POSITIVITY OF THE SYMMETRIC GROUP CHARACTERS IS AS HARD AS THE POLYNOMIAL TIME HIERARCHY

CHRISTIAN IKENMEYER[‡], IGOR PAK[⋆], AND GRETA PANOVA[†]

Abstract. We prove that deciding the vanishing of the character of the symmetric group is $\mathsf{C_{=}P}$-complete. We use this hardness result to prove that the the square of the character is not contained in $\#\mathsf{P}$, unless the polynomial hierarchy collapses to the second level. This rules out the existence of any (unsigned) combinatorial description for the square of the characters. As a byproduct of our proof we conclude that deciding positivity of the character is PP-complete under many-one reductions, and hence PH-hard under Turing-reductions.

1. Introduction

1.1. **Motivation.** Consider the following two classical identities from the representation theory of the symmetric group:

$$n! \;=\; \sum_{\lambda\vdash n} \big(\chi^\lambda(1)\big)^2\,, \quad \text{and} \tag{1.1.1}$$

$$n! \;=\; \sum_{\pi\in\mathfrak{S}_n} \big(\chi^\lambda(\pi)\big)^2 \quad \text{for all} \;\; \lambda\vdash n. \tag{1.1.2}$$

Here χ^λ is the irreducible character of the symmetric group $\mathfrak{S}_n$ of the representation indexed by λ, and $\chi^\lambda(\pi)\in\mathbb{Z}$ is its evaluation. Both identities arise in a similar manner, as squared norms of row and column vectors in the character table of $\mathfrak{S}_n$, see §5.4 for the context and generalizations.

Equalities such as these, are an invitation for combinatorialists to search for natural bijections between the sets of combinatorial objects counting both sides. In both cases, the LHS is the set $\mathfrak{S}_n$ of permutations of n symbols. For (1.1.1), the RHS is the set of pairs of *standard Young tableaux* of the same shape with n boxes. The bijection between the set of permutations and the set of pairs of Young tableaux is the celebrated *Robinson–Schensted correspondence*, which is fundamental in Algebraic Combinatorics, see [Sag01, Ch. 3] and [Sta99, §§7.11-14]. This correspondence has numerous generalizations and is studied widely across many areas of mathematics and applications, see e.g. [And76, BS17, DNV22, KP21, O'Con03, OW03].

Similarly, for (1.1.2), one would want to give a bijection between $\mathfrak{S}_n$ and a set of $n!$ many combinatorial objects that are partitioned naturally into subsets of sizes $\big(\chi^\lambda(\pi)\big)^2$. In this paper we prove that this approach would fail for the fundamental reason that the RHS of (1.1.2) does not admit such an interpretation. As the following theorem implies, it is unlikely that there exist "sets of $\big(\chi^\lambda(\pi)\big)^2$ many combinatorial objects" (see more on this below).

1.1.3. **Theorem.** *Let $\chi^2 : (\lambda,\pi)\mapsto \big(\chi^\lambda(\pi)\big)^2$, where $\lambda\vdash n$ and $\pi\in\mathfrak{S}_n$. If the function χ^2 is contained in the complexity class $\#\mathsf{P}$, then $\mathsf{coNP}=\mathsf{C_{=}P}$. Consequently[1], if $\chi^2\in\#\mathsf{P}$, then the polynomial hierarchy collapses to the second level:* $\mathsf{PH}=\Sigma_2^{\mathsf{p}}$.

Date: July 24, 2022.

[‡]Department of Computer Science, University of Liverpool, UK. Email: christian.ikenmeyer@liverpool.ac.uk.
[⋆]Department of Mathematics, UCLA, Los Angeles, CA 90095. Email: pak@math.ucla.edu.
[†]Department of Mathematics, USC, Los Angeles, CA 90089. Email: gpanova@usc.edu.

[1]Indeed, Tarui [Tar91] (see also [Gre93]), proves that $\mathsf{PH}\subseteq\mathsf{NP}^{\mathsf{C_{=}P}}$. Therefore, if $\mathsf{coNP}=\mathsf{C_{=}P}$, then $\mathsf{PH}\subseteq\mathsf{NP}^{\mathsf{C_{=}P}}=\mathsf{NP}^{\mathsf{coNP}}=\Sigma_2^{\mathsf{p}}$. Hence $\Sigma_2^{\mathsf{p}}=\mathsf{PH}$, as claimed in the second part of the theorem.

The assumption $\mathsf{PH} \neq \Sigma_2^{\mathrm{p}}$ in the theorem is a widely believed standard complexity theoretic assumption, which formally implies $\mathsf{P} \neq \mathsf{NP}$. From a combinatorial perspective, Theorem 1.1.3 is much stronger than just saying that the character squares are hard to compute. The theorem rules out that there exists any positive combinatorial interpretation for the character squares, even if "positive combinatorial interpretation" is interpreted in the widest possible sense. Large parts of Enumerative and Algebraic Combinatorics deal with finding explicit (positive) combinatorial interpretations of quantities, while impossibility results such as Theorem 1.1.3 are extremely rare, see §1.3.

