CORRELATION INEQUALITIES FOR LINEAR EXTENSIONS

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Abstract. We employ the combinatorial atlas technology to prove new correlation inequalities for the number of linear extensions of finite posets. These include the approximate independence of probabilities and expectations of values of random linear extensions, closely related to Stanley's inequality. We also give applications to the numbers of standard Young tableaux and to Euler numbers.

1. Introduction

If you really want to hear about it, the first thing you'll probably want to know about *Poset Theory* is that it was born and languished for decades without any tools whatsoever. It is still a question whether the area has caught up with the other, more established parts of Combinatorics, but by now it is definitely in possession of powerful tools of various nature, which makes it at least somewhat prominent, if not prosperous.

In this paper we obtain a melange of new correlation inequalities for the number of linear extensions of finite posets using the *combinatorial atlas* technology [CP21, CP22]. This is the tool we recently developed and whose power has yet to be fully exploited. Viewed individually, each new inequality shows that there is more to the story. Taken together, these inequalities offer a glimpse at the big world of poset inequalities that we are just beginning to understand.

1.1. **Linear extensions.** Let $P = (X, \prec)$ be a *partially ordered set* on the ground set X of size $|X| = n$, and with the partial order "$\prec$". A *linear extension* of P is a bijection $f : X \to \{1, \ldots, n\}$ that is order-preserving: $x \prec y$ implies $f(x) < f(y)$, for all $x, y \in X$. We denote by $\mathcal{E}(P)$ the set of linear extensions of P, and by $e(P) := |\,\mathcal{E}(P)|$ the number of linear extensions.

An element $x \in X$ is *minimal* if there is no $y \in X$ such that $y \prec x$. Denote by $\min(P) \subseteq X$ the subset of minimal elements in P. We use C_n and A_n to denote the n-element *chain* and *antichain*, respectively. For an element $x \in X$, we denote by $P - x$ the induced subposet of P on the subset $X - x$. See Section 2 for further poset notation.

Theorem 1.1 (cf. Theorem 6.3)**.** *Let $P = (X, \prec)$ be a poset with $|X| = n > 2$ elements. Let $x, y \in \min(X)$ be distinct minimal elements of P. Then:*

$$\frac{n}{n-1} \;\leq\; \frac{e(P)\,\cdot\,e(P-x-y)}{e(P-x)\,\cdot\,e(P-y)} \;\leq\; 2. \tag{1.1}$$

This correlation inequality is the most natural and the simplest to state. The lower bound in (1.1) is a special case of the *Fishburn inequality* (see §8.5), while the upper bound is new and is a special case of the upper bound in Theorem 6.3. Note that the lower bound is tight for $P = A_n$ and the upper bounds is tight for the linear sum $P = A_2 \oplus C_{n-2}$.

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Correlation inequalities are best understood in probabilistic notation. Let $\mathbb{P}$ and $\mathbb{E}$ be taken over the uniform random linear extension $f \in \mathcal{E}(P)$. The inequality (1.1) can be rewritten as

$$\frac{n}{n-1} \;\le\; \frac{\mathbb{P}[f(x)=1, f(y)=2]}{\mathbb{P}[f(x)=1]\,\cdot\,\mathbb{P}[f(y)=1]} \;\le\; 2. \tag{1.2}$$

The following theorem gives a similar upper bound for the covariances:

Theorem 1.2 (see §6.3). *Let $P=(X,\prec)$ be a finite poset, and let $x,y\in X$ be fixed poset elements. Then:*

$$\frac{\mathbb{E}[f(x)f(y)] \,+\, \mathbb{E}\big[\min\{f(x),f(y)\}\big]}{\mathbb{E}[f(x)]\,\cdot\,\mathbb{E}[f(y)]} \;\le\; 2\,. \tag{1.3}$$

See also §3.2 for an application to the second moment estimates.

1.2. **Stanley's inequality.** For linear extensions, one can use both elements and values to set up correlation inequalities. The following celebrated result by Stanley is foundational in the area:

Theorem 1.3 (*Stanley inequality* [Sta81, Thm 3.1]). *Let $P=(X,\prec)$ be a poset with $|X|=n$ elements. Fix $x\in X$ and let $2\le k\le n-1$. Denote by $e_k(P)$ the number of linear extensions $f\in\mathcal{E}(P)$ such that $f(x)=k$. Then:*

$$e_k(P)^2 \;\ge\; e_{k-1}(P)\,\cdot\,e_{k+1}(P)\,. \tag{1.4}$$

In probabilistic notation, Stanley's inequality can be restated as

$$\mathbb{P}[f(x)=k]^2 \;\ge\; \mathbb{P}[f(x)=k-1]\,\cdot\,\mathbb{P}[f(x)=k+1]\,. \tag{1.5}$$

The proof in [Sta81] uses the *Alexandrov–Fenchel inequality* applied to order polytopes, see §8.1. Stanley's inequality was famously extended by Kahn and Saks [KS84], to a prove a weak version of the $\frac{1}{3}$–$\frac{2}{3}$ *Conjecture*, see §8.3.

Stanley's inequality (1.5) has remained mysterious until recently, when the equality conditions has been established by Shenfeld and van Handel [SvH20] by advancing geometric arguments. In our paper [CP21], we gave a linear algebraic proof of Stanley's inequality, the equality conditions, and the generalization to weighted linear extensions, see also §8.1.

Let us single out two slightly nonstandard general consequences of Stanley's inequality:

Corollary 1.4 (see §7.1). *Let $P=(X,\prec)$ be a poset with $|X|=n$ elements. Fix $x\in X$ and let $1\le k\le n-1$. Then:*

$$\mathbb{P}[f(x)>k-1]\,\cdot\,\mathbb{P}[f(x)>k+1] \;\le\; \mathbb{P}[f(x)>k]^2\,. \tag{1.6}$$

and

$$\mathbb{P}[f(x)=1]\,\cdot\,\mathbb{P}[f(x)>1] \;\le\; \mathbb{P}[f(x)=2]\,. \tag{1.7}$$

For $k=1$, inequality (1.6) further simplifies to

$$\mathbb{P}[f(x)>2] \;\le\; \mathbb{P}[f(x)>1]^2\,. \tag{1.8}$$

We propose the following unusual extension of Stanley's inequality to general subsets of the ground set. Fix a nonempty subset $A\subseteq X$. For a linear extension $f\in\mathcal{E}(P)$, define

$$f(A) \;:=\; \big\{\,f(x)\,:\,x\in A\,\big\} \quad\text{and}\quad f_{\min}(A) \;:=\; \min f(A).$$

Note that $f_{\min}(A)=f(x)$ for all singletons $A=\{x\}$, where $x\in X$.

Conjecture 1.5 (*Extended Stanley inequality*)**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Fix a nonempty subset $A \subseteq X$, and let $2 \le k \le n-1$. Then:*

$$\mathbb{P}[f_{\min}(A) = k]^2 \;\ge\; \mathbb{P}[f_{\min}(A) = k-1] \cdot \mathbb{P}[f_{\min}(A) = k+1]. \tag{1.9}$$

To justify the conjecture, we prove that the inequalities (1.7) and (1.8) extend to all subsets (see also §8.10).

Theorem 1.6 (see §6.4)**.** *Let $P = (X, \prec)$ be a poset on $|X| \ge 2$ elements. Fix a nonempty subset $A \subseteq X$. Then:*

$$\mathbb{P}[f_{\min}(A) > 2] \;\le\; \mathbb{P}[f_{\min}(A) > 1]^2 \tag{1.10}$$

and

$$\mathbb{P}[f_{\min}(A) = 1] \cdot \mathbb{P}[f_{\min}(A) > 1] \;\le\; \mathbb{P}[f_{\min}(A) = 2]. \tag{1.11}$$

Although Conjecture 1.5 is written as a natural generalization of Stanley's Theorem 1.3, the inequalities (1.10) and (1.11) are best understood as

$$\mathbb{P}[1, 2 \notin f(A)] \;\le\; \mathbb{P}[1 \notin f(A)]^2 \qquad \text{and} \tag{1.12}$$

$$\mathbb{P}[1 \in f(A)] \cdot \mathbb{P}[1 \notin f(A)] \;\le\; \mathbb{P}[1 \notin f(A), 2 \in f(A)], \tag{1.13}$$

respectively. We show in §7.1, that Conjecture 1.5 implies Theorem 1.6.

Corollary 1.7 (cf. Lemma 6.4)**.** *Let $P = (X, \prec)$ be a poset on $|X| \ge 2$ elements, and let $A \subseteq X$ be a nonempty subset of elements. Then:*

$$\mathbb{P}[1, 2 \in f(A)] \cdot \mathbb{P}[1, 2 \notin f(A)] \;\le\; \mathbb{P}[1 \in f(A), 2 \notin f(A)]^2. \tag{1.14}$$

Proof. Multiply (1.12) for subsets A and $X \smallsetminus A$. Then use (1.13). □

1.3. **Multiple subsets.** We now consider correlation inequalities obtained by replacing poset elements with subsets. First, we have the following natural generalization of Theorem 1.2.

Theorem 1.8 (see §6.3)**.** *Let $P = (X, \prec)$ be a finite poset, and let $A, B \subseteq X$ be nonempty subsets. Then:*

$$\frac{\mathbb{E}\big[f_{\min}(A) f_{\min}(B)\big] + \mathbb{E}\big[f_{\min}(A \cup B)\big]}{\mathbb{E}\big[f_{\min}(A)\big] \cdot \mathbb{E}\big[f_{\min}(B)\big]} \;\le\; 2. \tag{1.15}$$

Let us emphasize that here A and B are *arbitrary* subsets of the ground set X. Next, we have the following variation on Corollary 1.7 for two subsets of minimal elements.

For an element $b \in X$, denote by $b\!\uparrow := \{x \in X \,:\, x \succcurlyeq b\}$ the *upper order ideal* generated by b. For a subset $B \subseteq X$, denote by $B\!\uparrow := \cup_{b \in B}\, b\!\uparrow$ the *upper closure* of B.

Theorem 1.9 (see §6.5)**.** *Let $P = (X, \prec)$ be a finite poset, and let $A, B \subset \min(P)$ be disjoint nonempty subsets of minimal elements. Then:*

$$\mathbb{P}\big[1 \in f(A), 2 \in f(A\!\uparrow)\big] \cdot \mathbb{P}\big[1 \in f(B), 2 \in f(B\!\uparrow)\big] \;\le\; \mathbb{P}\big[1 \in f(A), 2 \in f(B)\big]^2. \tag{1.16}$$

We conclude with the following three-subset variation:

Theorem 1.10 (see §6.5). *Let $P = (X, \prec)$ be a finite poset, and let $A, B, C \subset \min(P)$ be disjoint nonempty subsets of minimal elements. Then:*

$$\frac{\mathbb{P}\big[1 \in f(C), 2 \in f(C\!\uparrow)\big] \cdot \mathbb{P}\big[1 \in f(A), 2 \in f(B)\big]}{\mathbb{P}\big[1 \in f(A), 2 \in f(C)\big] \cdot \mathbb{P}\big[1 \in f(B), 2 \in f(C)\big]} \ \leq\ 2. \tag{1.17}$$

Moreover, there exists a permutation (A', B', C') of (A, B, C) such that

$$\frac{\mathbb{P}\big[1 \in f(C'), 2 \in f(C'\!\uparrow)\big] \cdot \mathbb{P}\big[1 \in f(A'), 2 \in f(B')\big]}{\mathbb{P}\big[1 \in f(A'), 2 \in f(C')\big] \cdot \mathbb{P}\big[1 \in f(B'), 2 \in f(C')\big]} \ \leq\ 1. \tag{1.18}$$

1.4. **Background and historical remarks.** In Combinatorics, correlation inequalities go back to the work of Kirchhoff on electrical networks. In modern notation, his result can be reformulated as saying that for all edges $e, e' \in E$, we have:

$$\mathbb{P}(e, e' \in T) \ \leq\ \mathbb{P}(e \in T) \cdot \mathbb{P}(e' \in T), \tag{1.19}$$

where the probability is over uniform spanning trees T in a given graph $G = (V, E)$. This idea has proved extremely influential over the past decades, leading to a long series of remarkable results.

