

Proof. Let $g(\alpha,\beta,\gamma)\ge 1$, for some $\alpha,\beta,\gamma\vdash n$ such that $\ell(\alpha),\ell(\beta),\ell(\gamma)\le k$. Recall from the introduction that $g\bigl(\delta_k,\delta_k,\delta_k\bigr)\ge 1$, where $\delta_k=(k^k)$ is the square shape. By the repeated alternating application of the monotonicity property (Theorem 2.1) and symmetry/conjugation properties (2.6), we have the following *long formula*:

$$\begin{aligned}
g(\alpha,\beta,\gamma) \ &\le\ g\big(\delta_k+\alpha,\delta_k+\beta,\delta_k+\gamma\big) \ = \ g\big(\delta_k\cup\alpha',\delta_k\cup\beta',\delta_k+\gamma\big)\\
&\le\ g\big((\delta_k+\beta)\cup\alpha',(\delta_k+\gamma)\cup\beta',\delta_k+\gamma+\alpha\big)\\
&\qquad\quad = \ g\big((\delta_k+\beta)\cup\alpha',(\delta_k+\beta)\cup\gamma',\delta_k\cup(\gamma+\alpha)'\big)\\
&\le\ g\big((\delta_k+\beta+\gamma)\cup\alpha',(\delta_k+\beta+\alpha)\cup\gamma',(\delta_k+\beta)\cup(\gamma+\alpha)'\big)\\
&\qquad\quad = \ g\big((\delta_k+\alpha)\cup(\beta+\gamma)',(\delta_k+\gamma)\cup(\beta+\alpha)',(\delta_k+\beta)\cup(\gamma+\alpha)'\big)\\
&\le\ g\big((\delta_k+\alpha+\beta)\cup(\beta+\gamma)',(\delta_k+\gamma+\alpha)\cup(\beta+\alpha)',(\delta_k+\beta+\gamma)\cup(\gamma+\alpha)'\big).
\end{aligned}$$

See Figure 5.1 below for an illustration.

Now, let $\alpha=\beta=\gamma$ and suppose $g(\alpha,\alpha,\alpha)=\mathrm{A}^s(n,k)$. From above,

$$g\big(\alpha,\alpha,\alpha\big) \ \le\ g\big((\delta_k+2\alpha)\cup(2\alpha)',(\delta_k+2\alpha)\cup(2\alpha)',(\delta_k+2\alpha)\cup(2\alpha)'\big) \ = \ g(\mu,\mu,\mu),$$

where $\mu:=\big(\delta_k+2\alpha\big)\cup(2\alpha)'\vdash 4n+k^2$. Since $\mu=\mu'$ and $d(\mu)\le k$, we conclude:

$$\mathrm{B}^{fs}(4n+k^2,k)\ \ge\ g(\mu,\mu,\mu)\ \ge\ g(\alpha,\alpha,\alpha)\ =\ \mathrm{A}^s(n,k),$$

as desired. $\square$

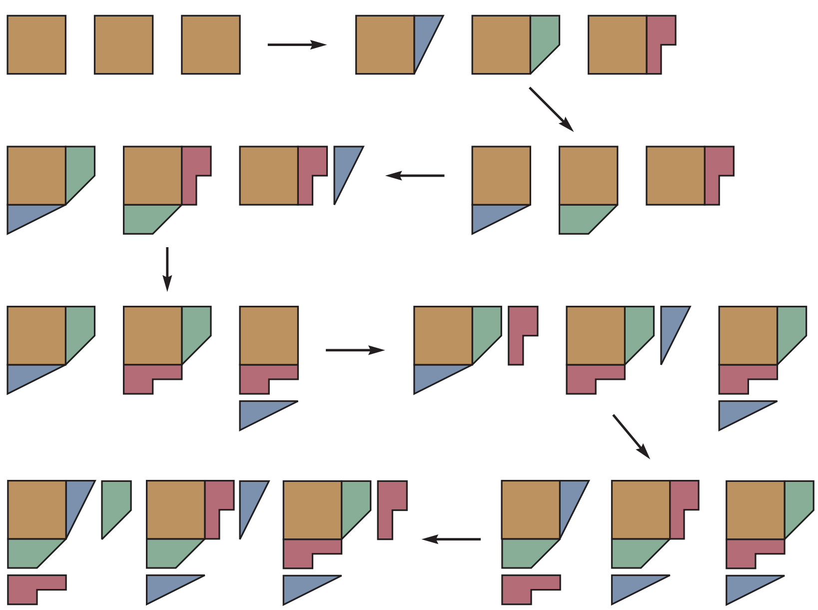


FIGURE 1. Graphic rendering of the long formula in the proof of Lemma 5.3.