Note also how close the upper and lower bounds are. Recall that the character square is in $\mathsf{GapP} = \#\mathsf{P} - \#\mathsf{P}$, is always nonnegative, and yet is not in $\#\mathsf{P}$ by the theorem unless the polynomial hierarchy collapses. Our proof goes via showing that deciding the vanishing of $\chi^\lambda(\pi)$ is $\mathsf{C_{=}P}$-complete:

1.1.4. **Theorem.** *The language* $\{(\lambda, \pi) \mid \chi^\lambda(\pi) = 0\}$ *is* $\mathsf{C_{=}P}$*-complete under many-one reductions.*

Theorem 1.1.3 then follows from Theorem 1.1.4 and Proposition 3.1.1. The result in the title is a direct consequence of the reduction in the proof of Theorem 1.1.4.

1.1.5. **Theorem.** *The language* $L = \{(\lambda, \pi) \mid \chi^\lambda(\pi) \geq 0\}$ *is* PP*-complete under many-one reductions. Consequently,* L *is* PH*-hard under Turing-reductions.*

Indeed, since $\mathsf{PH} \subseteq \mathsf{P}^{\mathsf{PP}}$ by [Toda89, Toda91], it immediately follows that L is PH-hard under Turing reductions:

$$\mathsf{PH} \ \subseteq\ \mathsf{P}^{\mathsf{PP}} \ \overset{\text{Thm. 1.1.5}}{\subseteq}\ \mathsf{P}^{\mathsf{P}^L} \ =\ \mathsf{P}^L .$$

This derives the second part of the theorem from the first part. As a side result we prove that computing the character is strongly GapP-complete, see Theorem 5.3.1.

1.2. **#P, GapP and combinatorial interpretations.** For a nondeterministic Turing machine M and a word $w \in \{0,1\}^*$ let $\mathrm{acc}_M(w)$ denote the number of accepting computation paths of M on input w. The complexity class $\#\mathsf{P}$ is defined as the class of those functions $f : \{0,1\}^* \to \mathbb{N}$ for which a nondeterministic Turing machine M exists with $\forall w \in \{0,1\}^* : f(w) = \mathrm{acc}_M(w)$.

For example, the famous *Littlewood–Richardson rule* states that the Littlewood–Richardson (LR) coefficient $c^\nu_{\lambda,\mu}$ equals the number of LR–tableaux of skew shape ν/λ and content μ, hence the map $(\lambda, \mu, \nu) \mapsto c^\nu_{\lambda,\mu}$ is in $\#\mathsf{P}$. Here we already see an interesting issue: This argument works if the partitions are given as their Young diagrams, i.e., the partitions are given in unary, because otherwise writing down a single LR-tableau would require exponential space. The LR-coefficient is in $\#\mathsf{P}$ for binary inputs, see e.g. [Nar06], which follows from their interpretation as the number of integer points in a certain polytope, and not the LR-tableaux. From the perspective of combinatorics, a "combinatorial interpretation" of the Littlewood–Richardson coefficient already follows from the former result. Theorem 1.1.3 works in unary and hence also in binary.

Let us also remark that $\#\mathsf{P}$ is the class of positive combinatorial interpretations if "positive combinatorial interpretation" is used in a very broad and all-encompassing sense. For example, all polynomial time computable nonnegative functions are in $\#\mathsf{P}$, for example the absolute value of the determinant of a binary matrix. Note that this means that a proof of the non-membership in $\#\mathsf{P}$ such as Theorem 1.1.3 is a very strong impossibility result, as it rules out also very complicated tableau constructions, including, e.g., those in [Bla17, TY08].

The complexity class $\mathsf{GapP} := \#\mathsf{P} - \#\mathsf{P}$ is defined as the class of differences of two $\#\mathsf{P}$ functions, i.e., $\mathsf{GapP} = \{f - g \mid f, g \in \#\mathsf{P}\}$. Let $\mathsf{GapP}_{\geq 0}$ denote the subset of nonnegative functions in GapP. Many interesting functions in algebraic combinatorics are known to be in $\mathsf{GapP}_{\geq 0}$, but conjectured to be in $\#\mathsf{P}$. See [IP22, Pak19] for many such functions arising from combinatorial inequalities. The most famous $\mathsf{GapP}_{\geq 0}$ functions are the subject of of Stanley's survey [Sta00] on positivity problems in algebraic combinatorics, where he asked for positive combinatorial interpretations of the plethysm, Kronecker, and Schubert coefficients. All these problems remain unresolved (cf. §5.1).

