

CORRELATION INEQUALITIES FOR LINEAR EXTENSIONS

SWEET HONG CHAN^{*} AND IGOR PAK[◊]

ABSTRACT. We employ the combinatorial atlas technology to prove new correlation inequalities for the number of linear extensions of finite posets. These include the approximate independence of probabilities and expectations of values of random linear extensions, closely related to Stanley’s inequality. We also give applications to the numbers of standard Young tableaux and to Euler numbers.

1. INTRODUCTION

If you really want to hear about it, the first thing you’ll probably want to know about *Poset Theory* is that it was born and languished for decades without any tools whatsoever. It is still a question whether the area has caught up with the other, more established parts of Combinatorics, but by now it is definitely in possession of powerful tools of various nature, which makes it at least somewhat prominent, if not prosperous.

In this paper we obtain a melange of new correlation inequalities for the number of linear extensions of finite posets using the *combinatorial atlas* technology [CP24, CP22]. This is the tool we recently developed and whose power has yet to be fully exploited. Viewed individually, each new inequality shows that there is more to the story. Taken together, these inequalities offer a glimpse at the big world of poset inequalities that we are just beginning to understand.

1.1. Linear extensions. Let $P = (X, \prec)$ be a *partially ordered set* on the ground set X of size $|X| = n$, and with the partial order “ \prec ”. A *linear extension* of P is a bijection $f : X \rightarrow \{1, \dots, n\}$ that is order-preserving: $x \prec y$ implies $f(x) < f(y)$, for all $x, y \in X$. We denote by $\mathcal{E}(P)$ the set of linear extensions of P , and by $e(P) := |\mathcal{E}(P)|$ the number of linear extensions.

An element $x \in X$ is *minimal* if there is no $y \in X$ such that $y \prec x$. Denote by $\min(P) \subseteq X$ the subset of minimal elements in P . We use C_n and A_n to denote the n -element *chain* and *antichain*, respectively. For an element $x \in X$, we denote by $P - x$ the induced subposet of P on the subset $X - x$. See Section 2 for further poset notations.

Theorem 1.1 (cf. Theorem 6.3). *Let $P = (X, \prec)$ be a poset with $|X| = n > 2$ elements. Let $x, y \in \min(P)$ be distinct minimal elements of P . Then:*

$$(1.1) \quad \frac{n}{n-1} \leq \frac{e(P) \cdot e(P - x - y)}{e(P - x) \cdot e(P - y)} \leq 2.$$

This correlation inequality is the most natural and the simplest to state. The lower bound in (1.1) is a special case of the *Fishburn inequality* (see §8.5), while the upper bound is new and is a special case of the upper bound in Theorem 6.3. Note that the lower bound is tight for $P = A_n$ and the upper bound is tight for the linear sum $P = A_2 \oplus C_{n-2}$.

Date: September 12, 2024.

2020 Mathematics Subject Classification. 05A20 (Primary), 06A07, 60C05 (Secondary).

Key words and phrases. poset, linear extension, combinatorial atlas, correlation inequality, log-concavity, standard Young tableau.

^{*}Department of Mathematics, Rutgers University, Piscataway, NJ, 08854. Email: sweehong.chan@rutgers.edu.

[◊]Department of Mathematics, UCLA, Los Angeles, CA, 90095. Email: pak@math.ucla.edu.

Correlation inequalities are best understood in probabilistic notation. Let \mathbb{P} and \mathbb{E} be taken over the uniform random linear extension $f \in \mathcal{E}(P)$. The inequality (1.1) can be rewritten as

$$(1.2) \quad \frac{n}{n-1} \leq \frac{\mathbb{P}[f(x) = 1, f(y) = 2]}{\mathbb{P}[f(x) = 1] \cdot \mathbb{P}[f(y) = 1]} \leq 2.$$

The following theorem gives a similar upper bound for the covariances:

Theorem 1.2 (see §6.3). *Let $P = (X, \prec)$ be a finite poset, and let $x, y \in X$ be fixed poset elements. Then:*

$$(1.3) \quad \frac{\mathbb{E}[f(x)f(y)] + \mathbb{E}[\min\{f(x), f(y)\}]}{\mathbb{E}[f(x)] \cdot \mathbb{E}[f(y)]} \leq 2.$$

We note that (1.3) is asymptotically tight; see Example 3.8. Also note that it follows from (1.3) that

$$(1.4) \quad \frac{\text{Cov}(f(x), f(y))}{\mathbb{E}[f(x)] \cdot \mathbb{E}[f(y)]} = \frac{\mathbb{E}[f(x)f(y)]}{\mathbb{E}[f(x)] \cdot \mathbb{E}[f(y)]} - 1 \leq 1.$$

See also §3.2 for an application to the second moment estimates.

1.2. Stanley inequality. For linear extensions, one can use both elements and values to set up correlation inequalities. The following celebrated result by Stanley is foundational in the area:

Theorem 1.3 (*Stanley inequality* [Sta81, Thm 3.1]). *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Fix $x \in X$ and let $2 \leq k \leq n-1$. Denote by $e_k(P)$ the number of linear extensions $f \in \mathcal{E}(P)$ such that $f(x) = k$. Then:*

$$(1.5) \quad e_k(P)^2 \geq e_{k-1}(P) \cdot e_{k+1}(P).$$

In probabilistic notation, Stanley's inequality can be restated as

$$(1.6) \quad \mathbb{P}[f(x) = k]^2 \geq \mathbb{P}[f(x) = k-1] \cdot \mathbb{P}[f(x) = k+1].$$

The proof in [Sta81] used the *Alexandrov–Fenchel inequality* applied to order polytopes, see §8.1. Stanley's inequality was famously extended by Kahn and Saks [KS84], to a prove a weak version of the $\frac{1}{3}-\frac{2}{3}$ Conjecture, see §8.3.

Stanley inequality (1.6) has remained mysterious until recently, when the equality conditions have been established by Shenfeld and van Handel [SvH23] by advancing geometric arguments. In our paper [CP24], we gave a linear algebraic proof of Stanley inequality, the equality conditions, and the generalization to weighted linear extensions, see also §8.1.

Let us single out two slightly nonstandard general consequences of Stanley inequality:

Corollary 1.4 (see §7.1). *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Fix $x \in X$ and let $1 \leq k \leq n-1$. Then:*

$$(1.7) \quad \mathbb{P}[f(x) > k]^2 \geq \mathbb{P}[f(x) > k-1] \cdot \mathbb{P}[f(x) > k+1].$$

and

$$(1.8) \quad \mathbb{P}[f(x) = 1] \cdot \mathbb{P}[f(x) > 1] \leq \mathbb{P}[f(x) = 2].$$

For $k = 1$, inequality (1.7) further simplifies to

$$(1.9) \quad \mathbb{P}[f(x) > 1]^2 \geq \mathbb{P}[f(x) > 2].$$

We propose the following unusual extension of Stanley inequality to general subsets of the ground set. Fix a nonempty subset $A \subseteq X$. For a linear extension $f \in \mathcal{E}(P)$, define

$$f(A) := \{f(x) : x \in A\} \quad \text{and} \quad f_{\min}(A) := \min f(A).$$

Note that $f_{\min}(A) = f(x)$ for all singletons $A = \{x\}$, where $x \in X$.

Conjecture 1.5 (*Extended Stanley inequality*). *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Fix a nonempty subset $A \subseteq X$, and let $2 \leq k \leq n - 1$. Then:*

$$(1.10) \quad \mathbb{P}[f_{\min}(A) = k]^2 \geq \mathbb{P}[f_{\min}(A) = k - 1] \cdot \mathbb{P}[f_{\min}(A) = k + 1].$$

To justify the conjecture, we prove that the inequalities (1.8) and (1.9) extend to all subsets (see also §8.10).

Theorem 1.6 (see §6.4). *Let $P = (X, \prec)$ be a poset on $|X| \geq 2$ elements. Fix a nonempty subset $A \subseteq X$. Then:*

$$(1.11) \quad \mathbb{P}[f_{\min}(A) > 1]^2 \geq \mathbb{P}[f_{\min}(A) > 2].$$

and

$$(1.12) \quad \mathbb{P}[f_{\min}(A) = 1] \cdot \mathbb{P}[f_{\min}(A) > 1] \leq \mathbb{P}[f_{\min}(A) = 2].$$

Although Conjecture 1.5 is written as a natural generalization of Stanley Theorem 1.3, the inequalities (1.11) and (1.12) are best understood as

$$(1.13) \quad \mathbb{P}[1 \notin f(A)]^2 \geq \mathbb{P}[1, 2 \notin f(A)] \quad \text{and}$$

$$(1.14) \quad \mathbb{P}[1 \in f(A)] \cdot \mathbb{P}[1 \notin f(A)] \leq \mathbb{P}[1 \notin f(A), 2 \in f(A)],$$

respectively. We show in §7.1, that Conjecture 1.5 implies Theorem 1.6. The following corollary is an immediate consequence of (1.13) and (1.14).

Corollary 1.7 (cf. Lemma 6.4). *Let $P = (X, \prec)$ be a poset on $|X| \geq 2$ elements, and let $A \subseteq X$ be a nonempty subset of elements. Then:*

$$(1.15) \quad \mathbb{P}[1 \in f(A), 2 \notin f(A)]^2 \geq \mathbb{P}[1, 2 \in f(A)] \cdot \mathbb{P}[1, 2 \notin f(A)].$$

Proof. Multiply (1.13) for subsets A and $X \setminus A$. Then use (1.14). □

1.3. Multiple subsets. We now consider other correlation inequalities obtained by replacing poset elements with subsets. First, we have the following natural generalization of Theorem 1.2.

Theorem 1.8 (see §6.3). *Let $P = (X, \prec)$ be a finite poset, and let $A, B \subseteq X$ be nonempty subsets. Then:*

$$(1.16) \quad \frac{\mathbb{E}[f_{\min}(A)f_{\min}(B)] + \mathbb{E}[f_{\min}(A \cup B)]}{\mathbb{E}[f_{\min}(A)] \cdot \mathbb{E}[f_{\min}(B)]} \leq 2.$$

Let us emphasize that here A and B are *arbitrary* subsets of the ground set X . Next, we have the following variation of Corollary 1.7 for two subsets of minimal elements.

For an element $b \in X$, denote by $b \uparrow := \{x \in X : x \succ b\}$ the *upper order ideal* generated by b . For a subset $B \subseteq X$, denote by $B \uparrow := \cup_{b \in B} b \uparrow$ the *upper closure* of B .

Theorem 1.9 (see §6.5). *Let $P = (X, \prec)$ be a finite poset, and let $A, B \subset \min(P)$ be disjoint nonempty subsets of minimal elements. Then:*

$$(1.17) \quad \mathbb{P}[1 \in f(A), 2 \in f(B)]^2 \geq \mathbb{P}[1 \in f(A), 2 \in f(A \uparrow)] \cdot \mathbb{P}[1 \in f(B), 2 \in f(B \uparrow)].$$

We conclude with the following three-subset variation:

Theorem 1.10 (see §6.5). *Let $P = (X, \prec)$ be a finite poset, and let $A, B, C \subset \min(P)$ be disjoint nonempty subsets of minimal elements. Then:*

$$(1.18) \quad \frac{\mathbb{P}[1 \in f(C), 2 \in f(C \uparrow)] \cdot \mathbb{P}[1 \in f(A), 2 \in f(B)]}{\mathbb{P}[1 \in f(A), 2 \in f(C)] \cdot \mathbb{P}[1 \in f(B), 2 \in f(C)]} \leq 2.$$

Moreover, there exists a permutation (A', B', C') of (A, B, C) such that

$$(1.19) \quad \frac{\mathbb{P}[1 \in f(C'), 2 \in f(C' \uparrow)] \cdot \mathbb{P}[1 \in f(A'), 2 \in f(B')]}{\mathbb{P}[1 \in f(A'), 2 \in f(C')] \cdot \mathbb{P}[1 \in f(B'), 2 \in f(C')]} \leq 1.$$

1.4. Background and historical remarks. In Combinatorics, correlation inequalities go back to the work of Kirchhoff on electrical networks. In modern notation, his result can be reformulated as saying that for all edges $e, e' \in E$, we have:

$$(1.20) \quad \mathbb{P}(e, e' \in T) \leq \mathbb{P}(e \in T) \cdot \mathbb{P}(e' \in T),$$

where the probability is over uniform spanning trees T in a given graph $G = (V, E)$. This idea has proved extremely influential over the past decades, leading to a long series of remarkable results.

