EFFECTIVE POSET INEQUALITIES

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Abstract. We prove a number of new inequalities for the numbers of linear extensions and order polynomials of finite posets. First, we generalize the *Björner–Wachs inequality* to inequalities on order polynomials and their q-analogues via direct injections and FKG inequalities, and establish several new inequalities on order polynomials.

Second, we generalize actions of Coxeter groups on restricted linear extensions, leading to vanishing and uniqueness conditions for the *generalized Stanley inequality*. Third, we generalize the *Sidorenko inequality* to posets with small chain intersections and give complexity theoretic applications.

1. Introduction

1.1. **Foreword.** There are two schools of thought on what to do when an *interesting combinatorial inequality* is established. The first approach would be to treat it as a tool to prove a desired result. The inequality can still be sharpened or generalized as needed, but this effort is aimed with applications as the goal and not about the inequality per se.

The second approach is to treat the inequality as a result of importance in its own right. The emphasis then shifts to finding the "right proof" in an attempt to understand, refine or generalize it, in which case we say that the inequality can be made *effective*. This is where the nature of the inequality intervenes — when both sides count combinatorial objects, the desire to relate these objects is overpowering.

The inequality can be made effective in several different ways. A *direct injection* can give it a combinatorial interpretation for the difference or prove the equality conditions. Such an injection can also be a work of art, inspiring and thought-provoking in the best case. Alternatively, a *technical proof* (say, probabilistic or algebraic), can establish tools for generalizations out of reach by direct combinatorial arguments.

Both types of proof are most impactful when presented in combination. Making comparisons between different approaches can lead to further results, new open problems, and is the source of wonder of the beauty and diversity of mathematics.

As the reader must have guessed, we aim to make effective several celebrated combinatorial inequalities for the numbers of linear extensions of finite posets:

- the *Björner–Wachs inequality*,
- the *Sidorenko inequality*, and
- the *generalized Stanley inequality*.

Although there is a certain commonality of tools and approaches, our investigation of these inequalities are largely independent, united by the goal of being effective, i.e. extending these inequalities with the goal of understanding them on a deeper level. In addition to injections, we also use *probabilistic* and *algebraic tools*, with some curious combinatorial twists.

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1.2. **Extensions and generalizations of the Björner–Wachs inequality.** Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. For each element $x \in X$, let $B(x) := \{y \in X : y \succcurlyeq x\}$ be the *upper order ideal* generated by x, and let $b(x) := |B(x)|$.

A *linear extension* of P is a bijection $f : X \to [n] = \{1, \ldots, n\}$, such that $f(x) < f(y)$ for all $x \prec y$. Denote by $\mathcal{E}(P)$ the set of linear extensions of P, and let $e(P) := |\mathcal{E}(P)|$.

Theorem 1.1 (Björner and Wachs [BW89, Thm 6.3])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. In the notation above, we have:*

$$e(P) \ \geq \ n! \cdot \prod_{x \in X} \frac{1}{b(x)} \,. \tag{1.1}$$

This inequality was popularized by Stanley who stated it without proof or a reference in [Sta12, Exc. 3.57].[1] When the poset is a tree rooted at the minimal element, the inequality in the theorem is an equality known in the literature as the *hook-length formula for trees*. A variation on the classical *hook-length formulas* for straight and shifted Young diagrams, this case is usually attributed to Don Knuth (1973), see e.g. [Bén12, SY89]. Although for other families of poset the lower bound on $e(P)$ given by (1.1) is relatively weak, nothing better is known in full generality, see e.g. [BP21, MPP18a, Pak21].

We start by recalling the original direct injective proof by Björner and Wachs of the inequality (1.1). This allows us to prove that the inequality is in #P (Theorem 1.12). We then obtain the following extension of Theorem 1.1.

Let $[k] := \{1, \ldots, k\}$. For an integer $t \geq 1$, denote by $\Omega(P, t)$ the number of order preserving maps $g : X \to [t]$, i.e. maps which satisfy $g(x) \leq g(y)$ for all $x \prec y$. This is the *order polynomial* corresponding to poset P, see e.g. [Sta12, §3.12].

Theorem 1.2. *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for every $t \in \mathbb{N}$, we have:*

$$\Omega(P, t) \ \geq \ t^r \, (t+1)^{n-r} \prod_{x \in X} \frac{1}{b(x)} \,, \tag{1.2}$$

where r is the number of maximal elements of P.

Let us note that

$$\Omega(P, t) \ \sim \ \frac{e(P)\, t^n}{n!} \quad \text{as} \quad t \to \infty \,. \tag{1.3}$$

Thus, Theorem 1.2 implies Theorem 1.1. Note also that $\Omega(P, t) = \Omega(P^*, t)$, where $P^* = (X, \prec^*)$ is the poset where relations are reversed: $x \prec y \ \Leftrightarrow \ y \prec^* x$. Thus, the theorem holds when maximal elements are replaced with minimal elements.

The tools we use to establish Theorem 1.2 are based on *Shepp's lattice*, and are extremely far-reaching. Notably, they allows us to establish the following *strict log-concavity* of the order polynomial:

Theorem 1.3 (= Theorem 4.8)**.** *Let $P = (X, \prec)$ be a finite poset. Then, for every integer $t \geq 2$, we have:*

$$\Omega(P, t)^2 \ > \ \Omega(P, t+1) \cdot \Omega(P, t-1).$$

We use this result to we obtain the asymptotic version of *Graham's conjecture* proved by Daykin–Daykin–Paterson in [DDP84] by a direct injective argument (Theorem 4.19). Our next result is a general lower bound on the order polynomial strengthening the asymptotic formula (1.3).

[1] Richard Stanley informed us that he indeed took it from [BW89] (personal communication, March 27, 2022).

Theorem 1.4. *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for every $t \in \mathbb{N}$, we have:*

$$\Omega(P, t) \ \geq \ \frac{e(P)\, t^n}{n!}\,. \tag{1.4}$$

Our proof of Theorem 1.4 uses a direct injection. Among other applications of this approach, we prove that (1.4) is an equality if and only if P is an antichain (Corollary 6.3).

Since the Björner–Wachs inequality (1.1) can be rather weak in various special cases, neither of Theorems 1.2 and 1.4 implies another (see Example 6.4). In a different direction, inequality (1.4) strengthens the trivial inequality

$$\Omega(P, t) \ \geq \ e(P) \cdot \binom{t}{n} \ = \ \frac{e(P)\; t\,(t-1)\cdots(t-n+1)}{n!}\,. \tag{1.5}$$

Here the RHS counts the number of injections $f : X \to [t]$, which are naturally mapped onto $\mathcal{E}(P)$. Note that (1.4) agrees with (1.5) in the leading term given also by (1.3), but is sharper in the second term of the asymptotics.

Finally, we include an unpublished remarkably simple proof of the Björner–Wachs inequality by Vic Reiner, via extension of the inequality to its q-analogue (Theorem 5.1). We then use our tools in §5.2 to obtain new inequalities for the *q-order polynomial*. Notably, we obtain the following *q-log-concavity*:

Theorem 1.5 (= Corollary 5.9)**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Define*

$$\Omega_q(P, t) \ := \ \sum_g \ q^{|g(X)|-n}$$

where the summation is over all order preserving maps *$g : X \to \{1, \ldots, t\}$, i.e. maps which satisfy $g(x) \leq g(y)$ for all $x \prec y$. Then, for every integer $t \geq 2$, we have:*

$$\Omega_q(P, t)^2 \ \geqslant_q \ \Omega_q(P, t+1) \cdot \Omega_q(P, t-1),$$

where the inequality holds coefficient-wise as a polynomial in q.

1.3. **Generalized Sidorenko inequality.** Below we give an equivalent but somewhat nonstandard reformulation of the *Sidorenko inequality* that makes it amenable for generalization. A more traditional version is given in Section 8.

A *chain* in a poset $P = (X, \prec)$ is a subset $\{x_1, \ldots, x_\ell\} \subseteq X$, such that $x_1 \prec x_2 \prec \ldots \prec x_\ell$. Denote by $\mathcal{C}(P)$ the set of chains in P.

Theorem 1.6 (Sidorenko [Sid91])**.** *Let $P = (X, \prec)$ and $Q = (X, \prec')$ be two posets on the same set with $|X| = n$ elements. Suppose*

$$\bigl|C \cap C'\bigr| \leq 1 \quad \text{for all } \ C \in \mathcal{C}(P), \ C' \in \mathcal{C}(Q). \tag{1.6}$$

Then:

$$e(P)\, e(Q) \ \geq \ n! \tag{1.7}$$

Natural examples of posets (P, Q) as in the theorem are the *permutation posets* $\bigl(P_\sigma, P_{\overline{\sigma}}\bigr)$, where $P_\sigma = ([n], \prec)$ is defined as

$$i \prec j \quad \Longleftrightarrow \quad i < j \ \text{ and } \ \sigma(i) < \sigma(j)\,, \quad \text{ for all } \ i, j \in [n].$$

and $\overline{\sigma} := \bigl(\sigma(n), \ldots, \sigma(1)\bigr)$. In this case P_σ is a *2-dimensional poset*, and $P_{\overline{\sigma}}$ is its *plane dual*.

Sidorenko's original proof used combinatorial optimization and proved also equality conditions for (1.7), see §8.2. In [BBS99], the authors gave an easy reduction to a special case of the (still open) *Mahler conjecture* for *convex corners*. That special case was resolved earlier by Saint-Raymond [StR81], and was reproved and further extended in a series of papers, see [AASS20, BBS99] for the context and the references.

In this paper we give a direct injective proof of the Sidorenko inequality (1.7), which allows us to prove that the inequality is in #P (Theorem 1.14). This completely resolves the open problem Morales and the last two authors in [MPP18b] of finding a combinatorial proof of (1.7). Although presented differently, our injection likely coincides with an injection of Gaetz and Gao [GG20+], see §9.7; the latter was discovered independently and generalized to other Coxeter groups.

Our proof can also be extended to give the following generalization of Theorem 1.6.

Theorem 1.7. *Let $P = (X, \prec)$ and $Q = (X, \prec')$ be two posets on the same set with $|X| = n$ elements. Suppose*

$$\big|C \cap C'\big| \leq k \quad \text{for all } \ C \in \mathcal{C}(P), \ C' \in \mathcal{C}(Q).$$

Then:

$$e(P)\, e(Q) \ \geq \ \frac{n!}{k^{n-k}\, k!}\,. \tag{1.8}$$

The proof, examples and applications of this result are given in Section 8.

1.4. **Generalized Stanley inequality.** We start with the following inspiring *Stanley inequality*:

Theorem 1.8 (Stanley [Sta81])**.** *Let $P = (X, \prec)$ be a poset on $|X| = n$ elements. For an element $x \in X$ and integer $1 \leq a \leq n$, let $\mathcal{E}(P, x, a)$ be the set of linear extensions $f \in \mathcal{E}(P)$ such that $f(x) = a$. Denote by $\mathrm{N}(P, x, a) := \big|\mathcal{E}(P, x, a)\big|$ the number of such linear extensions. Then:*

$$\mathrm{N}(P, x, a)^2 \ \geq \ \mathrm{N}(P, x, a+1) \cdot \mathrm{N}(P, x, a-1). \tag{1.9}$$

Stanley's original proof of this result is via reduction to the classical and very deep *Alexandrov–Fenchel* (AF-) *inequality* in convex geometry. While the latter has several proofs, see references in [CP22a, §7.1], none are elementary and direct even in the case of convex polytopes. With the aim to prove (1.9) by an elementary argument, the (somewhat technical) proof in [CP21] uses nothing but linear algebra. Finding a direct injective proof is a major open problem (see §9.12).

In the absence of an injective proof, the *equality conditions* can become *more difficult* than the original inequality, cf. §9.10. This is famously the case for the AF-inequality and many of its consequences. For the Stanley inequality, the equality conditions were discovered recently by Shenfeld and van Handel [SvH20], by a deep geometric argument.

Fortunately, part of the equality conditions called the *vanishing conditions*, are completely combinatorial. Denote by $\ell(x) := \big|\{y \in X \, : \, y \preccurlyeq x\}\big|$ and $b(x) := \big|\{y \in X \, : \, y \succcurlyeq x\}\big|$ the sizes of lower and upper ideals of $x \in X$, respectively.

Theorem 1.9 (Shenfeld and van Handel [SvH20, Lemma 15.2])**.** *Let $P = (X, \prec)$ be a poset on $|X| = n$ elements, let $x \in X$ and $1 \leq a \leq n$. Then $\mathrm{N}(P, x, a) > 0$ <u>if and only if</u> $\ell(x) \leq a$ and $b(x) \leq n - a + 1$.*

Note that by the Stanley inequality, if $\mathrm{N}(P, x, a) = 0$, then $\mathrm{N}(P, x, a+1) = 0$ or $\mathrm{N}(P, x, a-1) = 0$, so whenever the conditions in the theorem are not satisfied the equation (1.9) is an equality. We can now define the *generalized Stanley inequality*.

Theorem 1.10 (Stanley [Sta81])**.** *Let $P = (X, \prec)$ be a poset on $|X| = n$ elements. Fix elements $x, z_1, \ldots, z_k \in X$ and integers $a, c_1, \ldots, c_k \in [n]$; we write $\boldsymbol{z} = (z_1, \ldots, z_k)$ and $\mathbf{c} = (c_1, \ldots, c_k)$. Let $\mathcal{E}_{\boldsymbol{z}\mathbf{c}}(P, x, a)$ be the set of linear extensions $f \in \mathcal{E}(P)$ such that $f(x) = a$ and $f(z_i) = c_i$,* *for all* $1 \leq i \leq k$. *Denote by* $\mathrm{N}_{\boldsymbol{zc}}(P,x,a) := \left|\mathcal{E}_{\boldsymbol{zc}}(P,x,a)\right|$ *the number of such linear extensions. Then:*

$$\mathrm{N}_{\boldsymbol{zc}}(P,x,a)^2 \,\geq\, \mathrm{N}_{\boldsymbol{zc}}(P,x,a+1)\,\cdot\,\mathrm{N}_{\boldsymbol{zc}}(P,x,a-1). \tag{1.10}$$

We can now state the *vanishing conditions* for the generalized Stanley inequality. Without loss of generality, we can assume that numbers $\mathbf{c}$ are in increasing order, in which case we can assume that elements $\mathbf{z}$ form a chain (because $\mathrm{N}_{\mathbf{zc}}$ only counts linear extensions for which $z_1 \prec \cdots \prec z_k$). Let $x, y \in X$ be two poset elements such that $x \prec y$. Define $h(x,y) := \#\{z \in P, \text{ s.t. } x \prec z \prec y\}$.

Theorem 1.11. *Let* $P=(X,\prec)$ *be a poset on* $|X|=n$ *elements. Fix elements* $u_1 \prec \ldots \prec u_k \in X$ *and integers* $1 \leq a_1 < \ldots < a_k \leq n$*; we write* $\boldsymbol{u}=(u_1,\ldots,u_k)$ *and* $\boldsymbol{a}=(a_1,\ldots,a_k)$. *Let* $\mathcal{E}(P,\boldsymbol{u},\boldsymbol{a})$ *be the set of linear extensions* $f \in \mathcal{E}(P)$ *such that* $f(u_i)=a_i$, *for all* $1 \leq i \leq k$. *Then* $\left|\mathcal{E}(P,\boldsymbol{u},\boldsymbol{a})\right| > 0$ *<u>if and only if</u>*

$$\begin{aligned} &\ell(u_i) \,\leq\, a_i\,, \quad b(u_i) \leq n-a_i+1\,, \qquad \textit{for all} \quad 1 \,\leq\, i \,\leq\, k, \quad \textit{and} \\ &a_j \,-\, a_i \,>\, h(u_i,u_j) \qquad \textit{for all} \quad 1 \,\leq\, i \,<\, j \,\leq\, k. \end{aligned} \tag{1.11}$$

In Theorem 7.5, we also prove the *uniqueness conditions* for the problem, i.e. necessary and sufficient conditions for $\left|\mathcal{E}(P,\boldsymbol{u},\boldsymbol{a})\right|=1$. We postpone the statement until §7.4.

1.5. **Complexity implications.** We assume the reader is familiar with basic *Computational Complexity*, and refer to standard textbooks [AB09, MM11, Pap94] for definitions and notation. Here we follow the approach to inequalities proposed by the second author [Pak19, Pak22].

Recall the *counting complexity class* $\#\mathrm{P}$ of functions which count the number of objects whose membership is decided in polynomial time. Let $\textsc{GapP} = \#\mathrm{P} - \#\mathrm{P}$ be the closure of $\#\mathrm{P}$ under subtraction, see e.g. [For97]. Finally, let $\textsc{GapP}_{\geq 0} := \textsc{GapP} \cap \{u \geq 0\}$.

Clearly, $\#\mathrm{P} \subseteq \textsc{GapP}_{\geq 0}$, but it remains open whether this inclusion is proper. For example, the *Kronecker coefficients* $g(\lambda,\mu,\nu) \in \textsc{GapP}_{\geq 0}$. It is not known whether $g(\cdot) \in \#\mathrm{P}$, and this remains a major open problem in Algebraic Combinatorics, see e.g. [PP17].

As before, let $P=(X,\prec)$ be a poset on n elements. Clearly, the function $e: P \to e(P)$ is in $\#\mathrm{P}$, and is famously $\#\mathrm{P}$-complete [BW91]. In fact, the function $e(\cdot)$ is $\#\mathrm{P}$-complete even when restricted to permutation posets P_σ, $\sigma \in S_n$ and posets of height two, see [DP20]. Define

$$\xi(P) \,:=\, e(P)\,\cdot\,\prod_{x\in X} b(x) \,-\, n! \tag{1.12}$$

Observe that $\xi \in \textsc{GapP}_{\geq 0}$ by the definition and the Björner–Wachs inequality (1.1). In fact, the original injective proof of (1.1) easily implies the following effective version of the inequality:

Theorem 1.12. *The function* $\xi : P \to \mathbb{N}$ *defined by* (1.12) *is in* $\#\mathrm{P}$.