Closer to the subject of this paper, Stanley considered rows and column sums of the character table of $\mathfrak{S}_n$:

$$a_\lambda := \sum_{\mu \vdash n} \chi^\lambda(\mu) \qquad \text{and} \qquad b_\lambda := \sum_{\mu \vdash n} \chi^\mu(\lambda)\,, \tag{1.2.1}$$

respectively, see Problem 12 in [Sta00]. Here $\chi^\lambda(\mu)$ denotes the character value on permutations of cycle type μ. Viewed as a functions with unary input, it is easy to see that a_λ and b_λ are in $\mathsf{GapP}_{\geq 0}$. Stanley notes that $b_\lambda = \big|\{\omega \in \mathfrak{S}_n \mid \omega^2 = \sigma\}\big|$, where σ has cycle type λ, which implies that b_λ is in $\#\mathsf{P}$. Stanley asked for a positive combinatorial interpretation of a_λ, which remains an open problem (cf. §5.1). Theorem 1.1.3 could be seen as a critical reminder that there is the possibility that the desired combinatorial interpretations might not exist (cf. §5.6).

1.3. **Related work.** The amount of work on characters of the symmetric groups is much too large to be reviewed here, but let us note that they prominently appear in other fields, see e.g. [Dia88, Pau95, Ste94], and have remarkable applications, see e.g. [EFP11, MRS08]. On the other hand, the asymptotic proportion of zeros in the character table remains open, see complementary discussions of the same data in [Mil19, §1.2] and [PPV16, §8.5].

Hepler [Hep94] proved that the computation of $\chi^\lambda(\pi)$ is $\#\mathsf{P}$-hard under Turing reductions. He does not study the vanishing problem of $\chi^\lambda(\pi)$. The vanishing of the character $\chi^\lambda(\mu)$ was proved to be NP-hard in [PP17]. It is noteworthy that the result in [PP17] only holds for the problem where the input (λ, μ) is encoded in binary, i.e., instead of π the second parameter is just the cycle type μ in binary. Our results do not have such a restriction.

The relativizing closure properties of $\#\mathsf{P}$ have been characterized in [HVW95], which can be generalized to prove non-containment in $\#\mathsf{P}$ w.r.t. an oracle in several settings, see [IP22].

In the combinatorics literature, the notion of a "positive combinatorial interpretation" is used informally; these are also called *manifestly positive* combinatorial formulas, rules, expressions, etc. This is to emphasize the importance of positivity, as opposed to *signed combinatorial formulas*, which typically refers to formulas in (subsets of) GapP. A complexity theoretic approach in this setting was introduced in [Wilf82] (see also [Pak18]).

For characters $\chi^\lambda(\pi)$, the GapP formula is famously given by the *Murnaghan–Nakayama rule* as the difference is the number of certain *rim hook tableaux*, see e.g. [Sag01, §4.10] and [Sta99, §7.17]. In this context, [Sta84, Cor. 7.5] gave a simple sufficient condition for the vanishing $\chi^\lambda(\mu) = 0$.

For Kronecker coefficients $g(\lambda, \mu, \nu)$, the GapP formula is given in in [BI08] (see also [CDW12, PP17]). For GapP formulas of plethysm and Schubert coefficients, see [FI20] and [PS09, Prop. 17.3], respectively. In the context of *Geometric Complexity Theory* (GCT), the importance of being in $\#\mathsf{P}$ of plethysm and Kronecker coefficients was discussed in [Mul09]. Kahle and Michałek [KM18] prove that plethysm coefficients are not counting integer points in polytopes; this is a restricted notion compared to $\#\mathsf{P}$ of interest both in Algebraic Combinatorics and GCT.

Stanton and White [SW85] gave a generalization of the Robinson–Schensted correspondence for rim hook tableaux. This was used by White in [Whi83, Whi85] to obtain combinatorial proofs of two character identities: first, of a generalization of (1.1.1) given in (5.4.1), and then of (1.1.2), but both proofs use an explicit involution to cancel the signs.

Finally, the complexity classes that we study in this paper are all standard and have been studied in numerous papers. In particular, it is known that $\mathsf{C_{=}P} = \mathsf{coNQP}$ [FGHP99].

2. Preliminaries

2.1. **Notation.** Let $\{0,1\}^*$ denote the set of finite length sequences of zeros and ones. A subset $L \subseteq \{0,1\}^*$ is called a *language*. We write $\overline{L} := \{0,1\}^* \setminus L$ to denote the complement of L. For a set S let 2^S be its powerset, i.e., the set of all subsets of S. We write $\binom{S}{k}$ for the set of cardinality k subsets of S.