In Graph Theory, the classic examples of correlation inequalities for hereditary graph properties include the *Harris inequality* (1960), the *Kleitman inequality* (1966) and their far-reaching extensions: the *Ahlswede–Daykin* (1978) and the *FKG inequalities* (1971). We refer to [AS16, §6.1] for the introduction, and to [FS98] for a detailed overview.

In Statistical Physics, notable correlation inequalities include *Griffiths* (1967) and *Griffiths–Hurst–Sherman inequalities* (1970), all for the Ising model. We refer to [AB08, Ch. 15] for the background and combinatorial applications. See also [BBL09, KN10, Pem00] for the general theory of negative correlations.

In Enumerative and Algebraic Combinatorics, correlation inequalities are closely related to log-concavity (log-convexity) of sequences counting various combinatorial objects, see [Brä15, Sta89]. In the past few years, Huh and his coauthors introduced and explored a new algebro-geometric approach to the subject. We refer to [AHK18] for a major breakthrough proving a long-standing Mason's conjecture on the numbers of independent sets of a matroid, to [HSW22] for correlation inequalities for matroids, and to [Huh18, Huh22] for recent surveys.

In Poset Theory, besides Stanley's inequality (Theorem 1.3), notable correlation inequalities include the *Graham–Yao-Yao inequality* (1980), the *XYZ inequality* (1982), and the *Kahn–Saks inequality* [KS84], see an overview in [Fis92, Win86]. We also note the (conjectural) *cross–product inequality*, see §8.4.

As we mentioned earlier, this paper grew from our paper [CP21] where we introduced the combinatorial atlas technology, using an involved inductive argument based on linear algebra. See also [CP22] for the introduction, and for connections to Lorentzian polynomials and the Alexandrov–Fenchel inequality (see also §8.1).

In contrast with the Lorentzian polynomials approach, the combinatorial atlases are fundamentally noncommutative. This and the elementary nature of matrices makes the combinatorial atlas technology flexible enough to apply to linear extensions and obtain the inequalities that are not reachable by other means. We believe that this applies to [CP21, Thm 1.35] and most new inequalities for linear extension in this paper. See §8.2 for further discussion on this.

1.5. **Discussion.** Linear extensions of finite posets are somewhat different in spirit from graphs, matroids, etc., in that they have many more statistics worth studying. Of the fairly large pool of potential correlation inequalities we chose the ones that were accessible with the atlas technology. The following two examples were motivational for our work.

(1) It is instructive to compare our results with the *Huh–Schröter–Wang correlation inequality* for *bases of matroids* [HSW22], which can be stated as follows. Let $\mathcal{M} = (X, \mathcal{B})$ be a finite matroid of rank $d = \mathrm{rk}(\mathcal{M})$ given by the set of bases $\mathcal{B} \subseteq \binom{X}{d}$. Then, for every two distinct elements $u, v \in X$, we have:

$$\frac{\mathbb{P}[u \in B, v \in B] \cdot \mathbb{P}[u \notin B, v \notin B]}{\mathbb{P}[u \in B, v \notin B] \cdot \mathbb{P}[u \notin B, v \in B]} \;\leq\; 2\left(1 - \frac{1}{d}\right), \tag{1.20}$$

where the probability is over uniform $B \in \mathcal{B}$, see [HSW22, Thm 5]. The authors note that this implies the covariance bound

$$\mathrm{Cov}(u \in B, v \in B) \;<\; \mathbb{P}[u \in B] \cdot \mathbb{P}[v \in B], \tag{1.21}$$

for all fixed distinct elements $u, v \in X$. Although we are not aware of a formal connection, our upper bounds (1.2) and (1.3) have a similar in structure to (1.20) and (1.21), respectively.

Continuing the analogy, the constant 2 in (1.20) was improved to 1 in some special cases [HSW22, Thm 6]. It is not known whether it can be further improved in full generality, with some examples giving 8/7 ratio. Similarly, our tools do not allow the asymptotic improvement of the constant 2 in (1.3), but we do have examples giving the lower bound of 4/3 (Example 3.6).

(2) In a related earlier study of *random graph matchings*, Kahn [Kahn00, §4] introduced the notion of *approximate independence* to describe correlation inequalities of the type we consider in this paper. Let $G = (V, E)$ be a simple graph. A *matching* is a subset $M \subseteq E$ of pairwise nonadjacent edges. Let $\mathcal{M}$ denote the set of all matchings in G, and let $\mathcal{M}_k \subseteq \mathcal{M}$ be the set of matchings in G of size k.

Heilmann and Lieb (1972) famously showed that the sequence $\{|\mathcal{M}_k|\}$ is log-concave, which parallels Stanley's inequality (1.5). *Kahn's inequality* [Kahn00, Cor. 4.3] states that for all distinct vertices $u, v \in V$, we have:[1]

$$\frac{1}{2} \leq \frac{\mathbb{P}[u \notin M, v \notin M]}{\mathbb{P}[u \notin M] \cdot \mathbb{P}[v \notin M]} \;\leq\; 2, \tag{1.22}$$

where $M \in \mathcal{M}$ is a uniform random matching (cf. Example 3.9). Our correlation inequality (1.2) is modeled after this result.

1.6. **Paper structure.** We start with notations in Section 2. In Section 3, we present a number of examples and applications of correlation inequalities. Many of these are for illustrative purposes and can be skipped, but some give background and context for the type of correlation inequalities that we consider throughout the paper.

We present applications to Enumerative Combinatorics in Section 4, notably to inequalities between numbers of certain *standard Young tableaux*, and to polynomial inequalities for *Euler numbers*. Although these are results not mentioned in the introduction to avoid shifting the focus, the reader might find them curious (cf. §8.6). While we aim to be self-contained and include most definitions, some familiarity with the subject is helpful for the clarity and the motivation.

In Section 5, we give a linear algebraic setup to derive correlation inequalities for hyperbolic matrices. This section is completely self-contained. The hope is that our basic approach can be used to obtain other inequalities. A lengthy Section 6 contains proofs and occasional generalizations of results in the introduction. The key result there is Proposition 6.1 proved in [CP21] and used as a black box in this paper.

A short Section 7 has proofs of three implications of the log-concavity. All of these are straightforward and included for completeness. We conclude with final remarks and open problems in Section 8.

[1] In fact, Kahn's bound is stronger on both sides. We present a simplified version for clarity.

2. Notations

We use $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ = \{x \geq 0\}$ and $[n] = \{1, 2, \ldots, n\}$. For a poset $P = (X, \prec)$, denote by $P^* = (X, \prec^*)$ the *dual poset* with $x \prec^* y$ if and only if $y \prec x$, for all $x, y \in X$. Denote by $\min(P)$ and $\max(P)$ the set of minimal and maximal elements in P, respectively.

For posets $P = (X, \prec_P)$ and $Q = (Y, \prec_Q)$, the *parallel sum* $P + Q = (Z, \prec)$ is the poset on the disjoint union $Z = X \sqcup Y$, where elements of X retain the partial order of P, elements of Y retain the partial order of Q, and elements $x \in X$ and $y \in Y$ are incomparable. Similarly, the *linear sum* $P \oplus Q = (Z, \prec)$, where $x \prec y$ for every two elements $x \in X$ and $y \in Y$ and other relations as in the parallel sum. We refer to [Sta99, Ch. 3] and [Tro95] for more on poset definitions and notation.

We use capitalized bold letters to denote matrices, e.g. $\mathbf{M} = (\mathrm{M}_{ij})$. We also keep conventional index notations, so, e.g., $\big(\mathbf{M} + \mathbf{M}^2\big)_{ij}$ is the (i, j)-th entry of $\mathbf{M} + \mathbf{M}^2$. Similarly, we use small bold letters to denote vectors, e.g. $\mathbf{h} = (\mathrm{h}_1, \mathrm{h}_2, \ldots)$ and $\mathrm{h}_i = (\mathbf{h})_i$. For a subset $S \subseteq [n]$, the *characteristic vector* of S is the vector $\mathbf{v} \in \mathbb{R}^n$ such that $\mathrm{v}_i = 1$ if $i \in S$ and $\mathrm{v}_i = 0$ if $i \notin S$. We denote by $\mathbf{0}$ the zero vector $(0, \ldots, 0)$.

3. Examples and special cases

3.1. **Basic examples.** Here we consider several simple observations on correlation inequalities that apply to general posets. We recommend reading these first, before studying more elaborate examples with special families of posets.

Example 3.1. Suppose $\min(X) = \{x, y\}$, i.e. x, y are the only minimal elements of P. We then have $e(P) = e(P - x) + e(P - y)$. The upper bound of (1.1) is straightforward in this case:

$$\frac{e(P) \cdot e(P - x - y)}{e(P - x) \cdot e(P - y)} \ = \ \frac{e(P - x - y)}{e(P - x)} + \frac{e(P - x - y)}{e(P - y)} \ \leq \ 2,$$

where the inequality follows from $e(P - x - y) \leq \min\big\{e(P - x), e(P - y)\big\}$. See also §8.2 for a further discussion of this proof.

Example 3.2. Again, suppose $\min(X) = \{x, y\}$. In the notation of §1.2, we have:

$$\mathbb{P}\big[f(x) = 1\big] \ = \ \frac{e(P - x)}{e(P)}, \quad \mathbb{P}\big[f(x) > 1\big] \ = \ \frac{e(P - y)}{e(P)} \quad \text{and} \quad \mathbb{P}\big[f(x) = 2\big] \ = \ \frac{e(P - x - y)}{e(P)}.$$

Then (1.7) becomes

$$\frac{e(P - x - y) \cdot e(P)}{e(P - x) \cdot e(P - y)} \ \geq \ 1.$$

This inequality follows from and is asymptotically equivalent to the lower bound in (1.1).

Example 3.3. For a subset $A \subset \min(P)$ of minimal elements, define $A^\triangledown$ as follows:

$$A^\triangledown \ := \ \big\{x \in X \ : \ x \succcurlyeq y, \ y \in \min(P) \ \Rightarrow \ y \in A\big\}.$$

Equivalently, $A^\triangledown = A\!\uparrow \smallsetminus B\!\uparrow$, where $B = \min(P) - A$. Clearly, we have $A^\triangledown \subseteq A\!\uparrow$, and sometimes the subset $A^\triangledown$ is much smaller than $A\!\uparrow$. On the other hand, we have:

$$\mathbb{P}\big[1 \in f(A), 2 \in f(A\!\uparrow)\big] \ = \ \mathbb{P}\big[1 \in f(A), 2 \in f(A^\triangledown)\big].$$

This is because the only way we can have $f(y) = 2$ and $f(x) = 1$ for some $x \in A$, is if either $y \in \min(P)$, or y covers element x and no other element. This argument is useful in trying understand our correlation inequalities, as the next example shows.