Corollary 5.4. *For all $k\ge 1$, there is a constant $D_k>0$, such that*

$$\mathrm{B}^{fs}(n,k)\ \ge\ D_k\, n^{k^3-18k^2+102k-182} \qquad \textit{for all} \quad n\ge 1. \tag{5.4}$$

Proof. Combining Lemma 5.1 and Lemma 5.3, we have

$$\mathrm{B}^{fs}(12n+k^2,k)\ \ge\ \mathrm{A}(n,k). \tag{5.5}$$

Now Theorem 1.5 implies the result for $12\,|\,(n-k^2)$.

For general n, note that in the proof of Lemma 5.3, we can use

$$\mu\ :=\ \big(\delta_{k+c}+2\alpha\big)\cup(2\alpha)'\vdash 4n+(k+c)^2 \quad \text{for all} \ \ c\ge 1 \ \ \text{and} \ \ \ell(\alpha)\le k.$$

We can also replace δ_{k+c} with a chopped symmetric square by removing a symmetric partition of size $t\in\{0,1,3,4,\ldots,10,11,14\}$ from its bottom right corner. Since $g(\lambda,\lambda,\lambda)\ge 1$ for all $\lambda=\lambda'$, see [BB04, PPV16], we can then repeat the steps in the proof of Lemma 5.3 with the chopped square instead of the δ_k. This allows us to construct symmetric partitions μ of any size modulo 12, and note that $c = 5$ suffices.

We conclude that $g(\mu,\mu,\mu) \geq A(n,k)$ for some $\mu \vdash 12n+(k+c)^2-t$ with $k \leq d(\mu) \leq k+c$. This implies that $\mathrm{B}^{fs}(n,k) \geq A\bigl(\lfloor (n-k^2)/12\rfloor, k-5\bigr)$, and the bound follows. $\square$

Remark 5.5. A curious Conjecture 5.12 in [BRR17], claims that

$$g\bigl((\lambda+1)\cup 1,\, (\mu+1)\cup 1,\, (\nu+1)\cup 1\bigr) \,\geq\, g(\lambda,\mu,\nu) \quad \text{for all } \lambda,\mu,\nu \vdash n.$$

This would imply that $\mathrm{B}^{fs}(n+2,k) \geq \mathrm{B}^{fs}(n,k)$ and improve the lower order terms in (5.4).

Proof of Theorem 1.7. Fix $\varepsilon > 0$ and let $k = (2+\varepsilon)\sqrt{n}$. Let us show that for sufficiently large $n > N(\varepsilon)$, we have $\mathrm{A}(n,k) = \mathrm{K}(n)$. We follow [PPY19] in our presentation.

Recall that a sequence $\{\lambda\}$ of partitions is called *Plancherel* if $f^\lambda = \sqrt{n!}\, e^{-O(\sqrt{n})}$. Suppose that $g(\lambda,\mu,\nu) = \mathrm{K}(n)$. By Stanley's theorem (1.4) and the dimension bound, we have:

$$\sqrt{n!}\, e^{-O(\sqrt{n})} \,=\, \mathrm{K}(n) \,=\, g(\lambda,\mu,\nu) \,\leq\, f^\lambda \,\leq\, \sqrt{n!}$$

We conclude that all three sequences $\{\lambda\}$, $\{\mu\}$ and $\{\nu\}$ achieving the maximum $g(\lambda,\mu,\nu)$ are Plancherel. In fact, we can even fix two of these three sequences (see [PPY19, Thm 1.4]).

Now, by the VKLS Theorem (see [PPY19, Thm 1.3]), all three sequences must have *VKLS shape*. Without stating it explicitly, it follows from the definition that

$$\ell(\lambda),\, \ell(\mu),\, \ell(\nu) \,\leq\, 2\sqrt{n} \,+\, O(n^{1/6}) \,\leq\, (2+\varepsilon)\sqrt{n} \,=\, k\,,$$

for n large enough. Thus we have $\mathrm{A}(n,k) = \mathrm{K}(n)$ in that case. By (5.5), we conclude:

$$\mathrm{K}^{fs}\bigl((16+3\varepsilon)n\bigr) \,\geq\, \mathrm{K}^{fs}(12n+k^2) \,\geq\, \mathrm{B}^{fs}\bigl(12n+k^2,k\bigr) \,\geq\, \mathrm{A}(n,k) \,=\, \mathrm{K}(n) \,=\, \sqrt{n!}\, e^{-O(\sqrt{n})}\,,$$

for $k = (2+\varepsilon)\sqrt{n}$ as above and n large enough. Taking logs on both sides implies the result. $\square$

5.2. **Proof of Theorem 1.8.** We now use the iterated conjugation trick in the proof of Lemma 5.3 to give the first nontrivial lower bound for $g\bigl(\delta_k,\delta_k,\delta_k\bigr)$.