In Graph Theory, the classic examples of correlation inequalities for hereditary graph properties include the *Harris inequality* (1960), the *Kleitman inequality* (1966) and their far-reaching extensions: the *Ahlsvede–Daykin* (1978) and the *FKG inequalities* (1971). We refer to [AS16, §6.1] for the introduction, and to [FS98] for a detailed overview.

In Statistical Physics, notable correlation inequalities include *Griffiths* (1967) and *Griffiths–Hurst–Sherman inequalities* (1970), all for the Ising model. We refer to [AB08, Ch. 15] for the background and combinatorial applications. See also [BBL09, KN10, Pem00] for the general theory of negative correlations.

In Enumerative and Algebraic Combinatorics, correlation inequalities are closely related to log-concavity (log-convexity) of sequences counting various combinatorial objects, see [Brä15, Sta89]. In the past few years, Huh and his coauthors introduced and explored a new algebro-geometric approach to the subject. We refer to [AHK18] for a major breakthrough proving a long-standing Mason’s conjecture on the numbers of independent sets of a matroid, to [HSW22] for correlation inequalities for matroids, and to [Huh18, Huh23] for recent surveys. We also mention the remarkable papers [ALOV18, BH20] which used *Lorentzian polynomials* to obtain (stronger) ultra-log-concave inequalities for matroids.

In Poset Theory, besides Stanley inequality (Theorem 1.3), notable correlation inequalities include the *Graham–Yao–Yao inequality* (1980), the *XYZ inequality* (1982), and the *Kahn–Saks inequality* [KS84], see an overview in [Fis92, Win86]. We also note the (conjectural) *cross-product inequality*, see §8.4.

As we mentioned earlier, this paper grew from our paper [CP24] where we introduced the combinatorial atlas technology, using an involved inductive argument based on linear algebra. See also [CP22] for an introduction to this theory, and for connections to Lorentzian polynomials and the Alexandrov–Fenchel inequality (see also §8.1).

In contrast with the Lorentzian polynomial approach, combinatorial atlas is fundamentally noncommutative. This and the elementary nature of matrices makes the combinatorial atlas

technology flexible enough to apply to linear extensions and obtain the inequalities that are not reachable by other means. We believe that this applies to [CP24, Thm 1.35] and most new inequalities for linear extension in this paper. See §8.2 for further discussion on this.

1.5. Discussion. Linear extensions of finite posets are somewhat different in spirit from graphs, matroids, etc., in the type of statistics worth studying, and the difficulty of available tools, see [CP23b]. Of the fairly large pool of potential correlation inequalities we chose the ones that were accessible with the atlas technology. The following two examples were motivational for our work.

(1) It is instructive to compare our results with the *Huh–Schröter–Wang correlation inequality for bases of matroids* [HSW22], which can be stated as follows. Let $\mathcal{M} = (X, \mathcal{B})$ be a finite matroid of rank $d = \text{rk}(\mathcal{M})$ given by the set of bases $\mathcal{B} \subseteq \binom{X}{d}$. Then, for every two distinct elements $u, v \in X$, we have:

$$(1.21) \quad \frac{\mathbb{P}[u \in B, v \in B] \cdot \mathbb{P}[u \notin B, v \notin B]}{\mathbb{P}[u \in B, v \notin B] \cdot \mathbb{P}[u \notin B, v \in B]} \leq 2 \left(1 - \frac{1}{d}\right),$$

where the probability is over uniform $B \in \mathcal{B}$, see [HSW22, Thm 5]. The authors note that this implies the covariance bound

$$(1.22) \quad \text{Cov}(u \in B, v \in B) < \mathbb{P}[u \in B] \cdot \mathbb{P}[v \in B],$$

for all fixed distinct elements $u, v \in X$. Although we are not aware of a formal connection, our upper bounds (1.2) and (1.4) have a similar in structure to (1.21) and (1.22), respectively.

Continuing the analogy, the constant 2 in (1.21) was improved to 1 in some special cases [HSW22, Thm 6]. It is not known whether it can be further improved in full generality, with some examples giving 8/7 ratio. Similarly, our tools do not allow the asymptotic improvement of the constant 2. in (1.3), but we do have examples showing that here the constant 2 is asymptotically tight (Example 3.8).

(2) In a related earlier study of *random graph matchings*, Kahn [Kahn00, §4] introduced the notion of *approximate independence* to describe correlation inequalities of the type we consider in this paper. Let $G = (V, E)$ be a simple graph. A *matching* is a subset $M \subseteq E$ of pairwise nonadjacent edges. Let \mathcal{M} denote the set of all matchings in G , and let $\mathcal{M}_k \subseteq \mathcal{M}$ be the set of matchings in G of size k .

Heilmann and Lieb (1972) famously showed that the sequence $\{|\mathcal{M}_k|\}$ is log-concave, which parallels Stanley inequality (1.6). *Kahn inequality* [Kahn00, Cor. 4.3] states that for all distinct vertices $u, v \in V$, we have:¹

$$(1.23) \quad \frac{1}{2} \leq \frac{\mathbb{P}[u \notin M, v \notin M]}{\mathbb{P}[u \notin M] \cdot \mathbb{P}[v \notin M]} \leq 2,$$

where $M \in \mathcal{M}$ is a uniform random matching (cf. Example 3.9). Our correlation inequality (1.2) is modeled after this result.

1.6. Paper structure. We start with notations in Section 2. In Section 3, we present a number of examples and applications of correlation inequalities. Many of these are for illustrative purposes and can be skipped, but some give background and context for the type of correlation inequalities that we consider throughout the paper.

We present applications to Enumerative Combinatorics in Section 4, notably to inequalities between numbers of certain *standard Young tableaux*, and to polynomial inequalities for *Euler numbers*. Although these are results not mentioned in the introduction to avoid shifting the focus,

¹In fact, Kahn’s bound is stronger on both sides. We present a simplified version for clarity.

the reader might find them curious (cf. §8.6). While we aim to be self-contained and include most definitions, some familiarity with the subject is helpful for the clarity and the motivation.

In Section 5, we give a linear algebraic setup to derive correlation inequalities for hyperbolic matrices. This section is completely self-contained. The hope is that our basic approach can be used to obtain other inequalities. A lengthy Section 6 contains proofs and occasional generalizations of results in the introduction. The key result there is Proposition 6.1 proved in [CP24] and used as a black box in this paper.

A short Section 7 has proofs of three implications of the log-concavity. All of these are straightforward and included for completeness. We conclude with final remarks and open problems in Section 8.

2. NOTATIONS

We use $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{R}_+ = \{x \geq 0\}$ and $[n] = \{1, 2, \dots, n\}$. For a poset $P = (X, \prec)$, denote by $P^* = (X, \prec^*)$ the *dual poset* with $x \prec^* y$ if and only if $y \prec x$, for all $x, y \in X$. Denote by $\min(P)$ and $\max(P)$ the set of minimal and maximal elements in P , respectively. For $x, y \in X$, we write $x \parallel y$ if x is incomparable to y in P .

For posets $P = (X, \prec_P)$ and $Q = (Y, \prec_Q)$, the *parallel sum* $P + Q = (Z, \prec)$ is the poset on the disjoint union $Z = X \sqcup Y$, where elements of X retain the partial order of P , elements of Y retain the partial order of Q , and elements $x \in X$ and $y \in Y$ are incomparable. Similarly, the *linear sum* $P \oplus Q = (Z, \prec)$, where $x \prec y$ for every two elements $x \in X$ and $y \in Y$ and other relations as in the parallel sum. For an element $a \in X$, denote by $a \uparrow := \{x \in X : x \succcurlyeq a\}$ the *upper order ideal* generated by a . For a subset $A \subseteq X$, denote by $A \uparrow := \cup_{a \in A} a \uparrow$ the *upper closure* of A . We refer to [Sta99, Ch. 3] and [Tro95] for more on poset definitions and notation.

We use capitalized bold letters to denote matrices, e.g. $\mathbf{M} = (M_{ij})$. We also keep conventional index notations, so, e.g., $(\mathbf{M} + \mathbf{M}^2)_{ij}$ is the (i, j) -th entry of $\mathbf{M} + \mathbf{M}^2$. Similarly, we use small bold letters to denote vectors, e.g. $\mathbf{h} = (h_1, h_2, \dots)$ and $h_i = (\mathbf{h})_i$. For a subset $S \subseteq [n]$, the *characteristic vector* of S is the vector $\mathbf{v} \in \mathbb{R}^n$ such that $v_i = 1$ if $i \in S$ and $v_i = 0$ if $i \notin S$. We denote by $\mathbf{0}$ the zero vector $(0, \dots, 0)$.

3. EXAMPLES AND SPECIAL CASES

3.1. Basic examples. Here we consider several simple observations on correlation inequalities that apply to general posets. We recommend reading these first, before studying more elaborate examples with special families of posets.

Example 3.1. Suppose $\min(P) = \{x, y\}$, i.e. x, y are the only minimal elements of P . We then have $e(P) = e(P - x) + e(P - y)$. The upper bound of (1.1) is straightforward in this case:

$$\frac{e(P) \cdot e(P - x - y)}{e(P - x) \cdot e(P - y)} = \frac{e(P - x - y)}{e(P - x)} + \frac{e(P - x - y)}{e(P - y)} \leq 2,$$

where the inequality follows from $e(P - x - y) \leq \min\{e(P - x), e(P - y)\}$. See also §8.2 for a further discussion of this proof.

Example 3.2. Again, suppose $\min(P) = \{x, y\}$. In the notation of §1.2, we have:

$$\mathbb{P}[f(x) = 1] = \frac{e(P - x)}{e(P)}, \quad \mathbb{P}[f(x) > 1] = \frac{e(P - y)}{e(P)} \quad \text{and} \quad \mathbb{P}[f(x) = 2] = \frac{e(P - x - y)}{e(P)}.$$

Then (1.8) becomes

$$\frac{e(P - x - y) \cdot e(P)}{e(P - x) \cdot e(P - y)} \geq 1.$$

This inequality follows from and is asymptotically equivalent to the lower bound in (1.1).

Example 3.3. For a subset $A \subset \min(P)$ of minimal elements, define A^∇ as follows:

$$A^\nabla := \{x \in X : x \succ y, y \in \min(P) \Rightarrow y \in A\}.$$

Equivalently, $A^\nabla = A \uparrow \setminus B \uparrow$, where $B := \min(P) - A$. Clearly, we have $A^\nabla \subseteq A \uparrow$, and sometimes the subset A^∇ is much smaller than $A \uparrow$. On the other hand, we have:

$$\mathbb{P}[1 \in f(A), 2 \in f(A \uparrow)] = \mathbb{P}[1 \in f(A), 2 \in f(A^\nabla)].$$

This is because the only way we can have $f(y) = 2$ and $f(x) = 1$ for some $x \in A$, is if either $y \in \min(P)$, or y covers element x and no other element. This argument is useful in trying to understand our correlation inequalities, as the next example shows.

Example 3.4. Let $A \subset \min(P)$ and let $B := \min(P) - A$. By applying (1.15) to $A \uparrow$, we get:

$$(3.1) \quad \mathbb{P}[1 \in f(A), 2 \notin f(A \uparrow)]^2 \geq \mathbb{P}[1 \in f(A), 2 \in f(A \uparrow)] \cdot \mathbb{P}[1 \in f(B), 2 \notin f(A \uparrow)].$$

Alternatively, apply (1.17) with this choice of A, B . We get:

$$(3.2) \quad \mathbb{P}[1 \in f(A), 2 \in f(B)]^2 \geq \mathbb{P}[1 \in f(A), 2 \in f(A \uparrow)] \cdot \mathbb{P}[1 \in f(B), 2 \in f(B \uparrow)].$$

At first sight, the LHS of (3.1) \geq LHS of (3.2), while the RHS of (3.1) \leq RHS of (3.2). The argument in the previous example shows that both of these are equalities. In other words, the inequality (1.15) is equivalent to the inequality (1.17) in this case.

3.2. Second moment bound. In Theorem 1.2, letting $y = x$ gives the following curious bound on the second moment:

Corollary 3.5. *Let $P = (X, \prec)$ be a finite poset, and let $x \in X$ be a fixed element. Then:*

$$(3.3) \quad 1 \leq \frac{\mathbb{E}[f(x)^2]}{\mathbb{E}[f(x)]^2} < 2.$$

Here the lower bound is trivial: $0 \leq \text{VAR}(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2$ for every random variable Z . Note also that the corollary immediately implies a weaker version of (1.3):

$$(3.4) \quad \frac{\mathbb{E}[f(x)f(y)]}{\mathbb{E}[f(x)] \cdot \mathbb{E}[f(y)]} < 2.$$

This follows by combining the upper bound in (3.3) with $\text{Cov}(X, Y) \leq \text{VAR}(X) \cdot \text{VAR}(Y)$.