Example 3.4. Let $A \subset \min(P)$ and apply (1.14) to $A\uparrow$. We get:

$$\mathbb{P}[1 \in f(A), 2 \in f(A\uparrow)] \cdot \mathbb{P}[1 \in f(B), 2 \notin f(A\uparrow)] \ \le \ \mathbb{P}[1 \in f(A), 2 \notin f(A\uparrow)]^2. \tag{3.1}$$

Alternatively, let $B := \min(P) - A$ and apply (1.16). We get:

$$\mathbb{P}\big[1 \in f(A), 2 \in f(A\uparrow)\big] \cdot \mathbb{P}\big[1 \in f(B), 2 \in f(B\uparrow)\big] \ \le \ \mathbb{P}\big[1 \in f(A), 2 \in f(B)\big]^2. \tag{3.2}$$

At first sight, the LHS of (3.1) $\le$ LHS of (3.2), while the RHS of (3.1) $\ge$ RHS of (3.2). The argument in the previous example shows that both of these are equalities. In other words, the inequality (1.14) is equivalent to the inequality (1.16) in this case.

3.2. **Second moment bound.** In Theorem 1.2, letting $y = x$ gives the following curious bound on the second moment:

Corollary 3.5. *Let $P = (X, \prec)$ be a finite poset, and let $x \in X$ be a fixed element. Then:*

$$1 \ \le \ \frac{\mathbb{E}[f(x)^2]}{\mathbb{E}[f(x)]^2} \ < \ 2. \tag{3.3}$$

Here the lower bound is trivial: $0 \le \textrm{Var}(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2$ for every random variable Z. Note also that the corollary immediately implies a weaker version of (1.3):

$$\frac{\mathbb{E}[f(x)f(y)]}{\mathbb{E}[f(x)] \cdot \mathbb{E}[f(y)]} \ < \ 2. \tag{3.4}$$

This follows by combining the upper bound in (3.3) with $\operatorname{Cov}(X,Y) \le \textrm{Var}(X) \cdot \textrm{Var}(Y)$.

Example 3.6. Let $P := C_{n-1} + \{x\}$ be the parallel sum of two chains. We have:

$$\frac{\mathbb{E}[f(x)^2]}{\mathbb{E}[f(x)]^2} \ = \ \frac{\frac{1}{n}\sum_{k=1}^n k^2}{\big(\frac{1}{n}\sum_{k=1}^n k\big)^2} \ = \ \frac{4n+2}{3n+3} \ \to \ \frac{4}{3} \qquad \text{as} \qquad n \to \infty.$$

Thus, the constant in the upper bound (3.3) must be at least $4/3$.

Note that our proof of Theorem 1.2 follows from a construction similar to that used in [CP21] to rederive Stanley's Theorem 1.3. Since Corollary 3.5 is a simple consequence of Theorem 1.2, let us show that it follows from Stanley's theorem as well.

Proposition 3.7. *Let Z be a random variable on $\{1, 2, \ldots\}$ with a log-concave distribution. Then $\mathbb{E}[Z^2] \le 2\mathbb{E}[Z]^2$.*

Note that constant 2 in the proposition is tight, as shown by a geometric random variable $Z = \operatorname{Geo}(q)$, where we let $q \to 1$. We prove Proposition 3.7 in §7.2. On the other hand, we believe that the constant 2 in (3.3) can be lowered to $4/3$, as in Example 3.6.

Conjecture 3.8. *Let $P = (X, \prec)$ be a finite poset, and let $x \in X$ be a fixed element. Then $\mathbb{E}[f(x)^2] \le \frac{4}{3}\,\mathbb{E}[f(x)]^2$.*

Example 3.9. Proposition 3.7 is sharp in full generality, but can be weak in other instances. In notation of §1.5, let G be simple graph on $2n$ vertices. Denote by $\alpha_G(M)$ be the size of a random matching M, and let $\zeta_G(M) := n - \alpha_G$ be half of the number of vertices not covered by M. The Heilmann–Lieb theorem proves that ζ_G has a log-concave distribution. Proposition 3.7 then implies $\textrm{Var}(\zeta_G) = \mathbb{E}\big[\zeta_G^2\big] - \mathbb{E}[\zeta_G]^2 < \mathbb{E}[\zeta_G]^2$. On the other hand, [Kahn00, Cor. 4.2] gives a much stronger bound $\textrm{Var}(\zeta_G) \le \mathbb{E}[\zeta_G]$, which can then be used to obtain concentration inequalities.

3.3. **Posets with unique covers of minimal elements.** For elements $x, y \in X$ in a poset $P = (X, \prec)$, we say that element y *covers* x, if $x \prec y$, and there is no $v \in X$ s.t. $x \prec v \prec y$. For elements $x \prec y$ in X, we say that y is a *unique cover* of x, if

$(\Diamond)$ y covers x and does not cover any other elements in X.

The following correlation inequality is surprising even in the most simple special cases (see below).

Corollary 3.10. *Let $P = (X, \prec)$ be a finite poset, and let $x, y \in \min(P)$ be distinct minimal elements. Suppose element $v \in X$ is a unique cover of x, and $w \in X$ is a unique cover of y. Then:*

$$e(P - x - y)^2 \ \geq \ e(P - x - v) \cdot e(P - y - w). \tag{3.5}$$

Proof. In the notation of Theorem 1.9, let $A = \{x\}$ and $B = \{y\}$. Note that

$$\mathbb{P}\big[1 \in f(A), 2 \in f(A\uparrow)\big] \ = \ \mathbb{P}\big[f(x) = 1, f^{-1}(2) \succ x\big] \ \geq \ \mathbb{P}\big[f(x) = 1, f(v) = 2\big] \ = \ \frac{e(P - x - v)}{e(P)}\,.$$

Here the inequality follows from the definition of v as an element which satisfies $(\Diamond)$. By the same argument we have:

$$\mathbb{P}\big[1 \in f(B), 2 \in f(B\uparrow)\big] \ \geq \ \frac{e(P - y - w)}{e(P)}\,.$$

On the other hand, we also have:

$$\mathbb{P}\big[1 \in f(A), 2 \in f(B)\big] \ = \ \frac{e(P - x - y)}{e(P)}\,.$$

Now (1.16) implies the result. $\square$

We conclude this section with another four-element inequality:

Corollary 3.11. *Let $P = (X, \prec)$ be a finite poset, and let $x, y, z \in \min(P)$ be distinct minimal elements. Suppose element $u \in X$ is a unique cover of z. Then:*

$$e(P - u - z)\, e(P - x - y) \ \leq \ 2\, e(P - x - z)\, e(P - y - z). \tag{3.6}$$

Proof. Let $A = \{x\}$, $B = \{y\}$ and $C = \{z\}$. The corollary now follows from Theorem 1.10 and the argument in the proof of the Corollary 3.10. $\square$

4. Enumerative applications

4.1. **Standard Young tableaux.** Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition of n, write $\lambda \vdash n$. Here $\ell(\lambda)$ denotes the number of parts. A Young diagram is a set $Y_\lambda := \big\{(i,j) \in \mathbb{N}^2 \, : \, 1 \leq j \leq \lambda_i, 1 \leq i \leq \ell\big\}$. A *standard Young tableau* of shape λ is a bijection $f : Y_\lambda \to [n]$ which increases in both directions. Denote by $\operatorname{SYT}(\lambda)$ the set of standard Young tableaux of shape λ. The number $|\operatorname{SYT}(\lambda)|$ can be computed by the *hook-length formula*, see e.g. [Sta99, §7.21].

A *conjugate partition* $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is defined by $\lambda'_i = |\{i \, : \, \lambda_i \geq i\}$. Note that Young diagrams Y_λ and $Y_{\lambda'}$ are symmetric with respect to the $i = j$ reflection. Partition λ is called *self-conjugate* if $\lambda = \lambda'$. For partitions λ and μ, define the *sum* $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$ and the *union* $\lambda \cup \mu := (\lambda' + \mu')'$.

Two squares $x \prec y$ are called *adjacent* if $x = (i,j)$ and $y = (i, j+1)$ or $y = (i+1, j)$. The *corners* of λ are defined as elements $(i,j) \in X_\lambda$ such that $(i+1, j), (i, j+1) \notin Y_\lambda$. Denote by $\mathcal{C}(\lambda)$ the set of corners of λ. Similarly, the *boundary squares* of λ are defined as elements $(i,j) \in X_\lambda$ such that either $(i+1, j)$ or $(i, j+1)$ is not in Y_λ. Denote by $\mathcal{D}(\lambda)$ the set of boundary squares of λ, and note that $\mathcal{C}(\lambda) \subseteq \mathcal{D}(\lambda)$.

Let $\mu = (\mu_1, \mu_2, \ldots)$ be a partition such that $\mu_i \le \lambda_i$ for all $0 \le i \le \ell$. The set $Y_{\lambda/\mu} := Y_\lambda \smallsetminus Y_\mu$ is the *skew Young diagram* of (skew) shape λ/μ. Standard Young tableaux of skew shapes are defined similarly to the usual (straight) shapes. The number $|\operatorname{SYT}(\lambda/\mu)|$ of standard Young tableaux of shape λ/μ can be computed by the *Aitken–Feit determinant formula*, see e.g. [Sta99, §7.16].

4.2. **Removing the corners.** We now apply the correlation inequalities to posets corresponding to skew Young diagrams. As we do, we get both new and familiar inequalities (see below).

Corollary 4.1. *Let λ/μ be a skew shape, let $x, y \in \mathcal{C}(\lambda/\mu)$ be corners, and let $v, w \in \mathcal{D}(\lambda)$ be a boundary square adjacent to x and y, respectively. Then:*

$$|\operatorname{SYT}(\lambda/\mu - x - y)|^2 \;\ge\; |\operatorname{SYT}(\lambda/\mu - x - v)| \cdot |\operatorname{SYT}(\lambda/\mu - y - w)|. \tag{4.1}$$

In particular, if λ and μ are self-conjugate, $x = (i, j)$ and $y = (j, i)$, then:

$$|\operatorname{SYT}(\lambda/\mu - x - y)| \ge |\operatorname{SYT}(\lambda/\mu - x - v)|. \tag{4.2}$$

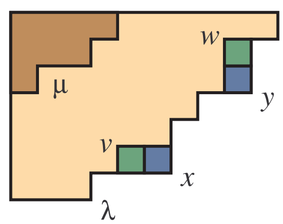

FIGURE 4.1. Skew shape λ/μ, where $\lambda = (10, 9, 9, 7, 6, 6, 3)$, $\mu = (4, 3, 1)$, corners $x, y \in \mathcal{C}(\lambda)$ and boundary squares $v, w \in \mathcal{D}(\lambda)$.

Proof. Let poset $\mathcal{P}_{\lambda/\mu} = (Y_{\lambda/\mu}, \prec)$ be defined by $(i, j) \preccurlyeq (i', j')$ if $i \le i'$ and $j \le j'$. The set of linear extensions $\mathcal{E}(\mathcal{P}_{\lambda/\mu})$ is in bijection with $\operatorname{SYT}(\lambda/\mu)$, so $e(\mathcal{P}_{\lambda/\mu}) = |\operatorname{SYT}(\lambda/\mu)|$. Note that minimal elements in the dual poset $\mathcal{P}^*_\lambda$ are the corners of λ: $\min(\mathcal{P}^*_\lambda) = \mathcal{C}(\lambda)$. Similarly, for skew shapes we have: $\min(\mathcal{P}^*_{\lambda/\mu}) = \mathcal{C}(\lambda) \smallsetminus \mathcal{C}(\mu)$.

In notation of §3.3, note that v and w are unique covers of x and y, respectively. Now Corollary 3.10 gives (4.1). For the second part, let $v = (p, q)$ and take $w := (q, p)$. Now (4.1) implies (4.2). ☐

Corollary 4.2. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition. We have the following log-concave inequality:*

$$\left|\operatorname{SYT}\left(\lambda/(a, 1^b)\right)\right|^2 \;\ge\; \left|\operatorname{SYT}\left(\lambda/(a+1, 1^{b-1})\right)\right| \cdot \left|\operatorname{SYT}\left(\lambda/(a-1, 1^{b+1})\right)\right|, \tag{4.3}$$

for all $1 < a < \lambda_1$ and $1 < b < \ell$.