We start with [PP17, Thm 1.2], which gives for $k = 2s^2$ that

$$g\bigl((2s)^{2s},(2s)^{2s},k^2\bigr) \,\geq\, C\, 2^{2s}\,(2s)^{-9/2} > 0$$

for some universal constant $C > 0.004$.

Observe that by conjugating two partitions we get

$$g\bigl((2s)^{2s},(2s)^{2s},2^k\bigr) \,=\, g\bigl((2s)^{2s},(2s)^{2s},k^2\bigr) \,>\, 0.$$

Let $s := 2^r$. We can repeatedly apply the combination of monotonicity and conjugation, from $4m = k = 2s^2$, to get

$$\begin{aligned} g\bigl((4m)^{4m},\,(4m)^{4m},\,(4m)^{4m}\bigr) \,&\geq\, g\bigl((2m)^{4m},\,(2m)^{4m},\,(2m)^{4m}\bigr) \,= \\ =\, g\bigl((4m)^{2m},\,(4m)^{2m},\,(2m)^{4m}\bigr) \,&\geq\, g\bigl((2m)^{2m},\,(2m)^{2m},\,(m)^{4m}\bigr) \,\geq \\ \geq\, \ldots \,\geq\, g\bigl((2s)^{2s},(2s)^{2s},k^2\bigr) \,&>\, C\, 2^{2s}\,(2s)^{-9/2}. \end{aligned}$$

Since $s = \sqrt{2k} = \sqrt{2}\, n^{1/4}$, we obtain the first part of Theorem 1.8.

For the caret shape τ_k, note that

$$\tau_k \,=\, \bigl(\delta_{k+1} + 2\rho_k\bigr) \,\cup\, 2\rho_k\,.$$

In notation of the proof of Lemma 5.3, let $\alpha := \rho_k$ and recall from (1.5) that $g(\alpha,\alpha,\alpha) > 0$. From the long formula in the proof, we have:

$$\begin{aligned} g\bigl(\tau_k,\tau_k,\tau_k\bigr) \,&=\, g\bigl((\delta_{k+1}+2\alpha)\cup 2\alpha, (\delta_{k+1}+2\alpha)\cup 2\alpha, (\delta_{k+1}+2\alpha)\cup 2\alpha\bigr) \\ &\geq\, \max\bigl\{\, g(\alpha,\alpha,\alpha),\, g(\delta_{k+1},\delta_{k+1},\delta_{k+1})\,\bigr\}. \end{aligned}$$

Now the second part of Theorem 1.8 follows from the first part. $\square$

6. Final remarks and open problems

6.1. The importance of Durfee square size in connection with the *vanishing of Kronecker coefficients* (i.e., whether they are nonzero), has long been understood in the literature. We refer to [BB04, §3] and [Dvir93, Reg80] for some notable examples.

6.2. There are many special cases of partitions with small Durfee square size (at most three), where the Kronecker coefficients are computed exactly, see e.g. [BvWZ10, BOR09, RW94, Tew15]. In all these cases the Kronecker coefficients are bounded by a constant. This is in sharp contrast with examples in [BV18, MRS21] and our lower bound in Theorem 1.5, suggesting that being bounded is a small numbers phenomenon.

6.3. Recall Murnaghan's *stability property*: the sequence $(a_0, a_1, a_2, \ldots)$ defined as

$$a_d \,=\, a_d(\lambda,\mu,\nu) \,:=\, g\big(\lambda+(d),\, \mu+(d),\, \nu+(d)\big)$$

is increasing and bounded. This phenomenon was recently generalized by Stembridge

$$a_d \,=\, a_d(\lambda,\mu,\nu;\,\alpha,\beta,\gamma) \,:=\, g\big(\lambda+d\alpha,\, \mu+d\beta, \nu+d\gamma\big) \quad \text{for} \quad g(\alpha,\beta,\gamma)=1.$$

The nondecreasing of $\{a_d\}$ follows from the monotonicity property (Theorem 2.1), while boundedness was proved by Sam and Snowden [SS16].

It is known that a_d are a quasi-polynomial in d [Man15], see also [BV18, Man16]. In view of Corollary 1.2 and Theorem 1.5, it would be interesting to give a combinatorial description of the degree of these quasi-polynomials. Let us mention that $a_d \ge d+1$ for all $g(\alpha,\beta,\gamma)>1$ [Ste14, Prop. 3.2].

Similarly, one can consider more general families of Kronecker coefficients

$$b_d \,=\, b_d(\lambda,\mu,\nu;\,\alpha,\beta,\gamma;\,\zeta,\xi,\eta) \,:=\, g\big((\lambda+d\alpha)\cup(d\zeta)',\, (\mu+d\beta)\cup(d\xi)',\, (\nu+d\gamma)\cup(d\eta)'\big),$$

where $g(\alpha,\beta,\gamma), g(\zeta,\xi,\eta)\ge 1$. In view of Theorem 1.4, we conjecture that b_d are again quasi-polynomial in d, for d large enough. In a special case when $\alpha=\beta=\gamma=\zeta=\xi=\eta=1$, this is the *hook stability* introduced in [BRR17]. Note that even characterizing the cases when these quasi-polynomials are nonzero is quite challenging (cf. [BRR17, §6.2]).