We use $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $[n] = \{1, \ldots, n\}$. We denote by $\mathbb{Z}_k = \{1, \ldots, k\}$ the set of integers modulo k. Let $\mathfrak{S}_n$ denote the group of permutations of $[n]$.

A *weak composition* of n is sequence of nonnegative integers whose entries sum up to n, a *strong composition* of n is a sequence of positive integers whose entries sum up to n. An *integer partition* λ of n, denoted $\lambda \vdash n$, is a sequence of weakly decreasing nonnegative integers $(\lambda_1, \lambda_2, \ldots)$ which sum up to n. We write $|\lambda| = \sum_i \lambda_i$. We call $\ell(\lambda) = \max\{i \mid \lambda_i > 0\}$ the *length* of λ.

We treat compositions and partitions as vectors with componentwise addition and with the simultaneous rescaling of all components. We write $\mathrm{sort}(\mathbf{a})$ for the tuple that has the same entries as $\mathbf{a}$, but they are permuted so that they appear in weakly decreasing order. We denote by a^b the sequence $(a, a, \ldots, a)$ with a appearing b times. We write $\mathbf{a} = (a_1, \ldots, a_\ell)$ and $\mathbf{b} = (b_1, \ldots, b_\ell)$ for compositions and $|\mathbf{a}| = a_1 + \ldots + a_\ell$ for their sum.

2.2. **Representation Theory.** Let $\chi^\lambda \in \mathbb{C}[\mathfrak{S}_n]$ be the complex irreducible character of $\mathfrak{S}_n$ corresponding to partition $\lambda \vdash n$, i.e., for $\pi \in \mathfrak{S}_n$ we have that $\chi^\lambda(\pi)$ equals the trace of the representation matrix corresponding to π in the irreducible $\mathfrak{S}_n$-representation (the so-called *Specht module*) of type λ. From this definition it immediately follows that $\chi^\lambda(\pi) = \chi^\lambda(\sigma)$ if π and σ are permutations that have the same cycle type μ, and we use this fact to define $\chi^\lambda(\mu)$ for a partition μ.

For a composition $\mathbf{a}$ of n, consider the Young subgroup $\mathfrak{S}_{\mathbf{a}} := \mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \ldots$ of $\mathfrak{S}_n$, where $\mathfrak{S}_{a_1}$ permutes only $\{1, \ldots, a_1\}$, S_{a_2} permutes only $\{a_1 + 1, \ldots, a_1 + a_2\}$, etc. The induced trivial representation $\mathrm{ind}_{\mathfrak{S}_{\mathbf{a}}}^{\mathfrak{S}_n} 1$ can be defined as the action of $\mathfrak{S}_n$ on the left cosets of $\mathfrak{S}_n/\mathfrak{S}_{\mathbf{a}}$, see [Sta99, §7.18]. This is equivalent to the action of $\mathfrak{S}_n$ on words with a_1 many 1s, a_2 many 2s etc by permuting their positions. Denote by $\phi^{\mathbf{a}}$ the character of this representation, then $\phi^{\mathbf{a}}(\pi) = \#\{u \mid u\pi = u\}$, the number of words fixed by π. A word u is fixed by π if and only if $u_i = u_j$ for all i, j in the same cycle of π. Thus the number of fixed words is equal to the number of ways we can label the cycles of π with $1, 2, \ldots$, so that the total number of elements in the cycles labeled by i is equal to a_i.

The *Frobenius character formula*, see e.g. [JK81, Eq. 2.3.8] (equivalent to the *Jacobi–Trudi identity*, see e.g. [Sta99, §7.16 and §7.18]), gives

$$\chi^\lambda \, = \sum_{\sigma \in \mathfrak{S}_{\ell(\lambda)}} \mathrm{sign}(\sigma)\, \phi^{\lambda + \sigma - \mathrm{id}}\,. \tag{2.2.1}$$

Here $\mathrm{id} = (1, 2, \ldots, \ell) \in \mathfrak{S}_\ell$ is the identity permutation, and $(\lambda + \sigma - \mathrm{id})$ denotes the composition $(\lambda_1 + \sigma_1 - 1, \lambda_2 + \sigma_2 - 2, \ldots, \lambda_\ell + \sigma_\ell - \ell)$. Also, in (2.2.1), for a composition $\mathbf{a}$ in the summation we let $\phi^{\mathbf{a}} := 0$ if $a_i < 0$ for some i, $\phi^{(\mathbf{a},0,\mathbf{b})} := \phi^{(\mathbf{a},\mathbf{b})}$, and $\phi^{(0)} := 1$.

3. Computational Complexity

3.1. **$\mathsf{C_=P}$ and the Collapse of the Polynomial Hierarchy.** We will use well-known complexity classes with oracle access to a language in the standard way, see e.g. [Pap94]. As it is common, the oracle language is written in the exponent. For a function $f : \{0,1\}^* \to \mathbb{Z}$ and an integer comparison operator $\sim$ we define the language $[f \sim 0] := \{w \in \{0,1\}^* \mid f(w) \sim 0\}$.