Proof. Following the argument in the proof above, let $\mathcal{P}_{\lambda/\mu} = (Y_{\lambda/\mu}, \prec)$ where $\mu = (a+1, 1^{b+1})$, and let $x = (1, a)$, $y = (b, 1)$, $v = (1, a+1)$, $w = (b+1, 1)$. Now Corollary 3.10 gives the result. ☐

Corollary 4.3. *Let λ/μ be a skew shape, let $x, y, z \in \mathcal{C}(\lambda/\mu)$ be corners, and let $u \in \mathcal{D}(\lambda)$ be a boundary square adjacent to z, respectively. Then:*

$$\frac{|\operatorname{SYT}(\lambda/\mu - u - z)| \cdot |\operatorname{SYT}(\lambda/\mu - x - y)|}{|\operatorname{SYT}(\lambda/\mu - x - z)| \cdot |\operatorname{SYT}(\lambda/\mu - y - z)|} \;\le\; 2. \tag{4.4}$$

The proof follows from Corollary 3.11 along the same lines as the proofs above. We omit the details.

4.3. **Related inequalities.** Corollary 4.1 is closely related to two other inequalities. The first is the *Okounkov inequality* conjectured in [Oko97, p. 269], and proved by Lam–Postnikov–Pylyavskyy [LPP07, Thm 4]:

$$\big|\operatorname{SYT}\big(\tfrac{\lambda+\nu}{2}\,/\,\tfrac{\mu+\alpha}{2}\big)\big|^2 \ \geq\ |\operatorname{SYT}(\lambda/\mu)|\cdot|\operatorname{SYT}(\nu/\alpha)|, \tag{4.5}$$

for all skew shapes λ/μ and ν/α, such that partitions $\lambda+\nu$ and $\mu+\alpha$ have even parts.[2]

In notation of Corollary 4.1, observe that for v to the left of x, and for w to the left of y (see Figure 4.1), the Okounkov inequality coincides with our inequality (4.1). On the other hand, even the inequality (4.2) does not follow from this argument.

The second is the *Fomin–Fulton–Li–Poon* (FFLP) *inequality* conjectured in [FFLP05, Conj. 2.7], and also proved in [LPP07, Thm 4]:

$$\begin{aligned}\big|\operatorname{SYT}\big(\operatorname{sort}_1(\lambda,\nu)/\operatorname{sort}_1(\mu,\alpha)\big)\big|\cdot\big|\operatorname{SYT}\big(\operatorname{sort}_2(\lambda,\nu)/\operatorname{sort}_2(\mu,\alpha)\big)\big|\\ \geq\ |\operatorname{SYT}(\lambda/\mu)|\cdot|\operatorname{SYT}(\nu/\alpha)|.\end{aligned} \tag{4.6}$$

Here for two partitions β and γ we define two other partitions $\operatorname{sort}_1(\beta,\gamma):=(\tau_1,\tau_3,\ldots)$ and $\operatorname{sort}_2(\lambda,\mu):=(\tau_2,\tau_4,\ldots)$, where $(\tau_1,\tau_2,\ldots):=\beta\cup\gamma$.

In notation of Corollary 4.1, observe that for v above x, and for w to the above y, the FFLP inequality coincides with our inequality (4.1). In this sense it is conjugate dual to the Okounkov inequality (4.5), cf. [LPP07]. On the other hand, for v to the left of x, and for w above y, the LHS of (4.6) has partitions of unequal size:

$$|\operatorname{SYT}(\lambda/\mu-y)|\cdot|\operatorname{SYT}(\lambda/\mu-x-y-v)|\ \geq\ |\operatorname{SYT}(\lambda/\mu-x-v)|\cdot|\operatorname{SYT}(\lambda/\mu-y-w)|. \tag{4.7}$$

4.4. **Hook walk.** Let $f\in\operatorname{SYT}(\lambda)$ be a random standard Young tableau of shape $\lambda\vdash n$. For a corner $x\in\mathcal{C}(\lambda)$, denote by

$$p(x)\ :=\ \frac{|\operatorname{SYT}(\lambda-x)|}{|\operatorname{SYT}(\lambda)|}$$

the probability distribution on $\mathcal{C}(\lambda)$ defined by the location of n in f. The celebrated *hook walk* [GNW79] shows how to sample from $p(x)$:

- Start at a random square $y\in Y_\lambda$.
- Move to a random square in the hook $H(y)$.
- Repeat until the walk reaches a corner $x\in\mathcal{C}(\lambda)$.

Here *the hook* is defined as $H(i,j):=\{(i,r)\,:\,j<r\leq\lambda_i\}\cup\{(s,j)\,:\,i<s\leq\lambda'_j\}$. Let $y\in\mathcal{C}(\lambda)$ be a different corner. One can think of (1.2) as follows:

$$\frac{n}{n-1}\ \leq\ \frac{\mathbb{P}[f(y)=n-1\,|\,f(x)=n]}{\mathbb{P}[f(y)=n]}\ \leq\ 2. \tag{4.8}$$

The lower bound in (4.8) is now straightforward. Indeed, all hook walk trajectories which arrive to y in $(\lambda-x)$, also arrive in λ, and the ratio is given by the ratio of sizes of diagrams in the starting location of the walk. The upper bound in (4.8) is less intuitive, however, even in this simple probabilistic model.

[2]The actual result in [LPP07] is much stronger, cf. §8.6 (see also [CP22+]).

4.5. **Bruhat order.** Let $\sigma \in S_n$ and define the *permutation poset* $P_\sigma = ([n], \prec)$ by letting

$$i \preccurlyeq j \quad \Leftrightarrow \quad i \leq j \ \text{ and } \ \sigma(i) \leq \sigma(j).$$

It is easy to see that $\mathcal{E}(P_\sigma) \subseteq S_n$ is the lower ideal of σ in the *weak Bruhat order* $\mathcal{B}_n = (S_n, \lhd)$, see [BW91, FW97]. The set of minimal elements in $\mathcal{P}_\sigma$ is the set of *minimal records* in σ:

$$\min(P_\sigma) \ = \ \big\{k \in [n] \, : \, \sigma(i) > \sigma(k) \ \text{ for all } \ i < k\big\}.$$

Fix two record minima $a, b \in \min(P_\sigma)$. We can then rewrite the correlation inequality (1.2) in terms of random permutations $\omega \in S_n$:

$$\frac{n}{n-1} \ \leq \ \frac{\mathbb{P}[\omega(a) = 1, \omega(b) = 2 \,|\, \omega \trianglelefteq \sigma]}{\mathbb{P}[\omega(a) = 1 \,|\, \omega \trianglelefteq \sigma] \, \cdot \, \mathbb{P}[\omega(a) = 2 \,|\, \omega \trianglelefteq \sigma]} \ \leq \ 2.$$

As in the introduction, the lower bound is an equality for $\sigma = (n, n-1, \ldots, 1)$ and the upper bound is an equality for $\sigma = (2, 1, 3, \ldots, n)$.

4.6. **Zigzag posets.** Let $P_n := (X, \prec)$ be the poset on n elements $X = \{x_1, y_1, x_2, y_2, x_3, \ldots\}$, with the only relations given by $x_1 \prec y_1 \succ x_2 \prec y_3 \succ x_3 \prec \ldots$ Note that P_n has height two and the set $\mathcal{E}(P_n)$ is in bijection with *alternating permutations* $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \sigma(5) < \ldots$ and the standard Young tableaux of a *zigzag shape* [Sta10].

It is well-known that $e(P_n) = E_n$ are the *Euler numbers*, which satisfy

$$\sum_{m=0}^{\infty} E_n \, \frac{t^n}{n!} \ = \ \tan(t) \, + \, \sec(t) \quad \text{and} \quad E_n \, \sim \, \frac{2^{n+2} \, n!}{\pi^{n+1}} \quad \text{as} \ \ n \to \infty,$$

see e.g. [OEIS, A000111], [FS09, §IV.6.1] and [Sta99, §1.6]. We also have:

$$\mathbb{P}[f(x_1) = k] \ = \ \frac{E_{n,k}}{E_k}\,, \quad \text{where} \quad E_{n,k} \ := \ \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} (-1)^i \binom{k}{2i+1} E_{n-2i-1}$$

are the *Entringer numbers*, see [OEIS, A008282] and [Sta10]. Stanley's Theorem 1.3 in this case gives log-concavity of the Entringer numbers:

$$E_{n,k}^2 \ \geq \ E_{n,k-1} \, \cdot \, E_{n,k+1}\,. \tag{4.9}$$

This inequality can be viewed as a degree two polynomial inequality for the Euler numbers. We refer to [B+19, GHMY21] and [CP21, §1.38] for more on (4.9) and its various generalizations.

Example 4.4. Let $n = 2m+1$, and note that $\min(P_n) = \{x_1, \ldots, x_{m+1}\}$. Fix $1 \leq k \leq m+1$ and consider a subset $A := \{x_1, \ldots, x_k\}$. We have:

$$\mathbb{P}\big[1 \in f(A)\big] \ = \ \frac{F_n(k)}{E_n}\,, \quad \mathbb{P}\big[1 \notin f(A)\big] \ = \ \frac{F_n(m-k+1)}{E_n}\,, \quad \mathbb{P}\big[1 \notin f(A), 2 \in f(A)\big] \ = \ \frac{G_n(k)}{E_n}\,,$$

where

$$F_n(k) \ := \ \sum_{i=1}^{k} \binom{n-1}{2i-2} E_{2i-2} \, E_{n-2i+1} \qquad \text{and}$$

$$G_n(k) \ := \ \sum_{i=1}^{k} \sum_{j=k+1}^{m+1} \binom{n-2}{2i-2, 2j-2i-1} E_{2i-2} \, E_{2j-2i-1} \, E_{n-2j+1}\,.$$

Substituting these into (1.11) gives a new degree four polynomial inequality for the Euler numbers:

$$F_n(k) \, \cdot \, F_n(m-k+1) \ \leq \ E_n \, \cdot \, G_n(k). \tag{4.10}$$

Note that for $k, n-k = \omega(1)$ both sides have the same leading asymptotics, which makes the inequality even more interesting.

Example 4.5. Let $n = 2m+1$ and $1 \le k \le \ell \le m+1$. Take $A = \{x_1, \ldots, x_k\}$ as above. We have:

$$\mathbb{P}\big[1, 2 \in f(A)\big] \ = \ \frac{H_n(k)}{E_n} \qquad \text{and} \qquad \mathbb{P}\big[1, 2 \notin f(A)\big] \ = \ \frac{H_n(m-k+1)}{E_n}\,,$$

where

$$H_n(k) \ := \ 2 \sum_{1 \le i < j \le k} \binom{n-2}{2i-2,\, 2j-2i-1} E_{2i-2}\, E_{2j-2i-1}\, E_{n-2j+1}\,.$$

Substituting this and the inequalities from the previous example into (1.14) gives a new degree six polynomial inequality for the Euler numbers:

$$H_n(k) \cdot H_n(m-k+1) \ \le \ G_n(k)^2. \tag{4.11}$$

5. Correlation inequalities from hyperbolic inequalities

In this short section we derive several correlation inequalities for hyperbolic matrices. Although we are motivated by applications to combinatorial atlases [CP21, CP22], the presentation here is self-contained and uses nothing but linear algebra.