Note that our proof of Theorem 1.2 follows from a construction similar to that used in [CP24] to rederive Stanley Theorem 1.3. Since Corollary 3.5 is a simple consequence of Theorem 1.2, let us show that it follows from Stanley theorem as well.

Proposition 3.6. *Let Z be a random variable on $\{1, 2, \dots\}$ with a log-concave distribution. Then $\mathbb{E}[Z^2] \leq 2\mathbb{E}[Z]^2$.*

Note that constant 2 in the proposition is tight, as shown by a geometric random variable $Z = \text{Geo}(q)$, where we let $q \rightarrow 1$. We prove Proposition 3.6 in §7.2. On the other hand, the constant 2 in (3.3) can be lowered in many cases.

Example 3.7. Let $P := C_{n-1} + \{x\}$ be the parallel sum of two chains of size $n - 1$ and 1. We have:

$$\frac{\mathbb{E}[f(x)^2]}{\mathbb{E}[f(x)]^2} = \frac{\frac{1}{n} \sum_{k=1}^n k^2}{\left(\frac{1}{n} \sum_{k=1}^n k\right)^2} = \frac{4n+2}{3n+3} \rightarrow \frac{4}{3} \quad \text{as } n \rightarrow \infty.$$

This example motivated us to conjecture that the bound $\frac{4}{3}$ is sharp. The following counterexample was given by Max Aires and Jeff Kahn.²

Example 3.8. Let $P := C_m + C_{n-m}$ be the parallel sum of two chains of size m and $n - m$, such that $m = o(n)$. Let x be the minimum element in the chain C_m . We have:

$$\mathbb{E}[f(x)] = \frac{1}{\binom{n}{m}} \sum_{k=1}^{n-m+1} k \binom{n-k}{m-1} \quad \text{and} \quad \mathbb{E}[f(x)^2] = \frac{1}{\binom{n}{m}} \sum_{k=1}^{n-m+1} k^2 \binom{n-k}{m-1}.$$

This gives

$$\frac{\mathbb{E}[f(x)^2]}{\mathbb{E}[f(x)]^2} \rightarrow 2 \quad \text{as } m, n \rightarrow \infty.$$

In particular, this shows that the upper bound in (3.3) and Theorem 1.2 is asymptotically tight.

Example 3.9. Proposition 3.6 is sharp in full generality, but can be weak in other instances. In the notation of §1.5, let G be simple graph on $2n$ vertices. Denote by $\alpha_G(M)$ be the size of a random matching M , and let $\zeta_G(M) := n - \alpha_G$ be half of the number of vertices not covered by M . The Heilmann–Lieb theorem proves that ζ_G has a log-concave distribution. Proposition 3.6 then implies $\text{VAR}(\zeta_G) = \mathbb{E}[\zeta_G^2] - \mathbb{E}[\zeta_G]^2 < \mathbb{E}[\zeta_G]^2$. On the other hand, [Kahn00, Cor. 4.2] gives a much stronger bound $\text{VAR}(\zeta_G) \leq \mathbb{E}[\zeta_G]$, which can then be used to obtain concentration inequalities (see e.g. [AS16, Ch. 7]).

3.3. Posets with unique covers of minimal elements. For elements $x, y \in X$ in a poset $P = (X, \prec)$, we say that element y *covers* x , if $x \prec y$, and there is no $v \in X$ s.t. $x \prec v \prec y$. For elements $x \prec y$ in X , we say that y is a *unique cover* of x , if

$$(\diamond) \quad y \text{ covers } x \text{ and does not cover any other elements in } X.$$

The following correlation inequality is surprising even in the most simple special cases (see below).

Corollary 3.10. *Let $P = (X, \prec)$ be a finite poset, and let $x, y \in \min(P)$ be distinct minimal elements. Suppose element $v \in X$ is a unique cover of x , and $w \in X$ is a unique cover of y . Then:*

$$(3.5) \quad e(P - x - y)^2 \geq e(P - x - v) \cdot e(P - y - w).$$

Proof. In the notation of Theorem 1.9, let $A = \{x\}$ and $B = \{y\}$. Note that

$$\mathbb{P}[1 \in f(A), 2 \in f(A \uparrow)] = \mathbb{P}[f(x) = 1, f^{-1}(2) \succ x] \geq \mathbb{P}[f(x) = 1, f(v) = 2] = \frac{e(P - x - v)}{e(P)}.$$

Here the inequality follows from the definition of v as an element which satisfies (\diamond) . By the same argument we have:

$$\mathbb{P}[1 \in f(B), 2 \in f(B \uparrow)] \geq \frac{e(P - y - w)}{e(P)}.$$

²July 2024, personal communication

On the other hand, we also have:

$$\mathbb{P}[1 \in f(A), 2 \in f(B)] = \frac{e(P - x - y)}{e(P)}.$$

Now (1.17) implies the result. \square

We conclude this section with another four-element inequality:

Corollary 3.11. *Let $P = (X, \prec)$ be a finite poset, and let $x, y, z \in \min(P)$ be distinct minimal elements. Suppose element $u \in X$ is a unique cover of z . Then:*

$$(3.6) \quad e(P - u - z) e(P - x - y) \leq 2 e(P - x - z) e(P - y - z).$$

Proof. Let $A = \{x\}$, $B = \{y\}$ and $C = \{z\}$. The corollary now follows from Theorem 1.10 and the argument in the proof of the Corollary 3.10. \square

4. ENUMERATIVE APPLICATIONS

4.1. Standard Young tableaux. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be an integer partition of n , write $\lambda \vdash n$. Here $\ell(\lambda)$ denotes the number of parts. A Young diagram is a set $Y_\lambda := \{(i, j) \in \mathbb{N}^2 : 1 \leq j \leq \lambda_i, 1 \leq i \leq \ell\}$. A *standard Young tableau* of shape λ is a bijection $f : Y_\lambda \rightarrow [n]$ which increases in both directions. Denote by $\text{SYT}(\lambda)$ the set of standard Young tableaux of shape λ . The number $|\text{SYT}(\lambda)|$ can be computed by the *hook-length formula*, see e.g. [Sta99, §7.21].

A *conjugate partition* $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ is defined by $\lambda'_i = |\{j : \lambda_j \geq i\}|$. Note that Young diagrams Y_λ and $Y_{\lambda'}$ are dual with respect to the $i = j$ reflection. Partition λ is called *self-conjugate* if $\lambda = \lambda'$. For partitions λ and μ , define the *sum* $\lambda + \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$ and the *union* $\lambda \cup \mu := (\lambda' + \mu')'$.

Two squares $x \prec y$ are called *adjacent* if $x = (i, j)$ and $y = (i, j + 1)$ or $y = (i + 1, j)$. The *corners* of λ are defined as elements $(i, j) \in Y_\lambda$ such that $(i + 1, j), (i, j + 1) \notin Y_\lambda$. Denote by $\mathcal{C}(\lambda)$ the set of corners of λ . Similarly, the *boundary squares* of λ are defined as elements $(i, j) \in Y_\lambda$ such that either $(i + 1, j)$ or $(i, j + 1)$ is not in Y_λ . Denote by $\mathcal{D}(\lambda)$ the set of boundary squares of λ , and note that $\mathcal{C}(\lambda) \subseteq \mathcal{D}(\lambda)$.

Let $\mu = (\mu_1, \mu_2, \dots)$ be a partition such that $\mu_i \leq \lambda_i$ for all $0 \leq i \leq \ell$. The set $Y_{\lambda/\mu} := Y_\lambda \setminus Y_\mu$ is the *skew Young diagram* of (skew) shape λ/μ . Standard Young tableaux of skew shapes are defined similarly to the usual (straight) shapes. The number $|\text{SYT}(\lambda/\mu)|$ of standard Young tableaux of shape λ/μ can be computed by the *Aitken–Feit determinant formula*, see e.g. [Sta99, §7.16]. The *corners* of λ/μ are defined as elements $(i, j) \in Y_{\lambda/\mu}$ such that $(i + 1, j), (i, j + 1) \notin Y_{\lambda/\mu}$.

4.2. Removing the corners. We now apply the correlation inequalities to posets corresponding to skew Young diagrams. As we do, we get both new and familiar inequalities (see below).

Corollary 4.1. *Let λ/μ be a skew shape, let $x, y \in \mathcal{C}(\lambda/\mu)$ be corners, and let $v, w \in \mathcal{D}(\lambda)$ be a boundary square adjacent to x and y , respectively. Then:*

$$(4.1) \quad |\text{SYT}(\lambda/\mu - x - y)|^2 \geq |\text{SYT}(\lambda/\mu - x - v)| \cdot |\text{SYT}(\lambda/\mu - y - w)|.$$

In particular, if λ and μ are self-conjugate, $x = (i, j)$ and $y = (j, i)$, then:

$$(4.2) \quad |\text{SYT}(\lambda/\mu - x - y)| \geq |\text{SYT}(\lambda/\mu - x - v)|.$$

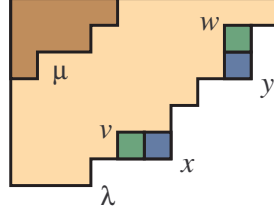


FIGURE 4.1. Skew shape λ/μ , where $\lambda = (10, 9, 9, 7, 6, 6, 3)$, $\mu = (4, 3, 1)$, corners $x, y \in \mathcal{C}(\lambda)$ and boundary squares $v, w \in \mathcal{D}(\lambda)$.

Proof. Let poset $\mathcal{P}_{\lambda/\mu} = (Y_{\lambda/\mu}, \prec)$ be defined by $(i, j) \prec (i', j')$ if $i \leq i'$ and $j \leq j'$. The set of linear extensions $\mathcal{E}(\mathcal{P}_{\lambda/\mu})$ is in bijection with $\text{SYT}(\lambda/\mu)$, so $e(\mathcal{P}_{\lambda/\mu}) = |\text{SYT}(\lambda/\mu)|$. Note that minimal elements in the dual poset \mathcal{P}_{λ}^* are the corners of λ : $\min(\mathcal{P}_{\lambda}^*) = \mathcal{C}(\lambda)$. Similarly, for skew shapes we have: $\min(\mathcal{P}_{\lambda/\mu}^*) = \mathcal{C}(\lambda) \setminus \mathcal{C}(\mu)$.

In notation of §3.3, note that v and w are unique covers of x and y , respectively. Now Corollary 3.10 gives (4.1). For the second part, let $v = (p, q)$ and take $w := (q, p)$. Now (4.1) implies (4.2). \square

Corollary 4.2. *Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition. We have the following log-concave inequality:*

$$(4.3) \quad |\text{SYT}(\lambda/(a, 1^b))|^2 \geq |\text{SYT}(\lambda/(a+1, 1^{b-1}))| \cdot |\text{SYT}(\lambda/(a-1, 1^{b+1}))|,$$

for all $1 < a < \lambda_1$ and $1 < b < \ell$.

Proof. Following the argument in the proof above, let $\mathcal{P}_{\lambda/\mu} = (Y_{\lambda/\mu}, \prec)$ where $\mu = (a+1, 1^{b+1})$, and let $x = (1, a)$, $y = (b, 1)$, $v = (1, a+1)$, $w = (b+1, 1)$. Now Corollary 3.10 gives the result. \square

Corollary 4.3. *Let λ/μ be a skew shape, let $x, y, z \in \mathcal{C}(\lambda/\mu)$ be corners, and let $u \in \mathcal{D}(\lambda)$ be a boundary square adjacent to z , respectively. Then:*

$$(4.4) \quad \frac{|\text{SYT}(\lambda/\mu - u - z)| \cdot |\text{SYT}(\lambda/\mu - x - y)|}{|\text{SYT}(\lambda/\mu - x - z)| \cdot |\text{SYT}(\lambda/\mu - y - z)|} \leq 2.$$

The proof follows from Corollary 3.11 along the same lines as the proofs above. We omit the details.