For a class Z of functions $\{0,1\}^* \to \mathbb{Z}$ and an integer comparison operator $\sim$ we define the decision class $[Z \sim 0] \subseteq 2^{\{0,1\}^*}$ via: $L \in [Z \sim 0]$ if and only if there exists $f \in Z$ with the property that for all $w \in \{0,1\}^*$ we have $w \in L$ if and only if $f(w) \sim 0$. Using this notation, we recall that $\mathsf{NP} = [\#\mathsf{P} > 0]$, $\mathsf{coNP} = [\#\mathsf{P} = 0]$, $\mathsf{C_=P} = [\mathsf{GapP} = 0]$, and $\mathsf{coC_=P} = [\mathsf{GapP} \neq 0]$. In particular, $\mathsf{coNP} \subseteq \mathsf{C_=P}$.

Recall that $\Sigma_0^{\mathrm{p}} = \mathsf{P}$, $\Sigma_{i+1}^{\mathrm{p}} = \mathsf{NP}^{\Sigma_i^{\mathrm{p}}}$, and that $\mathsf{PH} = \bigcup_{i \in \mathbb{N}} \Sigma_i^{\mathrm{p}}$. Moreover, for a class $A \subseteq 2^{\{0,1\}^*}$, recall that the complement class $\mathsf{co}A$ is defined via $L \in \mathsf{co}A$ if and only if $\overline{L} \in A$. For a language L we write $\langle L \rangle$ to be the class of all languages that are many-one reducible to L, for example $\mathsf{NP} =$

$\langle 3\text{SAT}\rangle$, where 3SAT is the language of all satisfiable Boolean formulas in 3CNF. A language $L \subseteq \{0,1\}^*$ is called $\mathsf{C_=P}$-hard under many-one reductions if $\mathsf{C_=P} \subseteq \langle L\rangle$. Our main application is the case where $L = [f = 0]$, where f is a function in GapP.

3.1.1. **Proposition.** *Given a function $f : \{0,1\}^* \to \mathbb{Z}$ with the property that $[f = 0]$ is $\mathsf{C_=P}$-hard under many-one reductions. Fix a function q such that $q(0) = 0$ and $q(x) > 0$ for all $x > 0$, for example $q(x) = x^2$ or $q(x) = |x|$. If $q(f) \in \#\mathsf{P}$, then $\mathsf{coNP} = \mathsf{C_=P}$ (and in particular $\mathsf{PH} = \Sigma_2^{\mathrm{p}}$).*

Proof. Note that $\mathsf{coNP} \subseteq \mathsf{C_=P}$ by definition. For the other direction, observe that

$$\mathsf{C_=P} \subseteq \langle [f = 0]\rangle = \mathsf{co}\langle [f \neq 0]\rangle = \mathsf{co}\langle [q(f) \neq 0]\rangle = \mathsf{co}\langle [q(f) > 0]\rangle \overset{q(f)\in\#\mathsf{P}}{\subseteq} \mathsf{coNP}. \qquad \square$$

Since $\mathsf{PH} \subseteq \mathsf{NP}^{\mathsf{C_=P}}$ (see ([Tar91], and also [Gre93]), we have that if $\mathsf{coNP} = \mathsf{C_=P}$, then $\mathsf{PH} \subseteq \mathsf{NP}^{\mathsf{C_=P}} = \mathsf{NP}^{\mathsf{coNP}} = \Sigma_2^{\mathrm{p}}$, and hence $\Sigma_2^{\mathrm{p}} = \mathsf{PH}$. As an aside, we remark that if $[f = 0]$ is $\mathsf{C_=P}$-hard *under Turing-reductions* only, then $q(f) \in \#\mathsf{P}$ also implies $\mathsf{PH} = \Sigma_2^{\mathrm{p}}$ via

$$\mathsf{PH} \;\subseteq\; \mathsf{NP}^{\mathsf{C_=P}} \;\subseteq\; \mathsf{NP}^{\mathsf{P}^{[f=0]}} \;=\; \mathsf{NP}^{[f=0]} \;=\; \mathsf{NP}^{[q(f)\neq 0]} \;\overset{q(f)\in\#\mathsf{P}}{\subseteq}\; \mathsf{NP}^{\mathsf{coNP}} \;=\; \Sigma_2^{\mathrm{p}}.$$

3.2. **3D- and 4D-matchings.** Recall the following standard counting problems, see [GJ79].

Problem #CircuitSAT:
- Input: A Boolean circuit C with n inputs.
- Output: The number of $w \in \{0,1\}^n$ with $C(w) = \text{true}$.