5.1. **Hyperbolic matrices.** Let $\mathbf{M} = (\mathrm{M}_{ij})$ be a symmetric $d \times d$ matrix with entries $\mathrm{M}_{ij} \in \mathbb{R}_+$. The matrix $\mathbf{M}$ is *hyperbolic* (satisfies *hyperbolic property*), if

$$\langle \mathbf{x}, \mathbf{M}\mathbf{y}\rangle^2 \ \ge \ \langle \mathbf{x}, \mathbf{M}\mathbf{x}\rangle\, \langle \mathbf{y}, \mathbf{M}\mathbf{y}\rangle \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d \ \text{ such that } \ \langle \mathbf{y}, \mathbf{M}\mathbf{y}\rangle \ge 0. \tag{Hyp}$$

Note that (Hyp) is equivalent to the matrix $\mathbf{M}$ having at most one positive eigenvalue, counting multiplicity (see e.g. [CP22, SvH19]).

For every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we employ the following shorthand

$$\mathbf{M}_{\mathbf{x}\mathbf{y}} \ := \ \langle \mathbf{x}, \mathbf{M}\mathbf{y}\rangle.$$

Note that $\mathbf{M}_{\mathbf{x}\mathbf{y}} = \mathbf{M}_{\mathbf{y}\mathbf{x}}$ since $\mathbf{M}$ is symmetric.

Lemma 5.1. *Let $\mathbf{M}$ be a nonnegative symmetric $d \times d$ matrix that satisfies* (Hyp)*, and let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d_+$ be nonnegative vectors. Then either $\mathbf{M}\mathbf{x} = \mathbf{M}\mathbf{y} = \mathbf{M}\mathbf{z} = \mathbf{0}$, or there exists $\mathbf{v} \in \mathbb{R}^d_+$ s.t. $\langle \mathbf{x} + \varepsilon\, \mathbf{v}, \mathbf{M}(\mathbf{x} + \varepsilon\, \mathbf{v})\rangle > 0$ for all $\varepsilon > 0$.*

Proof. We split the proof into two cases. First, suppose that $\mathbf{M}$ has a positive eigenvalue $\lambda > 0$ with a corresponding eigenvector $\mathbf{v} \in \mathbb{R}^d_+$. Then we get

$$\begin{aligned} \langle \mathbf{x} + \varepsilon\, \mathbf{v}, \mathbf{M}(\mathbf{x} + \varepsilon\, \mathbf{v})\rangle \ &= \ \langle \mathbf{x}, \mathbf{M}\mathbf{x}\rangle \ + \ 2\,\varepsilon\, \langle \mathbf{x}, \mathbf{M}\mathbf{v}\rangle \ + \ \varepsilon^2\, \langle \mathbf{v}, \mathbf{M}\mathbf{v}\rangle \\ &\ge \ \varepsilon^2\, \langle \mathbf{v}, \mathbf{M}\mathbf{v}\rangle \ = \ \varepsilon^2\, \lambda\, \langle \mathbf{v}, \mathbf{v}\rangle \ > \ 0, \end{aligned}$$

as desired.

Now suppose that all eigenvalues of g $\mathbf{M}$ are nonpositive. Since $\mathbf{M}_{\mathbf{x}\mathbf{x}} = \langle \mathbf{x}, \mathbf{M}\mathbf{x}\rangle \ge 0$ by the non-negativity of $\mathbf{M}$ and $\mathbf{x}$, this implies that $\mathbf{M}\mathbf{x} = \mathbf{0}$. Analogously, we have $\mathbf{M}\mathbf{y} = \mathbf{M}\mathbf{z} = \mathbf{0}$, which completes the proof. $\square$

The following lemma is a useful consequence of (Hyp), and is inspired by the inequality in [Sch14, Lemma 7.4.1] for mixed volumes.[3]

[3] Our original proof was more complicated. The simplified proof below was suggested to us by Ramon van Handel (Nov. 2022, personal communication).

Lemma 5.2. *Let* $\mathbf{M}$ *be a nonnegative symmetric* $d \times d$ *matrix that satisfies* (Hyp), *and let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d_+$ *be nonnegative vectors. Then:*

$$\big(\mathbf{M_{yz}}\,\mathbf{M_{xx}} - \mathbf{M_{xy}}\,\mathbf{M_{xz}}\big)^2 \ \le\ \big(\mathbf{M^2_{xy}} - \mathbf{M_{xx}}\,\mathbf{M_{yy}}\big)\,\big(\mathbf{M^2_{xz}} - \mathbf{M_{xx}}\,\mathbf{M_{zz}}\big). \tag{5.1}$$

Proof. Fix $\mathbf{x} \in \mathbb{R}^d_+$ and let $\mathrm{Q} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be the bilinear form given by

$$\mathrm{Q}(\mathbf{y}, \mathbf{z}) \ :=\ \mathbf{M_{yz}}\,\mathbf{M_{xx}} - \mathbf{M_{xy}}\,\mathbf{M_{xz}}\,.$$

Then (Hyp) implies that $\mathrm{Q}(\mathbf{y}, \mathbf{y})$ is a nonnegative quadratic form. It then follows from Cauchy–Schwarz inequality that

$$\mathrm{Q}(\mathbf{y}, \mathbf{z})^2 \ \le\ \mathrm{Q}(\mathbf{y}, \mathbf{y})\,\mathrm{Q}(\mathbf{z}, \mathbf{z})\,.$$

This is equivalent to (5.1), which completes the proof. □

5.2. **The implications.** The following lemma is a simple consequence of Lemma 5.2.

Lemma 5.3. *Let* $\mathbf{M}$ *be a nonnegative symmetric* $d \times d$ *matrix that satisfies* (Hyp), *and let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d_+$ *be nonnegative vectors. Then:*

$$\mathbf{M_{yy}}\,\mathbf{M^2_{xz}} \ +\ \mathbf{M_{zz}}\,\mathbf{M^2_{xy}} \ \le\ 2\mathbf{M_{xy}}\,\mathbf{M_{xz}}\,\mathbf{M_{yz}}\,. \tag{5.2}$$

Proof. The lemma clearly follows if $\mathbf{M}\mathbf{x} = \mathbf{M}\mathbf{y} = \mathbf{M}\mathbf{z} = \mathbf{0}$, so assume that this is not the case. Then, by Lemma 5.1, we can substitute $\mathbf{x}$ with $\mathbf{x}^\varepsilon$ and then takes the limit $\varepsilon \to 0$ if necessary, so we can additionally assume that $\mathbf{M_{xx}} > 0$. Expanding the squares in (5.1), we get:

$$\begin{aligned} &\mathbf{M^2_{yz}}\,\mathbf{M^2_{xx}} \ -\ 2\mathbf{M_{xx}}\,\mathbf{M_{xy}}\,\mathbf{M_{xz}}\,\mathbf{M_{yz}} \ +\ \mathbf{M^2_{xy}}\,\mathbf{M^2_{xz}} \\ &\qquad \le\ \mathbf{M^2_{xy}}\,\mathbf{M^2_{xz}} \ +\ \mathbf{M^2_{xx}}\,\mathbf{M_{yy}}\,\mathbf{M_{zz}} \ -\ \mathbf{M_{xx}}\,\mathbf{M_{yy}}\,\mathbf{M^2_{xz}} \ -\ \mathbf{M_{xx}}\,\mathbf{M_{zz}}\,\mathbf{M^2_{xy}}\,. \end{aligned}$$

Cancelling the term $\mathbf{M^2_{xy}}\,\mathbf{M^2_{xz}}$ from both sides, and then dividing both sides by $\mathbf{M_{xx}}$, we get:

$$\mathbf{M^2_{yz}}\,\mathbf{M_{xx}} \ -\ 2\mathbf{M_{xy}}\,\mathbf{M_{xz}}\,\mathbf{M_{yz}} \ \le\ \mathbf{M_{xx}}\,\mathbf{M_{yy}}\,\mathbf{M_{zz}} \ -\ \mathbf{M_{yy}}\,\mathbf{M^2_{xz}} \ -\ \mathbf{M_{zz}}\,\mathbf{M^2_{xy}}\,. \tag{5.3}$$

This is equivalent to

$$\mathbf{M_{yy}}\,\mathbf{M^2_{xz}} \ +\ \mathbf{M_{zz}}\,\mathbf{M^2_{xy}} \ +\ \mathbf{M_{xx}}\big(\mathbf{M^2_{yz}} \ -\ \mathbf{M_{yy}}\,\mathbf{M_{zz}}\big) \ \le\ 2\mathbf{M_{xy}}\,\mathbf{M_{xz}}\,\mathbf{M_{yz}}\,.$$

By (Hyp), we have $\mathbf{M^2_{yz}} \ge \mathbf{M_{yy}}\,\mathbf{M_{zz}}$. This and the above inequality imply the result. □

Remark 5.4. Note that (5.2) implies (Hyp) by substitution $\mathbf{z} \leftarrow \mathbf{x}$. The inequality (5.3) can be rewritten symmetrically as:

$$\det \begin{bmatrix} \mathbf{M_{xx}} & \mathbf{M_{xy}} & \mathbf{M_{xz}} \\ \mathbf{M_{xy}} & \mathbf{M_{yy}} & \mathbf{M_{yz}} \\ \mathbf{M_{xz}} & \mathbf{M_{yz}} & \mathbf{M_{zz}} \end{bmatrix} \ \ge\ 0\,. \tag{5.4}$$

It can be viewed as the counterpart of *Shephard's inequality* for mixed volumes, see e.g. [vH21].

Lemma 5.5. *Let* $\mathbf{M}$ *be a nonnegative symmetric* $d \times d$ *matrix that satisfies* (Hyp), *and let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d_+$ *be nonnegative vectors. Then:*

$$\mathbf{M_{zz}}\,\mathbf{M_{xy}} \ \le\ 2\,\mathbf{M_{xz}}\,\mathbf{M_{yz}}\,. \tag{5.5}$$

Proof. Note that the lemma clearly holds if $\mathbf{M_{xy}} = 0$, so we can assume that $\mathbf{M_{xy}} > 0$. The lemma now follows by removing the term $\mathbf{M_{yy}}\,\mathbf{M^2_{xz}}$ from the LHS of (5.2) and dividing both sides by $\mathbf{M_{xy}}$. □

The upper bound in (5.5) can be improved in some cases.

Lemma 5.6. *Let* $\mathbf{M}$ *be a nonnegative symmetric* $d \times d$ *matrix that satisfies* (Hyp), *and let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}_+^d$ *be nonnegative vectors. Then at least two out of these three inequalities hold:*

$$\mathbf{M}_{\mathbf{xx}}\, \mathbf{M}_{\mathbf{yz}} \ \leq \ \mathbf{M}_{\mathbf{xy}}\, \mathbf{M}_{\mathbf{xz}}\ , \quad \mathbf{M}_{\mathbf{yy}}\, \mathbf{M}_{\mathbf{xz}} \ \leq \ \mathbf{M}_{\mathbf{xy}}\, \mathbf{M}_{\mathbf{yz}} \quad \text{or} \quad \mathbf{M}_{\mathbf{zz}}\, \mathbf{M}_{\mathbf{xy}} \ \leq \ \mathbf{M}_{\mathbf{xz}}\, \mathbf{M}_{\mathbf{yz}}\,.$$

Proof. Suppose to the contrary that at least two of these inequalities are false. Without loss of generality we assume the two inequalities are

$$\mathbf{M}_{\mathbf{xx}}\, \mathbf{M}_{\mathbf{yz}} \ > \ \mathbf{M}_{\mathbf{xy}}\, \mathbf{M}_{\mathbf{xz}} \qquad \text{and} \qquad \mathbf{M}_{\mathbf{yy}}\, \mathbf{M}_{\mathbf{xz}} \ > \ \mathbf{M}_{\mathbf{xy}}\, \mathbf{M}_{\mathbf{yz}}\,.$$

Taking the product of these two inequalities gives $\mathbf{M}_{\mathbf{xx}}\, \mathbf{M}_{\mathbf{yy}} > \mathbf{M}_{\mathbf{xy}}^2$. This contradicts (Hyp) and completes the proof. □

Remark 5.7. In conditions of Lemma 5.6, there are cases such that exactly two out of these three inequalities hold. For example, the lower bound in (1.1) is an example in which $\mathbf{M}_{\mathbf{zz}}\, \mathbf{M}_{\mathbf{xy}} > \mathbf{M}_{\mathbf{xz}}\, \mathbf{M}_{\mathbf{yz}}$, with the choice of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ as in the proof of Theorem 6.3.