A more general approach to stability for Lie groups can be found in [Par19].

6.4. One can give explicit lower bounds on the constants C_k in (1.3). For that, in notation of the proof of Theorem 1.5, we need to use the integral volume G_k of the *three-way transportation polytopes* $T_k(1)$. These polytopes are highly symmetric, so the lower bounds are especially simple and can be found in [Ben14], see also [Bar17] for a survey. See also a recent explicit lower bound in [BR07, Ex. 2.6], on the (usual) volume of $T_k(1)$. To put these bounds into context, recall the natural upper bound used in [PP20] in this case. The main result in [Ben14] (and in greater generality in [Bar17]), is that these upper bounds are asymptotically sharp.

6.5. For the uniform random partitions $\lambda\vdash n$, we have $\ell(\lambda)=\Theta(\sqrt{n}\log n)$ and $d(\lambda)=\Theta(\sqrt{n})$, see e.g. [DP19] and references therein. This implies that the bounds in (1.1) and (1.2) are useful only for partitions with relatively few rows and small Durfee square size, respectfully.

6.6. Define

$$\mathrm{B}(n,k) \,:=\, \max\,\big\{ g(\lambda,\mu,\nu) \,:\, \lambda,\mu,\nu\vdash n \ \text{ and } \ d(\lambda),d(\mu),d(\nu)\le k\big\}.$$

Comparing the bounds in Theorem 1.5 and Theorem 1.4, it would be natural to believe that the upper bound on $\mathrm{B}(n,k)$ in (1.2) is closer to the truth than the lower bound in (1.3).

Conjecture 6.1. *There is a universal constant $c>0$ such that*

$$\mathrm{B}(n,k) \,\ge\, n^{4k^3-ck^2} \quad \textit{for all} \ \ n,k\ge 1\,.$$

6.7. We believe that the Kronecker coefficients in Theorem 1.8 grow much faster than our lower bounds suggest. The following conjecture immediately implies Conjecture 1.6 improving upon Theorem 1.7.

Conjecture 6.2. *We have:*

$$g(\rho_k, \rho_k, \rho_k) \ = \ \sqrt{n!}\, e^{-O(n)} \quad \textit{where} \quad n = \tbinom{k}{2}, \qquad \textit{and}$$
$$g(\delta_\ell, \delta_\ell, \delta_\ell) \ = \ \sqrt{n!}\, e^{-O(n)} \quad \textit{where} \quad n = \ell^2.$$

6.8. Let $n = \binom{k}{2}$. Define

$$\mathrm{F}(n) \, := \, \max \big\{ g(\rho_k\,, \rho_k\,, \lambda) \ : \ \lambda \vdash n \ \text{ and } \ \lambda = \lambda' \big\}.$$

It would be interesting to find sharp lower bounds on $\mathrm{F}(n)$.

It was shown in [PP17, §4.2], that $g(\rho_k\,, \rho_k\,, \lambda) = e^{\Omega(\sqrt{n})}$ for two-row partitions $\lambda = (n/2, n/2)$, where n is even. For self-conjugate λ, it was only shown recently in [BBS21, §5] using *modular representation theory*, that the $\mathrm{F}(n)$ is unbounded.

Combined with the lower bound for Littlewood–Richardson coefficients given in [PPY19, Thm 1.5], Theorem 5.11 in [BBS21] implies that $\mathrm{F}(n) = e^{\Omega(\sqrt{n})}$. This is nowhere close to Conjecture 6.2, but gives us a hope that there might be more tools to be discovered.

6.9. In [MPP20], there is a tight asymptotic bound $g\big(\delta_{2s}, \delta_{2s}, (n-k,k)\big) = \Theta\big(2^{\sqrt{2k}}/k^{3/2}\big)$ in the case when $k/n \in (0, 1/2)$. However, this bound cannot be applied when $k = n/2 + o(n)$, so the bound from [PP17] is still the best known lower bound in this case.

Acknowledgements. We are grateful to Sasha Barvinok, Chris Bowman, Christian Ikenmeyer, Rosa Orellana, Fedya Petrov and Mike Zabrocki for helpful discussions and remarks on the subject. We also thank Stefan Trandafir for reminding us of the polynomial growth problem for fixed Durfee size. We thank the anonymous referee for many helpful remarks leading to improved exposition. Both authors were partially supported by the NSF.

We dedicate this paper to the memory of Christine Bessenrodt who passed away earlier this year. May her memory be a blessing and may her life be an inspiration for all of us.

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