Problem #3DM:
- Input: A subset $E \subseteq \mathbb{Z}_k^3$.
- Output: The number of $M \in \binom{E}{k}$ such that $\forall \{(x,y,z),(x',y',z')\} \in \binom{M}{2}$ we have: $x \neq x', y \neq y'$ and $z \neq z'$.

Problem #4DM:
- Input: A subset $E \subseteq \mathbb{Z}_k^4$.
- Output: The number of $M \in \binom{E}{k}$ such that $\forall \{(w,x,y,z),(w',x',y',z')\} \in \binom{M}{2}$ we have: $w \neq w', x \neq x', y \neq y'$ and $z \neq z'$.

We will use a parsimonious polynomial-time reduction R from #CircuitSAT to #3DM, defined as the composition of three known parsimonious polynomial time reductions. First, take the classical *Tseytin transformation* (see e.g. Example 8.3 in [Pap94, page 163]), which is a parsimonious polynomial time reduction from #CircuitSAT to #3SAT. Next, take Schaefer's parsimonious reduction [Sch78] from #3SAT to #1-in-3SAT: replace $x \vee y \vee z$ by one-in-three$(\neg x, u_1, u_2)$ $\wedge$ one-in-three(y, u_2, u_3) $\wedge$ one-in-three$(\neg z, u_3, u_4)$. Finally, take Young's parsimonious reduction from #3DM to #1-in-3SAT, defined via a promise problem called 1+3DM, see [You20].

3.3. **Ordered set partitions.** Let $\mathbf{a} = (a_1, \ldots, a_m)$ be a positive integer sequence and $\mathbf{b} = (b_1, \ldots, b_\ell)$ be a nonnegative integer sequence, both with the same total sum: $|\mathbf{a}| = |\mathbf{b}|$. An *ordered set partition with item sizes* $\mathbf{a}$ *and bin sizes* $\mathbf{b}$ is a tuple $\overrightarrow{K} = (K_1, \ldots, K_\ell)$ of pairwise disjoint subsets $K_1, \ldots, K_\ell \subseteq [m]$, such that

$$\bigcup_{i=1}^{\ell} K_i \;=\; [m] \quad \text{and} \quad \sum_{j \in K_i} a_j \;=\; b_i \quad \text{for all } 1 \leq i \leq \ell. \tag{3.3.1}$$

We use $\mathcal{P}(\mathbf{a},\mathbf{b})$ to denote the set of ordered set partitions with with item sizes $\mathbf{a}$ and bin sizes $\mathbf{b}$, and let $\mathrm{P}(\mathbf{a},\mathbf{b}) = \big|\mathcal{P}(\mathbf{a},\mathbf{b})\big|$. Here we will assume that $a_i > 0$ for all i and $P(\mathbf{a},\mathbf{b}) = 0$ if $b_i < 0$ for some i. All set partitions considered here will be ordered.

Problem #SetPartition:

- Input: $(\mathbf{a},\mathbf{b}) \in \mathbb{N}^\ell \times \mathbb{N}^m$.
- Output: The number of $\overrightarrow{K}$ that satisfy (3.3.1).

In other words, $\#\textsc{SetPartition}(\mathbf{a},\mathbf{b}) = \mathrm{P}(\mathbf{a},\mathbf{b})$.

4. Main result

In this section we prove Theorem 1.1.4 and Theorem 1.1.5. Combined with Proposition 3.1.1, Theorem 1.1.4 immediately implies Theorem 1.1.3.

4.1. **Characters and set partitions.** We start by translating our problem from the language of characters of $\mathfrak{S}_n$ into the language of ordered set partitions.

4.1.1. **Lemma.** *The characters of the induced representation ϕ^ν evaluated at a conjugacy class of type α are equal to the number of ordered set partitions of α into sets of sizes ν. That is,*

$$\phi^\nu(\alpha) = \mathrm{P}(\alpha,\nu).$$

Proof. As explained in §2.2, the evaluation $\phi^\nu(\alpha)$ is equal to the number of words u with ν_i letters i for $i = 1,\ldots,\ell(\nu)$, which are fixed under permuting the positions of their entries by a permutation π of cycle type $\alpha = (\alpha_1,\ldots,\alpha_m)$. Thus, the positions (elements of π) in the same cycle have the same letter. Let the cycles of π be $c_1,\ldots,c_m$ of lengths $\alpha_1,\ldots,\alpha_m$ respectively. Let $K_i = \{j : u|_{c_j} = (i,\ldots,i)\}$ be the set of cycles on which u has value i. Then $(K_1,\ldots,K_{\ell(\nu)})$ is an ordered set partition with item sizes $\alpha_1,\alpha_2,\ldots$ and bin sizes $\nu_1,\nu_2,\ldots$ Conversely, such a set partition determines the word u uniquely, and so $\mathrm{P}(\alpha,\nu) = \phi^\nu(\alpha)$. □