6. Proof of correlation inequalities

In this section we prove the inequalities in the introduction. The proofs follow the approach in [CP21], and are based on observation that certain matrices associated with posets satisfy (Hyp).

6.1. **The setup.** Fix a poset $P = (X, \prec)$ with $|X| = n$ elements, and fix an element $a \in X$. For every $k \geq 0$ and $S \subseteq \mathcal{E}(P)$, we write

$$\mathrm{N}_k(S) \ := \ \big|\{f \in S \, : \, f(a) = k\}\big|.$$

To simplify the notation, we write

$$\mathrm{N}_k\big(f(x) = 1\big) \quad \text{ to mean } \quad \mathrm{N}_k\big(\{f \in \mathcal{E}(P) \, : \, f(x) = 1\}\big).$$

When the underlying poset P is potentially ambiguous, we will write $\mathrm{N}_k^{\langle P \rangle}$ instead of N_k.

Let Z_{up} and Z_{down} be two distinct copies of $X - \{a\}$, let $Z := Z_{\mathrm{up}} \cup Z_{\mathrm{down}}$, let $d = |Z| = 2(n-1)$, and let $k \in \{2, \ldots, n-1\}$. We denote by $\mathbf{M} := \mathbf{M}(P, a, k)$ the symmetric $d \times d$ matrix given by

- $\mathrm{M}_{xy} \ := \ \mathrm{N}_{k+1}\big(f(x) = 1, f(y) = 2\big)$ if $x, y \in \min(Z_{\mathrm{down}})$, $x \neq y$,
- $\mathrm{M}_{xy} \ := \ \mathrm{N}_{k-1}\big(f(x) = n, f(y) = n-1\big)$ if $x, y \in \max(Z_{\mathrm{up}})$, $x \neq y$,
- $\mathrm{M}_{xy} \ := \ \mathrm{N}_{k}\big(f(x) = 1, f(y) = n\big)$ if $x \in \min(Z_{\mathrm{down}})$, $y \in \max(Z_{\mathrm{up}})$,
- $\mathrm{M}_{xx} \ := \ \mathrm{N}_{k+1}\big(f(x) = 1\big) \ - \ \mathrm{N}_{k+1}\big(f(x) = 2\big)$ if $x \in \min(Z_{\mathrm{down}})$,
- $\mathrm{M}_{xx} \ := \ \mathrm{N}_{k-1}\big(f(x) = n\big) \ - \ \mathrm{N}_{k-1}\big(f(x) = n-1\big)$ if $x \in \max(Z_{\mathrm{up}})$,
- $\mathrm{M}_{xy} \ := \ 0$ otherwise.

Proposition 6.1 ([CP21, Prop 14.9])**.** *The matrix* $\mathbf{M}$ *satisfies* (Hyp).

We will also use the following combinatorial properties of the matrix $\mathbf{M}$. By a direct calculation, the diagonal entry M_{xx} for $x \in \min(Z_{\mathrm{down}})$, is equal to

$$\begin{aligned} \mathrm{M}_{xx} \ &= \ \mathrm{N}_{k+1}\big(f(x) = 1\big) \ - \ \mathrm{N}_{k+1}\big(f(x) = 2, x \,||\, f^{-1}(1)\big) \\ &= \ \mathrm{N}_{k+1}\big(f(x) = 1\big) \ - \ \mathrm{N}_{k+1}\big(f(x) = 1, x \,||\, f^{-1}(2)\big) \\ &= \ \mathrm{N}_{k+1}\big(f(x) = 1, x \prec f^{-1}(2)\big). \end{aligned} \tag{6.1}$$

By a similar argument, for a $x \in \max(Z_{\text{up}})$, we have:

$$\mathrm{M}_{xx} \;=\; \mathrm{N}_{k-1}\big(f(x) = n,\, x \succ f^{-1}(n-1)\big). \tag{6.2}$$

Lemma 6.2. *If $x \in \min(Z_{\text{down}})$, then:*

$$\sum_{y \in Z_{\text{down}}} \mathrm{M}_{xy} \;=\; \mathrm{N}_{k+1}\big(f(x)=1\big) \qquad \textit{and} \qquad \sum_{y \in Z_{\text{up}}} \mathrm{M}_{xy} \;=\; \mathrm{N}_{k}\big(f(x)=1\big). \tag{6.3}$$

Similarly, if $x \in \min(Z_{\text{up}})$, then:

$$\sum_{y \in Z_{\text{down}}} \mathrm{M}_{xy} \;=\; \mathrm{N}_{k}\big(f(x)=n\big) \qquad \textit{and} \qquad \sum_{y \in Z_{\text{up}}} \mathrm{M}_{xy} \;=\; \mathrm{N}_{k-1}\big(f(x)=n\big). \tag{6.4}$$

Proof. This follows from a direct calculation. The details are straightforward. □

6.2. **Correlations for deletion operations.** Let $P = (X, \prec)$ be a poset with n elements. Fix $k \geq 1$ and $a \in X$. As in the introduction, denote by $e_k(P) := \mathrm{N}_k^{\langle P \rangle}$ the number of linear extensions $f \in \mathcal{E}(P)$ such that $f(a) = k$. We start with the following correlation inequality extending the upper bound in Theorem 1.1.

Theorem 6.3. *Let $P = (X, \prec)$ be a poset with $|X| = n > 2$ elements. Fix an element $a \in X$ and integer $1 \leq k \leq n-2$. Then, for every distinct minimal elements $x, y \in \min(X - a)$, we have:*

$$e_k(P)\, e_k(P - x - y) \;\leq\; 2\, e_k(P - x)\, e_k(P - y). \tag{6.5}$$

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ be the characteristic vectors of $x, y \in Z_{\text{down}}$. Similarly, let $\mathbf{z} \in \mathbb{R}^d$ be the characteristic vector of Z_{up}. Let $\mathbf{M} := \mathbf{M}(P, a, k)$ be the matrix in Proposition 6.1. Note that

$$\mathbf{M}_{\mathbf{xy}} \;=\; \mathrm{N}_{k+1}\big(f(x) = 1, f(y) = 2\big) \;=\; e_{k-1}(P - x - y)$$

and

$$\mathbf{M}_{\mathbf{xz}} \;=\; \sum_{z \in Z_{\text{up}}} \mathrm{M}_{xz} \;=_{(6.3)}\; \mathrm{N}_k\big(f(x) = 1\big) \;=\; e_{k-1}(P - x).$$

We have $\mathbf{M}_{\mathbf{yz}} = e_{k-1}(P - y)$ by the same reasoning. Finally, note that

$$\mathbf{M}_{\mathbf{zz}} \;=\; \sum_{v \in Z_{\text{up}}} \sum_{w \in Z_{\text{up}}} \mathrm{M}_{vw} \;=_{(6.4)}\; e_{k-1}(P).$$

Substituting these equations into (5.5), gives

$$e_{k-1}(P)\, e_{k-1}(P - x - y) \;\leq\; 2\, e_{k-1}(P - x)\, e_{k-1}(P - y),$$

Letting $k \leftarrow k + 1$ implies the result. □

Proof of Theorem 1.1. Let $P' := P + \{a\}$ be the parallel sum of the poset P and the single element $\{a\}$. Note that

$$\begin{aligned} e_{k-1}(P') \;&=\; e(P), \qquad & e_{k-1}(P' - x - y) \;&=\; e(P - x, y), \\ e_{k-1}(P' - x) \;&=\; e(P - x), \qquad & e_{k-1}(P' - y) \;&=\; e(P - y). \end{aligned}$$

The theorem now follows by applying Theorem 6.3 to the poset P'. □

6.3. **Covariances for random linear extensions.** We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. We create a new poset $P' := (X', \prec')$ as follows. Let ground set $X' := X \cup \{x, y\}$ be obtained by adding two new elements x, y. Let the partial order $\prec'$ be the closure of the partial order $\prec$ and the extra relations

$$x \prec' v \quad \text{for every} \quad v \in A, \qquad y \prec' w \quad \text{for every} \quad w \in B. \tag{6.6}$$

Note that x and y are minimal elements of P'.

Clearly, $e(P' - x, y) = e(P)$. Note that, for every linear extension $f \in \mathcal{E}(P) = \mathcal{E}(P' - x - y)$, there are exactly $f_{\min}(B)$ ways to add y to form a linear extension of $P' - x$. This implies that

$$e(P' - x) \ = \ \sum_{i \geq 1} i \, \cdot \, \mathrm{N}^{\langle P \rangle}\big(f_{\min}(B) = i\big) \ = \ \mathbb{E}\big[f_{\min}(B)\big] \, \cdot \, e(P).$$

By the same reasoning, we have

$$e(P' - y) \ = \ \mathbb{E}\big[f_{\min}(A)\big] \, \cdot \, e(P).$$

Finally, note that for every linear extension $f \in \mathcal{E}(P) = \mathcal{E}(P' - x - y)$, there are exactly $f_{\min}(A)\, f_{\min}(B) + \min\{f_{\min}(A), f_{\min}(B)\}$ ways to add x and y to form a linear extension of P'. This implies that

$$\begin{aligned} e(P') \ &= \ \sum_{i \geq 1} \sum_{j \geq 1} \big(ij \, + \, \min\{i, j\}\big) \, \cdot \, \mathrm{N}^{\langle P \rangle}\big(f_{\min}(A) = i, \, f_{\min}(B) = j\big) \\ &= \ \Big(\mathbb{E}\big[f_{\min}(A)\, f_{\min}(B)\big] \ + \ \mathbb{E}\big[f_{\min}(A \cup B)\big]\Big) \, \cdot \, e(P). \end{aligned}$$

Combining all these equations, we get

$$\frac{e(P') \, \cdot \, e(P' - x - y)}{e(P' - x) \, \cdot \, e(P' - y)} \ = \ \frac{\mathbb{E}\big[f_{\min}(A)\, f_{\min}(B)\big] \, + \, \mathbb{E}\big[f_{\min}(A \cup B)\big]}{\mathbb{E}\big[f_{\min}(A)\big]\, \mathbb{E}\big[f_{\min}(B)\big]}\,.$$

The result now follows from the upper bound in Theorem 1.1 applied to the poset P'. $\square$

6.4. **Stanley type inequalities for subsets.** As in §6.2, we start with the following result extending Corollary 1.7 (that we have not proved yet).

Lemma 6.4. *Let $P = (X, \prec)$ be a poset on $|X| = n \geq 3$ elements, let $a \in X$, and let $A \subseteq X - a$ be a nonempty subset. Then:*

$$\mathrm{N}_k\big(1 \in f(A), 2 \in f(A\!\uparrow)\big) \, \cdot \, \mathrm{N}_k(1, 2 \notin f(A)) \ \leq \ \mathrm{N}_k\big(1 \notin f(A), 2 \in f(A)\big)^2,$$

for every $3 \leq k \leq n$.