Similarly, define $\zeta : P \times \mathbb{N} \to \mathbb{N}$

$$\zeta(P,t) \,:=\, \Omega(P,t)\,n! \,-\, e(P)\,t^n\,. \tag{1.13}$$

We can now give an effective version of (1.4):

Theorem 1.13. *The function* $\zeta : P \to \mathbb{N}$ *defined by* (1.13) *is in* $\#\mathrm{P}$.

For every $\sigma \in S_n$, let $\eta : S_n \to \mathbb{Z}$ be defined as follows:

$$\eta(\sigma) \,:=\, e(P_\sigma)\,e\big(P_{\overline{\sigma}}\big) \,-\, n! \tag{1.14}$$

Observe that $\eta \in \textsc{GapP}_{\geq 0}$ by the definition and the Sidorenko inequality (1.7). In fact, our injective proof of (1.7) can be used to obtain the following result:

Theorem 1.14. *The function* $\eta : S_n \to \mathbb{N}$ *defined by* (1.14) *is in* $\#\mathrm{P}$.

For the vanishing conditions of the generalized Stanley inequality, the implications are completely straightforward:

Corollary 1.15. *In the conditions of Theorem 1.11, deciding whether* $\left|\mathcal{E}(P,\boldsymbol{u},\boldsymbol{a})\right|>0$ *is in* P. *Moreover, when* $\left|\mathcal{E}(P,\boldsymbol{u},\boldsymbol{a})\right|>0$, *a linear extension* $f\in\mathcal{E}(P,\boldsymbol{u},\boldsymbol{a})$ *can be found in polynomial time.*

We conclude with a corollary of Theorem 7.5.

Corollary 1.16. *In the conditions of Theorem 1.11, deciding whether* $\left|\mathcal{E}(P,\boldsymbol{u},\boldsymbol{a})\right|=1$ *is in* P.

1.6. **Structure of the paper.** The paper is written in a straightforward manner, as we devote different sections to proofs of different results. These proofs are completely independent and largely self-contained. We are hoping they will appeal to a diverse readership.

We start with a short Section 2, where give some basic notation used throughout the paper. In Section 3, we recall the original direct injective proof of the Björner–Wachs inequality (1.1) via direct injection (cf. Theorem 1.12). Here we introduce promotions of linear extensions, a tool which will also be used later in the paper (Sections 7 and 8).

In a lengthy Section 4, we use Shepp's lattice to prove Theorem 1.3 and other inequalities for the order polynomial. The second half of this section is motivated by connection and applications to the *Kahn–Saks Conjecture* (Conjecture 4.12) and the *Graham Conjecture* (Theorem 4.19), as we prove special cases of both of them.

In the next Section 5, we present an elegant proof by Reiner of the Björner–Wachs inequality. We then prove a q-analogue of *Shepp's inequality* for the q-analogue of the order polynomial, by using the remarkable *q-FKG inequality* by Björner. We continue with the general lower bound on the order polynomial (Section 6), and prove Theorem 1.4 by a direct injection.

In Section 7, we prove the vanishing conditions (Theorem 1.11) and uniqueness conditions (Theorem 7.5), from which Corollaries 1.15 and 1.16 easily follow. Our proof is based on an algebraic approach of Coxeter group action on linear extensions, see §9.11 for some history of the subject.

In Section 8, we give an injective proof of the Sidorenko inequality, and prove its extension Theorem 1.7. We then derive Theorem 1.14 which is surprisingly nontrivial given the many other proofs of the inequality (see §9.7). We conclude with Section 9 containing lengthy historical remarks and open problems.

2. Basic definitions and notation

In a poset $P=(X,\prec)$, elements $x,y\in X$ are called *parallel* or *incomparable* if $x\not\prec y$ and $y\not\prec x$. We write $x\parallel y$ in this case. Element $x\in X$ is said to *cover* $y\in X$, if $y\prec x$ and there are no elements $z\in X$ such that $y\prec z\prec x$.

A *chain* is a subset $C\subset X$ of pairwise comparable elements. The *height* of poset $P=(X,\prec)$ is the maximum size of a chain. An *antichain* is a subset $A\subset X$ of pairwise incomparable elements. The *width* of poset $P=(X,\prec)$ is the size of the maximal antichain.

A *dual poset* is a poset $P^*=(X,\prec^*)$, where $x\prec^* y$ if and only if $y\prec x$.

A *disjoint sum* $P+Q$ of posets $P=(X,\prec)$ and $Q=(Y,\prec')$ is a poset on $(X\cup Y,\prec^\diamond)$, where the relation $\prec^\diamond$ coincides with $\prec$ and $\prec'$ on X and Y, and $x\parallel y$ for all $x\in X$, $y\in Y$.

A *linear sum* $P\oplus Q$ of posets $P=(X,\prec)$ and $Q=(Y,\prec')$ is a poset on $(X\cup Y,\prec^\diamond)$, where the relation $\prec^\diamond$ coincides with $\prec$ and $\prec'$ on X and Y, and $x\prec^\diamond y$ for all $x\in X$, $y\in Y$.

A *product* $P\times Q$ of posets $P=(X,\prec)$ and $Q=(Y,\prec^*)$ is a poset on $(X\times Y,\prec^\diamond)$, where the relation $(x,y)\preccurlyeq^\diamond(x',y')$ if and only if $x\preccurlyeq x'$ and $y\preccurlyeq^* y'$, for all $x,x'\in X$ and $y,y'\in Y$.

Posets constructed from one-element posets by recursively taking disjoint and linear sums are called *series-parallel*. Both *n-chain* C_n and *n-antichain* A_n are examples of series-parallel posets.

For a subset $Y \subset X$, a *restriction* of the poset $= (X, \prec)$ to $X \smallsetminus Y$ is a subposet $(X \smallsetminus Y, \prec)$ of P, which we denote by $P \smallsetminus Y$ and $P|_{X \smallsetminus Y}$.

For a poset $P = (X, \prec)$, a function $f : X \to \mathbb{R}$ is called *$\prec$-increasing* if $f(x) \leq f(y)$ for all $x \preccurlyeq y$; such functions are also called *weakly order-preserving* in a different context. The *$\prec$-decreasing* functions are defined analogously.

Throughout the paper we use $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{P} = \mathbb{N}_{\geq 1} = \{1, 2, \ldots\}$ and $[n] = \{1, \ldots, n\}$.

3. Injective proof of the Björner–Wachs inequality

In this short section we recap the original proof by Björner and Wachs. We do this both as a warmup and as a way to introduce some definitions and ideas that will prove useful throughout the paper. As a quick application, we obtain the proof of Theorem 1.12. The reader well familiar with [BW89] can skip this section.

Denote by $\mathcal{S} = \mathcal{S}(P)$ the set of all bijections $\sigma : X \to [n]$, so that $\mathcal{E}(P) \subseteq \mathcal{S}(P)$. Denote by $\mathcal{B} = \mathcal{B}(P)$ the set of maps $g : X \to X$ such that $g(x) \succcurlyeq x$ for all $x \in X$. The inequality (1.1) can then be written as:

$$(*) \qquad \big|\mathcal{S}(P)\big| \,\leq\, \big|\mathcal{E}(P)\big| \cdot \big|\mathcal{B}(P)\big|.$$

We prove $(*)$ by a direct injection $\Phi : \mathcal{S} \to \mathcal{E} \times \mathcal{B}$ defined as follows.

We say that a bijection $f : X \to [n]$ is *sorted* on a subset $Y \subseteq X$, if $f(x) < f(y)$ for all $x, y \in Y$ such that $x \prec y$. Fix a linear extension $\alpha \in \mathcal{E}(P)$ and label the elements of X naturally according to α, so that $x_i = \alpha^{-1}(i)$. For every $\sigma \in \mathcal{S}$, proceed with the following *sorting algorithm* for the elements $x_n, \ldots, x_1$ in this order. At the k-th step, take the element $x = x_{n-k+1}$ and let $f(x) := \sigma(x)$. If $\sigma(x)$ is the smallest of $\{f(y), y \in B(x)\}$, do nothing. Otherwise, start the *demotion* of x by swapping its value $f(x)$ with the smallest $f(x')$, where $x' \in B(x)$. Repeat this with x', etc., until all elements in $B(x)$ are sorted. Let $g(x) := y$ be the element in $B(x)$ where x is demoted to (i.e. y is the largest element in $B(x)$ affected by the demotion).

At end of the sorting algorithm, we obtain a bijection $f : X \to [n]$ that is sorted on the whole X, i.e. $f \in \mathcal{E}(P)$. We also obtain a map $g \in \mathcal{B}(P)$. Define $\Phi(\sigma) := (f, g)$.

Proposition 3.1 ([BW89])**.** *The map $\Phi : \mathcal{S}(P) \to \mathcal{E}(P) \times \mathcal{B}(P)$ defined above is an injection.*

Proof. Define the *inverse construction* as follows. Proceed through the reverse order of the elements $x_1, \ldots, x_n$. At $(n-k)$-th step, take the element $x = x_{n-k}$ and let $y = g(x)$. At this step, the bijection $f : X \to [n]$ is sorted on $B(x)$.

Start the *promotion* of y by swapping $f(y)$ with the maximal $f(y')$, over all $x \preccurlyeq y' \prec y$, until eventually $f(y)$ is promoted to the element x. Denote by $\sigma \in \mathcal{S}(P)$ the result of this iterated promotion and define a map $\Psi(f, g) := \sigma$.

Now observe that for all $\sigma \in \mathcal{S}(P)$ we have $\Psi(\Phi(\sigma)) = \sigma$, since the map Ψ retraces each step of Φ by the properties of promotion and demotion. This implies that Φ is an injection and completes the proof of the claim. □

Proof of Theorem 1.12. Let $\mathcal{H}(P) \subseteq \mathcal{E}(P) \times \mathcal{B}(P)$ be the set of pairs $(f, g) \in \mathcal{E}(P) \times \mathcal{B}(P)$, such that $\Phi(\Psi(f, g)) \neq (f, g)$. By definition, $\big|\mathcal{H}(P)\big| = \xi(P)$. Since both Φ and Ψ are computable in polynomial time, then so is the membership in $\mathcal{H}(P)$. This proves the result. □

A series-parallel poset $P = (X, \prec)$ is called an *ordered forest* if it is a disjoint union of rooted trees, where each tree is rooted it its unique minimal element.

Proposition 3.2 ([BW89])**.** *The Björner–Wachs inequality* (1.1) *is an equality <u>if and only if</u> P is an ordered forest.*

Proof. As mentioned in the introduction, for the "if" direction, the equality can be easily proved by induction, see e.g. [Bén12, SY89]. For the "only if" direction, let $x, y, z \in X$ be a poset elements such that $x \prec z$, $y \prec z$, $x \,\|\, y$, and such that both elements x, y are covered by z. We claim that (1.1) is a strict inequality in this case.

In the notation above, choose $g \in \mathcal{B}(P)$ such that $g(x) = g(y) = z$ and $g(s) = s$ for all $s \in X \smallsetminus \{x, y\}$. It is easy to see that there exists $f \in \mathcal{E}(P)$, such that $f(x) = k-1$, $f(y) = k$ and $f(z) = k+1$, for some $1 < k < n$. Assume that x precedes y in the natural labeling. Applying Ψ we see that after promoting $f(x)$ to z, the result is no longer a linear extension on $B(y)$ and thus Ψ is not defined there. Thus Φ is not a bijection, which proves the claim.

Finally, observe that P is an ordered forest if and only if every poset element covers at most one element. This proves the result. $\square$

4. Bounding order polynomial by the FKG inequality

In this section we prove the bound on the order polynomial from Theorem 1.2 using an inductive approach and an application of the FKG inequality on the Shepp's lattice.

4.1. **Shepp's lattice and the FKG inequality.** Recall that a *lattice* $\mathcal{L} := (L, \prec^{\diamond})$ is a partially ordered set on L, such that every $a, b \in L$ has a unique least upper bound called *join* $a \vee b$, and a unique greatest lower bound called *meet* $a \wedge b$. A lattice is called *distributive* if

$$a \wedge (b \vee c) \ = \ (a \wedge b) \vee (a \wedge c) \quad \text{ for all } \ a, b, c \in L.$$

A function $\mu : L \to \mathbb{R}_{\geq 0}$ is called *log-supermodular* if

$$\mu(a)\, \mu(b) \ \leq \ \mu(a \wedge b)\, \mu(a \vee b) \quad \text{ for all } \ a, b \in L.$$

Fix a positive integer $t > 0$. Let $P = (X, \prec)$ be a poset on $|X| = n$ elements, let $X = Y \sqcup Z$ be a partition of X into two disjoint subsets. *Shepp's lattice* $\mathcal{L} = \mathcal{L}_{Y,Z,t} := (L, \prec^{\diamond})$ is defined as

$$L \ := \ \big\{ \boldsymbol{v} = (v_x)_{x \in X} \ : \ 1 \ \leq \ v_x \ \leq \ t \big\},$$

and let

$$\boldsymbol{v} \ \preccurlyeq^{\diamond} \ \boldsymbol{w} \quad \Longleftrightarrow \quad \begin{cases} v_y \ \leq \ w_y & \text{for all } \ y \in Y \\ v_z \ \geq \ w_z & \text{for all } \ z \in Z \end{cases}$$

Let $\mu = \mu_{Y,Z} : L \to \{0, 1\}$ be a function defined as

$$\mu(\boldsymbol{v}) = 1 \quad \Longleftrightarrow \quad v_x \ \leq \ v_{x'} \quad \text{for all } \ x \preccurlyeq x' \quad \text{such that } \quad \left[\begin{array}{l} x, x' \in Y, \quad \text{or} \\ x, x' \in Z. \end{array}\right.$$

Theorem 4.1 ([She80]). *Let $P = (X, \prec)$ be a finite poset, and let $X = Y \sqcup Z$ be a partition of the ground set X into two disjoint subsets. Then $\mathcal{L}_{Y,Z,t}$ is a distributive lattice, and $\mu_{Y,Z}$ is a log-supermodular function.*

This beautiful result is relatively little known; we include a short proof for completeness.

Proof. It follows from the definition of $\prec^{\diamond}$, that for all $\boldsymbol{v}, \boldsymbol{w} \in L$, we have:

$$\begin{aligned} (\boldsymbol{v} \wedge \boldsymbol{w})_y \ &= \ \min\{v_y, w_y\}, \qquad (\boldsymbol{v} \wedge \boldsymbol{w})_z \ = \ \max\{v_z, w_z\}, \\ (\boldsymbol{v} \vee \boldsymbol{w})_y \ &= \ \max\{v_y, w_y\}, \qquad (\boldsymbol{v} \vee \boldsymbol{w})_z \ = \ \min\{v_z, w_z\}, \end{aligned} \tag{4.1}$$

where $y \in Y$ and $z \in Z$. Now note that, for all real numbers $\alpha, \beta, \gamma \in \mathbb{R}$,

$$\min\{\alpha, \max\{\beta, \gamma\}\} \ = \ \max\{\min\{\alpha, \beta\}, \min\{\alpha, \gamma\}\}. \tag{4.2}$$

It then follows from (4.1) and (4.2) that $\mathcal{L}$ is a distributive lattice.

To show that μ is a log-supermodular function, it suffices to verify the cases when $\mu(\boldsymbol{v}) = \mu(\boldsymbol{w}) = 1$. Let $y, y' \in Y$ be such that $y \prec y'$. Note that $v_y \leq v_{y'}$ and $w_y \leq w_{y'}$. Then we have:

$$(\boldsymbol{v} \wedge \boldsymbol{w})_y \ = \ \min\{v_y, w_y\} \ \leq \ \min\{v_{y'}, w_{y'}\} \ = \ (\boldsymbol{v} \wedge \boldsymbol{w})_{y'}\,.$$

Similarly, we have $(\boldsymbol{v} \wedge \boldsymbol{w})_z \leq (\boldsymbol{v} \wedge \boldsymbol{w})_{z'}$ for all $z \prec z'$, $z, z' \in Z$. Therefore, $\mu(\boldsymbol{v} \wedge \boldsymbol{w}) = 1$. Analogously, we also have $\mu(\boldsymbol{v} \vee \boldsymbol{w}) = 1$, and the proof is complete. □

Remark 4.2. Shepp's lattice $\mathcal{L}$ used in this section should not be confused with another lattice defined in [She82] by Shepp. Both lattices share the same ground set but have different partial orders, and the partial order of the lattice in [She82] was specifically chosen to prove the *XYZ inequality*.

Now recall the classical *FKG inequality*, see e.g. [AS16, §6.2].

Theorem 4.3 (FKG inequality, [FKG71])**.** *Let $\mathcal{L} = (L, \prec)$ be a finite distributive lattice, and let $\mu : L \to \mathbb{R}_{\geq 0}$ be a log-supermodular function. Then, for every $\prec^\diamond$-decreasing functions $g, h : L \to \mathbb{R}_{\geq 0}$, we have:*

$$E(\mathbf{1})\, E(gh) \ \geq \ E(g)\, E(h), \tag{4.3}$$

where

$$E(g) \ = \ E_\mu(g) \ := \ \sum_{x \in L} g(x)\, \mu(x)\,,$$

and function $\mathbf{1} : L \to \mathbb{R}$ is given by $\mathbf{1}(x) = 1$ for all $x \in L$.