4.3. Related inequalities. Corollary 4.1 is closely related to two other inequalities. The first is the *Okounkov inequality* conjectured in [Oko97, p. 269], and proved by Lam–Postnikov–Pylyavskyy [LPP07, Thm 4]. In a special case, the Okounkov inequality gives:

$$(4.5) \quad \left| \text{SYT} \left(\frac{\lambda + \nu}{2} / \frac{\mu + \alpha}{2} \right) \right|^2 \geq |\text{SYT}(\lambda/\mu)| \cdot |\text{SYT}(\nu/\alpha)|,$$

for all skew shapes λ/μ and ν/α , such that partitions $\lambda + \nu$ and $\mu + \alpha$ have even parts, cf. §8.6 and [CP23a].

In the notation of Corollary 4.1, observe that for v to the left of x , and for w to the left of y (see Figure 4.1), the Okounkov inequality coincides with our inequality (4.1). On the other hand, even the inequality (4.2) does not follow from this argument.

The second is the *Fomin–Fulton–Li–Poon (FFLP) inequality* conjectured in [FFLP05, Conj. 2.7], and also proved in [LPP07, Thm 4]. Again, in a special case, this inequality gives:

$$(4.6) \quad \begin{aligned} & \left| \text{SYT}(\text{sort}_1(\lambda, \nu) / \text{sort}_1(\mu, \alpha)) \right| \cdot \left| \text{SYT}(\text{sort}_2(\lambda, \nu) / \text{sort}_2(\mu, \alpha)) \right| \\ & \geq |\text{SYT}(\lambda/\mu)| \cdot |\text{SYT}(\nu/\alpha)|. \end{aligned}$$

Here for two partitions β and γ we define two other partitions $\text{sort}_1(\beta, \gamma) := (\tau_1, \tau_3, \dots)$ and $\text{sort}_2(\lambda, \mu) := (\tau_2, \tau_4, \dots)$, where $(\tau_1, \tau_2, \dots) := \beta \cup \gamma$.

In notation of Corollary 4.1, observe that for v above x , and for w to the above y , the FFLP inequality coincides with our inequality (4.1). In this sense it is a conjugate dual to the Okounkov inequality (4.5), cf. [LPP07]. On the other hand, for v to the left of x , and for w above y , the LHS of (4.6) has partitions of unequal size:

$$(4.7) \quad |\text{SYT}(\lambda/\mu - y)| \cdot |\text{SYT}(\lambda/\mu - x - y - v)| \geq |\text{SYT}(\lambda/\mu - x - v)| \cdot |\text{SYT}(\lambda/\mu - y - w)|.$$

In summary, the inequality (4.7) follows from known inequalities in some cases (where it extends to inequalities for symmetric functions), and is new in other cases (where it does not).

4.4. Hook walk. Let $f \in \text{SYT}(\lambda)$ be a random standard Young tableau of shape $\lambda \vdash n$. For a corner $x \in \mathcal{C}(\lambda)$, denote by

$$p(x) := \frac{|\text{SYT}(\lambda - x)|}{|\text{SYT}(\lambda)|}$$

the probability distribution on $\mathcal{C}(\lambda)$ defined by the location of n in f . The celebrated *hook walk* [GNW79] shows how to sample from $p(x)$:

- Start at a random square $y \in Y_\lambda$.
- Move to a random square in the hook $H(y)$.
- Repeat until the walk reaches a corner $x \in \mathcal{C}(\lambda)$.

Here *the hook* is defined as $H(i, j) := \{(i, r) : j < r \leq \lambda_i\} \cup \{(s, j) : i < s \leq \lambda'_j\}$. Let $y \in \mathcal{C}(\lambda)$ be a different corner. One can think of (1.2) as follows:

$$(4.8) \quad \frac{n}{n-1} \leq \frac{\mathbb{P}[f(y) = n-1 \mid f(x) = n]}{\mathbb{P}[f(y) = n]} \leq 2.$$

The lower bound in (4.8) is now straightforward. Indeed, all hook walk trajectories which arrive to y in $(\lambda - x)$, also arrive in λ , and the ratio is given by the ratio of sizes of diagrams in the starting location of the walk. The upper bound in (4.8) does not seem to follow directly from this probabilistic model.

4.5. Bruhat order. Let $\sigma \in S_n$ and define the *permutation poset* $P_\sigma = ([n], \prec)$ by letting

$$i \prec j \iff i \leq j \text{ and } \sigma(i) \leq \sigma(j).$$

It is easy to see that $\mathcal{E}(P_\sigma) \subseteq S_n$ is the lower ideal $[\text{id}, \sigma]$ of σ in the *weak Bruhat order* $\mathcal{B}_n = (S_n, \triangleleft)$, see [BW91, FW97]. The set of minimal elements in \mathcal{P}_σ is the set of *minimal records* in σ :

$$\min(P_\sigma) = \{k \in [n] : \sigma(i) > \sigma(k) \text{ for all } i < k\}.$$

Fix two record minima $a, b \in \min(P_\sigma)$. We can then rewrite the correlation inequality (1.2) in terms of random permutations $\omega \in S_n$:

$$\frac{n}{n-1} \leq \frac{\mathbb{P}[\omega(a) = 1, \omega(b) = 2 \mid \omega \trianglelefteq \sigma]}{\mathbb{P}[\omega(a) = 1 \mid \omega \trianglelefteq \sigma] \cdot \mathbb{P}[\omega(a) = 2 \mid \omega \trianglelefteq \sigma]} \leq 2.$$

As in the introduction, the lower bound is an equality for $\sigma = (n, n-1, \dots, 1)$ and the upper bound is an equality for $\sigma = (2, 1, 3, \dots, n)$.

4.6. Zigzag posets. Let $P_n := (X, \prec)$ be the poset on n elements $X = \{x_1, y_1, x_2, y_2, x_3, \dots\}$, with the only relations given by $x_1 \prec y_1 \succ x_2 \prec y_2 \succ x_3 \prec \dots$. Note that P_n has height two and the set $\mathcal{E}(P_n)$ is in bijection with *alternating permutations* $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \sigma(5) < \dots$ and the standard Young tableaux of a *zigzag shape* [Sta10].

It is well-known that $e(P_n) = E_n$ are the *Euler numbers*, which satisfy

$$\sum_{m=0}^{\infty} E_n \frac{t^m}{m!} = \tan(t) + \sec(t) \quad \text{and} \quad E_n \sim \frac{2^{n+2} n!}{\pi^{n+1}} \quad \text{as } n \rightarrow \infty,$$

see e.g. [OEIS, A000111], [FS09, §IV.6.1] and [Sta99, §1.6]. We also have:

$$\mathbb{P}[f(x_1) = k] = \frac{E_{n,k}}{E_n}, \quad \text{where} \quad E_{n,k} := \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} (-1)^i \binom{k}{2i+1} E_{n-2i-1}$$

are the *Entringer numbers*, see [OEIS, A008282] and [Sta10]. Stanley Theorem 1.3 in this case gives log-concavity of the Entringer numbers:

$$(4.9) \quad E_{n,k}^2 \geq E_{n,k-1} \cdot E_{n,k+1}.$$

This inequality can be viewed as a degree two polynomial inequality for the Euler numbers. We refer to [B+19, GHMY23] and [CP24, §1.38] for more on (4.9) and its various generalizations.

Example 4.4. Let $n = 2m + 1$, and note that $\min(P_n) = \{x_1, \dots, x_{m+1}\}$. Fix $1 \leq k \leq m + 1$ and consider a subset $A := \{x_1, \dots, x_k\}$. We have:

$$\mathbb{P}[1 \in f(A)] = \frac{F_n(k)}{E_n}, \quad \mathbb{P}[1 \notin f(A)] = \frac{F_n(m-k+1)}{E_n}, \quad \mathbb{P}[1 \notin f(A), 2 \in f(A)] = \frac{G_n(k)}{E_n},$$

where

$$F_n(k) := \sum_{i=1}^k \binom{n-1}{2i-2} E_{2i-2} E_{n-2i+1} \quad \text{and} \\ G_n(k) := \sum_{i=1}^k \sum_{j=k+1}^{m+1} \binom{n-2}{2i-2, 2j-2i-1} E_{2i-2} E_{2j-2i-1} E_{n-2j+1}.$$

Substituting these into (1.12) gives a new degree four polynomial inequality for the Euler numbers:

$$(4.10) \quad F_n(k) \cdot F_n(m-k+1) \leq E_n \cdot G_n(k).$$

Note that for $k, n-k = \omega(1)$ both sides have the same leading asymptotics, which makes the inequality even more interesting.

Example 4.5. Let $n = 2m + 1$ and $1 \leq k \leq \ell \leq m + 1$. Take $A = \{x_1, \dots, x_k\}$ as above. We have:

$$\mathbb{P}[1, 2 \in f(A)] = \frac{H_n(k)}{E_n} \quad \text{and} \quad \mathbb{P}[1, 2 \notin f(A)] = \frac{H_n(m-k+1)}{E_n},$$

where

$$H_n(k) := 2 \sum_{1 \leq i < j \leq k} \binom{n-2}{2i-2, 2j-2i-1} E_{2i-2} E_{2j-2i-1} E_{n-2j+1}.$$

Substituting this and the inequalities from the previous example into (1.15) with $A \leftarrow X \setminus A$ gives a new degree six polynomial inequality for the Euler numbers:

$$(4.11) \quad G_n(k)^2 \geq H_n(k) \cdot H_n(m-k+1).$$

5. CORRELATION INEQUALITIES FROM HYPERBOLIC INEQUALITIES

In this short section we derive several correlation inequalities for hyperbolic matrices. Although we are motivated by applications to combinatorial atlas [CP22, CP24], the presentation here is self-contained and uses nothing but linear algebra.

5.1. Hyperbolic matrices. Let $\mathbf{M} = (M_{ij})$ be a symmetric $d \times d$ matrix with entries $M_{ij} \in \mathbb{R}_+$. The matrix \mathbf{M} is *hyperbolic* (satisfies *hyperbolic property*), if

$$(\text{Hyp}) \quad \langle \mathbf{x}, \mathbf{M}\mathbf{y} \rangle^2 \geq \langle \mathbf{x}, \mathbf{M}\mathbf{x} \rangle \langle \mathbf{y}, \mathbf{M}\mathbf{y} \rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d \text{ such that } \langle \mathbf{y}, \mathbf{M}\mathbf{y} \rangle \geq 0.$$

Note that (Hyp) is equivalent to the matrix \mathbf{M} having at most one positive eigenvalue, counting multiplicity (see e.g. [CP22, SvH19]).

For every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we employ the following shorthand

$$\mathbf{M}_{\mathbf{xy}} := \langle \mathbf{x}, \mathbf{M}\mathbf{y} \rangle.$$

Note that $\mathbf{M}_{\mathbf{xy}} = \mathbf{M}_{\mathbf{yx}}$ since \mathbf{M} is symmetric.

Lemma 5.1. *Let \mathbf{M} be a nonnegative symmetric $d \times d$ matrix that satisfies (Hyp), and let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}_+^d$ be nonnegative vectors. Then either $\mathbf{M}\mathbf{x} = \mathbf{M}\mathbf{y} = \mathbf{M}\mathbf{z} = \mathbf{0}$, or there exists $\mathbf{v} \in \mathbb{R}_+^d$ such that $\langle \mathbf{x} + \varepsilon \mathbf{v}, \mathbf{M}(\mathbf{x} + \varepsilon \mathbf{v}) \rangle > 0$ for all $\varepsilon > 0$.*

Proof. We split the proof into two cases. First, suppose that \mathbf{M} has a positive eigenvalue $\lambda > 0$ with a corresponding eigenvector $\mathbf{v} \in \mathbb{R}_+^d$. Then we get

$$\begin{aligned} \langle \mathbf{x} + \varepsilon \mathbf{v}, \mathbf{M}(\mathbf{x} + \varepsilon \mathbf{v}) \rangle &= \langle \mathbf{x}, \mathbf{M}\mathbf{x} \rangle + 2\varepsilon \langle \mathbf{x}, \mathbf{M}\mathbf{v} \rangle + \varepsilon^2 \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \\ &\geq \varepsilon^2 \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle = \varepsilon^2 \lambda \langle \mathbf{v}, \mathbf{v} \rangle > 0, \end{aligned}$$

as desired.