Proof. Let $\mathbf{x} \in \mathbb{R}^d$ be the characteristic vector of $A \subseteq Z_{\text{down}}$, and let $\mathbf{y} \in \mathbb{R}^d$ be the characteristic vector of $(X - A - a) \subseteq Z_{\text{down}}$. Let $\mathbf{M} := \mathbf{M}(P, a, k)$ be the matrix in Proposition 6.1. Then we have:

$$\mathbf{M}_{\mathbf{xx}} \ = \ \sum_{x \in A} \Big[\mathrm{N}_{k+1}\big(f(x) = 1, x \prec f^{-1}(2)\big) \, + \, \sum_{y \in A - x} \mathrm{N}_{k+1}\big(f(x) = 1, \, f(y) = 2\big)\Big].$$

Let $B := \min(P) \smallsetminus A$ be the set of minimal elements of P that are not contained in A. Note that the sum above is then equal to

$$\begin{aligned} \mathbf{M}_{\mathbf{xx}} \ &= \ \sum_{x \in A} \mathrm{N}_{k+1}\big(f(x) = 1, \, f^{-1}(2) \notin B\big) \ = \ \mathrm{N}_{k+1}\big(1 \in f(A), \, 2 \notin f(B)\big) \\ &= \ \mathrm{N}_{k+1}\big(1 \in f(A), \, 2 \in f(A\!\uparrow)\big). \end{aligned} \tag{6.7}$$

Here the second equality follows from the definition of B and of the upper closure $A\!\uparrow$.

By the same argument as in (6.7), we have:

$$\mathbf{M}_{\mathbf{yy}} \, = \, \mathrm{N}_{k+1}\big(1 \in f(B), f^{-1}(2) \notin A\big) \, = \, \mathrm{N}_{k+1}\big(1, 2 \notin f(A)\big). \tag{6.8}$$

Finally, note that

$$\mathbf{M}_{\mathbf{xy}} \, = \, \mathbf{M}_{\mathbf{yx}} \, = \, \sum_{y \in B} \sum_{x \in A} \mathrm{N}_{k+1}\big(f(y) = 1, \, f(x) = 2\big) \, = \, \mathrm{N}_{k+1}\big(1 \in f(B), 2 \in f(A)\big).$$

By the definition of B, we then get

$$\mathbf{M}_{\mathbf{xy}} \, = \, \mathrm{N}_{k+1}\big(1 \notin f(A), 2 \in f(A)\big). \tag{6.9}$$

Combining (6.7), (6.8), (6.9) into (Hyp), and substituting $k \leftarrow k-1$, we obtain the result. $\square$

Proof of the first part of Theorem 1.6. Let $P' := P + \{a\}$ be the parallel sum of the poset P and the single element $\{a\}$. Note that $\mathrm{N}_k^{\langle P' \rangle} = \mathrm{N}^{\langle P \rangle}$. Applying Lemma 6.4 to the poset P', we get

$$\mathrm{N}\big(1 \in f(A), 2 \in f(A\!\uparrow)\big) \cdot \mathrm{N}\big(1, 2 \notin f(A)\big) \, \leq \, \mathrm{N}\big(1 \notin f(A), 2 \in f(A)\big)^2. \tag{6.10}$$

On the other hand, we have:

$$\begin{aligned} \mathrm{N}\big(1 \in f(A), 2 \in f(A\!\uparrow)\big) \, &= \, \mathrm{N}\big(1 \in f(A)\big) \, - \, \mathrm{N}\big(1 \in f(A), 2 \notin f(A)\big) \\ &= \, \mathrm{N}\big(1 \in f(A)\big) \, - \, \mathrm{N}\big(1 \notin f(A), 2 \in f(A)\big). \end{aligned} \tag{6.11}$$

Here the first inequality is by $A \subset \min(P)$ and the definition of the upper closure $A\!\uparrow$. The second equality is obtained by swapping $1 \leftrightarrow 2$ in f.

Substituting (6.11) into (6.10) and dividing both sides of the inequality by $e(P)^2$, we get a probabilistic inequality:

$$\big[\mathbb{P}\big(1 \in f(A)\big) - \mathbb{P}\big(1 \notin f(A), 2 \in f(A)\big)\big] \cdot \mathbb{P}\big(1, 2 \notin f(A)\big) \, \leq \, \mathbb{P}\big(1 \notin f(A), 2 \in f(A)\big)^2. \tag{6.12}$$

Let $\alpha := \mathbb{P}\big(1 \notin f(A)\big)$, $\beta := \mathbb{P}\big(1, 2 \notin f(A)\big)$. Note that

$$\mathbb{P}\big(1 \in f(A)\big) \, = \, 1 - \alpha, \qquad \mathbb{P}\big(1 \notin f(A), 2 \in f(A)\big) \, = \, \alpha - \beta.$$

Substituting this into (6.12), we get

$$(1 - 2\alpha + \beta)\, \beta \, \leq \, (\alpha - \beta)^2,$$

which is equivalent to $\beta \leq \alpha^2$. This gives (1.12), as desired. $\square$

To finish the proof of Theorem 1.6 we need the following lemma.

Lemma 6.5. *Let $P = (X, \prec)$ be a poset on $|X| = n \geq 3$ elements, let $a \in X$, and let $A \subseteq X - a$ be a nonempty subset. Then:*

$$\mathrm{N}_k\big(1 \in f(A), 2 \in f(A\!\uparrow)\big) \cdot e_k(P) \, \leq \, \mathrm{N}_k\big(1 \in f(A)\big)^2,$$

for every $3 \leq k \leq n$.

Proof. Let $\mathbf{x} \in \mathbb{R}_+^d$ be the characteristic vector of $A \subseteq Z_{\text{down}}$, and let $\mathbf{y} \in \mathbb{R}_+^d$ be the characteristic vector of Z_{down}. Finally, let $\mathbf{M} := \mathbf{M}(P, a, k)$ be the matrix in Proposition 6.1. It then follows from the same argument as in (6.7) that

$$\mathbf{M}_{\mathbf{xx}} \, = \, \mathrm{N}_{k+1}\big(1 \in f(A), 2 \in f(A\!\uparrow)\big).$$

Note also that

$$\mathbf{M}_{\mathbf{yy}} \, = \, \sum_{v \in Z_{\text{down}}} \sum_{w \in Z_{\text{down}}} \mathrm{M}_{vw} \, =_{(6.3)} \, e_{k+1}(P).$$

Finally, note that

$$\mathbf{M}_{\mathbf{x}\mathbf{y}} \ = \ \sum_{x\in A}\sum_{y\in X} \mathrm{N}_{k+1}\big(f(x)=1,\, f(y)=2\big) \ = \ \mathrm{N}_{k+1}\big(1\in f(A)\big).$$

Combining these three equations with (Hyp), and substituting $k \leftarrow k-1$ implies the result. $\square$

Proof of the second part of Theorem 1.6. As before, let $P' := P + \{a\}$ be the parallel sum of the poset P and the single element $\{a\}$, and note that $\mathrm{N}_k^{\langle P'\rangle} = \mathrm{N}^{\langle P\rangle}$. Applying Lemma 6.5 to the poset P', we get

$$\mathrm{N}\big(1\in f(A), 2\in (A\!\uparrow)\big) \cdot e(P) \ \leq \ \mathrm{N}\big(1\in f(A)\big)^2.$$

The equality (6.11) gives

$$\Big[\mathrm{N}\big(1\in f(A)\big) - \mathrm{N}\big(1\notin f(A), 2\in f(A)\big)\Big] \cdot e(P) \ \leq \ \mathrm{N}\big(1\in f(A)\big)^2.$$

Dividing both sides of the inequality by $e(P)^2$, we get a probabilistic version:

$$\mathbb{P}\big(1\in f(A)\big) - \mathbb{P}\big(1\notin f(A), 2\in f(A)\big) \ \leq \ \mathbb{P}\big(1\in f(A)\big)^2.$$

This inequality is equivalent to (1.13), as desired. $\square$

6.5. **Multiple subsets.** Let $P=(X,\prec)$ be a poset on $|X|=n$ elements and fix $a\in X$. From this point on, let $A_1, A_2, A_3 \subset \min(P-a)$ be disjoint subsets of minimal elements. Denote by $\mathbf{x}_i \in \mathbb{R}^d$, $1\leq i\leq 3$, the characteristic vectors of $A_i \subseteq Z_{\text{down}}$. Let $\mathbf{M} := \mathbf{M}(P,a,k)$ be the matrix in Proposition 6.1.

Lemma 6.6. *For all $1\leq i,j\leq 3$, we have:*

$$\mathbf{M}_{\mathbf{x}_i\mathbf{x}_j} \ = \ \begin{cases} \mathrm{N}_{k+1}\big(1\in f(A_i),\, 2\in f(A_i\!\uparrow)\big) & \textit{if } \ i=j, \\ \mathrm{N}_{k+1}\big(1\in f(A_i),\, 2\in f(A_j)\big) & \textit{otherwise.}\end{cases} \tag{6.13}$$

Proof. Using (6.7) with $\mathbf{x}\leftarrow\mathbf{x}_i$ and $A\leftarrow A_i$ we have:

$$\mathbf{M}_{\mathbf{x}_i\mathbf{x}_i} \ = \ \mathrm{N}_{k+1}\big(1\in f(A_i), 2\in f(A_i\!\uparrow)\big).$$

Similarly, for $i\neq j$, we have:

$$\mathbf{M}_{\mathbf{x}_i\mathbf{x}_j} \ = \ \sum_{x\in A_i}\sum_{y\in A_j} \mathrm{N}_{k+1}\big(f(x)=1, f(y)=2\big) \ = \ \mathrm{N}_{k+1}\big(1\in f(A_i),\, 2\in f(A_j)\big),$$

which completes the proof. $\square$

Proof of Theorem 1.9. By (Hyp), we have:

$$\mathbf{M}_{\mathbf{x}_1\mathbf{x}_1}\,\mathbf{M}_{\mathbf{x}_2\mathbf{x}_2} \ \leq \ \mathbf{M}^2_{\mathbf{x}_1\mathbf{x}_2}\,.$$

Using (6.13) in this inequality, we get:

$$\mathrm{N}_{k+1}\big(1\in f(A_1), 2\in f(A_1\!\uparrow)\big)\cdot \mathrm{N}_{k+1}\big(1\in f(A_2), 2\in f(A_2\!\uparrow)\big) \ \leq \ \mathrm{N}_{k+1}\big(1\in f(A_1), 2\in f(A_2)\big)^2.$$

As before, let $P' := P+\{a\}$ be the parallel sum of the poset P and the single element $\{a\}$, and note that $\mathrm{N}_{k+1}^{\langle P'\rangle} = \mathrm{N}^{\langle P\rangle}$. This and the substitution $A_1\leftarrow A$, $A_2\leftarrow B$, gives

$$\mathrm{N}\big(1\in f(A), 2\in f(A\!\uparrow)\big)\cdot \mathrm{N}\big(1\in f(B), 2\in f(B\!\uparrow)\big) \ \leq \ \mathrm{N}\big(1\in f(A), 2\in f(B)\big)^2,$$

as desired. $\square$

Proof of Theorem 1.10. Substituting $\mathbf{x} \leftarrow \mathbf{x}_1$, $\mathbf{y} \leftarrow \mathbf{x}_2$, $\mathbf{z} \leftarrow \mathbf{x}_3$ and $A_1 \leftarrow A$, $A_2 \leftarrow B$, $A_3 \leftarrow C$, into Lemma 5.5 and Lemma 6.6, we get

$$\begin{aligned} &\mathrm{N}_{k+1}\big(1 \in f(C), 2 \in f(C\!\uparrow)\big) \cdot \mathrm{N}_{k+1}\big(1 \in f(A), 2 \in f(B)\big) \\ &\quad \leq\ 2\,\mathrm{N}_{k+1}\big(1 \in f(A), 2 \in f(C)\big) \cdot \mathrm{N}_{k+1}\big(1 \in f(B), 2 \in f(C)\big). \end{aligned}$$

As before, let $P' := P + \{a\}$ be the parallel sum of the poset P and the single element $\{a\}$, and repeat the substitution argument $\mathrm{N}_{k+1} \leftarrow \mathrm{N}$ as above. The first inequality (1.17) now follows by dividing both sides of the equation by $e(P)^2$.