Furthermore, the inequality (4.3) also holds when both g, h are $\prec^\diamond$-increasing. On the other hand, when g is $\prec^\diamond$-decreasing and h is $\prec^\diamond$-increasing, the inequality (4.3) is reversed.

Below we apply the FKG inequality to Shepp's lattice to prove several inequalities for the order polynomial.

4.2. **Correlation inequalities.** Let $P = (X, \prec)$ be a poset on n elements. As above, denote by $\mathcal{S} = \mathcal{S}(P)$ the set of bijections $f : X \to [n]$. By an abuse of notation, for every (not necessarily distinct) elements $u, v \in X$, we write

$$\{u \preccurlyeq v\} \qquad \text{as a shorthand for the collection} \qquad \{f \in \mathcal{S} \,:\, f(u) \leq f(v)\}\,.$$

One can write the set of linear extensions $\mathcal{E}(P)$ as the intersection of collections $\{u \preccurlyeq v\}$, for all pairs $u \prec v$ in P. Conversely, every such intersection is a set of linear extensions of the corresponding poset. The language of *collections* is technically useful for our purposes.

Let $X = Y \sqcup Z$ be a *partition* of X into two disjoint subsets. A collection C is called *Y-minimizing* w.r.t. the partition $Y \sqcup Z$ if C is an intersection of collections of the form $\{y \preccurlyeq z\}$, for $y \in Y$ and $z \in Z$. Similarly, a collection C is called *Y-maximizing* w.r.t. the partition $Y \sqcup Z$ if C is an intersection of collections of the form $\{z \preccurlyeq y\}$, for $y \in Y$ and $z \in Z$. By a slight abuse of notation, we write $\Omega(C, t)$ to denote the order polynomial of a poset given by the collection C.

For the rest of this section, let A be the collection given by

$$A \ := \bigcap_{y \prec y',\ y, y' \in Y} \{y \preccurlyeq y'\} \ \cap \bigcap_{z \prec z',\ z, z' \in Z} \{z \preccurlyeq z'\},$$

the collection of events involving only elements of Y or only elements of Z.

Lemma 4.4 ([She80, Eq. (2.12)])**.** *In the notation above, let C, C' be Y-minimizing collections w.r.t. partition $X = Y \sqcup Z$. Then, for every integer $t > 0$, we have:*

$$\Omega\big(C \cap C' \cap A, t\big) \cdot \Omega\big(A, t\big) \quad \geq \quad \Omega\big(C \cap A, t\big) \cdot \Omega\big(C' \cap A, t\big)\,.$$

If C is Y-minimizing and C' is Y-maximizing, then the above inequality is reversed.

In the probabilistic language, it says that the order polynomial satisfies *positive correlation* for intersections of Y-minimizing collections (viewed as events). We should note that in [She80] this result was not singled out and appears as an equation in the middle of the proof of the main result. We again include the proof for completeness.

Proof of Lemma 4.4. Let $\mathcal{L} = (L, \prec^\diamond)$ be Shepp's lattice defined in §4.1. Let $g, h : L \to \{0,1\}$ be given by

$$g(\boldsymbol{v}) \ := \ \begin{cases} 1 & \text{if } \ v_y \leq v_z \quad \forall \{y \preccurlyeq z\} \in C \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \quad h(\boldsymbol{v}) \ := \ \begin{cases} 1 & \text{if } \ v_y \leq v_z \quad \forall \{y \preccurlyeq z\} \in C' \\ 0 & \text{otherwise.} \end{cases}$$

Let us prove that g is $\prec^\diamond$-decreasing. It suffices to show that

$$g(\boldsymbol{w}) = 1 \quad \text{and} \quad \boldsymbol{v} \prec^\diamond \boldsymbol{w} \quad \Longrightarrow \quad g(\boldsymbol{v}) = 1.$$

Note that for every $y \in Y$ and $z \in Z$ such that $y \preccurlyeq z$, we have:

$$v_y \ \leq \ w_y \ \leq \ w_z \ \leq \ v_z,$$

where the first and the third inequality is because $\boldsymbol{v} \prec^\diamond \boldsymbol{w}$, and the second inequality is because $g(\boldsymbol{w}) = 1$. This implies that $g(\boldsymbol{v}) = 1$, and thus g is a $\prec^\diamond$-decreasing function. By the same reasoning, we also have that function h is $\prec^\diamond$-decreasing.

Finally, note that

$$\Omega(C \cap C' \cap A, t) \ = \ E(gh), \quad \Omega(A, t) \ = \ E(\mathbf{1}), \quad \Omega(C \cap A, t) \ = \ E(g), \quad \Omega(C' \cap A, t) \ = \ E(h),$$

The lemma now follows from the FKG inequality (Theorem 4.3). $\square$

We can now apply this result in the more traditional notation of order polynomials of posets.

Lemma 4.5. *Let $P = (X, \prec)$ be a poset, and let $x, y \in X$ be minimal elements. Then, for every integer $t > 0$, we have:*

$$\Omega\big(P, t\big) \cdot \Omega\big(P \smallsetminus \{x, y\}, t\big) \ \geq \ \Omega\big(P \smallsetminus x, t\big) \cdot \Omega\big(P \smallsetminus y, t\big),$$

where by $P \smallsetminus x$, $P \smallsetminus y$ and $P \smallsetminus \{x,y\}$ denote the subposets of P restricted to $X - x$, $X - y$ and $X - x - y$, respectively.

Proof. Note that x and y are incomparable elements. Let $Y := \{x, y\}$ and $Z := X \smallsetminus Y$ be the partition of X. Consider the Y-minimizing collections C and C' given by

$$C \ := \bigcap_{z \in B(x) - x} \{x \preccurlyeq z\} \quad \text{and} \quad C' \ := \bigcap_{z' \in B(y) - y} \{y \preccurlyeq z'\}, \tag{4.4}$$

where $B(x)$ and $B(y)$ are upper order ideals of elements x and y, respectively. Observe that

$$\Omega(P, t) \ = \ \Omega(C \cap C' \cap A, t), \qquad \Omega(P \smallsetminus x, t) \ = \ \frac{1}{t}\, \Omega(C' \cap A, t),$$
$$\Omega(P \smallsetminus y, t) \ = \ \frac{1}{t}\, \Omega(C \cap A, t), \qquad \Omega(P \smallsetminus \{x, y\}, t) \ = \ \frac{1}{t^2}\, \Omega(A, t).$$

The lemma now follows from Lemma 4.4 and the equations above. $\square$

4.3. **Lower bounds.** We are now ready to prove Theorem 1.2 by induction. The following lemma established the induction step from which the theorem follows.

Lemma 4.6. *Let $P = (X, \prec)$ be a finite poset, and let $x \in X$ be a minimal element. Assume that $b(x) > 1$. Then we have:*

$$\frac{\Omega(P,t)}{t+1} \ \geq \ \frac{\Omega(P \smallsetminus x, t)}{b(x)}\,. \tag{4.5}$$

Proof. Let $Q = (Y, \prec)$ be a finite poset. By labeling the elements in Y with m distinct integers from $[t]$, we can write

$$\Omega(Q,t) \ = \ \sum_{m=1}^{n} \big|\mathcal{I}_m(Q)\big| \binom{t}{m}, \tag{4.6}$$

where $|\mathcal{I}_m(Q)|$ is the number of ascending chains

$$\varnothing \ = \ I_0 \ \subset \ I_1 \ \subset \ I_2 \ \subset \ \ldots \ \subset \ I_m \ = \ Q$$

of upper order ideals in P, s.t. $I_i \smallsetminus I_{i-1} \neq \varnothing$ for all $1 \leq i \leq m$.

Let $n := |X|$ be the number of elements in X. Denote by $P' = P \smallsetminus \{x\}$ the induced poset on $X \smallsetminus x$. Suppose that x is a unique minimal element of P, and let $P' = P \smallsetminus x$. Note that $b(x) = n$ in this case. Summing over all possible values of x, we obtain (4.5):

$$\begin{aligned}\Omega(P,t) \ &= \ \sum_{k=1}^{t} \Omega(P',k) \ =_{(4.6)} \ \sum_{m=1}^{n-1} \big|\mathcal{I}_m(P')\big| \sum_{k=m}^{t} \binom{k}{m} \ = \ \sum_{m=1}^{n-1} \big|\mathcal{I}_m(P')\big| \binom{t+1}{m+1} \\ &\geq \ \frac{t+1}{n} \sum_{m=1}^{n-1} \big|\mathcal{I}_m(P')\big| \binom{t}{m} \ =_{(4.6)} \ \frac{t+1}{n}\, \Omega(P',t)\,.\end{aligned}$$

Here the inequality follows from

$$\binom{t+1}{m+1} \ = \ \frac{t+1}{m+1}\binom{t}{m} \ \geq \ \frac{t+1}{n}\binom{t}{m} \qquad \text{for all} \ \ m \leq n-1.$$

Suppose now that $x \in X$ is not a unique minimal element. Let $y \in X$, $y \neq x$ be another minimal element in P. By Lemma 4.5, we have:

$$\frac{\Omega(P,t)}{\Omega(P',t)} \ \geq \ \frac{\Omega(P \smallsetminus y, t)}{\Omega(P' \smallsetminus y, t)}\,. \tag{4.7}$$

Now proceed by induction to remove all minimal elements in P incomparable to x, until element x becomes the unique minimal element. Applying the inequality (4.7) repeatedly, we obtain:

$$\frac{\Omega(P,t)}{\Omega(P',t)} \ \geq \ \cdots \ \geq \ \frac{t+1}{b(x)}\,.$$

This proves (4.5) in full generality. □

Proof of Theorem 1.2. We prove the inequality (1.2) by induction. First, suppose that $b(x) = 1$, so x is the maximal element in P. Then $\Omega(P,t) = t\,\Omega(P',t)$, since we can choose the value $f(x) \in [t]$ independently of other values. The inequality (1.2) follows then. For $b(x) > 1$, Lemma 4.6 gives the step of induction and complete the proof. □

4.4. **Log-concavity.** The main result of this subsection is the log-concavity of the evaluation of the order polynomial.

Theorem 4.7. *Let $P = (X, \prec)$ be a finite poset. Then, for every integer $t \geq 2$, we have:*

$$\Omega(P,t)^2 \ \geq \ \Omega(P,t+1) \cdot \Omega(P,t-1).$$

As for other poset inequalities, one can ask about equality conditions in Theorem 4.7. Turns out, the log-concavity in the theorem is always strict, see Theorem 4.8 below. The proof of both results use the same approach, but the strict log-concavity is built on top of the non-strict version and is a bit more involved. Thus, we start with the easier result for clarity.

Proof of Theorem 4.7. Let $Y = X$ and $Z = \varnothing$. Let $\mathcal{L} = (L, \prec^\diamond)$ be Shepp's lattice defined in §4.1, and let $t \geq 3$. Let $g, h : L \to \{0,1\}$ be two functions given by

$$g(\boldsymbol{v}) \ := \ \begin{cases} 1 & \text{if } \ v_x \geq 2 \quad \text{for all } \ x \in X \\ 0 & \text{otherwise,} \end{cases} \qquad h(\boldsymbol{v}) \ := \ \begin{cases} 1 & \text{if } \ v_x \leq t-1 \quad \text{for all } \ x \in X \\ 0 & \text{otherwise.} \end{cases}$$

To prove that g is $\prec^\diamond$-increasing, it suffices to show that

$$g(\boldsymbol{v}) = 1 \quad \text{and} \quad \boldsymbol{v} \prec^\diamond \boldsymbol{w} \quad \Longrightarrow \quad g(\boldsymbol{w}) = 1.$$

Note that, for every $x \in Y = X$, we have:

$$w_x \ \geq \ v_x \ \geq \ 2,$$

where the first inequality is because $\boldsymbol{v} \prec^\diamond \boldsymbol{w}$, and the second inequality is because $g(\boldsymbol{v}) = 1$. This implies that $g(\boldsymbol{w}) = 1$, and thus g is a $\prec^\diamond$-increasing function. By an analogous reasoning we also have that h is $\prec^\diamond$-decreasing.

Now note that

$$\begin{aligned} E(gh) \ &= \ \big|\big\{\boldsymbol{v} \in L \, : \, 2 \leq v_x \leq t-1 \quad \text{for all} \quad x \in X\big\}\big| \ = \ \Omega(P, t-2), \\ E(g) \ &= \ \big|\big\{\boldsymbol{v} \in L \, : \, 2 \leq v_x \quad \text{for all} \quad x \in X\big\}\big| \ = \ \Omega(P, t-1), \\ E(h) \ &= \ \big|\big\{\boldsymbol{v} \in L \, : \, v_x \leq t-1 \quad \text{for all} \quad x \in X\big\}\big| \ = \ \Omega(P, t-1), \\ E(\mathbf{1}) \ &= \ |L| \ = \ \Omega(P,t). \end{aligned} \tag{4.8}$$

It then follows from the FKG inequality (Theorem 4.3), that

$$\Omega(P,t-2) \cdot \Omega(P,t) \ \leq \ \Omega(P,t-1) \cdot \Omega(P,t-1),$$

and the theorem now follows by substituting $t \to t+1$. □

4.5. **Strict log-concavity.** We can now prove that the inequality in Theorem 4.7 is always strict, by applying the FKG inequality in a more careful manner. This theorem will also proved useful in §4.8 to establish the strict asymptotic version of Graham's Conjecture 4.19.

Theorem 4.8. *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for every integer $t \geq 2$, we have:*

$$\Omega(P,t)^2 \ \geq \ \left(1 + \frac{1}{(t+1)^{n+1}}\right) \Omega(P,t+1) \ \Omega(P,t-1). \tag{4.9}$$

Let $Y = X$ and $Z = \varnothing$. Let $\mathcal{L} = (L, \prec^\diamond)$ be Shepp's lattice, let $\mu = \mu_{Y,Z} : L \to \{0,1\}$ be the log-supermodular function defined in §4.1, and let $t \geq 3$. Without loss of generality, assume that $X = [n]$ and that this is a *natural labeling* of X, i.e. $i < j$ for all $i \prec j$.

For all $1 \le i \le n$, let $g_i, h_i : L \to \{0,1\}$ be two functions given by

$$g_i(\boldsymbol{v}) \ := \ \begin{cases} 1 & \text{if } \ v_i \ge 2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad h_i(\boldsymbol{v}) \ := \ \begin{cases} 1 & \text{if } \ v_i \le t-1 \\ 0 & \text{otherwise.} \end{cases}$$

In notation of the proof of Theorem 4.7, we have $g = g_1 \cdots g_n$ and $h = h_1 \cdots h_n$.

It follows from the same argument above, that g_i are $\prec^\diamond$-increasing, while h_i are $\prec^\diamond$-increasing, for all $1 \le i \le n$. We now show that g_i and h_i are log-supermodular functions. Indeed, note that

$$v_i \ge 2 \ \text{ and } \ w_i \ge 2 \qquad \Longleftrightarrow \qquad \max\{v_i, w_i\} \ge 2 \ \text{ and } \ \min\{v_i, w_i\} \ge 2.$$

This implies that $g_i(\boldsymbol{v})\, g_i(\boldsymbol{w}) \ = \ g_i(\boldsymbol{v} \wedge \boldsymbol{w})\, g_i(\boldsymbol{v} \vee \boldsymbol{w})$, as desired. The same argument implies that h_i is also a log-modular function.

Lemma 4.9.

$$\frac{E_\mu(g_1 \cdots g_n \cdot h)}{E_\mu(g_1 \cdots g_n)} \ \le \ \frac{E_\mu(g_n h)}{E_\mu(g_n)}. \tag{4.10}$$

Proof. Let $\mu_i : L \to \mathbb{R}$ be given by $\mu_i := (g_i \cdots g_n)\mu$, for all $1 \le i \le n$, and let $\mu_{n+1} := \mu$. Note that function μ_i is a log-supermodular since it is a product of log-modular and log-supermodular functions. Therefore, for all $2 \le i \le n$, we have:

$$\frac{E_{\mu_i}(g_{i-1} h)}{E_{\mu_i}(g_{i-1})} \ \le \ \frac{E_{\mu_i}(h)}{E_{\mu_i}(\mathbf{1})} \ = \ \frac{E_{\mu_{i+1}}(g_i h)}{E_{\mu_{i+1}}(g_i)}\,, \tag{4.11}$$

where the inequality is by the FKG inequality (Theorem 4.3) applied to the $\prec^\diamond$-increasing function g_{i-1}, to the $\prec^\diamond$-decreasing function h, and to the log-supermodular function μ_i. We conclude:

$$\frac{E_\mu(g_1 \cdots g_n \cdot h)}{E_\mu(g_1 \cdots g_n)} \ = \ \frac{E_{\mu_2}(g_1 h)}{E_{\mu_2}(g_1)} \le \ \frac{E_{\mu_{n+1}}(g_n h)}{E_{\mu_{n+1}}(g_n)} \ = \ \frac{E_\mu(g_n h)}{E_\mu(g_n)}\,,$$

where the inequality is by consecutive applications of (4.11). □

Now, let $\eta_i := (h_i \cdots h_n)\mu$ for all $1 \le i \le n$, and let $\eta_{n+1} := \mu$. Again, note that η_i is a log-supermodular function. Observe that $E_{\eta_i}[g_n] = E_{\eta_{i+1}}[g_n h_i]$ for all $2 \le i \le n$. This implies that

$$\frac{E_\mu(g_n h)}{E_\mu(g_n)} \ = \ \prod_{i=2}^{n+1} \frac{E_{\eta_i}(g_n h_{i-1})}{E_{\eta_i}(g_n)}\,. \tag{4.12}$$

We apply two different inequalities to the RHS of (4.12). First, for all $2 \le i \le n$, we have:

$$\frac{E_{\eta_i}(g_n h_{i-1})}{E_{\eta_i}(g_n)} \ \le \ \frac{E_{\eta_i}(h_{i-1})}{E_{\eta_i}(\mathbf{1})} \ = \ \frac{E_\mu(h_{i-1} h_i \ldots h_n)}{E_\mu(h_i \ldots h_n)}\,, \tag{4.13}$$

where the inequality is due to the FKG inequality (Theorem 4.3) applied to the $\prec^\diamond$-increasing function g_n, to the $\prec^\diamond$-decreasing function h_{i-1}, and to the log-supermodular function η_i. Although (4.13) holds for $i = n+1$ by the same argument, we will use the following stronger inequality instead.