Now suppose that all eigenvalues of \mathbf{M} are nonpositive. Then this implies, $\mathbf{M}_{\mathbf{xx}} = \langle \mathbf{x}, \mathbf{M}\mathbf{x} \rangle < 0$ whenever $\mathbf{x} \notin \ker \mathbf{M}$. On the other hand, we have $\mathbf{M}_{\mathbf{xx}} = \langle \mathbf{x}, \mathbf{M}\mathbf{x} \rangle \geq 0$ by the non-negativity of \mathbf{M} and \mathbf{x} . Combining these two observations, we get that $\mathbf{M}\mathbf{x} = \mathbf{0}$. Analogously, we have $\mathbf{M}\mathbf{y} = \mathbf{M}\mathbf{z} = \mathbf{0}$, which completes the proof. \square

The following lemma is a useful consequence of (Hyp), and is inspired by the inequality in [Sch14, Lemma 7.4.1] for mixed volumes.³

Lemma 5.2. *Let \mathbf{M} be a nonnegative symmetric $d \times d$ matrix that satisfies (Hyp), and let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}_+^d$ be nonnegative vectors. Then:*

$$(5.1) \quad (\mathbf{M}_{\mathbf{yz}} \mathbf{M}_{\mathbf{xx}} - \mathbf{M}_{\mathbf{xy}} \mathbf{M}_{\mathbf{xz}})^2 \leq (\mathbf{M}_{\mathbf{xy}}^2 - \mathbf{M}_{\mathbf{xx}} \mathbf{M}_{\mathbf{yy}}) (\mathbf{M}_{\mathbf{xz}}^2 - \mathbf{M}_{\mathbf{xx}} \mathbf{M}_{\mathbf{zz}}).$$

Proof. Fix $\mathbf{x} \in \mathbb{R}_+^d$ and let $Q : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the bilinear form given by

$$Q(\mathbf{y}, \mathbf{z}) := \mathbf{M}_{\mathbf{yz}} \mathbf{M}_{\mathbf{xx}} - \mathbf{M}_{\mathbf{xy}} \mathbf{M}_{\mathbf{xz}}.$$

Then (Hyp) implies that $Q(\mathbf{y}, \mathbf{y})$ is a nonnegative quadratic form. It then follows from the Cauchy–Schwarz inequality that

$$Q(\mathbf{y}, \mathbf{z})^2 \leq Q(\mathbf{y}, \mathbf{y}) Q(\mathbf{z}, \mathbf{z}).$$

This is equivalent to (5.1), which completes the proof. \square

³Our original proof was more complicated. The simplified proof below was suggested to us by Ramon van Handel (Nov. 2022, personal communication).

5.2. The implications. The following lemma is a simple consequence of Lemma 5.2.

Lemma 5.3. *Let \mathbf{M} be a nonnegative symmetric $d \times d$ matrix that satisfies (Hyp), and let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}_+^d$ be nonnegative vectors. Then:*

$$(5.2) \quad \mathbf{M}_{yy} \mathbf{M}_{xz}^2 + \mathbf{M}_{zz} \mathbf{M}_{xy}^2 \leq 2 \mathbf{M}_{xy} \mathbf{M}_{xz} \mathbf{M}_{yz}.$$

Proof. The lemma clearly follows if $\mathbf{M}\mathbf{x} = \mathbf{M}\mathbf{y} = \mathbf{M}\mathbf{z} = \mathbf{0}$, so assume that this is not the case. Then, by Lemma 5.1, we can substitute \mathbf{x} with $\mathbf{x} + \varepsilon \mathbf{v}$ and then takes the limit $\varepsilon \rightarrow 0$ if necessary, so we can additionally assume that $\mathbf{M}_{xx} > 0$. Expanding the squares in (5.1), we get:

$$\begin{aligned} & \mathbf{M}_{yz}^2 \mathbf{M}_{xx}^2 - 2 \mathbf{M}_{xx} \mathbf{M}_{xy} \mathbf{M}_{xz} \mathbf{M}_{yz} + \mathbf{M}_{xy}^2 \mathbf{M}_{xz}^2 \\ & \leq \mathbf{M}_{xy}^2 \mathbf{M}_{xz}^2 + \mathbf{M}_{xx}^2 \mathbf{M}_{yy} \mathbf{M}_{zz} - \mathbf{M}_{xx} \mathbf{M}_{yy} \mathbf{M}_{xz}^2 - \mathbf{M}_{xx} \mathbf{M}_{zz} \mathbf{M}_{xy}^2. \end{aligned}$$

Cancelling the term $\mathbf{M}_{xy}^2 \mathbf{M}_{xz}^2$ from both sides, and then dividing both sides by \mathbf{M}_{xx} , we get:

$$(5.3) \quad \mathbf{M}_{yz}^2 \mathbf{M}_{xx} - 2 \mathbf{M}_{xy} \mathbf{M}_{xz} \mathbf{M}_{yz} \leq \mathbf{M}_{xx} \mathbf{M}_{yy} \mathbf{M}_{zz} - \mathbf{M}_{yy} \mathbf{M}_{xz}^2 - \mathbf{M}_{zz} \mathbf{M}_{xy}^2.$$

This is equivalent to

$$\mathbf{M}_{yy} \mathbf{M}_{xz}^2 + \mathbf{M}_{zz} \mathbf{M}_{xy}^2 + \mathbf{M}_{xx} (\mathbf{M}_{yz}^2 - \mathbf{M}_{yy} \mathbf{M}_{zz}) \leq 2 \mathbf{M}_{xy} \mathbf{M}_{xz} \mathbf{M}_{yz}.$$

By (Hyp), we have $\mathbf{M}_{yz}^2 \geq \mathbf{M}_{yy} \mathbf{M}_{zz}$. This and the above inequality imply the result. \square

Remark 5.4. Note that (5.2) implies (Hyp) by substitution $\mathbf{z} \leftarrow \mathbf{x}$. The inequality (5.3) can be rewritten symmetrically as:

$$(5.4) \quad \det \begin{bmatrix} \mathbf{M}_{xx} & \mathbf{M}_{xy} & \mathbf{M}_{xz} \\ \mathbf{M}_{xy} & \mathbf{M}_{yy} & \mathbf{M}_{yz} \\ \mathbf{M}_{xz} & \mathbf{M}_{yz} & \mathbf{M}_{zz} \end{bmatrix} \geq 0.$$

It can be viewed as the counterpart of *Shephard's inequality* for mixed volumes, see e.g. [vH23].

Lemma 5.5. *Let \mathbf{M} be a nonnegative symmetric $d \times d$ matrix that satisfies (Hyp), and let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}_+^d$ be nonnegative vectors. Then:*

$$(5.5) \quad \mathbf{M}_{zz} \mathbf{M}_{xy} \leq 2 \mathbf{M}_{xz} \mathbf{M}_{yz}.$$

Proof. Note that the lemma clearly holds if $\mathbf{M}_{xy} = 0$, so we can assume that $\mathbf{M}_{xy} > 0$. The lemma now follows by removing the term $\mathbf{M}_{yy} \mathbf{M}_{xz}^2$ from the LHS of (5.2) and dividing both sides by \mathbf{M}_{xy} . \square

The upper bound in (5.5) can be improved in some cases.

Lemma 5.6. *Let \mathbf{M} be a nonnegative symmetric $d \times d$ matrix that satisfies (Hyp), and let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}_+^d$ be nonnegative vectors. Then at least two out of these three inequalities hold:*

$$\mathbf{M}_{xx} \mathbf{M}_{yz} \leq \mathbf{M}_{xy} \mathbf{M}_{xz}, \quad \mathbf{M}_{yy} \mathbf{M}_{xz} \leq \mathbf{M}_{xy} \mathbf{M}_{yz} \quad \text{or} \quad \mathbf{M}_{zz} \mathbf{M}_{xy} \leq \mathbf{M}_{xz} \mathbf{M}_{yz}.$$

Proof. Suppose to the contrary that at least two of these inequalities are false. Without loss of generality we assume the two inequalities are

$$\mathbf{M}_{xx} \mathbf{M}_{yz} > \mathbf{M}_{xy} \mathbf{M}_{xz} \quad \text{and} \quad \mathbf{M}_{yy} \mathbf{M}_{xz} > \mathbf{M}_{xy} \mathbf{M}_{yz}.$$

Taking the product of these two inequalities gives $\mathbf{M}_{xx} \mathbf{M}_{yy} > \mathbf{M}_{xy}^2$. This contradicts (Hyp) and completes the proof. \square

Remark 5.7. In conditions of Lemma 5.6, there are cases such that exactly two out of these three inequalities hold. For example, the lower bound in (1.1) is an example in which $\mathbf{M}_{zz} \mathbf{M}_{xy} > \mathbf{M}_{xz} \mathbf{M}_{yz}$, with the choice of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ as in the proof of Theorem 6.3.

6. PROOF OF CORRELATION INEQUALITIES

In this section we prove the inequalities in the introduction. The proofs follow the approach in [CP24], and are based on the observation that certain matrices associated with posets satisfy (Hyp).

6.1. The setup. Fix a poset $P = (X, \prec)$ with $|X| = n$ elements, and fix an element $a \in X$. For every $k \geq 0$ and $S \subseteq \mathcal{E}(P)$, we write

$$N_k(S) := |\{f \in S : f(a) = k\}|.$$

To simplify the notation, we write

$$N_k(f(x) = 1) \quad \text{to mean} \quad N_k(\{f \in \mathcal{E}(P) : f(x) = 1\}).$$

When the underlying poset P is potentially ambiguous, we will write $N_k^{(P)}$ instead of N_k .

Let Z_{up} and Z_{down} be two distinct copies of $X - \{a\}$, let $Z := Z_{\text{up}} \cup Z_{\text{down}}$, let $d = |Z| = 2(n-1)$, and let $k \in \{2, \dots, n-1\}$. We denote by $\mathbf{M} := \mathbf{M}(P, a, k)$ the symmetric $d \times d$ matrix given by

- $M_{xy} := N_{k+1}(f(x) = 1, f(y) = 2)$ if $x, y \in \min(Z_{\text{down}})$, $x \neq y$,
- $M_{xy} := N_{k-1}(f(x) = n, f(y) = n-1)$ if $x, y \in \max(Z_{\text{up}})$, $x \neq y$,
- $M_{xy} := N_k(f(x) = 1, f(y) = n)$ if $x \in \min(Z_{\text{down}})$, $y \in \max(Z_{\text{up}})$,
- $M_{xx} := N_{k+1}(f(x) = 1) - N_{k+1}(f(x) = 2)$ if $x \in \min(Z_{\text{down}})$,
- $M_{xx} := N_{k-1}(f(x) = n) - N_{k-1}(f(x) = n-1)$ if $x \in \max(Z_{\text{up}})$,
- $M_{xy} := 0$ otherwise.

Proposition 6.1 ([CP24, Prop 14.9]). *The matrix \mathbf{M} defined above satisfies (Hyp).*

We will also use the following combinatorial properties of the matrix \mathbf{M} . By a direct calculation, the diagonal entry M_{xx} for $x \in \min(Z_{\text{down}})$, is equal to

$$\begin{aligned} M_{xx} &= N_{k+1}(f(x) = 1) - N_{k+1}(f(x) = 2, x \parallel f^{-1}(1)) \\ (6.1) \quad &= N_{k+1}(f(x) = 1) - N_{k+1}(f(x) = 1, x \parallel f^{-1}(2)) \\ &= N_{k+1}(f(x) = 1, x \prec f^{-1}(2)), \end{aligned}$$

where $x \parallel y$ means that x is incomparable to y in P . By a similar argument, for $x \in \max(Z_{\text{up}})$, we have:

$$(6.2) \quad M_{xx} = N_{k-1}(f(x) = n, x \succ f^{-1}(n-1)).$$

Lemma 6.2. *If $x \in \min(Z_{\text{down}})$, then:*

$$(6.3) \quad \sum_{y \in Z_{\text{down}}} M_{xy} = N_{k+1}(f(x) = 1) \quad \text{and} \quad \sum_{y \in Z_{\text{up}}} M_{xy} = N_k(f(x) = 1).$$

Similarly, if $x \in \max(Z_{\text{up}})$, then:

$$(6.4) \quad \sum_{y \in Z_{\text{down}}} M_{xy} = N_k(f(x) = n) \quad \text{and} \quad \sum_{y \in Z_{\text{up}}} M_{xy} = N_{k-1}(f(x) = n).$$

Proof. This follows from a direct calculation. The details are straightforward. \square

6.2. Correlations for deletion operations. Let $P = (X, \prec)$ be a poset with n elements. Fix $k \geq 1$ and $a \in X$. As in the introduction, denote by $e_k(P) := N_k^{(P)}$ the number of linear extensions $f \in \mathcal{E}(P)$ such that $f(a) = k$. We start with the following correlation inequality extending the upper bound in Theorem 1.1.