The second inequality (1.18) follows from a similar argument applied to Lemma 5.6. We omit the details for brevity. □

7. Log-concavity implications

7.1. **Proof of Corollary 1.4.** Both inequalities in the corollary follow immediately from the following two general log-concavity results. The proofs are straightforward and included here for completeness.[4]

Lemma 7.1. *Let $\{p_1, \ldots, p_n\}$ be a log-concave sequence of real numbers $p_k \geq 0$. Let $s_k := p_k + p_{k+1} + \ldots + p_n$, for all $1 \leq k \leq n$. Then $\{s_1, \ldots, s_n\}$ is also log-concave.*

Proof. After expanding and simplifying the terms, the log-concavity $s_{k-1}s_{k+1} \leq s_k^2$ is equivalent to

$$\sum_{i=k}^{n-1} p_{k-1}\,p_{i+1} \ \leq\ \sum_{i=k}^{n} p_k\,p_i.$$

By the log-concavity of $\{p_i\}$, we have $p_{k-1}\,p_{i+1} \ \leq\ p_k\,p_i$ for every $i \geq k$. Thus the difference between the RHS and LHS of the inequality above is at least $p_k\,p_n \geq 0$. □

Lemma 7.2. *Let $\{p_1, \ldots, p_n\}$ be a log-concave sequence of real numbers $p_i \geq 0$. Then:*

$$p_1\,(p_2 + \ldots + p_n) \ \leq\ p_2\,(p_1 + \ldots + p_{n-1} + p_n).$$

Proof. By the log-concavity of $\{p_i\}$, we have $p_1\,p_i \ \leq\ p_2\,p_{i-1}$ for every $i \geq 3$. Thus, the difference between the RHS and LHS of the inequality in the lemma is at least $p_2\,p_n \geq 0$. □

7.2. **Proof of Proposition 3.7.** We prove the proposition by analogy with lemmas above, as a polynomial inequality.[5]

Lemma 7.3. *Let $\{p_1, \ldots, p_n\}$ be a log-concave sequence of real numbers $p_i \geq 0$. Then:*

$$\big(p_1 + p_2 + \ldots + p_n\big)\big(1^2 p_1 + 2^2 p_2 + \ldots + n^2 p_n\big) \ \leq\ 2\big(1 p_1 + 2 p_2 + \ldots + n\,p_n\big)^2.$$

Proof. Write the difference of the RHS and LHS as the sum $\alpha_2 + \alpha_3 + \ldots + \alpha_{2n}$ over terms $p_i p_j$ with a fixed sum $i + j$. We have:

$$\alpha_{k+1} \ = \ p_1 p_k \big(4k - 1 - k^2\big) \ + \ p_2 p_{k-1} \big(8(k-1) - 2^2 - (k-1)^2\big) \ + \ \ldots$$

Observe that the coefficients in α_{k+1} start negative and become positive. The total sum of the coefficients in α_{k+1} is equal to $\frac{1}{3}k(k+1)(k+2) - \frac{1}{6}k(k+1)(2k+1) > 0$. By the log-concavity, we have $p_1 p_k \leq p_2 p_{k-1} \leq \ldots$ We conclude that the positive terms in the sum α_{k+1} dominate the negative terms, which implies the result. □

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[4]For example, a continuous version of Lemma 7.1 is well-known in the Economics literature, see [BB05, Thm 1].

[5]Our original proof was more complicated. The proof below was suggested to us by Fëdor Petrov (Oct. 2022, personal communication).

8. Final remarks

8.1. The geometric approach via the Alexandrov–Fenchel inequality was introduced by Stanley [Sta81] and further explored in [KS84, vH21]. More recently, this approach was employed to study the equality conditions [MS22+, SvH20].

The vanishing conditions in Stanley's inequality (Theorem 1.3) have purely combinatorial proofs, see [SvH20, Lem. 15.2]. See also generalizations in [CPP21b, Thm 8.5], [CPP22b, Thm 1.12] and [MS22+, Thm 5.3]. The same holds for the uniqueness conditions [CPP22b, Thm 7.5].

It remains open whether Stanley's inequality has a combinatorial proof. Formally, it is not known whether the difference of the RLS and the LHS is in #P. The same holds for our correlation inequalities as the combinatorial atlas technology is inherently ineffective. This is in contrast with other poset inequalities, see e.g. [CPP22b]. We refer to the lengthy discussion of these problems in [Pak22, §6].

8.2. One can ask if the atlas technology we use largely as a black box in this paper is really necessary to derive our results? We certainly believe this to be the case, since there are so few other tools. For now, let us offer a word of caution to the reader accepting the challenge: while our inequalities can certainly appear intuitively obvious at least in special cases, this impression is largely deceptive in the generality of all finite posets.

For example, the proof of the upper bound of (1.1) given in Example 3.1 for the case of exactly two minimal elements, is elementary indeed. Thus one might be tempted to extend this proof to general posets. We hope you succeed, but keep in mind that the only proof we have is as a corollary of a difficult Theorem 6.3.

This "intuitively obvious" phenomenon is both quite old and completely understandable. For example, *Winkler's inequality* [Win82, Thm 1] certainly appears that way:

$$\mathbb{E}\big[f(x)\big] \ \leq\ \mathbb{E}\big[f(x)\,\big|\, f(x) > f(y)\big] \quad \text{for every incomparable } \ x,y \in X\,. \tag{8.1}$$

To this day, the only known proof of (8.1) is the original proof which uses the XYZ inequality.

8.3. The famous $\frac{1}{3}$–$\frac{2}{3}$ *Conjecture* states that for every finite poset $P = (X, \prec)$ there are elements $x, y \in X$, such that

$$\frac{1}{3} \ \leq\ \mathbb{P}\big[f(x) < f(y)\big] \ \leq\ \frac{2}{3}\,.$$

This conjecture was stated independently by Kislitsyn (1968) and Fredman (1875). and studied in a long series of papers.

As we mentioned in the introduction, a version of the conjecture with weaker constants was first proved in [KS84] using the extension of Stanley's inequality (1.5). See also [BFT95] for the currently best known bound.

8.4. Let $P = (X, \prec)$ be a finite poset. Fix distinct elements $x, y, z \in X$. For $k, \ell \geq 1$, denote

$$\mathcal{F}(k,\ell) \ :=\ \{f \in \mathcal{E}(P) \,:\, f(y) - f(x) = k,\ f(z) - f(y) = \ell\},$$

and let $\mathrm{F}(k,\ell) \,:=\, \big|\mathcal{F}(k,\ell)\big|$. The *cross-product conjecture* (CPC) by Brightwell–Felsner–Trotter [BFT95, Thm 3.2] states that

$$\mathrm{F}(k,\ell) \cdot \mathrm{F}(k+1,\ell+1) \ \leq\ \mathrm{F}(k+1,\ell) \cdot \mathrm{F}(k,\ell+1), \tag{CPC}$$

for every $k, \ell \geq 1$. The $k = \ell = 1$ case was proved in [BFT95, Thm 3.2], and the case of width two posets was proved in [CPP22a]. This remains one of the most challenging open problems in the area. As the authors lamented, "something more powerful seems to be needed" to prove the general form of CPC.

8.5. Let $P = (X, \prec)$ be a poset, and let $A, B \subset X$ be upper ideals. We denote by $e(A)$ the number of linear extensions of the poset $(A, \prec)$. *Fishburn's inequality* [Fis84] states:

$$\frac{|A \cup B|! \cdot |A \cap B|!}{|A|! \cdot |B|!} \ \le \ \frac{e(A \cup B) \cdot e(A \cap B)}{e(A) \cdot e(B)} \,. \tag{8.2}$$

Note that this is a generalization of the lower bound in (1.1), obtained by taking $A := X - x$ and $B := X - y$. In a joint work with Panova, we recently rederive this special case in a remark [CPP22b, §9.8]. Both the original proof in [Fis84, §2] and our proof use the FKG inequality.

8.6. In the context of §4.2, Fishburn's inequality for straight shapes was studied by Björner [Bjö11, §6], as the following correlation inequality:

$$|\operatorname{SYT}(\mu)| \cdot |\operatorname{SYT}(\nu)| \ \le \ |\operatorname{SYT}(\mu \vee \nu)| \cdot |\operatorname{SYT}(\mu \wedge \nu)| \,, \tag{8.3}$$

where $\vee$ and $\wedge$ refer to the union and intersection of the corresponding Young diagrams.

It was also pointed out it in [Pak22, §7.4], that (8.3) is also a corollary of the very general *Lam–Pylyavskyy inequality* [LP07, Thm 4.5] which applies to skew shapes:

$$|\operatorname{SYT}(\mu/\alpha)| \cdot |\operatorname{SYT}(\nu/\beta)| \ \le \ \left|\operatorname{SYT}\left(\mu \vee \nu \,/\, \alpha \vee \beta\right)\right| \cdot \left|\operatorname{SYT}\left(\mu \wedge \nu \,/\, \alpha \wedge \beta\right)\right|. \tag{8.4}$$

This result was reproved and further extended in [LPP07] by an algebraic argument. In a followup paper [CP22+], we show how (8.4) and its generalizations can be proved via the Ahlswede–Daykin inequality.

8.7. In the context of §4.4, recall that there is no analogue of the hook walk for skew Young diagrams, or in fact any direct combinatorial way to sample from $\operatorname{SYT}(\lambda/\mu)$, see e.g. [MPP18, §10.3]. It would be interesting to find a direct combinatorial proof (1.1) in this case.

8.8. In conditions of Theorem 1.1, when $x \in \min(P)$ and $y \in \max(P)$, the lower bound in (1.1) reverses direction:

$$\frac{e(P) \cdot e(P-x-y)}{e(P-x) \cdot e(P-y)} \ \le \ \frac{n}{n-1} \,, \tag{8.5}$$

see [CPP22b, §9.8]. It would be interesting to find a lower bound for the LHS of (8.5) similar to the RHS of (1.1). We conjecture that the lower bound $1/2$ holds in this case. For example, let $P = C_{n-2} + C_2$ be a parallel sum (disjoint union) of two chains, and let $C_2 = \{x, y \, : \, x \prec y\}$. Then the LHS of (8.5) is equal to $\frac{n}{2(n-1)}$.

8.9. The combinatorial atlas technology does prove the equality conditions in some cases, such as for Stanley's inequality (1.4), see [CP21, §1.18]. While this paper does not explore the equality conditions for any of our correlation inequalities, we believe this is an important direction to pursue. Finding equality conditions for both inequalities in Theorem 1.1 would be especially interesting, as would be the equality conditions for Corollary 1.7 and for Theorem 1.9.

8.10. Both Conjectures 1.5 and 3.8 are very speculative. It would be interesting to check them computationally for sufficiently large posets. One way to motivate Conjecture 3.8 is to note that there are sharp constraints on the distribution of $f(x)$ beyond log-concavity. Notably, the equality conditions in [SvH20, Thm 15.3], see also [CP21, Thm 1.39], do not allow a geometric distribution (cf. the discussion in §3.2).

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