Lemma 4.10.

$$\frac{E_{\eta_{n+1}}(g_n h_n)}{E_{\eta_{n+1}}(g_n)} \ \le \ \left(1 - \frac{1}{t^{n+1}}\right) \frac{E_\mu(h_n)}{E_\mu(\mathbf{1})}. \tag{4.14}$$

Proof. By a direct calculation, the claim is equivalent to showing that

$$\frac{E_\mu(g_n)\, E_\mu(h_n) \ - \ E_\mu(g_n h_n)\, E_\mu(\mathbf{1})}{E_\mu(g_n)\, E_\mu(h_n)} \ \ge \ \frac{1}{t^{n+1}}\,.$$

Let $g'_n, h'_n : L \to \{0,1\}$ be given by $g'_n(\boldsymbol{v}) := 1 - g_n(\boldsymbol{v})$ and $h'_n(\boldsymbol{v}) := 1 - h_n(\boldsymbol{v})$. Then we have:

$$g'_n(\boldsymbol{v}) \ = \ \begin{cases} 1 & \text{if } \ v_n \ = \ 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad h'_n(\boldsymbol{v}) \ = \ \begin{cases} 1 & \text{if } \ v_n \ = \ t \\ 0 & \text{otherwise.} \end{cases}$$

By the linearity of expectations, the claim is then equivalent to showing that

$$\frac{E_\mu(g'_n)\, E_\mu(h'_n) \ - \ E_\mu(g'_n h'_n)\, E_\mu(\mathbf{1})}{E_\mu(g_n)\, E_\mu(h_n)} \ \geq \ \frac{1}{t^{n+1}}. \tag{4.15}$$

Now note that, since n is a maximal element of P, we have:

$$\begin{aligned} E_\mu(g'_n) \ &= \ \big|\big\{\boldsymbol{v} \in L \, : \, v_n \ = \ 1\big\}\big| \ \geq \ 1, \\ E_\mu(h'_n) \ &= \ \big|\big\{\boldsymbol{v} \in L \, : \, v_n \ = \ t\big\}\big| \ = \ \Omega(P \smallsetminus \{n\}, t), \\ E_\mu(g'_n h'_n) \ &= \ \big|\big\{\boldsymbol{v} \in L \, : \, v_n \ = \ 1 \ = \ t\big\}\big| \ = \ 0, \\ E_\mu(g_n) \ &= \ \big|\big\{\boldsymbol{v} \in L \, : \, v_n \ \geq \ 2\big\}\big| \ \leq \ t\, \Omega(P \smallsetminus \{n\}, t), \\ E_\mu(h_n) \ &= \ \big|\big\{\boldsymbol{v} \in L \, : \, v_n \ \leq \ t-1\big\}\big| \ \leq \ \Omega(P, t) \ \leq \ t^n. \end{aligned}$$

The inequalities above directly imply (4.15). □

Proof of Theorem 4.8. Combining (4.10), (4.12), (4.13), and (4.14), we get:

$$\frac{E_\mu(gh)}{E_\mu(g)} \ \leq \ \left(1 \ - \ \frac{1}{t^{n+1}}\right) \frac{E_\mu(h)}{E_\mu(\mathbf{1})}\,.$$

Using the values from (4.8), we obtain:

$$\frac{\Omega(P, t-2)}{\Omega(P, t-1)} \ \leq \ \left(1 \ - \ \frac{1}{t^{n+1}}\right) \frac{\Omega(P, t-1)}{\Omega(P, t)}\,.$$

The theorem now follows by substituting $t \leftarrow t+1$. □

Remark 4.11. The term $\big(1 + 1/t^{n+1}\big)$ in (4.14) is far from optimal and can be improved in many cases. In particular, note that in the proof of Lemma 4.10 we used a separate calculation for an element n in $X = [n]$. Making this calculation for a general element $x \in [n]$, gives a lower bound with the term $\big(1 + C/t^{\ell(x)+b(x)}\big)$, for some $C > 0$. Thus $x = n$ is the *least optimal choice* for the lower bound, and is made for clarity.

4.6. **Kahn–Saks conjecture.** The following interesting conjecture can be found in the solution to Exc. 3.163(b) in [Sta12].

Conjecture 4.12 (Kahn–Saks monotonicity conjecture). *For a poset $P = (X, \prec)$ with $|X| = n$ elements, the scaled order polynomial $\Omega(P, t)/t^n$ is weakly decreasing on $\mathbb{N}_{\geq 1}$.*

As Stanley points out in [Sta12, Exc. 3.163(a)], the conjecture holds for t large enough, since the coefficient $[t^{n-1}]\,\Omega(P, t) > 0$. Curiously, the proof is based on an elegant direct injection. Now, to fully appreciate the power of this conjecture, let us derive from it the following unusual extension of Theorem 1.2.

Theorem 4.13. *Let $P = (X, \prec)$, let $\max(P) \subseteq X$ be the subset of maximal elements, and let $r := |\max(P)|$ be the number of maximal elements. If Conjecture 4.12 holds, then we have:*

$$\Omega(P, t) \ \geq \ t^r \prod_{x \in X \smallsetminus \max(P)} \left(\frac{t}{b(x)} \ + \ \frac{1}{2}\right). \tag{4.16}$$

Compared to (1.2), the inequality (4.16) adds $\frac{1}{2}$ to every term in the product. It would be interesting to prove this result unconditionally.[2]

Proof. Denote

$$F_m(t) \ := \ \frac{1}{t^m} \sum_{k=1}^{t} k^m \,.$$

Let us prove now, that if the Kahn–Saks monotonicity conjecture holds, then we have:

$$\Omega(P,t) \ \geq \ \prod_{x\in X} F_{b(x)-1}(t)\,. \tag{4.17}$$

To see this, first suppose that $r=1$, so the poset P has a unique maximal element x. Thus, $b(x)=n$ in this case. The number of order preserving functions for which x has value k is equal to $\Omega(P \smallsetminus x, t-k+1)$. We have:

$$\Omega(P,t) \ = \ \sum_{k=1}^{t} \Omega(P \setminus x, k) \ \geq \ \sum_{k=1}^{t} \Omega(P \smallsetminus x, t)\, \frac{k^{n-1}}{t^{n-1}} \ = \ \Omega(P \smallsetminus x, t)\, F_{b(x)-1}(t),$$

where the inequality follows from the conjectured monotonicity. When $r \geq 2$, the rest of the proof of (4.17) follows verbatim the proof of Theorem 1.2.

Now note the following bounded version of the *Faulhaber's formula*:

$$F_m(t) \ \geq \ \frac{t}{m} \, + \, \frac{1}{2} \quad \text{for all} \ \ m \geq 2.$$

This inequality is well-known and can be easily proved by induction. Substituting it into (4.17), gives the result. $\square$

In support of this conjecture we prove the following partial result.

Proposition 4.14. *Let $P=(X,\prec)$ be a poset with $|X|=n$ elements, and let $k,t \in \mathbb{N}_{\geq 1}$. Then:*

$$\frac{1}{t^n}\, \Omega(P,t) \ \geq \ \frac{1}{(kt)^n}\, \Omega(P,kt)\,.$$

Moreover, there is an injection which shows that the function $\Omega(P,t)\,k^n - \Omega(P,kt) \in \#\mathrm{P}$.

Proof. Let $f \in \mathbf{\Omega}(P,kt)$. Consider an increasing function $g \in \mathbf{\Omega}(P,t)$ given by

$$g(x) \ := \ \left\lfloor \frac{f(x)-1}{k} \right\rfloor \, + \, 1,$$

and let $\beta : X \to \{0,1,\ldots,k-1\}$ be given as the residue of $f(x)$ modulo k. It is clear that the pair (g,β) uniquely determines f. Then $\Omega(P,t)\,k^n - \Omega(P,kt)$ is the number of pairs (g,β), such that the map $h : X \to [n]$ given by $h(x) := k(g(x)-1)+\beta(x)+1$ is not a linear extension, i.e. if $h(x) > h(y)$ for some $x \prec y$. The last condition can be verified in polynomial time, proving that the difference is in #P. $\square$

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[2]It would be even more interesting to *disprove* it, perhaps.

4.7. **Reverse monotonicity.** The following result at first appears counterintuitive until one realizes that it's trivial asymptotically, when $t \to \infty$. Just like the Kahn–Saks monotonicity conjecture, the small values of t is where the difficulty occurs.

Theorem 4.15. *Let $P = (X, \prec)$ be a finite poset of width w. Then the function $\Omega(P,t)/t^w$ is weakly increasing on all $t \in \mathbb{N}_{\geq 1}$.*

The proof is based on yet another application of the FKG inequality in the following lemma of independent interest.

Lemma 4.16. *Let $P = (X, \prec)$ be a finite poset and let $t \geq k \geq 1$ be positive integers. Then, for every minimal element x of P, we have:*

$$\frac{\Omega(P,k)}{\Omega(P,t)} \;\leq\; \frac{\Omega(P \setminus x, k)}{\Omega(P \setminus x, t)}\,. \tag{4.18}$$

Proof. Let $Y = \{x\}$ and $Z = X \setminus \{x\}$. When x is incomparable to every element of Z, we have

$$\frac{\Omega(P,k)}{\Omega(P,t)} \;=\; \frac{k\,\Omega(P \setminus x, k)}{t\,\Omega(P \setminus x, t)}\,, \tag{4.19}$$

and the result follows.

Thus, without loss of generality we can assume that $x \prec z$ for some $z \in Z$. Let $g, h : L \to \{0,1\}$ be given by

$$g(\boldsymbol{v}) \;:=\; \begin{cases} 1 & \text{if } \; v_x \,\leq\, v_z \;\; \text{ for all } \;\; z \in B(x),\, z \neq x \\ 0 & \text{otherwise,} \end{cases}$$

$$h(\boldsymbol{v}) \;:=\; \begin{cases} 1 & \text{if } \; v_z \leq k \;\; \text{ for all } \;\; z \in Z \\ 0 & \text{otherwise.} \end{cases}$$

To show that g is $\prec^\diamond$-decreasing, it suffices to check that,

$$g(\boldsymbol{w}) = 1 \quad \text{and} \quad \boldsymbol{v} \prec^\diamond \boldsymbol{w} \quad \Longrightarrow \quad g(\boldsymbol{v}) = 1.$$

Note that for every $z \in Z$ such that $x \prec z$, we have:

$$v_x \;\leq\; w_x \;\leq\; w_z \;\leq\; v_z,$$

where the first and the third inequality is because $\boldsymbol{v} \prec^\diamond \boldsymbol{w}$, and the second inequality is because $g(\boldsymbol{w}) = 1$. This implies that $g(\boldsymbol{v}) = 1$, and thus g is a $\prec^\diamond$-decreasing function.

Similarly, to show that h is $\prec^\diamond$-increasing, it suffices to check that

$$h(\boldsymbol{v}) = 1 \quad \text{and} \quad \boldsymbol{v} \prec^\diamond \boldsymbol{w} \quad \Longrightarrow \quad h(\boldsymbol{w}) = 1.$$

Note that for every $z \in Z$, we have:

$$w_z \;\leq\; v_z \;\leq\; k,$$

where the first inequality is because $\boldsymbol{v} \prec^\diamond \boldsymbol{w}$, and the second inequality is because $h(\boldsymbol{v}) = 1$. This implies that $h(\boldsymbol{w}) = 1$, and thus h is a $\prec^\diamond$-increasing function.

Now observe that the um $E(gh)$ counts $\boldsymbol{v} \in L$ for which $v_x \leq k$. Indeed, by the assumption there exist $z \in Z$, such that $x \prec z$. This implies

$$v_x \;\leq\; v_z \;\leq\; k,$$

where the first inequality is because $g(\boldsymbol{v}) = 1$, and the second inequality is because $h(\boldsymbol{v}) = 1$. Thus, $E(gh)$ counts the number of order preserving maps $f : X \to [k]$, i.e. $E(gh) = \Omega(P,k)$. It is also straightforward to verify that

$$E(g) \,=\, \Omega(P,t), \quad E(h) \,=\, t\,\Omega(P \setminus x, k) \quad \text{and} \quad E(\mathbf{1}) \,=\, t\,\Omega(P \setminus x, t).$$

The lemma now follows from the FKG inequality (Theorem 4.3). $\square$

Proof of Theorem 4.15. Let H be a maximal antichain in P of width w. Note that Lemma 4.16 can also be applied to maximal elements x of P, by considering the dual poset P^*. Now, by consecutively removing elements in $X \setminus H$, by Lemma 4.16, we get

$$\frac{\Omega(P,k)}{\Omega(P,t)} \;\leq\; \frac{\Omega(P',k)}{\Omega(P',t)} \;=\; \frac{k^w}{t^w}\,,$$

where $P' = P|_H$ is the subposet of P restricted to H. This implies the result. □

We conclude with another conjecture motivated by (4.19) in the proof of Lemma 4.16.

Conjecture 4.17. *Let $P = (X, \prec)$ be a finite poset, and let $t \geq k \geq 1$ be positive integers. Then there exists $x \in X$, such that*

$$\frac{\Omega(P,k)}{\Omega(P,t)} \;\geq\; \frac{k\,\Omega(P\setminus x,k)}{t\,\Omega(P\setminus x,t)}. \tag{4.20}$$

Proposition 4.18. *Conjecture 4.17 implies Conjecture 4.12.*

The proof of this proposition follows the proof of the theorem above.

4.8. **Daykin–Daykin–Paterson inequality.** Let $P = (X, \prec)$ on $|X| = n$ elements. Fix an element $x \in X$ and an integer $t \geq 1$. Denote by $\Omega(P,t;x,a)$ the number of order preserving maps $g : X \to [t]$, such that $g(x) = a$. The following result resolves a conjecture by Graham in [Gra83, p. 129], made by analogy with Stanley's inequality (1.9).

Theorem 4.19 (Daykin, Daykin and Paterson [DDP84], formerly Graham's conjecture)**.** *Let $P = (X, \prec)$ be a finite poset, and let $x \in X$, let $a, t \in \mathbb{N}_{\geq 1}$, and suppose $1 < a < t$. Then:*

$$\Omega(P,t;x,a)^2 \;\geq\; \Omega(P,t;x,a+1)\,\cdot\,\Omega(P,t;x,a-1). \tag{4.21}$$

The proof in [DDP84] is based on a direct injection (see §9.6). Curiously, we can use Theorem 4.8 to show that (4.21) holds asymptotically.

Corollary 4.20. *Let $P = (X, \prec)$ be a finite poset, and let $x \in X$. Then, for every integer $a \geq 1$, there exists $T(P,x,a) > 0$, such that for all $t > T(P,x,a)$, we have:*

$$\Omega(P,t;x,a)^2 \;\geq\; \Omega(P,t;x,a+1)\,\cdot\,\Omega(P,t;x,a-1).$$

Furthermore, if x is incomparable to any other element of P, then the inequality above is strict for sufficiently large t.

Proof. Fix an integer $a \geq 1$. First note that, if x is incomparable to every other element in P, then equality in fact occurs in the inequality (4.21). So we can assume that x is comparable to some other element y of P, and we will further assume that $y \prec x$, as the proof for the other case is analogous. Denote by $D := \{y \in X \,:\, y \preccurlyeq x\}$ the lower order ideal of x, and let $d := |D|$. Note that $D - x$ is a non-empty set by assumption. Now observe that $\Omega(P,t;x,a)$ is a polynomial in t with the leading term

$$\Omega(D-x,a)\,\frac{e(P \smallsetminus D)}{(n-d)!}\; t^{n-d}\,.$$

Indeed, for every $\prec$-increasing function $g : X \to [t]$, we have $g(y) \leq a$ for all $y \in D$, which explains the term $\Omega(D-x,a)$. For the remaining elements $z \in X \smallsetminus D$, we have no such restrictions as $t \to \infty$, and the number of such functions is asymptotically $\sim e(P \smallsetminus D)\, t^{n-d}/(n-d)!$

Therefore, as $t \to \infty$, the leading coefficient of the polynomial

$$\Omega(P,t;x,a)^2 \,-\, \Omega(P,t;x,a+1)\,\cdot\,\Omega(P,t;x,a-1)$$

is equal to

$$\Big[\Omega(D-x,a)^2 \,-\, \Omega(D-x,a+1)\,\Omega(D-x,a-1)\Big]\,\frac{e(P-D)}{(n-d)!}\,.$$

This is strictly positive by Theorem 4.8 (note that $D-x$ is a non-empty poset by assumption), which implies the result. □

5. Bounds on the q-analogue

In this section we study the q-order polynomial generalization. First, we present Reiner's short proof of the Björner–Wachs inequality. Then, we give q-analogue of Shepp's inequality and study its consequences.