Theorem 6.3. *Let $P = (X, \prec)$ be a poset with $|X| = n > 2$ elements. Fix an element $a \in X$ and integer $1 \leq k \leq n - 2$. Then, for every distinct minimal elements $x, y \in \min(X - a)$, we have:*

$$(6.5) \quad e_k(P) e_k(P - x - y) \leq 2 e_k(P - x) e_k(P - y).$$

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ be the characteristic vectors of $x, y \in \min(Z_{\text{down}})$. Similarly, let $\mathbf{z} \in \mathbb{R}^d$ be the characteristic vector of Z_{up} . Let $\mathbf{M} := \mathbf{M}(P, a, k)$ be the matrix in Proposition 6.1. Note that

$$\mathbf{M}_{\mathbf{x}\mathbf{y}} = N_{k+1}(f(x) = 1, f(y) = 2) = e_{k-1}(P - x - y)$$

and

$$\mathbf{M}_{\mathbf{x}\mathbf{z}} = \sum_{z \in Z_{\text{up}}} M_{xz} \stackrel{(6.3)}{=} N_k(f(x) = 1) = e_{k-1}(P - x).$$

We have $\mathbf{M}_{\mathbf{y}\mathbf{z}} = e_{k-1}(P - y)$ by the same reasoning. Finally, note that

$$\mathbf{M}_{\mathbf{z}\mathbf{z}} = \sum_{v \in Z_{\text{up}}} \sum_{w \in Z_{\text{up}}} M_{vw} \stackrel{(6.4)}{=} e_{k-1}(P).$$

Substituting these equations into (5.5), gives

$$e_{k-1}(P) e_{k-1}(P - x - y) \leq 2 e_{k-1}(P - x) e_{k-1}(P - y),$$

Letting $k \leftarrow k + 1$ implies the result. \square

Proof of Theorem 1.1. Let $P' := P + \{a\}$ be the parallel sum of the poset P and the single element $\{a\}$. Note that

$$\begin{aligned} e_{k-1}(P') &= e(P), & e_{k-1}(P' - x - y) &= e(P - x - y), \\ e_{k-1}(P' - x) &= e(P - x), & e_{k-1}(P' - y) &= e(P - y). \end{aligned}$$

The theorem now follows by applying Theorem 6.3 to the poset P' . \square

6.3. Covariances for random linear extensions. We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. We create a new poset $P' := (X', \prec')$ as follows. Let the ground set $X' := X \cup \{x, y\}$ be obtained by adding two new elements x, y . Let the partial order \prec' be the closure of the partial order \prec and the extra relations

$$(6.6) \quad x \prec' v \text{ for every } v \in A, \quad y \prec' w \text{ for every } w \in B.$$

Note that x and y are minimal elements of P' .

Clearly, $e(P' - x - y) = e(P)$. Note that, for every linear extension $f \in \mathcal{E}(P) = \mathcal{E}(P' - x - y)$, there are exactly $f_{\min}(B)$ ways to add y to form a linear extension of $P' - x$. This implies that

$$e(P' - x) = \sum_{i \geq 1} i \cdot N^{(P)}(f_{\min}(B) = i) = \mathbb{E}[f_{\min}(B)] \cdot e(P).$$

By the same reasoning, we have

$$e(P' - y) = \mathbb{E}[f_{\min}(A)] \cdot e(P).$$

Finally, note that for every linear extension $f \in \mathcal{E}(P) = \mathcal{E}(P' - x - y)$, there are exactly $f_{\min}(A)f_{\min}(B) + \min\{f_{\min}(A), f_{\min}(B)\}$ ways to add x and y to form a linear extension of P' . This implies that

$$\begin{aligned} e(P') &= \sum_{i \geq 1} \sum_{j \geq 1} (ij + \min\{i, j\}) \cdot N^{(P)}(f_{\min}(A) = i, f_{\min}(B) = j) \\ &= \left(\mathbb{E}[f_{\min}(A)f_{\min}(B)] + \mathbb{E}[f_{\min}(A \cup B)] \right) \cdot e(P). \end{aligned}$$

Combining all these equations, we get

$$\frac{e(P') \cdot e(P' - x - y)}{e(P' - x) \cdot e(P' - y)} = \frac{\mathbb{E}[f_{\min}(A)f_{\min}(B)] + \mathbb{E}[f_{\min}(A \cup B)]}{\mathbb{E}[f_{\min}(A)] \mathbb{E}[f_{\min}(B)]}.$$

The result now follows from the upper bound in Theorem 1.1 applied to the poset P' . \square

6.4. Stanley type inequalities for subsets. As in §6.2, we start with the following result implying Corollary 1.7.

Lemma 6.4. *Let $P = (X, \prec)$ be a poset on $|X| = n \geq 3$ elements, let $a \in X$, and let $A \subseteq X - a$ be a nonempty subset. Then:*

$$N_k(1 \notin f(A), 2 \in f(A))^2 \geq N_k(1 \in f(A), 2 \in f(A \uparrow)) \cdot N_k(1, 2 \notin f(A)),$$

for every $3 \leq k \leq n$.

Proof. Let $\mathbf{x} \in \mathbb{R}^d$ be the characteristic vector of $A \subseteq Z_{\text{down}}$, and let $\mathbf{y} \in \mathbb{R}^d$ be the characteristic vector of $(X - A - a) \subseteq Z_{\text{down}}$. Let $\mathbf{M} := \mathbf{M}(P, a, k)$ be the matrix in Proposition 6.1. Let $A' := A \cap \min(P)$. Then it follows from (6.1) that:

$$\mathbf{M}_{\mathbf{x}\mathbf{x}} = \sum_{x \in A'} \left[N_{k+1}(f(x) = 1, x \prec f^{-1}(2)) + \sum_{y \in A', y \parallel x} N_{k+1}(f(x) = 1, f(y) = 2) \right].$$

Since the terms in the sums above are only nonzero if $x, y \in \min(P)$, it then follows that

$$\mathbf{M}_{\mathbf{x}\mathbf{x}} = \sum_{x \in A} \left[N_{k+1}(f(x) = 1, x \prec f^{-1}(2)) + \sum_{y \in A, y \parallel x} N_{k+1}(f(x) = 1, f(y) = 2) \right].$$

This is then equal to

$$(6.7) \quad \mathbf{M}_{\mathbf{x}\mathbf{x}} = N_{k+1}(1 \in f(A), 2 \in f(A \uparrow)),$$

which follows from the definition of the upper closure $A \uparrow$.

By the same argument that proves (6.7), we have:

$$\begin{aligned} (6.8) \quad \mathbf{M}_{\mathbf{y}\mathbf{y}} &= \sum_{x \notin A} \left[N_{k+1}(f(x) = 1, x \prec f^{-1}(2)) + \sum_{y \notin A, y \parallel x} N_{k+1}(f(x) = 1, f(y) = 2) \right] \\ &\geq N_{k+1}(1, 2 \notin f(A)). \end{aligned}$$

Now, let $B' := \min(P) \setminus A$. Then

$$\mathbf{M}_{\mathbf{x}\mathbf{y}} = \mathbf{M}_{\mathbf{y}\mathbf{x}} = \sum_{y \in B'} \sum_{x \in A'} N_{k+1}(f(y) = 1, f(x) = 2) = N_{k+1}(1 \in f(B'), 2 \in f(A')).$$

By the definition of B' , we then get

$$(6.9) \quad \mathbf{M}_{\mathbf{xy}} = N_{k+1}(1 \notin f(A), 2 \in f(A')) \leq N_{k+1}(1 \notin f(A), 2 \in f(A)).$$

Combining (6.7), (6.8), (6.9) into (Hyp), and substituting $k \leftarrow k - 1$, we obtain the result. \square

Proof of the first part of Theorem 1.6. Let $P' := P + \{a\}$ be the parallel sum of the poset P and the single element $\{a\}$. Note that $N_k^{(P')} = N^{(P)}$. Applying Lemma 6.4 to the poset P' , we get

$$(6.10) \quad N(1 \notin f(A), 2 \in f(A))^2 \geq N(1 \in f(A), 2 \in f(A\uparrow)) \cdot N(1, 2 \notin f(A)).$$

Now let $A' := A \cap \min(P)$ and $B' := \min(P) \setminus A$. Note that

$$(6.11) \quad \begin{aligned} N(1 \in f(A), 2 \in f(A\uparrow)) &= N(1 \in f(A)) - N(1 \in f(A), 2 \in f(B')) \\ &= N(1 \in f(A)) - N(1 \in f(B'), 2 \in f(A')) \\ &\geq N(1 \in f(A)) - N(1 \notin f(A), 2 \in f(A)). \end{aligned}$$

Here the first equality is by the definition of the upper closure $A\uparrow$, the second equality is obtained by swapping $1 \leftrightarrow 2$ in f , and the third inequality is by the same argument that proves (6.9).

Substituting (6.11) into (6.10) and dividing both sides of the inequality by $e(P)^2$, we get a probabilistic inequality:

$$(6.12) \quad \mathbb{P}(1 \notin f(A), 2 \in f(A))^2 \geq [\mathbb{P}(1 \in f(A)) - \mathbb{P}(1 \notin f(A), 2 \in f(A))] \cdot \mathbb{P}(1, 2 \notin f(A)).$$

Let $\alpha := \mathbb{P}(1 \notin f(A))$, $\beta := \mathbb{P}(1, 2 \notin f(A))$. Note that

$$\mathbb{P}(1 \in f(A)) = 1 - \alpha, \quad \mathbb{P}(1 \notin f(A), 2 \in f(A)) = \alpha - \beta.$$

Substituting this into (6.12), we get

$$(\alpha - \beta)^2 \geq (1 - 2\alpha + \beta)\beta,$$

which is equivalent to $\alpha^2 \geq \beta$. This gives (1.13), as desired. \square

To finish the proof of Theorem 1.6 we need the following lemma.

Lemma 6.5. *Let $P = (X, \prec)$ be a poset on $|X| = n \geq 3$ elements, let $a \in X$, and let $A \subseteq X - a$ be a nonempty subset. Then:*

$$N_k(1 \in f(A))^2 \geq N_k(1 \in f(A), 2 \in f(A\uparrow)) \cdot e_k(P),$$

for every $3 \leq k \leq n$.

Proof. Let $\mathbf{x} \in \mathbb{R}_+^d$ be the characteristic vector of $A \subseteq Z_{\text{down}}$, and let $\mathbf{y} \in \mathbb{R}_+^d$ be the characteristic vector of Z_{down} . Finally, let $\mathbf{M} := \mathbf{M}(P, a, k)$ be the matrix in Proposition 6.1. It then follows from the same argument as in (6.7) that

$$\mathbf{M}_{\mathbf{xx}} = N_{k+1}(1 \in f(A), 2 \in f(A\uparrow)).$$

Note also that

$$\mathbf{M}_{\mathbf{yy}} = \sum_{v \in Z_{\text{down}}} \sum_{w \in Z_{\text{down}}} M_{vw} \stackrel{(6.3)}{=} e_{k+1}(P).$$

Now let $A' := A \cap \min(P)$, and note that

$$\mathbf{M}_{\mathbf{xy}} \stackrel{(6.3)}{=} \sum_{x \in A'} N_{k+1}(f(x) = 1) = N_{k+1}(1 \in f(A')) = N_{k+1}(1 \in f(A)).$$

Combining these three equations with (Hyp), and substituting $k \leftarrow k - 1$ implies the result. \square

Proof of the second part of Theorem 1.6. As before, let $P' := P + \{a\}$ be the parallel sum of the poset P and the single element $\{a\}$, and note that $N_k^{(P')} = N^{(P)}$. Applying Lemma 6.5 to the poset P' , we get

$$N(1 \in f(A))^2 \geq N(1 \in f(A), 2 \in f(A \uparrow)) \cdot e(P).$$

The inequality (6.11) gives

$$N(1 \in f(A))^2 \geq \left[N(1 \in f(A)) - N(1 \notin f(A), 2 \in f(A)) \right] \cdot e(P).$$

Dividing both sides of the inequality by $e(P)^2$, we get a probabilistic version:

$$\mathbb{P}(1 \in f(A))^2 \geq \mathbb{P}(1 \in f(A)) - \mathbb{P}(1 \notin f(A), 2 \in f(A)).$$

This inequality is equivalent to (1.14), as desired. \square

6.5. Multiple subsets. Let $P = (X, \prec)$ be a poset on $|X| = n$ elements and fix $a \in X$. From this point on, let $A_1, A_2, A_3 \subset \min(P - a)$ be disjoint subsets of minimal elements. Denote by $\mathbf{x}_i \in \mathbb{R}^d$, $1 \leq i \leq 3$, the characteristic vectors of $A_i \subseteq Z_{\text{down}}$. Let $\mathbf{M} := \mathbf{M}(P, a, k)$ be the matrix in Proposition 6.1.