4.1.2. **Proposition.** *Let $\lambda \vdash n$ with $\ell(\lambda) \le \ell$, and let α be a composition of n. Then*

$$\chi^\lambda(\alpha) = \sum_{\sigma\in\mathfrak{S}_{\ell(\lambda)}} \mathrm{sign}(\sigma)\ \mathrm{P}(\alpha,\lambda+\sigma-\mathrm{id}).$$

Proof. This follows directly from equation (2.2.1) and Lemma 4.1.1. □

4.1.3. **Lemma.** *Let $\mathbf{a}$ and $\mathbf{b}$ be two positive sequences with equal sums, and let $\mathbf{b}$ have ℓ parts. Let $p = \ell+1$, $\lambda = \mathrm{sort}(p\mathbf{b})$ and $\alpha = p\mathbf{a} + e_1 - e_2$, where $e_i = (0,\ldots,0,1,0,\ldots)$ is the i'th standard basis vector. Then*

$$\chi^\lambda(\alpha) = \sum_{i=1}^{\ell} \mathrm{P}\big(\overline{\mathbf{a}},\mathbf{b}-(a_1+a_2)e_i\big) - \sum_{i=1}^{\ell-1} \mathrm{P}\big(\overline{\mathbf{a}},\mathbf{b}-a_1e_i-a_2e_{i+1}\big). \tag{4.1.4}$$

Proof. Without loss of generality, assume $b_1 \ge b_2 \ge \ldots$ We apply Proposition 4.1.2 with the given partitions. Consider a set partition of $\alpha = (pa_1+1, pa_2-1, pa_3,\ldots)$ into bins of sizes $pb_i+\sigma_i-i$ for $i = 1,\ldots,\ell$. Since $p|\alpha_i$ for $i \ne 1,2$, we must have that at most two of the sum sets are not divisible by p, and so $\sigma_j \equiv j \pmod p$ for all but possibly two values of j corresponding to the bins containing α_1 and α_2.

We have two possibilities. In the first case, both α_1,α_2 are in the same set (bin), of size $\lambda_i+\sigma_i-i$ for some i. Since $\alpha_1+\alpha_2 = p(a_1+a_2)$ and $\alpha_i = pa_i$ for all other is, the bin size must be divisible by p. Thus $0 \equiv \lambda_i+\sigma_i-i \equiv pb_i+\sigma_i-i \pmod p$ for all i and so $\sigma = \mathrm{id}$. Choosing in which set the $\alpha_1+\alpha_2$ go gives us the left big summation in (4.1.4).

In the second case, α_1,α_2 are in two different sets (bins), say t and r, whose sums must then be $\equiv 1,-1 \pmod p$ respectively. Since all other item sizes are divisible by p, we must have $\lambda_i+\sigma_i-i = pb_i+\sigma_i-i \equiv 0 \pmod p$ for $i \ne r,t$. Thus $\sigma_i = i$ for $i \ne t,r$ and we must have $\sigma_t = r$ and $\sigma_r = t$. Then $\lambda_t+r-t \equiv +1 \pmod p$ and $\lambda_r+t-r \equiv -1 \pmod p$. Since $1 \le r,t \le p-1$, we must have $r = t+1$, and we arrive in the other big summation, with α_1 in set t and α_2 in set $t+1$. This completes the proof. □

4.1.5. **Proposition.** *Let* $\mathbf{c}$ *and* $\mathbf{d}$ *be two sequences of nonnegative integers, such that* $|\mathbf{c}| = |\mathbf{d}| + 6$. *Then there are partitions* λ *and* α *of size* $O(\ell|\mathbf{c}|)$ *determined in linear time, such that*

$$\chi^\lambda(\alpha) \, = \, \mathrm{P}\big(\mathbf{c}, \overline{\mathbf{d}}\big) \, - \, \mathrm{P}\big(\mathbf{c}, \overline{\mathbf{d}'}\big),$$

where $\overline{\mathbf{d}} := (2, 4, d_1, d_2, \ldots)$ *and* $\overline{\mathbf{d}'} := (1, 5, d_1, d_2, \ldots)$.

Proof. We will use Lemma 4.1.3 with the following construction. Set $m := \max\{c_1, \ldots, d_1, \ldots\} + 4$. Let $\mathbf{a} := (2, m, m-3, c_1, c_2, \ldots)$ and $\mathbf{b} := (m+4, m+1, d_1, d_2, \ldots)$. Now construct λ and α as in Lemma 4.1.3. Note that $b_i \geq a_1 + a_2 = m+2$ only for $i = 1$, and $b_{i+1} \geq m = a_2$ only for $i = 1$, so the only nonzero terms in equation (4.1.4) are the summands for $i = 1$.