5.1. **Reiner's inequality.** Most recently, Vic Reiner shared with us the following elegant approach to the Björner–Wachs inequality which we reproduce with his permission.[3]

Theorem 5.1 (Reiner, 2022)**.** *Let $P=(X,\prec)$ be a poset with $|X|=n$ elements. Denote by $\mathcal{R}(P)$ the set of all weakly order-preserving maps $g:X\to\mathbb{N}$, i.e. $g(x)\le g(y)$ for all $x\prec y$. Let $|g|:=\sum_{x\in X} g(x)$. Then:*

$$\sum_{g\in\mathcal{R}(P)} q^{|g|} \;\geqslant_q\; \prod_{x\in X}\frac{1}{1-q^{b(x)}}\,, \tag{5.1}$$

where the inequality between two power series is coefficient-wise.

Theorem 1.1 follows from *P-partition theory* of Stanley, see [Sta12, §3.15]. Indeed, recall that

$$\sum_{g\in\mathcal{R}(P)} q^{|g|} \;=\; \frac{F_P(q)}{(1-q)(1-q^2)\cdots(1-q^n)}\,,$$

such that $F_P(1)=e(P)$. Here $F_P(q)$ denotes the sum of $q^{\mathrm{maj}(f)}$ over all $f\in\mathcal{E}(P)$, see [Sta12] for the details.[4] Taking $0<q<1$, multiplying both sides of (5.1) by $(1-q)(1-q^2)\cdots(1-q^n)$, and taking the limit $q\to 1-$, gives the Björner–Wachs inequality (1.1).

Proof of Theorem 5.1. Interpret the RHS of (5.1) as the GF for maps $g\in\mathcal{R}(P)$ which are obtained as a nonnegative integer linear combination of the characteristic functions of upper order ideals:

$$g \;=\; \sum_{x\in X} m(x)\,\chi_{B(x)}\,, \quad \text{where} \;\; m(x)\in\mathbb{N} \;\; \text{for all} \;\; x\in X.$$

Note that characteristic functions $\chi_{B(x)}$ are linearly independent because in the standard basis χ_y, $y\in X$, the transition matrix is unitriangular. Since now $|g|=\sum_{x\in X} m(x)\,b(x)$, the result follows immediately. □

Example 5.2. Consider a poset $P=(X,\prec)$ with $X=\{a,b,c,d\}$ and $a\prec\{b,c\}\prec d$, so $P\simeq C_2\times C_2$. The RHS of (5.1) as in the proof above is the GF for $g\in\mathcal{R}(P)$, such that $g(a)+g(d)\ge g(b)+g(c)$. Not all $g\in\mathcal{R}(P)$ satisfy this property, e.g. $g(a)=0$, $g(b)=g(c)=g(d)=1$ does not.

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[3] Vic Reiner, personal communication, March 17, 2022.

[4] The notation in [Sta12] is different but equivalent; we change it for simplicity since the *major index* plays only tangential role in this paper. There is also a minor subtlety here, that *Stanley's P-partition theory* needs to be applied to a *natural labeling* of X, cf. §6.

Remark 5.3. In principle, there is a way to convert the natural injection as in the proof above into an injection as in Proposition 3.1. The idea is to make the multiplication by $(1-q)\cdots(1-q^n)$ to be effective by using the *involution principle* of Garsia and Milne [GM81]. See also [Gre88] which comes closest in this special case. Note that the resulting maps tend to be hard to compute, sometimes provably so, see e.g. [KP09].

5.2. **q-order polynomial.** For an integer $t \geq 1$, define

$$\Omega_q(P,t) \,:=\, \sum_g \, q^{|g|-n}$$

where the summation is over all order preserving maps $g : X \to [t] = \{1,\ldots,t\}$, i.e. maps which satisfy $g(x) \leq g(y)$ for all $x \prec y$. This is the *q-order polynomial* corresponding to poset P, see e.g. [Cha16]. Let us emphasize that here q is a formal variable, while $t \geq 1$ is an integer.

Theorem 5.4 (q-analogue of Shepp's inequality)**.** *Let A be the collection defined in §4.2, and let C, C' be Y-minimizing collections w.r.t. partition $X = Y \sqcup Z$. Then,*

$$\Omega_q\big(C\cap C'\cap A, t\big) \cdot \Omega_q\big(A,t\big) \quad \geqslant_q \quad \Omega_q\big(C\cap A, t\big) \cdot \Omega_q\big(C'\cap A, t\big)\,,$$

where the inequality holds coefficient-wise as a polynomial in q, for all integer $t \geq 1$.

The proof follows the original proof in [She80], with the following *q-FKG inequality* by Björner [Bjö11]. Let $L := (L, \prec^\diamond)$ be a distributive lattice. A function $r : L \to \mathbb{R}_{\geq 0}$ is called *modular* if

$$r(a) \,+\, r(b) \;=\; r(a\wedge b) \,+\, r(a \vee b) \qquad \text{for every } \ a,b\in L.$$

Theorem 5.5 (q-FKG inequality, [Bjö11, Thm 2.1])**.** *Let $\mathcal{L} = (L,\prec)$ be a finite distributive lattice, let $\mu : L \to \mathbb{R}_{\geq 0}$ be a log-supermodular function, and let $r : L \to \mathbb{R}_{\geq 0}$ be a modular function. Then, for every pair of $\prec^\diamond$-decreasing functions $g,h : L \to \mathbb{R}_{\geq 0}$, we have:*

$$E_q(\mathbf{1})\, E_q(gh) \;\geqslant_q\; E_q(g)\, E_q(h),$$

where the inequality holds coefficient-wise as a polynomial in q, where

$$E_q(g) \;=\; E_q(g;\mu,r) \;:=\; \sum_{x\in L} g(x)\,\mu(x)\, q^{r(x)},$$

and $\mathbf{1} : L \to \mathbb{R}$ is given by $\mathbf{1}(x) = 1$ for all $x \in L$.

Remark 5.6. Note that Theorem 2.1 in [Bjö11] assumes that $r : L \to \mathbb{R}_{\geq 0}$ is the *rank function* of the lattice L. It is however straightforward to show that the same proof still works when applied to any modular function r.

Proof of Theorem 5.4. The proof follows the same argument as in the proof of Lemma 4.4, with the FKG inequality being replaced with Theorem 5.5 applied to the modular function $r : L \to \mathbb{R}_{\geq 0}$ given by $r(\boldsymbol{v}) \,:=\, \sum_{x\in X} v_x\,.$ □

Corollary 5.7. *Let $P = (X,\prec)$ be a poset, and let $x,y \in X$ be minimal elements. Then, for all $t \in \mathbb{N}_{\geq 1}$ and $q \in \mathbb{R}_+$, we have:*

$$\Omega_q\big(P,t\big) \,\cdot\, \Omega_q\big(P \smallsetminus \{x,y\}, t\big) \;\geq\; \Omega_q\big(P\smallsetminus x, t\big) \,\cdot\, \Omega_q\big(P \smallsetminus y, t\big).$$

Proof. Denote $(n)_q := 1 + q + \ldots + q^{n-1}$. Let C and C' be as in (4.4), and A be as in (4.2). Observe that

$$\Omega_q(C \cap C' \cap A, t) \ = \ \Omega_q(P, t), \qquad \Omega_q(C' \cap A, t) \ = \ q(t)_q\, \Omega_q(P \smallsetminus x, t),$$
$$\Omega_q(C \cap A, t) \ = \ q(t)_q\, \Omega_q(P \smallsetminus y, t), \qquad \Omega_q(A, t) \ = \ q^2(t)_q^2\, \Omega_q(P \smallsetminus \{x, y\}, t).$$

The conclusion of the lemma now follows from Theorem 5.4 and the equation above. □

Remark 5.8. Note that our proof does not show that the inequality in Corollary 5.7 holds coefficient-wise as a polynomial in q, since the derivation involves canceling the term $q(t)_q$. It remains to be seen if a q-analogue of Theorem 1.2 exists, which hinges on finding an appropriate q-analogue for Lemma 4.5.

We also have the following *q-log-concavity* for order polynomials.

Corollary 5.9. *Let $P = (X, \prec)$ be a finite poset. Then, for every integer $t \geq 2$, we have:*

$$\Omega_q(P, t)^2 \ \geqslant_q \ \Omega_q(P, t+1) \cdot \Omega_q(P, t-1),$$

where the inequality holds coefficient-wise as a polynomial in q.

Proof. The proof follows the same argument as in the proof of Theorem 4.7, with the FKG inequality being replaced with Theorem 5.5 applied to the modular function $r : L \to \mathbb{R}_{\geq 0}$ given by $r(\boldsymbol{v}) := \sum_{x \in X} v_x$. □

6. Bounding the order polynomial by injection

Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Denote by $\boldsymbol{\Omega}(P, t)$ the set of order preserving maps $P \to [t]$, so that $\Omega(P, t) = |\boldsymbol{\Omega}(P, t)|$. Fix a *natural labeling* of X, i.e. write $X = \{x_1, \ldots, x_n\}$, where $i < j$ for all $x_i \prec x_j$.

For a sequence $(a_1, \ldots, a_k)$ of distinct integers, a *standardization* is a permutation $\sigma = (\sigma_1, \ldots, \sigma_k) \in S_k$ with integers in the same relative order:

$$a_i < a_j \ \Leftrightarrow \ \sigma_i < \sigma_j \quad \text{for all} \quad 1 \leq i < j \leq k.$$

For example, the standardization of $(4, 7, 6, 3)$ is $(2, 4, 3, 1) \in S_4$.

Proof of Theorem 1.4. We construct an injection

$$\Psi : \mathcal{E}(P) \times [t]^n \ \to \ \boldsymbol{\Omega}(P, t) \times S_n.$$

One can think of $[t]^n$ as an *ordered set partition*

$$[n] \ = \ B_1 \sqcup \ldots \sqcup B_t,$$

where $B_i \subseteq [n]$ can be empty. We use $\beta = (B_1, \ldots, B_t)$ to denote this ordered set partition.

Let $f \in \mathcal{E}(P)$ be a linear extension, and $\beta = (B_1, \ldots, B_t)$ be an ordered set partition as above. Denote $b_i := |B_i|$, where $1 \leq i \leq t$. Let $\alpha = (a_1, \ldots, a_n) \in [t]^n$ be a weakly increasing sequence

$$\big(1, \ldots, 1,\ 2, \ldots, 2,\ \ \ldots\ \ ,\ t, \ldots, t\big) \quad \text{with } b_i \text{ copies of } i, \ \text{ for all } 1 \leq i \leq t.$$

By abuse of notation, we also use α to denote a function $\alpha : [n] \to [t]$ given by $\alpha(i) := a_i$.

Define a function $g : X \to [t]$ as $g(x_i) := \alpha\big(f(x_i)\big)$, so that elements $f^{-1}(1), \ldots, f^{-1}(b_1)$ are assigned value 1, elements $f^{-1}(b_1 + 1), \ldots, f^{-1}(b_1 + b_2)$ are assigned value 2, etc. Observe that $g \in \boldsymbol{\Omega}(P, t)$ since f is increasing with respect to the poset order, and α is a weakly increasing function.

Next, define a permutation $\sigma \in S_n$ as follows. For each i, let $g^{-1}(i) = \big\{x_{i_1}, \ldots, x_{i_k}\big\}$, where $i_1 < \ldots < i_k$ and $k = b_i$ by construction. Let $s^{(i)} \in S_k$ be the standardization of the sequence

$\big(f(x_{i_1}),\ldots,f(x_{i_k})\big)$. Now, rearrange the elements in B_i according to $s^{(i)}$, obtaining a sequence γ_i whose standardization is $s^{(i)}$. The permutation σ is then obtained by concatenating the resulting sequences, i.e. $\sigma := \gamma_1\gamma_2\ldots\gamma_t \in S_n$. Finally, define $\Psi(f,\beta) := (g,\sigma)$.

To prove that Ψ is an injection, we construct an inverse map Ψ^{-1}. Let $g \in \Omega(P,t)$ and $\sigma \in S_n$. Denote $c_i := |g^{-1}(i)|$, for all $1 \le i \le t$. Let $\tau \in [t]^n$ be the sorted sequence of values that the function g takes, i.e.

$$\tau \ := \ \big(1,\ldots,1,\ 2,\ldots,2,\ \ \ldots\ \ ,\ t,\ldots,t\big) \quad \text{with } c_i \text{ copies of } i, \ \text{ for all } 1 \le i \le t.$$

Note that τ is the weakly increasing. For each i, let

$$C_i \ := \ \big\{\sigma_{c_1+\ldots+c_{i-1}+1}\,,\ \ldots\ ,\ \sigma_{c_1+\ldots+c_i}\big\}$$

consisting of a block of size c_i of entries from σ. Denote by $\pi = (C_1,\ldots,C_t)$ the resulting ordered partition.

Finally, define a function $h : X \to [n]$ obtained by rearranging the values on the c_i elements in $g^{-1}(i)$ according to the ordering in

$$\big(\sigma_{c_1+\ldots+c_{i-1}+1}\,,\ \ldots\ ,\ \sigma_{c_1+\ldots+c_i}\big),$$

i.e., so that their standardizations are the same permutations. Let us emphasize that h is not necessarily a linear extension for general (g,σ) as above.

Now take $\Psi^{-1} := (h,\pi)$, and observe that

$$\Psi^{-1}\big(\Psi(f,\beta)\big) \ = \ (f,\beta)$$

by construction. This completes the proof. □

Example 6.1. Let us illustrate the construction of $\Psi(f,\beta) = (g,\sigma)$ in the proof above. Let $P = (X,\prec)$ be a poset on $n = 7$ elements as in Figure 6.1, where $X = \{x_1,\ldots,x_7\}$ with the partial order $\prec$ increasing downwards. Note that we chose a natural labeling, see above.

Suppose $t = 3$. Let $f \in \mathcal{E}(P)$ be a linear extension as in the figure, and let $\beta = (B_1,B_2,B_3)$, where $B_1 = \{2,3,7\}$, $B_2 = \{4,6\}$ and $B_3 = \{1,5\}$. Then we have $\alpha = (1,1,1,2,2,3,3)$ and the order preserving function g is given as in the figure. Then, standardize the values

$$\big(f(x_1),f(x_3),f(x_5)\big) = (2,1,3) \ \longrightarrow \ (2,1,3)\,,$$
$$\big(f(x_2),f(x_7)\big) = (4,5) \ \longrightarrow \ (1,2)\,, \qquad \big(f(x_4),f(x_6)\big) = (6,7) \ \longrightarrow \ (1,2).$$

Permute the elements within B_1,B_2,B_3 accordingly to get $\gamma_1 = (3,2,7)$, $\gamma_2 = (4,6)$ and $\gamma_3 = (1,5)$. Concatenating these, we obtain $\sigma = (3,2,7,4,6,1,5) \in S_7$.

In the opposite direction, let $g' \in \mathbf{\Omega}(P,3)$ be as in Figure 6.1, and let $\sigma = (7,1,3,2,5,4,6)$. Then $\tau = (1,1,2,2,3,3,3)$, so $c_1 = c_2 = 2$ and $c_3 = 3$. This gives $C_1 = \{1,7\}$, $C_2 = \{2,3\}$ and $C_3 = \{4,5,6\}$. The corresponding reduced permutations are then $(2,1)$, $(2,1)$ and $(2,1,3)$, respectively, giving a map $h : X \to [t]$. Finally, note that $h \notin \mathcal{E}(P)$ in this case.

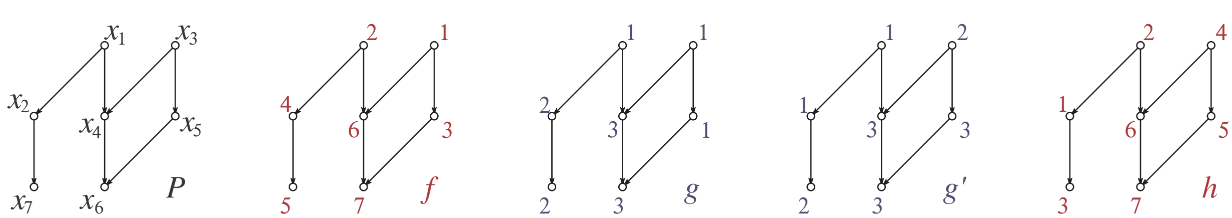

Figure 6.1. An example of the injection Ψ and the inverse map Ψ^{-1}.

Proof of Theorem 1.13. In the notation of the proof above, let $(g,\sigma) \in \mathbf{\Omega}(P,t) \times S_n$ and let $(h,\pi) = \Psi^{-1}(g,\sigma)$. By construction, we have $(h,\pi) \in \mathcal{E}(P) \times [t]^n$ if and only if $h \in \mathcal{E}(P)$. Thus, the function $\zeta(P,t)$ is equal to the number of $(g,\sigma) \in \mathbf{\Omega}(P,t) \times S_n$ such that $h \notin \mathcal{E}(P)$. Since Ψ^{-1} can be computed in polynomial time, this implies the result. $\square$

Example 6.2. Let $P = A_n$ be an antichain on n elements. Then we have $e(A_n) = n!$ and $\Omega(A_n,t) = t^n$. In this case both (1.2) and (1.4) are equalities. Similarly, let $P = C_n$ be a chain of n elements. Then we have $e(C_n) = 1$ and $\Omega(P,t) = \binom{t+n-1}{n}$. In this case, the lower bound (1.2) is slightly better than (1.4).

In a different direction, here are equality conditions for (1.4) in Theorem 1.4.

Corollary 6.3. *Let $P = (X,\prec)$ be a poset on $|X| = n$ elements. Then*

$$\Omega(P,t) \ = \ e(P)\,\frac{t^n}{n!}$$

for some $t \in \mathbb{N}_{\geq 1}$ if and only if $P = A_n$ is an n-antichain.