Lemma 6.6. *For all $1 \leq i, j \leq 3$, we have:*

$$(6.13) \quad \mathbf{M}_{\mathbf{x}_i \mathbf{x}_j} = \begin{cases} N_{k+1}(1 \in f(A_i), 2 \in f(A_i \uparrow)) & \text{if } i = j, \\ N_{k+1}(1 \in f(A_i), 2 \in f(A_j)) & \text{otherwise.} \end{cases}$$

Proof. Using (6.7) with $\mathbf{x} \leftarrow \mathbf{x}_i$ and $A \leftarrow A_i$ we have:

$$\mathbf{M}_{\mathbf{x}_i \mathbf{x}_i} = N_{k+1}(1 \in f(A_i), 2 \in f(A_i \uparrow)).$$

Similarly, for $i \neq j$, we have:

$$\mathbf{M}_{\mathbf{x}_i \mathbf{x}_j} = \sum_{x \in A_i} \sum_{y \in A_j} N_{k+1}(f(x) = 1, f(y) = 2) = N_{k+1}(1 \in f(A_i), 2 \in f(A_j)),$$

which completes the proof. \square

Proof of Theorem 1.9. By (Hyp), we have:

$$\mathbf{M}_{\mathbf{x}_1 \mathbf{x}_2}^2 \geq \mathbf{M}_{\mathbf{x}_1 \mathbf{x}_1} \mathbf{M}_{\mathbf{x}_2 \mathbf{x}_2}.$$

Using (6.13) in this inequality, we get:

$$N_{k+1}(1 \in f(A_1), 2 \in f(A_2))^2 \geq N_{k+1}(1 \in f(A_1), 2 \in f(A_1 \uparrow)) \cdot N_{k+1}(1 \in f(A_2), 2 \in f(A_2 \uparrow)).$$

As before, let $P' := P + \{a\}$ be the parallel sum of the poset P and the single element $\{a\}$, and note that $N_{k+1}^{(P')} = N^{(P)}$. This and the substitution $A_1 \leftarrow A$, $A_2 \leftarrow B$, give

$$N(1 \in f(A), 2 \in f(B))^2 \geq N(1 \in f(A), 2 \in f(A \uparrow)) \cdot N(1 \in f(B), 2 \in f(B \uparrow)),$$

as desired. \square

Proof of Theorem 1.10. Substituting $\mathbf{x} \leftarrow \mathbf{x}_1$, $\mathbf{y} \leftarrow \mathbf{x}_2$, $\mathbf{z} \leftarrow \mathbf{x}_3$ and $A_1 \leftarrow A$, $A_2 \leftarrow B$, $A_3 \leftarrow C$, into Lemma 5.5 and Lemma 6.6, we get

$$\begin{aligned} & N_{k+1}(1 \in f(C), 2 \in f(C \uparrow)) \cdot N_{k+1}(1 \in f(A), 2 \in f(B)) \\ & \leq 2 N_{k+1}(1 \in f(A), 2 \in f(C)) \cdot N_{k+1}(1 \in f(B), 2 \in f(C)). \end{aligned}$$

As before, let $P' := P + \{a\}$ be the parallel sum of the poset P and the single element $\{a\}$, and repeat the substitution argument $N_{k+1} \leftarrow N$ as above. The first inequality (1.18) now follows by dividing both sides of the equation by $e(P)^2$.

The second inequality (1.19) follows from a similar argument applied to Lemma 5.6. We omit the details for brevity. \square

7. LOG-CONCAVITY IMPLICATIONS

7.1. Proof of Corollary 1.4. Both inequalities in the corollary follow immediately from the following two general log-concavity results. The proofs are straightforward and included here for completeness.⁴

Lemma 7.1. *Let $\{p_1, \dots, p_n\}$ be a log-concave sequence of real numbers $p_k \geq 0$. Let $s_k := p_k + p_{k+1} + \dots + p_n$, for all $1 \leq k \leq n$. Then $\{s_1, \dots, s_n\}$ is also log-concave.*

Proof. After expanding and simplifying the terms, the log-concavity $s_{k-1}s_{k+1} \leq s_k^2$ is equivalent to

$$\sum_{i=k}^{n-1} p_{k-1} p_{i+1} \leq \sum_{i=k}^n p_k p_i.$$

By the log-concavity of $\{p_i\}$, we have $p_{k-1} p_{i+1} \leq p_k p_i$ for every $i \geq k$. Thus the difference between the RHS and LHS of the inequality above is at least $p_k p_n \geq 0$. \square

Lemma 7.2. *Let $\{p_1, \dots, p_n\}$ be a log-concave sequence of real numbers $p_i \geq 0$. Then:*

$$p_1(p_2 + \dots + p_{n-1} + p_n) \leq p_2(p_1 + \dots + p_{n-1} + p_n).$$

Proof. By the log-concavity of $\{p_i\}$, we have $p_1 p_i \leq p_2 p_{i-1}$ for every $i \geq 3$. Thus, the difference between the RHS and LHS of the inequality in the lemma is at least $p_2 p_n \geq 0$. \square

7.2. Proof of Proposition 3.6. We prove the proposition by analogy with the lemmas above, as a polynomial inequality.⁵

Lemma 7.3. *Let $\{p_1, \dots, p_n\}$ be a log-concave sequence of real numbers $p_i \geq 0$. Then:*

$$(p_1 + p_2 + \dots + p_n)(1^2 p_1 + 2^2 p_2 + \dots + n^2 p_n) \leq 2(1p_1 + 2p_2 + \dots + np_n)^2.$$

Proof. Write the difference of the RHS and LHS as the sum $\alpha_2 + \alpha_3 + \dots + \alpha_{2n}$ over terms $p_i p_j$ with a fixed sum $i + j$. We have:

$$\alpha_{k+1} = p_1 p_k (4k - 1 - k^2) + p_2 p_{k-1} (8(k-1) - 2^2 - (k-1)^2) + \dots$$

Observe that the coefficients in α_{k+1} start negative and become positive. The total sum of the coefficients in α_{k+1} is equal to $\frac{1}{3}k(k+1)(k+2) - \frac{1}{6}k(k+1)(2k+1) > 0$. By the log-concavity of $\{p_i\}$, we have $p_1 p_k \leq p_2 p_{k-1} \leq \dots$. We conclude that the positive terms in the sum α_{k+1} dominate the negative terms, which implies the result. \square

⁴For example, a continuous version of Lemma 7.1 is well-known in the Economics literature, see [BB05, Thm 1].

⁵Our original proof of the lemma was more complicated. The proof below was suggested to us by Fëdor Petrov (Oct. 2022, personal communication).

8. FINAL REMARKS

8.1. The geometric approach via the Alexandrov–Fenchel inequality was introduced by Stanley [Sta81] and further explored in [KS84, vH23]. More recently, this approach was employed to study the equality conditions [CP23c, MS24, SvH23, vHYZ24]. The vanishing conditions in Stanley inequality (Theorem 1.3) have purely combinatorial proofs, see [SvH23, Lem. 15.2]. See also generalizations in [CPP23a, Thm 8.5], [CPP23b, Thm 1.12] and [MS24, Thm 5.3]. The same holds for the uniqueness conditions [CPP23b, Thm 7.5].

It remains open whether Stanley inequality (1.5) has a combinatorial proof. Formally, it is not known whether the difference of the RHS and the LHS is in $\#P$, although a negative result of this type was recently obtained in [CP23c] for the generalized Stanley’s inequality under standard complexity assumptions. This problem is similarly open for our new correlation inequalities, since the combinatorial atlas technology is inherently ineffective. This is in contrast with other poset inequalities, see [CPP23b]. We refer to the lengthy discussion of these problems in [Pak24, §6].

8.2. One can ask if the atlas technology we use largely as a black box in this paper is really necessary to derive our results? We certainly believe this to be the case, since there are so few other tools. For now, let us offer a word of caution to the reader accepting the challenge: while our inequalities can certainly appear intuitively obvious at least in special cases, this impression is largely deceptive in the generality of all finite posets.

For example, the proof of the upper bound of (1.1) given in Example 3.1 for the case of exactly two minimal elements, is elementary indeed. Thus one might be tempted to extend this proof to general posets. We hope you succeed, but keep in mind that the only proof we have is as a corollary of a difficult Theorem 6.3.

This “intuitively obvious” phenomenon is both quite old and completely understandable. For example, *Winkler’s inequality* [Win82, Thm 1] certainly appears that way:

$$(8.1) \quad \mathbb{E}[f(x)] \leq \mathbb{E}[f(x) \mid f(x) > f(y)] \quad \text{for every incomparable } x, y \in X.$$

To this day, the only known proof of (8.1) is the original proof which uses the XYZ inequality.

8.3. The famous $\frac{1}{3} - \frac{2}{3}$ *Conjecture* states that for every finite poset $P = (X, \prec)$ there are elements $x, y \in X$, such that

$$\frac{1}{3} \leq \mathbb{P}[f(x) < f(y)] \leq \frac{2}{3}.$$

This conjecture was stated independently by Kislitsyn (1968) and Fredman (1875). and studied in a long series of papers, see e.g. [CP23b, §13] for a detailed overview of the literature.

As we mentioned in the introduction, a version of the conjecture with weaker constants was first proved in [KS84] using the extension of Stanley inequality (1.6). See also [BFT95] for the currently best known bound.

8.4. Let $P = (X, \prec)$ be a finite poset. Fix distinct elements $x, y, z \in X$. For $k, \ell \geq 1$, denote

$$\mathcal{F}(k, \ell) := \{f \in \mathcal{E}(P) : f(y) - f(x) = k, f(z) - f(y) = \ell\},$$

and let $F(k, \ell) := |\mathcal{F}(k, \ell)|$. The *cross-product conjecture* (CPC) by Brightwell–Felsner–Trotter [BFT95, Thm 3.2] states that

$$(CPC) \quad F(k, \ell) \cdot F(k+1, \ell+1) \leq F(k+1, \ell) \cdot F(k, \ell+1),$$

for every $k, \ell \geq 1$. The $k = \ell = 1$ case was proved in [BFT95, Thm 3.2], and the case of width two posets was proved in [CPP22]. This remains one of the most challenging open problems in the area. A constant ratio bound was recently given in [CPP24]. As Brightwell–Felsner–Trotter lamented, “something more powerful seems to be needed” to prove the general form of (CPC).

8.5. Let $P = (X, \prec)$ be a poset, and let $A, B \subset X$ be upper ideals. We denote by $e(A)$ the number of linear extensions of the poset (A, \prec) . *Fishburn's inequality* [Fis84] states:

$$(8.2) \quad \frac{|A \cup B|! \cdot |A \cap B|!}{|A|! \cdot |B|!} \leq \frac{e(A \cup B) \cdot e(A \cap B)}{e(A) \cdot e(B)}.$$

Note that this is a generalization of the lower bound in (1.1), obtained by taking $A := X - x$ and $B := X - y$. In a joint work with Panova, we recently rederive this special case in a remark [CPP23b, §9.8]. Both the original proof in [Fis84, §2] and our proof use the FKG inequality. We also note that Fishburn's inequality (8.2) is a special case of an inequality of Shepp [She80, Thm 2], see e.g. [Bri88, Lemma 10] for this derivation.

8.6. In the context of §4.2, Fishburn's inequality for straight shapes was studied by Björner [Bjö11, §6], as the following correlation inequality:

$$(8.3) \quad |\text{SYT}(\mu)| \cdot |\text{SYT}(\nu)| \leq |\text{SYT}(\mu \vee \nu)| \cdot |\text{SYT}(\mu \wedge \nu)|,$$

where \vee and \wedge refer to the union and intersection of the corresponding Young diagrams.

It was also pointed out in [Pak24, §7.4], that (8.3) is also a corollary of the very general *Lam–Pylyavskyy inequality* [LP07, Thm 4.5] which applies to skew shapes:

$$(8.4) \quad |\text{SYT}(\mu/\alpha)| \cdot |\text{SYT}(\nu/\beta)| \leq |\text{SYT}(\mu \vee \nu / \alpha \vee \beta)| \cdot |\text{SYT}(\mu \wedge \nu / \alpha \wedge \beta)|.$$

This result was reproved and further extended in [LPP07] by an algebraic argument. In a followup paper [CP23a], we show how (8.4) and its generalizations can be proved via the Ahlswede–Daykin inequality [AD78].