Proof. The "if" part is clear. For the "only if" part, suppose that $P \neq A_n$. Then there are $x_i, x_{i+1} \in X$ such that $x_i \prec x_{i+1}$, and x_{i+1} covers x_i. Without loss of generality, we can assume that x_i is a minimal element.

It follows from the proof of Theorem 1.4, that equality in (1.4) holds if and only if Ψ is a bijection. In particular, for all $(g,\sigma) \in \mathbf{\Omega}(P,t) \times S_n$ the map h in $\Psi^{-1}(g,\sigma) = (h,\pi)$ must be a linear extension. Now take an order preserving map g, such that $g(x_1) = \ldots = g(x_{i+1}) = 1$, and $\sigma = (i,i+1)$. Then we have $h(x_i) = i+1$ and $h(x_{i+1}) = i$, so that $h \notin \mathcal{E}(P)$ is not a linear extension. Thus, map Ψ is not a bijection in this case. This completes the proof. $\square$

Example 6.4. Let $P_n = C_1 \oplus A_{n-1}$ be an ordered tree poset consisting of one minimal element and $(n-1)$ maximal elements. Then $e(P) = (n-1)!$ and the bound (1.1) is an equality. Observe that

$$\Omega(P_n,t) \ = \ 1^{n-1} \, + \, 2^{n-1} \, + \, \ldots \, + \, t^{n-1} \ = \ \frac{t^n}{n} + \frac{t^{n-1}}{2} + O(t^{n-1}).$$

The general inequality (1.4) gives $\Omega(P_n,t) \geq \frac{t^n}{n}$, while (1.2) gives a stronger bound:

$$\Omega(P_n,t) \ \geq \ \frac{1}{n}\big(t^n + t^{n-1}\big).$$

Asymptotically, both lower bounds are not tight in the second order term. Compare this to (4.17) which conjecturally gives a sharp bound.

Remark 6.5. Neither of the Theorems 1.2 and 1.4 imply each other. Note that the leading coefficient of $\Omega(P,t)$ is $e(P)/n!$, and the Björner–Wachs inequality (1.1) is an equality only for ordered forests (Proposition 3.2). Thus, for large values of t, the lower bound in Theorem 1.4 asymptotically better.

On the other hand, the lower bound in Theorem 1.4 cannot be improved to $t^r\,(t+1)^{n-r}\,e(P)/n!$ as Theorem 1.2 might suggest. Indeed, for the poset P_n as in the example above, we have:

$$\Omega(P_n,2) \ = \ 1^{n-1} + 2^{n-1} \ < \ \frac{2 \cdot 3^{n-1}}{n} \quad \text{ for } \ n \geq 3.$$

Finally, let us mention that the order polynomial $\Omega(P,t)$ can have negative coefficients, implying that (1.4) does not follow directly from the leading term of $t^n e(P)/n!$ For example, recall that

$$\Omega(P_5,t) \ = \ 1^4 \, + \, 2^4 \, + \, \ldots \, + \, t^4 \ = \ \frac{1}{30}\big(6t^5 + 15t^4 + 10t^3 - t\big),$$

and note the negative coefficient in t.

7. Restricted linear extensions

In this section we use an algebraic approach to obtain vanishing and uniqueness conditions for the generalized Stanley inequality. We also present a direct combinatorial argument for the uniqueness conditions.

7.1. **Background.** Before we proceed to generalizations, let us recall some definition and results about group action on the set $\mathcal{E}(P)$ of linear extensions. In our presentation we follow Stanley's survey [Sta09].

Let $X = (X, \prec)$ be a poset on $|X| = n$ elements. *Promotion* $\partial : \mathcal{E}(P) \to \mathcal{E}(P)$ is a bijection on linear extensions defined as follows. For $f \in \mathcal{E}(P)$, let $t_1 \prec \ldots \prec t_r$ be a maximal chain in P such that $f(t_1), f(t_2), \ldots, f(t_r)$ is lexicographically smallest. Define $f\partial \in \mathcal{E}(P)$ as

$$f\partial(x) = \begin{cases} f(t_{i+1}) - 1 & \text{if } \ x = t_i \ \text{ for some } \ i < r\,, \\ n & \text{if } \ x = t_r\,, \\ f(x) - 1 & \text{otherwise.} \end{cases}$$

We think of ∂ as an operator applied on the right, and write $\partial : f \mapsto f\partial$.

Evacuation $\varepsilon : \mathcal{E}(P) \to \mathcal{E}(P)$ is a another operator on linear extensions defined as follows. Denote by ∂_i as the promotion on a poset obtained by restriction to elements with f-values $1, \ldots, i$, so that $\partial_n = \partial$ and $\partial_1 = 1$. Then ε is defined as the composition $\varepsilon := \partial_n \circ \ldots \circ \partial_1$, and we write $f\varepsilon = f\partial_n \cdots \partial_1$.

The promotion and evacuation maps can be interpreted using group actions on linear extensions as follows.

Let $\mathrm{G}_n = \langle \tau_1, \ldots, \tau_{n-1} \rangle$ be an infinite Coxeter group with the relations

$$\tau_1^2 = \ldots = \tau_{n-1}^2 = 1 \qquad \text{and} \qquad \tau_i \tau_j = \tau_j \tau_i \quad \text{for all} \quad |i - j| > 1\,. \tag{7.1}$$

Note that the symmetric group S_n is a quotient of G_n. We also define elements $\delta_2, \ldots, \delta_n = \delta \in \mathrm{G}_n$ as follows:

$$\delta_k := \tau_1 \tau_2 \cdots \tau_{k-1} \quad \text{for} \ \ 1 < k \leq n\,, \quad \text{and} \quad \gamma := \delta_n \delta_{n-1} \cdots \delta_2\,.$$

Note that $\mathrm{G}_n = \langle \delta_2, \ldots, \delta_n \rangle$, and that γ is an involution: $\gamma^2 = 1$, see e.g. [Sta09, Lemma 2.2].

With every linear extension $f \in \mathcal{E}(P)$ we associate a word $\boldsymbol{x}_f = x_1 \ldots x_n \in X^*$, such that $f(x_i) = i$ for all $1 \leq i \leq n$. In the notation of the previous section, this says that $X = \{x_1, \ldots, x_n\}$ is a natural labeling corresponding to f.

We can now define the action of G_n on $\mathcal{E}(P)$ as the right action on the words $\boldsymbol{x}_f$, $f \in \mathcal{E}(P)$. For $\boldsymbol{x}_f = x_1 \ldots x_n$ as above, let

$$(x_1 \ldots x_n)\, \tau_i \ := \ \begin{cases} x_1 \ldots x_n, & \text{if } \ x_i \prec x_{i+1}\,, \\ x_1 \ldots x_{i+1} x_i \ldots x_n\,, & \text{if } \ x_i \parallel x_{i+1}\,. \end{cases} \tag{7.2}$$

Observe that if $1 = i_1 < i_2 < \cdots < i_r \leq n$ are the indices of the lexicographically smallest maximal chain in the linear extension f, then

$$(\boldsymbol{x}_f)\, \delta \ = \ (x_1 \ldots x_n)\, \delta \ = \ x_2 \ldots x_{i_2 - 1} x_{i_1} \ldots x_{i_r - 1} x_{i_{r-1}} \ldots x_{i_r} \ = \ \boldsymbol{x}_{f\partial}\,,$$

where $\delta = \delta_n$ as above and $\boldsymbol{x}_f \delta = \boldsymbol{x}_{f\partial}$ is the promotion operator.

Proposition 7.1 (see e.g. [AKS14, Prop. 4.1])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then group G_n acts transitively on $\mathcal{E}(P)$.*

Remark 7.2. The proposition is a folklore result repeatedly rediscovered in different contexts. For the early proofs and connections to Markov chains, see [KK91, Mat91]. For a brief overview of generalizations and further references, we refer to the discussion which follows Prop. 1.2 in [DK21].

7.2. **Generalization to restricted posets.** Let $P = (X, \prec)$. Fix a sequence of k elements $\boldsymbol{u} = (u_1, \ldots, u_k) \in X$ and a sequence of k distinct integers $\boldsymbol{a} = (a_1, \ldots, a_k)$, such that $1 \leq a_1 < \ldots < a_k \leq n$. A *restricted linear extension* with respect to $(\boldsymbol{u}, \boldsymbol{a})$ is a linear extension $f \in \mathcal{E}(P)$ such that $f(u_i) = a_i$ for all $1 \leq i \leq k$. As in the introduction, we denote this set by $\mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$.

Our first goal is to modify and generalize Proposition 7.1 for restricted linear extensions. For simplicity assume that $a_i + 1 < a_{i+1}$ for all $1 \leq i < k$. Otherwise, we can identify elements $u_i \sim u_{i+1}$ and consider the equivalent problem for the so obtained smaller poset. We also assume $a_1 > 1$ and $a_k < n$, since otherwise the corresponding element u_1 or u_n should be minimal/maximal and can be removed from the poset, again reducing the problem. We also assume that $u_i \prec u_{i+1}$ in the poset.

Let $A := \{a_1, \ldots, a_k\}$ and $A' = \{a_1 - 1, \ldots, a_k - 1\}$, so $A \cap A' = \varnothing$ by the assumption. Let

$$\mathrm{H}_n(\boldsymbol{a}) = \langle \tau_i, \sigma_r \ : \ 1 \leq i < n, \, i \notin A \cup A', \, 1 \leq r \leq k \rangle$$

be an infinite group with relations as in (7.1) and $\sigma_r^2 = 1$ for all $1 \leq r \leq k$. This is a free product of several infinite Coxeter groups which acts on $\mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$ as follows.

First, for all $i \notin A \cup A'$, $1 \leq i < n$, the action of τ_i defined in (7.2) can be restricted to act on $\mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$. Next, for all $1 \leq r \leq k$ and $j = a_i$ define the action of σ_i on $\mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$:

$$(x_1 \ldots x_n)\, \sigma_i \ := \ \begin{cases} x_1 \ldots x_{j+1} x_j x_{j-1} \ldots x_n & \text{if} \quad x_{j-1} \parallel x_j\,, \ \ x_{j-1} \parallel x_{j+1} \ \text{ and } \ x_j \parallel x_{j+1}\,, \\ x_1 \ldots x_{j-1} x_j x_{j+1} \ldots x_n & \text{otherwise}. \end{cases}$$

Here we continue using our convention of association of $\boldsymbol{x}_f$ with $f \in \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$.

Note that when $x_{j-1} \parallel x_j$ and $x_j \parallel x_{j+1}$ we have

$$\boldsymbol{x}\, \sigma_i \ = \boldsymbol{x}\, \tau_j \tau_{j-1} \tau_j \ = \ \boldsymbol{x}\, \tau_{j-1} \tau_j \tau_{j-1}\,.$$

However, when $x_{j-1} \prec x_j$ and $x_j \parallel x_{j+1}$ we have $\boldsymbol{x}\, \sigma_i \ = \ \boldsymbol{x}$, but

$$\boldsymbol{x}\, \tau_{j-1} \tau_j \tau_{j-1} \ \neq \ \boldsymbol{x},$$

since x_j has been moved to $(j+1)$-st position. The same property holds when $x_{j-1} \parallel x_j$ and $x_j \prec x_{j+1}$.

Example 7.3. Let us note that the action of $\mathrm{H}_n(\boldsymbol{a})$ on $\mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$ is not necessarily transitive. For example, let $P = \big(X, \prec\big)$, where $X = \{x, y, u_1, z, u_2\}$, be a poset isomorphic to $C_3 + C_1 + C_1$ with $u_1 \prec z \prec u_2$ and x, y incomparable to $\{u_1, z, u_2\}$. Now, when $a_1 = 2$ and $a_2 = 4$, the action of group $\mathrm{H}_5(\boldsymbol{a})$ has two orbits: $\{x u_1 z u_2 y\}$ and $\{y u_1 z u_2 x\}$. This shows that to generalize Proposition 7.1 we need to enlarge group $\mathrm{H}_5(\boldsymbol{a})$.

Let $\widehat{\mathrm{G}}_n = \big\langle \tau_{ij} \ : \ 1 \leq i < j \leq n \big\rangle$ be an infinite group with relations

$$\begin{aligned} \tau_{ij}^2 \ &= 1 \qquad \text{for all} \quad 1 \leq i < j \leq n, \\ \tau_{ij}\, \tau_{k\ell} \ &= \ \tau_{k\ell}\, \tau_{ij} \qquad \text{for all} \quad i < k < \ell < j \ \text{ or } \ i < j < k < \ell. \end{aligned} \tag{7.3}$$

Define the action of $\widehat{\mathrm{G}}_n$ on $\mathcal{E}(P)$ as

$$(x_1 \ldots x_n)\, \tau_{ij} \ := \ \begin{cases} x_1 \ldots x_j \ldots x_i \ldots x_n & \text{if } x_i \parallel y \text{ and } x_j \parallel y \text{ for all } y \in \{x_{i+1}, \ldots, x_{j-1}\}, \\ x_1 \ldots x_n & \text{otherwise}. \end{cases}$$

In the notation above, we have $\tau_{i\,i+1} = \tau_i$, so $\mathrm{G}_n \subset \widehat{\mathrm{G}}_n$ is a subgroup. For brevity, we write τ_i for $\tau_{i\,i+1}$ from this point on.

Finally, let $\widehat{\mathrm{G}}_n(\boldsymbol{a})$ be a subgroup of $\widehat{\mathrm{G}}$ defined as follows:

$$\widehat{\mathrm{G}}_n(\boldsymbol{a}) \ := \ \big\langle \tau_{ij} \ : \ i, j \notin A, \, 1 \leq i < j \leq n \big\rangle.$$

Theorem 7.4. *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Fix a chain of k elements $\boldsymbol{u} = (u_1, \ldots, u_k) \in X$ and an increasing sequence of k distinct integers $\boldsymbol{a} = (a_1, \ldots, a_k)$. Then group $\widehat{\mathrm{G}}_n(\boldsymbol{a})$ defined above acts transitively on $\mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$.*

Proof. Suppose $\mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a}) \neq \varnothing$. Fix $f \in \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$ and write $\boldsymbol{x}_0 = x_1 \ldots x_n$ corresponding to the natural labeling $f(x_i) = i$. For every $\boldsymbol{y} = y_1 \ldots y_n$ corresponding to a linear extension $g \in \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$, let $\operatorname{inv}(\boldsymbol{y})$ be the number of inversions in the permutation $f(\boldsymbol{y}) := \big(f(y_1), \ldots, f(y_n)\big)$. We claim that unless $\boldsymbol{y} = \boldsymbol{x}_0$, we can use operators in $\widehat{\mathrm{G}}_n(\boldsymbol{a})$ to decrease $\operatorname{inv}(\boldsymbol{y})$. Using this recursively, we can then reach $\boldsymbol{x}_0$ as the unique element for which $\operatorname{inv}(\boldsymbol{x}_0) = 0$.

Consider the permutation $w := \big(f(y_1), \ldots, f(y_n)\big)$. If $w \neq \mathbf{1}$, there exist elements $f(y_i), f(y_{i+1}) \in w$ such that $f(y_i) > f(y_{i+1})$. Then we have $y_i \parallel y_{i+1}$. We call such (y_i, y_{i+1}) a *descending pair*. Suppose there is a descending pair with $y_i, y_{i+1} \notin \boldsymbol{u}$. Then $\tau_i \in \mathrm{H}_n(\boldsymbol{a})$, and $\boldsymbol{y}\,\tau_i = \ldots y_{i+1} y_i \ldots$ Therefore, we have $\operatorname{inv}(\boldsymbol{y}\,\tau_i) = \operatorname{inv}(\boldsymbol{y}) - 1$, which proves the claim in this case.

In the remaining cases, every descending pair involves at least one element from $\boldsymbol{u}$, which are fixed points of the labeling f. Suppose there are two adjacent descending pairs, i.e. $f(y_{i-1}) > f(y_i) > f(y_{i-1})$ and $y_i \in \boldsymbol{u}$. Then we have $\operatorname{inv}(\boldsymbol{y}\tau_{i-1\,i+1}) = \operatorname{inv}(\boldsymbol{y}) - 3$, which prove the claim in this case.

Finally, suppose every descending pair involves at least one element from $\boldsymbol{u}$, and none are adjacent. Let $i_1 - 1$ be the last descent of w. Then $i_1 - 1 \in \boldsymbol{a}$, i.e. $i_1 - 1 = a_t$ for some $a_t \in \boldsymbol{a}$, and we have $a_t = w_{a_t} > w_{i_1}$. To see this, suppose the contrary that the last descent in w is at $i_1 - 1 = a_t - 1$, so $w_{i_1 - 1} > w_{a_t} = a_t$ and all elements of w after a_t are increasing. Since a_t is a fixed point and we have $n - a_t$ positions after a_t filled with numbers larger than a_t, i.e. from the interval $\{a_t + 1, \ldots, n\}$, we must have $w_i = i$ for $i = a_t, \ldots, n$, so all values $> a_t$ appear after it. Thus $w_{a_t - 1} < a_t = w_{a_t}$, reaching a contradiction, and so the last descent is at a_t.

Let $m < i_1 - 1$ be the largest value for which $w_m > i_1 - 1 = a_t$. Such value exists since by the reasoning above at least one of $\{i_1, \ldots, n\}$ appears before $i_1 - 1$. Now, form a sequence $i_1 > i_2 > \ldots > i_r > i_0 = m$, such that i_j is the largest index smaller than i_{j-1} such that $w_{i_j} < w_{i_{j-1}}$. Note that by similar interval arguments we must have that $f(y_{i_j}) < i_j$ and they do not hit any of the elements in $\boldsymbol{a}$. Note that these indices give a maximal increasing subsequence in $w_{m+1} \ldots w_n$ which ends at w_{i_1}.

Now apply $\tau_{m\, i_r}\, \tau_{i_r\, i_{r-1}} \cdots \tau_{i_2\, i_1}$, which is nontrivial as it transposes the element y_m, incomparable to all elements in positions $\in [m+1, i_1]$, with the elements $y_{i_r}, \ldots$ which are also incomparable with the elements in the corresponding interval. Note that this is the cycle permutation $(m, i_1, i_2, \ldots)$ so that y_m moves to position i_1, and the other elements slide down. This give a linear extension where the elements sliding to left bypass only elements of larger value of f, and hence respect the partial order. Since $f(y_m)$ is larger than the elements it jumps over, the resulting permutation has fewer inversions. This proves the claim in that case and completes the proof of the theorem. $\square$

7.3. **Vanishing conditions.** For $\boldsymbol{a} = (a_1, \ldots, a_k)$, let $\boldsymbol{a}^{\langle i \rangle} := (a_1, \ldots, a_i + 1, \ldots, a_k)$. By definition, the operator

$$\tau_{a_i} \,:\, \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a}) \cup \mathcal{E}\big(P, \boldsymbol{u}, \boldsymbol{a}^{\langle i \rangle}\big) \;\to\; \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a}) \cup \mathcal{E}\big(P, \boldsymbol{u}, \boldsymbol{a}^{\langle i \rangle}\big)$$

is an involution. For $i < j$, let

$$\delta_{i\,j} \,:=\, \tau_i\, \tau_{i+1} \cdots \tau_{j-1} \quad \text{and} \quad \delta_{j\,i} \,:=\, \tau_{j-1} \cdots \tau_{i+1}\, \tau_i$$

be the *promotion operator* starting at position i and ending in position j, and the *demotion operator* starting at position j and ending in position i.

Proof of Theorem 1.11. Without loss of generality, we can assume that poset $P = (X, \prec)$ has a unique minimal element $\widehat{0}$ and unique maximal element $\widehat{1}$. Since $f\big(\widehat{0}\big) = 1$ and $f\big(\widehat{1}\big) = n$ for every linear extension $f \in \mathcal{E}(P)$, we can also add $u_0 = \widehat{0}$ and $u_{k+1} = \widehat{1}$ to the chain $u_1 \prec \ldots \prec u_k$, and set $a_0 = 1$, $a_{k+1} = n$. Equation (1.11) then simplifies to

$$a_j - a_i > h(u_i, u_j) \quad \text{for all} \quad 0 \le i < j \le k+1. \tag{7.4}$$

First, let us show that inequalities (7.4) always hold. Indeed, in every word $\boldsymbol{x}_f$ corresponding to a linear extension $f \in \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$ with $f(u_i) = a_i$, we must have the elements from $(u_i, u_j)_P$ lie between u_i and u_j, and hence $a_j - a_i > h(u_i, u_j)$.

In the opposite direction, assume that the inequalities (7.4) hold for all $0 \le i < j \le k+1$. To prove that $\mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a}) \ne \varnothing$, proceed by induction on k. For $k = 1$, let α be a word obtained from totally ordering of the poset interval $(\widehat{0}, u_1)$, and β be a word obtained from totally ordering of the poset interval $(u_1, \widehat{1})$. Order the remaining elements of $P - u_1$ into a word γ, and then insert u_1 at position a_1 in the concatenation $\alpha\gamma\beta$. Since $u_1 \parallel \gamma$, $a_1 > |\alpha|$, and $n - a_1 > |\beta|$, this is a linear extension in $\mathcal{E}(P, u_1, a_1)$.

Suppose now that the result holds for all sequences of length $k \ge 1$, and let $\boldsymbol{y} \in \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$. Now let u_{k+1} be another element and a_{k+1} satisfy the conditions in the statement. Suppose that the position of u_{k+1} is at $a' \ne a_{k+1}$. Let us show that if $a' < a_{k+1}$, then we can move u_{k+1} to position $a' + 1$ without moving the other u's, and if $a' > a_{k+1}$ we can move u_{k+1} one position down. Repeating this we will eventually get u_{k+1} at a position $a' = a_{k+1}$, to obtain the desired linear extension.

From now on we act with the group G, and the promotion and demotion operators δ_{ij} to transform $\boldsymbol{y}$.

Let $a' < a_{k+1}$. Since $n - a' > n - a_{k+1} \ge h(u_{k+1}, \widehat{1})$, there must be at least one element in $\boldsymbol{y}$ appearing after u_{k+1} which is incomparable to u_{k+1}; denote by z the first such element, at position t. Then the elements between u_{k+1} and z are incomparable with z, since they must be $\succ u_{k+1}$. Inserting z immediately before u_{k+1} , i.e. forming the word $\boldsymbol{y}\delta_{ta'}$, then respects the partial order and shifts u_{k+1} to position $a' + 1$. Note that this transformation does not move $u_1, \ldots, u_k$ since we assume that $u_k \prec u_{k+1}$, which implies that $a_k < a$.

Suppose now that $a' > a_{k+1}$. Then $a' > h(\widehat{0}, u_{k+1}) + 1$, so there is an element before u_{k+1} which is incomparable to u_{k+1}. Let z_0 be the last such element, and suppose that it is at position $i_0 \in (a_{r-1}, a_r)$. Note that the elements between z_0 and u_{k+1} in $\boldsymbol{y}$ must be incomparable to z_0, since by minimality they must be all $\prec u_{k+1}$. If z_0 appears after u_k, then we obtain $\boldsymbol{y}\delta_{i_0a'}$, where z is inserted after u_{k+1} and u_{k+1} shifts to position $a'-1$. Otherwise, since $a'-a_k > h(u_{k+1}, u_k)+1$, there is an element z_k between u_k and u_{k+1} in $\boldsymbol{y}$ such that z_k is incomparable to either u_k or u_{k+1}. Since $z_k \ne z_0$, we must have $z_k \parallel u_k$, and let this z_k be the first such element after u_k at position i_k.

In general, for every $t \in [r, k]$ we define z_t at position $i_t < a'$ to be the first element after u_t , such that $z_t \parallel u_t$. Note that such element exists, which is seen as follows. Since $h(u_t, u_{k+1})+1 < a'-a_t$ there is an element z between u_t and u_{k+1} incomparable to at least one of them. However, for all such $z \prec u_{k+1}$, so we must have $z \parallel u_t$. Next, observe that z_t is incomparable to all elements in $\boldsymbol{y}$ appearing between u_t and z_t. Now transform $\boldsymbol{y}$ as follows. First, let $\boldsymbol{y}^0 := \boldsymbol{y}\delta_{i_0\, a'}$, and note that here z_0 is sent to position a' and all elements in between have been shifted down one position. Let $a'_t := a_t - 1$ and $i'_t := i_t - 1$ be the positions of u_t and z_t in $\boldsymbol{y}^0$. Next, let $\boldsymbol{y}^1 = \boldsymbol{y}^0\, \delta_{i'_r\, a'_r}$ which moves z_r before u_r, so the position of u_r is restored to a_r as well as all elements between them. Suppose that $i_r \in (a_{p-1}, a_p)$. Let then $\boldsymbol{y}^2 := \boldsymbol{y}^1\, \delta_{i'_p\, a'_p}$, so the element z_p is demoted to the position before u_p, and thus all other elements at positions $[a_p, i_p]$ have now restored their original position from $\boldsymbol{y}$. Continuing this way, if $i_p \in (a_{q-1}, a_q)$ we obtain $\boldsymbol{y}^3 := \boldsymbol{y}^2\, \delta_{i'_q\, a'_q}$ and so on until we have shifted all elements $u_r, \ldots, u_k$ to their positions $a_r, \ldots, a_k$. Also, u_{k+1} is at position $a' - 1$, which is what we needed to show. This completes the proof. $\square$

Proof of Corollary 1.15. The first part follows trivially from (1.11) or, equivalently, its simplified version (7.4). For the second part, note that the proof above is completely constructive and builds $f \in \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$ in polynomial time starting with a linear extension $g \in \mathcal{E}(P)$. The details are straightforward. □

7.4. **Uniqueness conditions.** In the next lemma, we show that, given a linear extension $f \in \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$, we can check if such f is unique in polynomial time.

In the notation of Theorem 1.11, let $v_i := f^{-1}(a_i - 1)$ and $w_i := f^{-1}(a_i + 1)$ for $1 \le i \le k$. We adopt the convention that $v_1 = \widehat{0}$ if $a_1 = 1$, and $w_k = \widehat{1}$ if $a_k = n$. For $1 \le i \le j \le n$, let

$$f^{-1}[i,j] \ := \ \big\{ f^{-1}(i) \, , \, \ldots \, , f^{-1}(j) \big\}.$$

Theorem 7.5. *In the notation of Theorem 1.11, let $f \in \mathcal{E}(P, \boldsymbol{a}, \boldsymbol{u})$ be a linear extension as in the theorem. Then $|\mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})| = 1$ <u>if and only if</u> the following conditions hold:*

(1) $f^{-1}[a_i + 1, a_{i+1} - 1]$ *forms a chain in P for every $1 \le i \le k$, and*

(2) *There are no $1 \le i \le j \le k$, such that $\{v_i, w_j\} \parallel f^{-1}[a_i, a_j]$.*

Proof. For the ⇒ direction, note that (1) follows directly. Indeed, recall that $e(P) > 1$ unless P is a chain. Therefore if the restriction of P to $\big\{f^{-1}(a_i + 1), \ldots, f^{-1}(a_{i+1} - 1)\big\}$ is not a chain, then there is more than one linear extension over these elements, which extends to the desired linear extension $g \in \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$, $g \ne f$. For (2), suppose to the contrary that $v_i, w_j \parallel f^{-1}[a_i, a_j]$. Then swapping the value of $f(v_i)$ and $f(w_j)$ via $\boldsymbol{x}_f \tau_{a_i - 1\, a_j + 1}$ we get a new linear extension, a contradiction.

For the ⇐ direction, suppose that (1) and (2) hold and there exists $g \in \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$ for some $g \ne f$. By Theorem 7.4 we have that there is an element $\pi \in \widehat{\mathrm{G}}_n(\boldsymbol{a})$, s.t. $\boldsymbol{x}_f \pi = \boldsymbol{x}_g$. Write π as the minimal (reduced) product of transpositions $\pi = \tau_{r_1 s_1} \cdots$ which act nontrivially. So $\boldsymbol{x}_f \tau_{r_1 s_1} \ne \boldsymbol{x}_f$ and thus $\{f^{-1}(r_1), f^{-1}(s_1)\} \parallel f^{-1}[r_1 + 1, s_1 - 1]$. Since the elements in $f^{-1}[a_i + 1, a_{i+1} - 1]$ form a chain we must have that $r_1 = a_i - 1$ for some i and that $s_1 = a_j + 1$ for some j. Note that by definition $r_1 < s_1$ and $r_1, s_1 \notin \boldsymbol{a}$. Thus $\{f^{-1}(a_i - 1), f^{-1}(a_j + 1)\} \parallel f^{-1}[a_i, a_j]$, and so condition (2) does not hold, a contradiction. □

Proof of Corollary 1.16. By the first part of Corollary 1.15, we can decide if $|\mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})| > 0$ in polynomial time. By the second part of the same corollary, we can find a linear extension $f \in \mathcal{E}(P, \boldsymbol{u}, \boldsymbol{a})$ in polynomial time. By Theorem 7.5, we can decide if such f is unique in polynomial time. □

8. Injective proof of the Sidorenko inequality

8.1. **Preliminaries.** Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Denote by $P|_J$ the restriction of P to a subset $J \subset X$. We write $P - y$ to denote the restriction $P|_{X - y}$. Denote by $P^* = (X, \prec^*)$ the *dual poset* :

$$x \prec^* y \quad \Longleftrightarrow \quad y \prec x \, , \quad \text{for all } \ x, y \in X.$$

Clearly, $e(P^*) = e(P)$.

Denote by $\mathcal{C}(P)$ the set of chains, and by $\mathcal{A}(P)$ the set of antichains in P. The *comparability graph* $\mathrm{Com}(P) = (X, E)$ is defined by $E = \big\{(x, y) \ : \ x \prec y, \ \text{where } x, y \in X\big\}$. Note that the chains in P are *cliques* (complete subgraphs) in $\mathrm{Com}(P)$. Similarly, the antichains in P are *stable* (independent) *sets* in $\mathrm{Com}(P)$.

Throughout this section, we think of the *promotion* in a different way, as a map from linear extensions to chains in the poset. Formally, for $f \in \mathcal{E}(P)$, let $x_1 = f^{-1}(1)$. For $i > 1$, let $x_i \in X$

be an element with the smallest value of f on $\{y \,:\, x_{i-1} \prec y\}$. This gives a *promotion chain* $C = \big[x_1 \to x_2 \to \ldots \to x_\ell\big] \in \mathcal{C}(P)$, which can also be viewed as the *DFS path* in the *Hasse diagram* of P. Denote by $\Phi : \mathcal{E}(P) \to \mathcal{C}(P)$ the map $\Phi(f) = C$.

Lemma 8.1. *For all $P = (X, \prec)$ and $y \in X$ we have:*

$$e(P - y) \,=\, \big|\big\{g \in \mathcal{E}(P) \,:\, y \in \Phi(g)\big\}\big|$$

Proof. Consider a bijection

$$\varphi : \mathcal{E}(P - y) \longrightarrow \big\{g \in \mathcal{E}(P) \,:\, y \in \Phi(f)\big\}$$

defined as follows. Let $f \in \mathcal{E}(P - y)$, and let $[y \to x_1 \to \ldots \to x_k]$ be the promotion path in the upper order ideal $B(y) = \{x \in X \,:\, x \succ y\}$. Define $g = \varphi(f) \in \mathcal{E}(P)$ as follows. Let $g(y) := f(x_1)$, $g(x_i) := f(x_{i+1})$ for $1 \le i < k$, and $g(x_k) := n$. Observe that in P^* we now have $\Phi(f) = \big[x_k \to \ldots \to x_1 \to y \to \ldots\big]$. Reversing the role of P and P^* implies the result. □

Corollary 8.2 (see [EHS89]). *For every antichain $A \in \mathcal{A}(P)$ we have*

$$\sum_{y \in A} e(P - y) \,\le\, e(P). \tag{8.1}$$

Furthermore, when $A \subseteq X$ is the set of minimal elements, the inequality (8.1) is an equality.

Proof. Note that for every $C \in \mathcal{C}(P)$ and $A \in \mathcal{A}(P)$, we have $|A \cap C| \le 1$. Thus, we have:

$$\sum_{y \in A} e(P - y) \,=\, \big|\big\{f \in \mathcal{E}(P) \,:\, |\Phi(f) \cap A| = 1\big\}\big| \,\le\, |\mathcal{E}(P)| \,=\, e(P),$$

which proves (8.1). For the second part, note that for every $f \in \mathcal{E}(P)$, the promotion path $\Phi(f)$ starts with the minimal element in A. This implies that (8.1) is an equality, as desired. □

Remark 8.3. Lemma 8.1 is implicit in [EHS89], which only discusses equality cases (cf. Corollary 8.2). By [Sid91, Thm 4], map Φ gives the *linear extension flow* through $\mathrm{Com}(P)$ viewed as directed network. Although Sidorenko gives a combinatorial construction of this flow in [Sid91, Rem 1.2], this construction is also inexplicit.

Let us mention that second part of Corollary 8.2 implies by induction that $e(P)$ depends only on the comparability graph $\mathrm{Com}(P)$, see [EHS89, Sta09]. The same holds for the order polynomial $\Omega(P, t)$, and can be proved using Ehrhart polynomials [Sta86], cf. §9.3. Alternatively, this result can be shown via certain "turning upside-down" flips discussed in [Sta12, Exc. 3.163].

8.2. **Sidorenko's inequality.** As in the introduction, let $P = (X, \prec)$ and $Q = (X, \prec')$ be two posets on the same ground set, such that $|C \cap C'| \le 1$ for all $C \in \mathcal{C}(P)$ and $C' \in \mathcal{C}(Q)$. Then $\mathcal{C}(P) \subseteq \mathcal{A}(Q)$ and $\mathcal{A}(P) \subseteq \mathcal{C}(Q)$, by definition.

Lemma 8.4 (cf. [Sid91, Lemma 10]). *For all P and Q as above, we have:*

$$\sum_{y \in X} e(P - y)\, e(Q - y) \,\le\, e(P)\, e(Q).$$

Proof. We have:

$$\begin{aligned}
\sum_{y\in X} e(P-y)\, e(Q-y) \ &=_{\text{Lem } 8.1} \sum_{y\in X} e(Q-y) \sum_{C\in\mathcal{C}(P)\,:\,C\ni y} |\{f\in\mathcal{E}(P)\,:\,\Phi(f)=C\}| \\
&= \sum_{C\in\mathcal{C}(P)} \sum_{y\in C} e(Q-y)\,\cdot\,|\{f\in\mathcal{E}(P)\,:\,\Phi(f)=C\}| \\
&\leq_{\text{Cor. } 8.2} \sum_{C\in\mathcal{C}(P)} e(Q)\,\cdot\,|\{f\in\mathcal{E}(P)\,:\,\Phi(f)=C\}| \\
&\leq\ e(Q) \sum_{C\in\mathcal{C}(P)} |\{f\in\mathcal{E}(P)\,:\,\Phi(f)=C\}| \quad = \ e(P)\, e(Q).
\end{aligned}$$

Here in the third line, Cor. 8.2 applies to poset Q, since every chain in P is an antichain in Q. □

Proof of Theorem 1.6. The theorem follows from Lemma 8.4, by induction on $n=|X|$. □

Corollary 8.5 ([Sid91, Thm 11]). *In notation of Theorem 1.6, the inequality (1.7) is an equality* <u>*if and only if*</u> *P is a* series-parallel poset*.*

The result is well-known and follows easily by tracing back the inequalities in the proof of Lemma 8.4. We omit the details.