8.7. In the context of §4.4, recall that there is no analogue of the hook walk for skew Young diagrams, or in fact any direct combinatorial way to sample from $\text{SYT}(\lambda/\mu)$, see e.g. [MPP18, §10.3]. It would be interesting to find a direct combinatorial proof (1.1) in this case.

8.8. In conditions of Theorem 1.1, when $x \in \min(P)$ and $y \in \max(P)$, the lower bound in (1.1) reverses direction:

$$(8.5) \quad \frac{e(P) \cdot e(P - x - y)}{e(P - x) \cdot e(P - y)} \leq \frac{n}{n-1},$$

see [CPP23b, §9.8]. It would be interesting to find a lower bound for the LHS of (8.5) similar to the RHS of (1.1). For example, let $P = C_{n-2} + C_2$ be a parallel sum (disjoint union) of two chains, and let $C_2 = \{x, y : x \prec y\}$. Then the LHS of (8.5) is equal to $\frac{n}{2(n-1)}$. We conjecture that this lower bound is tight:

Conjecture 8.1. *Let $x \in \min(P)$ and $y \in \max(P)$. We have:*

$$\frac{e(P) \cdot e(P - x - y)}{e(P - x) \cdot e(P - y)} \geq \frac{1}{2}.$$

8.9. The combinatorial atlas technology does prove the equality conditions in some cases, such as for Stanley inequality (1.5), see [CP24, §1.18]. While this paper does not explore the equality conditions for any of our correlation inequalities, we believe this is an important direction to pursue. Finding equality conditions for both inequalities in Theorem 1.1 would be especially interesting, as would be the equality conditions for Corollary 1.7 and for Theorem 1.9.

8.10. Conjectures 1.5 and 8.1 are very speculative. It would be interesting to check them computationally for sufficiently large posets.

Acknowledgements. We are grateful to Pasha Galashin, Nikita Gladkov, Alejandro Morales, Greta Panova, Fëdor Petrov, Michael Saks, Yair Shenfeld and Ramon van Handel for helpful discussions and remarks on the subject. We are especially thankful to Max Aires and Jeff Kahn for showing us Example 3.8. The first author was partially supported by the Simons Foundation. Both authors were partially supported by the NSF.

REFERENCES

- [AHK18] Karim Adiprasito, June Huh and Eric Katz, Hodge theory for combinatorial geometries, *Annals of Math.* **188** (2018), 381–452.
- [AB08] Rudolf Ahlswede and Vladimir Blinovskiy, *Lectures on advances in combinatorics*, Springer, Berlin, 2008, 314 pp.
- [AD78] Rudolf Ahlswede and David E. Daykin, An inequality for the weights of two families of sets, their unions and intersections, *Z. Wahrsch. Verw. Gebiete* **43** (1978), 183–185.
- [AS16] Noga Alon and Joel H. Spencer, *The probabilistic method* (Fourth ed.), John Wiley, Hoboken, NJ, 2016, 375 pp.
- [ALOV18] Nima Anari, Kuikui Liu, Shayan Oveis Gharan and Cynthia Vinzant, Log-concave polynomials III: Mason’s Ultra-log-concavity conjecture for independent sets of matroids, *Proc. AMS* **152** (2024), 1969–1981.
- [BB05] Mark Bagnoli and Ted Bergstrom, Log-concave probability and its applications, *Econom. Theory* **26** (2005), 445–469; see also a 1989 draft version at deepblue.lib.umich.edu/handle/2027.42/100687
- [B+19] Carolina Benedetti, Rafael S. González D’León, Christopher R. H. Hanusa, Pamela E. Harris, Apoorva Khare, Alejandro H. Morales and Martha Yip, A combinatorial model for computing volumes of flow polytopes, *Trans. AMS* **372** (2019), 3369–3404.
- [Bjö11] Anders Björner, A q -analogue of the FKG inequality and some applications, *Combinatorica* **31** (2011), 151–164.
- [BW91] Anders Björner and Michelle L. Wachs, Permutation statistics and linear extensions of posets, *J. Combin. Theory A* **58** (1991), 85–114.
- [BBL09] Julius Borcea, Petter Brändén and Thomas M. Liggett, Negative dependence and the geometry of polynomials, *J. Amer. Math. Soc.* **22** (2009), 521–567.
- [Brä15] Petter Brändén, Unimodality, log-concavity, real-rootedness and beyond, in *Handbook of enumerative combinatorics*, CRC Press, Boca Raton, FL, 2015, 437–483.
- [BH20] Petter Brändén and June Huh, Lorentzian polynomials, *Annals of Math.* **192** (2020), 821–891.
- [Bri88] Graham R. Brightwell, Linear extensions of infinite posets, *Discrete Math.* **70** (1988), 113–136.
- [BFT95] Graham R. Brightwell, Stefan Felsner and William T. Trotter, Balancing pairs and the cross product conjecture, *Order* **12** (1995), 327–349.
- [CP22] Swee Hong Chan and Igor Pak, Introduction to the combinatorial atlas, *Expo. Math.* **40** (2022), 1014–1048.
- [CP23a] Swee Hong Chan and Igor Pak, Multivariate correlation inequalities for P -partitions, *Pacific J. Math.* **323** (2023), 223–252.
- [CP23b] Swee Hong Chan and Igor Pak, Linear extensions of finite posets, preprint (2023), 55 pp.; [arXiv:2311.02743](https://arxiv.org/abs/2311.02743).
- [CP23c] Swee Hong Chan and Igor Pak, Equality cases of the Alexandrov–Fenchel inequality are not in the polynomial hierarchy, to appear in *Forum Math. Pi*; preprint (2023), 35 pp.; [arXiv:2309.05764](https://arxiv.org/abs/2309.05764).
- [CP24] Swee Hong Chan and Igor Pak, Log-concave poset inequalities, *Jour. Assoc. Math. Res.* **2** (2024), 53–153.
- [CPP22] Swee Hong Chan, Igor Pak and Greta Panova, The cross-product conjecture for width two posets, *Trans. AMS* **375** (2022), 5923–5961.
- [CPP23a] Swee Hong Chan, Igor Pak and Greta Panova, Extensions of the Kahn–Saks inequality for posets of width two, *Combinatorial Theory* **3** (2023), no. 1, Paper No. 8, 34 pp.
- [CPP23b] Swee Hong Chan, Igor Pak, and Greta Panova, Effective poset inequalities, *SIAM J. Discrete Math.* **37** (2023), 1842–1880.
- [CPP24] Swee Hong Chan, Igor Pak and Greta Panova, On the cross-product conjecture for the number of linear extensions, *Canad. J. Math.*, published online Jan. 19, 2024, 28 pp.

- [FW97] Stefan Felsner and Lorenz Wernisch, Markov chains for linear extensions, the two-dimensional case, in *Proc. 8th SODA* (1997), 239–247.
- [Fis84] Peter C. Fishburn, A correlational inequality for linear extensions of a poset, *Order* **1** (1984), 127–137.
- [Fis92] Peter C. Fishburn, Correlation in partially ordered sets, *Discrete Appl. Math.* **39** (1992), 173–191.
- [FS98] Peter C. Fishburn and Lawrence A. Shepp, The Ahlswede–Daykin theorem, in *Numbers, information and complexity*, Kluwer, Boston, MA, 2000, 501–516.
- [FS09] Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge Univ. Press, Cambridge, UK, 2009, 810 pp.
- [FFLP05] Sergey Fomin, William Fulton, Chi-Kwong Li and Yiu-Tung Poon, Eigenvalues, singular values, and Littlewood–Richardson coefficients, *Amer. J. Math.* **127** (2005), 101–127.
- [GHMY23] Rafael S. González D’León, Christopher R. H. Hanusa, Alejandro H. Morales and Martha Yip, Column convex matrices, G -cyclic orders, and flow polytopes, *Discrete Comput. Geom.* **70** (2023), 1593–1631.
- [GNW79] Curtis Greene, Albert Nijenhuis and Herbert S. Wilf, A probabilistic proof of a formula for the number of Young tableaux of a given shape, *Adv. Math.* **31** (1979), 104–109.
- [Huh18] June Huh, Combinatorial applications of the Hodge–Riemann relations, in *Proc. ICM Rio de Janeiro*, vol. IV, World Sci., Hackensack, NJ, 2018, 3093–3111.
- [Huh23] June Huh, Combinatorics and Hodge theory, in *Proc. ICM 2022* (virtual), Vol. 1, EMS Press, Berlin, 2023, 212–239.
- [HSW22] June Huh, Benjamin Schröter and Botong Wang, Correlation bounds for fields and matroids, *Jour. Eur. Math. Soc.* **24** (2022), 1335–1351.
- [Kahn00] Jeff Kahn, A normal law for matchings, *Combinatorica* **20** (2000), 339–391.
- [KN10] Jeff Kahn and Michael Neiman, Negative correlation and log-concavity, *Random Structures Algorithms* **37** (2010), 367–388.
- [KS84] Jeff Kahn and Michael Saks, Balancing poset extensions, *Order* **1** (1984), 113–126.
- [LPP07] Thomas Lam, Alexander Postnikov and Pavlo Pylyavskyy, Schur positivity and Schur log-concavity, *Amer. J. Math.* **129** (2007), 1611–1622.
- [LP07] Thomas Lam and Pavlo Pylyavskyy, Cell transfer and monomial positivity, *J. Algebraic Combin.* **26** (2007), 209–224.
- [MS24] Zhao Yu Ma and Yair Shenfeld, The extremals of Stanley’s inequalities for partially ordered sets, *Adv. Math.* **436** (2024), Paper No. 109404, 72 pp.
- [MPP18] Alejandro H. Morales, Igor Pak and Greta Panova, Hook formulas for skew shapes I. q -analogues and bijections, *J. Combin. Theory, Ser. A* **154** (2018), 350–405.
- [Oko97] Andrei Okounkov, Log-concavity of multiplicities with application to characters of $U(\infty)$, *Adv. Math.* **127** (1997), 258–282.
- [Pak24] Igor Pak, What is a combinatorial interpretation?, in *Open Problems in Algebraic Combinatorics*, AMS, Providence, RI, 2024, 191–260.
- [Pem00] Robin Pemantle, Towards a theory of negative dependence, *J. Math. Phys.* **41** (2000), 1371–1390.
- [Sch14] Rolf Schneider, *Convex bodies: the Brunn–Minkowski theory* (second edition), Cambridge Univ. Press, Cambridge, UK, 2014, 736 pp.
- [SvH19] Yair Shenfeld and Ramon van Handel, Mixed volumes and the Bochner method, *Proc. AMS* **147** (2019), 5385–5402.
- [SvH23] Yair Shenfeld and Ramon van Handel, The extremals of the Alexandrov–Fenchel inequality for convex polytopes, *Acta Math.* **231** (2023), 89–204.
- [She80] Lawrence A. Shepp, The FKG inequality and some monotonicity properties of partial orders, *SIAM J. Algebraic Discrete Methods* **1** (1980), 295–299.
- [OEIS] Neil J. A. Sloane, *The online encyclopedia of integer sequences*, oeis.org
- [Sta81] Richard P. Stanley, Two combinatorial applications of the Aleksandrov–Fenchel inequalities, *J. Combin. Theory, Ser. A* **31** (1981), 56–65.
- [Sta89] Richard P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in *Graph theory and its applications*, New York Acad. Sci., New York, 1989, 500–535.
- [Sta99] Richard P. Stanley, *Enumerative Combinatorics*, vol. 1 (second edition) and vol. 2, Cambridge Univ. Press, 2012 and 1999.
- [Sta10] Richard P. Stanley, A survey of alternating permutations, in *Combinatorics and graphs*, AMS, Providence, RI, 2010, 165–196.
- [Tro95] William T. Trotter, Partially ordered sets, in *Handbook of combinatorics*, vol. 1, Elsevier, Amsterdam, 1995, 433–480.

- [vH23] Ramon van Handel, Shephard's inequalities, Hodge–Riemann relations, and a conjecture of Fedotov, in *Lecture Notes in Math.* **2327**, Springer, Cham, 2023, 337–354.
- [vHYZ24] Ramon van Handel, Alan Yan and Ximeng Zeng, The extremals of the Kahn–Saks inequality, *Adv. Math.* **456** (2024), Paper No. 109892, 38 pp.
- [Win82] Peter M. Winkler, Average height in a partially ordered set, *Discrete Math.* **39** (1982), 337–341.
- [Win86] Peter M. Winkler, Correlation and order, in *Combinatorics and ordered sets*, AMS, Providence, RI, 1986, 151–174.