Proof of Theorem 1.7. We can rewrite the proof of Lemma 8.4 as follows:

$$\sum_{y\in X} e(P-y)\, e(Q-y) \ = \ \sum_{f\in\mathcal{E}(P)} \sum_{g\in\mathcal{E}(Q)} \sum_{y\in\Phi(P)\cap\Phi(Q)} 1 \ \leq \ k\, e(P)\, e(Q).$$

The result now follows by induction on $n\geq k$, with the base $n=k$ trivial. □

Proof of Theorem 1.14. Let $\beta: S_n\to\mathcal{E}(P_\sigma)\times\mathcal{E}\big(P_{\overline{\sigma}}\big)$ be the injection defined implicitly by the proof of Theorem 1.6 above. First, observe that β is computable in polynomial time. Indeed, by induction, it is a composition of maps β_i each consisting of applying maps Φ to posets corresponding to partial permutations $\sigma_i:=(\sigma(1),\ldots,\sigma(i))$ and its dual $\overline{\sigma_i}$, see the proof of Lemma 8.1.

Second, whenever defined, the inverse map β^{-1} can be computed by the proof of Lemma 8.1, since the inverse of Φ on P is a map Φ on P^*. On the other hand, at each stage, the decision if the inverse of β_i exists reduces to a problem whether a given antichain in the $Q_i:=P_{\overline{\sigma_i}}$ is a *cut*, i.e. it intersects every chain in Q_i. This is a special case of directed graph connectivity problem, and thus in P. Putting this together implies that we can decide in polynomial time if $(f,g)\in\beta(S_n)$, for all $f\in\mathcal{E}(P_\sigma)$ and $g\in\mathcal{E}\big(P_{\overline{\sigma}}\big)$.

In summary, the function $\eta(\sigma)$ counts the number of pairs of linear extensions (f,g) as above, such that $(f,g)\notin\beta(S_n)$. Since the problem whether $(f,g)\in\beta(S_n)$ can be decided in polynomial time, this completes the proof. □

9. Final remarks and open problems

9.1. **Björner–Wachs inequality.** In total, we include three proofs of the Björner–Wachs inequality: the original injective proof in §3, the probabilistic proof via Shepp's inequality in §4, and Reiner's proof via q-analogue in §5. Another proof was given by Hammett and Pittel in [HP08, Cor. 2], who seemed unaware of the origin of the problem despite having [BW89] among the references. Although somewhat lengthy and technical, their proof is completely self-contained and is based on a geometric probability argument. It is similar in spirit to Reiner's proof, but without benefits of the brevity.

9.2. **Order polynomial.** There is surprisingly little literature on the order polynomials given that they emerge naturally in both P-partition theory and discrete geometry. We refer to [Joc14] for order polynomials in the case of symmetric posets, which are of independent interest, and to [LT19] for some computations.

It seems, there are more conjectures and open problems than results in the subject. It addition to the Kahn–Saks Conjecture 4.12, we have our own Conjecture 4.17. We should warn the reader that there seem to be insufficient effort towards testing of these conjectures, so it would be interesting to obtain more computational evidence.

9.3. **Ehrhart polynomial.** It is a classical observation by Stanley [Sta86], that the order polynomial $\Omega(P, t+1)$ is the *Ehrhart polynomial* of the corresponding order polytope $\mathcal{O}_P$:

$$\operatorname{Ehr}(\mathcal{O}_P, t) \, = \, \Omega(P, t+1).$$

This allows one to translate the results from combinatorial to geometric language ad vice versa.

Notably, our Example 6.4 is motivated by Stanley's `MathOverflow` observation[5] that the order polynomial $\Omega(C_1 \oplus A_m, t+1)$ can have negative coefficients for $m \geq 20$. We refer to [LT19] for more on this example and to [Liu19] for the background on non-negative Ehrhart polynomials and further references. We refer to [Cha16] for q-Ehrhart polynomials, and to [KS17] for further results.

9.4. **Geometric form of the Kahn–Saks conjecture.** One can ask if a version of the Kahn–Saks Conjecture 4.12 holds for general integral polytopes:

$$\text{Is} \quad \operatorname{Ehr}(Q, t-1)/t^d \quad \text{weakly decreasing for all} \quad Q \in \mathbb{R}^d \quad \text{and} \quad t \in \mathbb{N}_{\geq 1}\,? \tag{9.1}$$

First, recall the example of *Reeve's tetrahedron* with vertices at

$$(0,0,0)\,, \quad (1,0,0)\,, \quad (0,1,0) \quad \text{and} \quad (1,1,h),$$

see e.g. [BR07, Ex. 3.23] and [GW93, §4.1]. In this case, the Ehrhart polynomial has negative signs, and the scaled Ehrhart polynomial is non-monotone for large values of h. This shows that the *geometric Kahn–Saks conjecture* (9.1) does not hold for general lattice polytopes.

On the other hand, it is rather plausible that (9.1) holds for *antiblocking* (*corner*) *polytopes* (see e.g. [Sch72, §5.9]) with integer vertices. If true, this would imply the Kahn–Saks Conjecture 4.12. Indeed, although the order polytope $\mathcal{O}_P$ is not antiblocking, the *stable set* (*chain*) *polytope* $\mathcal{C}_P$ is both altiblocking and has the same Ehrhart polynomial by Stanley's theorem: $\operatorname{Ehr}(\mathcal{C}_P, t) = \operatorname{Ehr}(\mathcal{O}_P, t)$, see [Sta86].

Finally, let us mention that the proof of Proposition 4.14 can be modified to show that

$$\frac{1}{t^d} \operatorname{Ehr}(Q, t-1) \, \geq \, \frac{1}{(kt)^d} \operatorname{Ehr}(Q, kt-1)\,,$$

for all antiblocking polytopes $Q \in \mathbb{R}^d$ with integer vertices. This gives some credence to our speculation (9.1) in this case.

9.5. **Log-concavity and q-log-concavity.** The log-concavity for order polynomials proved in Theorem 4.7 is somewhat different from other log-concave inequalities, see e.g. [Brä15, Huh18, CP21, Sta89]. The q-log-concavity in Corollary 5.9 is also classical albeit less studied, see e.g. [Kra89, Ler90, Sag92].

[5]Richard P. Stanley, mathoverflow.net/q/200574 (March 20, 2015).

9.6. **Graham's conjecture.** We learned that of Daykin–Daykin–Paterson paper [DDP84, Thm 2] proving Graham's conjecture (Theorem 4.19) by accident, while revising the paper. We chose to keep our Corollary 4.20 as a nice application of our tools. Most recently, the first and second authors found a new proof of Theorem 4.19 based on the *Ahlswede–Daykin inequality*, and further generalized this inequality to a multivariate version [CP22b, §9].

9.7. **Sidorenko inequality.** Note that another combinatorial proof of Sidorenko's inequality (Theorem 1.6) was independently found in [GG20, §4.1], where the authors gave an elegant explicit construction of a surjection proving (1.7). Unfortunately, the proof of correctness of that surjection is technical and cannot be easily inverted to obtain the desired injection. More precisely, the authors give a explicit surjection $\alpha : \mathcal{E}(P_\sigma) \times \mathcal{E}\big(P_{\overline{\sigma}}\big) \to S_n$. Unfortunately, the proof in [GG20] is technical and indirect, so an explicit injection requires further effort.

As we mentioned in the introduction, our injection β defined implicitly in the proof of Theorem 1.6 likely coincides with an explicit injection in [GG20+], since both essentially reverse engineer and make effective the original proof by Sidorenko [Sid91]. The connection with the argument in [StR81] and the surjection in [MPP18b] in the case of *Fibonacci posets* remains unclear.

We also conjecture that the function $u : S_n \to \mathbb{N}$ defined by (1.14) is #P-complete. The conjecture would follow if #P-completeness was proved for self-dual 2-dimensional posets $P \simeq \overline{P}$. Unfortunately, the construction in [DP20] is too specialized and technical to obtain this result.

Finally, there a q-analogue of Sidorenko's inequality in [GG20, Cor. 3] generalizing q-equality for the series-parallel posets given in [Wei12]. See also [KS17] for the definition of $e_q(P)$ for general P based on the P-partition theory, and [BW91] for many other results on $e_q(P)$.

9.8. **Mixed Sidorenko inequality.** In [BBS99], Bollobás, Brightwell and Sidorenko showed how to obtain Sidorenko's Theorem 1.6 via a known special case of *Mahler's Conjecture*. Most recently, Artstein-Avidan, Sadovsky and Sanyal extended this approach in [AASS20] to obtain the following remarkable generalization of the Sidorenko inequality.

For two posets $P = (X, \prec)$ and $Q = (X, \prec')$ on the same set, *mixed linear extensions* are triples (f, g, J), where $J \subset [n]$, $f \in \mathcal{E}(P|_J)$, and $g \in \mathcal{E}\big(Q|_{\overline{J}}\big)$. Denote by $e_k(P, Q)$ the number of such triples with $|J| = k$, i.e.

$$e_k(P,Q) \ := \sum_{J \in \binom{[n]}{k}} e(P|_J)\, e\big(Q|_{\overline{J}}\big).$$

Theorem 9.1 ([AASS20, Thm 6.2])**.** *Let P, Q, S, T be four posets on the same ground set, such that $|C \cap C'| \le 1$ and $|D \cap D'| \le 1$, for all $C \in \mathcal{C}(P)$, $C' \in \mathcal{C}(Q)$, $D \in \mathcal{C}(S)$ and $D' \in \mathcal{C}(T)$. Then we have:*

$$e_k(P,Q)\, e_k(S,T) \ \ge \ n! \binom{n}{k}. \tag{9.2}$$

It would be interesting to find a combinatorial proof of this result. It would be even more interesting to find a direct injective proof, and conclude that the function giving the difference of the two sides of (9.2) is in #P. The results in [IP22] suggest that this might not be possible. Finally, does the *mixed Sidorenko inequality* (9.2) have an upper bound similar to that in [BBS99]?

9.9. **Complexity of correlation inequalities.** By taking the limit $t \to \infty$ in Lemma 4.5, we obtain:

$$(n-1) \cdot e(P) \cdot e\big(P \smallsetminus \{x,y\}\big) \ \ge \ n \cdot e\big(P \smallsetminus x\big) \cdot e\big(P \smallsetminus y\big). \tag{$*$}$$

It would be interesting to see if this inequality can be proved injectively. Is the function giving the difference of the two sides of this inequality in #P?

Note that, by applying the negative-correlation version of the FKG inequality to the proof of Lemma 4.5, we obtain the following result:

Lemma 9.2. *Let $P = (X, \prec)$ be a poset, let $x \in X$ be a minimal element, and let $y \in X$ be a maximal element such that y does not cover x. Then, for every integer $t > 0$, we have:*

$$\Omega\big(P, t\big) \cdot \Omega\big(P \smallsetminus \{x, y\}, t\big) \ \le \ \Omega\big(P \smallsetminus x, t\big) \cdot \Omega\big(P \smallsetminus y, t\big).$$

By taking the limit $t \to \infty$ in the lemma, we get the inequality opposite to $(*)$:

$$(**) \qquad (n-1) \cdot e(P) \cdot e\big(P \smallsetminus \{x, y\}\big) \ \le \ n \cdot e\big(P \smallsetminus x\big) \cdot e\big(P \smallsetminus y\big).$$

Of course, the element y was minimal in $(*)$ and is maximal in $(**)$, but these inequalities are striking in appearance. Again, it would be interesting to see if this inequality can be proved injectively.

9.10. **Vanishing and uniqueness conditions.** Note that the vanishing conditions for the Stanley inequality are a special case of the *equality conditions*, which are fully described in [SvH20] and reproved in [CP21]. For example, Corollary 8.2 and Corollary 8.5 give further examples of equality conditions, with a simple proof in both cases via direct injection. When there is no injective proof, the equality condition can become a major challenge. On the other hand, the vanishing and uniqueness conditions tend to be much easier to establish using either combinatorial or geometric tools (see [EG15]).

For example, for the *Kahn–Saks inequality* generalizing Stanley's inequality, the equality conditions remain open in full generality. See, however, [CPP21b, §8] for the vanishing conditions of the Kahn–Saks inequality, proved also via the promotion technology. See also [CPP21a, CPP21b], for the equality conditions of the Kahn–Saks and cross-product inequalities for posets of width two. Finally, let us mention Lemma 14.6 in [CP21], which is yet another variation on Theorem 7.4 and proved by a direct combinatorial argument using promotions.

In a different direction, sometimes the equality conditions are trivial as the natural inequalities are always strict except for some degenerate cases. This is the case with the *XYZ inequality* [Fis84], and the *log-concavity* (Theorem 4.8) discussed above.

The uniqueness conditions are studied less frequently than vanishing and equality conditions, since they tend to be harder. For example, there is no description of the uniquely colorable graphs, and this remains a major open problem [CZ20]. Notable positive results include uniqueness conditions for the *Kostka numbers* [BZ90] and for the *Littlewood–Richardson coefficients* [BI13, Prop. 3.13].

9.11. **Poset dynamics.** Promotions, demotions and evacuations were defined by Schützenberger in [Sch72], and this approach has been immensely influential leading to the *RSK Algorithm* and the *Edelman–Greene bijection*, among other things. Group theoretic approach in the context of combinatorics of words were developed by Lascoux and Schützenberger, and specifically in the generality of posets were introduced by Haiman [Hai92] and Malvenuto–Reutenauer [MR94]. See also [KB96] for a related approach in the context of semistandard Young tableaux, and [Sta09] for an extensive survey.

It would be interesting to find a generalization of the evacuation ε which would preserve its involution property. Such "restricted evacuation" might give rise to "restricted domino linear extensions" which would be of independent interest, see e.g. [Sta09, §3].

The extension of promotion to general bijections $X \to [n]$ was obtained in [DK20]. Can our group action on restricted linear extensions be generalized in this direction? Note that we have only limited understanding if the group action can be applied to study the order polynomial. See however [Hop20] for some elegant product formulas in some special cases.

The promotion operators were used in [AKS14] to define a Markov chain on the set $\mathcal{E}(P)$ of linear extensions of a given poset P. See also [RS20] where a related Markov chain was shown to be mixing in time $O(n \log n)$. It would be interesting to see if these results can be generalized to show that the restricted linear extensions in $\mathcal{E}(P, \boldsymbol{x}, \boldsymbol{a})$ can be sampled in polynomial time.

Finally, let us mention a curious loop-free listing algorithm in [CW95]. Is there a similar algorithm for the restricted linear extensions?

9.12. **Injections and matchings for the Stanley inequality.** As we mentioned in the introduction, it remains a major open problem whether Stanley's inequality (1.9) can be proved by a direct injection, see e.g. [CP21, §17.17]. Formally, in the notation of Theorem 1.8, let

$$\rho(P,x,a) \ := \ \mathrm{N}(P,x,a)^2 \, - \, \mathrm{N}(P,x,a+1) \cdot \mathrm{N}(P,x,a-1).$$

Open Problem 9.3. *Is* $\rho \in \#\mathrm{P}$*?*

At this point, it is even hard to guess which way the answer would go. While some of us believe the answer should be negative, others disagree. The only thing certain is that none of the positive proofs in [CP21, Sta81] imply a positive answer, while the negative results in [IP22] are not even close to resolving the problem. Since part of the motivation behind our algebraic approach aimed at resolving this problem, let us propose the following approach.

We would like to give an injection proving Stanley's inequality (1.9). Consider the following family of elements of the group G from Section 7 whose actions would be good candidates for such an injection. Let $\mathcal{G} = (V \sqcup W, E)$ be a bipartite graph, where $V = \mathcal{E}(P,x,a) \times \mathcal{E}(P,x,a)$ and $W = \mathcal{E}(P,x,a-1) \times \mathcal{E}(P,x,a+1)$. We define the set E of edges as follows.

Let $\pi \in \mathcal{E}(P,x,a-1)$ and $\sigma \in \mathcal{E}(P,x,a+1)$, so that $(\pi,\sigma) \in W$. For every element y which appears after x in π and before x in σ, that is $i := \pi^{-1}(y) > a-1$ and $j := \sigma^{-1}(y) < a+1$, we apply the promotion/demotion operators on the chain starting/ending at y in σ and π, respectively. Let $\bar{\delta}_j := \tau_{n-1} \cdots \tau_j$. Then in the word $\delta_i \pi$, the chain starting at y is pushed up, so that x is moved to position a. Similarly, in the word $\bar{\delta}_j \sigma$, the chain ending at y is pushed down, so that x is moved to position a. Thus $(\delta_i \pi, \bar{\delta}_j \sigma) \in \mathcal{E}(P;a;x) \times \mathcal{E}(P;a;x) = V$, and we connect it to (π,σ) by an edge. Note that by the pigeonhole principle, there are at least two possibilities for elements y, and thus there will be at least one edge, however it is not necessarily true that the degree of every (π,σ) is at least 2.

Conjecture 9.4. *Let* $\mathcal{G} = (V \sqcup W, E)$ *be the graph defined above. Then there exists a maximal matching which covers all vertices in* W*.*

This matching will be the desired injection and imply the Stanley inequality. By itself, the conjecture would not imply that $\rho \in \#\mathrm{P}$. For that, the injection would need to be computable in polynomial time.

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