We thus obtain

$$\chi^\lambda(\alpha) \, = \, \mathrm{P}\big((m-3, c_1, \ldots), (2, m+1, d_1, \ldots)\big) \, - \, \mathrm{P}\big((m-3, c_1, \ldots), (m+2, 1, d_1, \ldots)\big).$$

Since $m - 3 > d_i$ for all i, the item of size $m-3$ can only go into the bins of sizes $m+1$ and $m+2$, respectively. Therefore, we have:

$$\chi^\lambda(\alpha) \, = \, \mathrm{P}\big((c_1, \ldots), (2, 4, d_1, \ldots)\big) \, - \, \mathrm{P}\big((c_1, \ldots), (5, 1, d_1, \ldots)\big),$$

and the proof is complete. □

4.2. **The join of two 3D-matchings.** Let $\mathbb{Z}_k := \{1, \ldots, k\}$ be the set $[k]$ with addition modulo k. In particular, we have $\mathbb{Z}_k \subseteq \mathbb{Z}_u$ as sets for $u \geq k$. We write $+_k$ or $+_u$ depending on whether we use addition modulo k or u.

Let $E \subseteq \mathbb{Z}_k^3$. For $u \geq k$ define the *padding* $E_u \subseteq \mathbb{Z}_u^3$ via $E_u := E \cup \{(x, x, x) \mid x > k\}$. Clearly $\#3\mathrm{DM}(E) = \#3\mathrm{DM}(E_u)$ for every $u \geq k$.

Given two subsets $E \subseteq \mathbb{Z}_k^3$ and $E' \subseteq \mathbb{Z}_{k'}^3$, let $u := 1 + \max\{k, k'\}$. We define the $\mathsf{join}(E, E') := (J, H, H')$ to be the following 3-tuple (J, H, H'):

- $J := \{(x, x, y, z) \mid (x, y, z) \in E_u\} \; \cup \; \{(x +_u 1, x, y, z) \mid (x, y, z) \in E'_u\} \; \subseteq \mathbb{Z}_u^4$,
- $H := (u, u, u, u)$,
- $H' := (1, u, u, u)$.

Note that $H \in J$ and $H' \in J$. Moreover, H and H' are the only hyperedges in J that have u as their last coordinate. This construction is illustrated in Figure 1.

4.2.1. **Lemma.** *Given two subsets* $E \subseteq \mathbb{Z}_k^3$ *and* $E' \subseteq \mathbb{Z}_{k'}^3$, *let* $(J, H, H') = \mathsf{join}(E, E')$. *Then* $\#4\mathrm{DM}(J \setminus \{H'\}) = \#3\mathrm{DM}(E)$ *and* $\#4\mathrm{DM}(J \setminus \{H\}) = \#3\mathrm{DM}(E')$.

Proof. Clearly $\#4\mathrm{DM}(J \setminus \{H'\}) \geq \#3\mathrm{DM}(E)$, because a 3D-matching $M \subseteq E$ can be converted to a 4D-matching by converting each hyperedge (x, y, z) to (x, x, y, z), and adding the special hyperedge (u, u, u, u). Analogously one shows $\#4\mathrm{DM}(J \setminus \{H\}) \geq \#3\mathrm{DM}(E')$.

The reverse is also true, which can be seen as follows. If (x, x, y, z) is contained in a 4D-matching $M \subseteq J$, then M cannot contain a hyperedge with (first, second) coordinate $(x +_u 1, x)$, hence M contains a hyperedge with (first, second) coordinate $(x +_u 1, x +_u 1)$. This argument is repeated and we see that for all $w \in \mathbb{Z}_u$ the hyperedge in M with first coordinate w has second coordinate w. On the other hand, if $(x +_u 1, x, y, z)$ is contained in a 4D-matching $M \subseteq J$, then M cannot contain a hyperedge with (first, second) coordinate $(x +_u 1, x +_u 1)$, hence it contains a hyperedge with (first, second) coordinate $(x +_u 2, x +_u 1)$. This argument is repeated and we see that for all $w \in \mathbb{Z}_u$ the hyperedge in M with first coordinate $w +_u 1$ has second coordinate w. □

4.3. **An auxiliary SetPartition instance.** We follow the ideas of [GJ79, p. 96] rather closely.

Given two subsets $E \subseteq \mathbb{Z}_k^3$ and $E' \subseteq \mathbb{Z}_{k'}^3$, with E covering $\mathbb{Z}_k^3$ and E' covering $\mathbb{Z}_{k'}^3$, let $\mathsf{join}(E, E') = (J, H, H')$ with $J \subset \mathbb{Z}_u^4$. Note that J covers $\mathbb{Z}_u^4$. We now describe how from (J, H, H') one constructs a SetPartition instance $(\mathbf{a}, \mathbf{b})$.

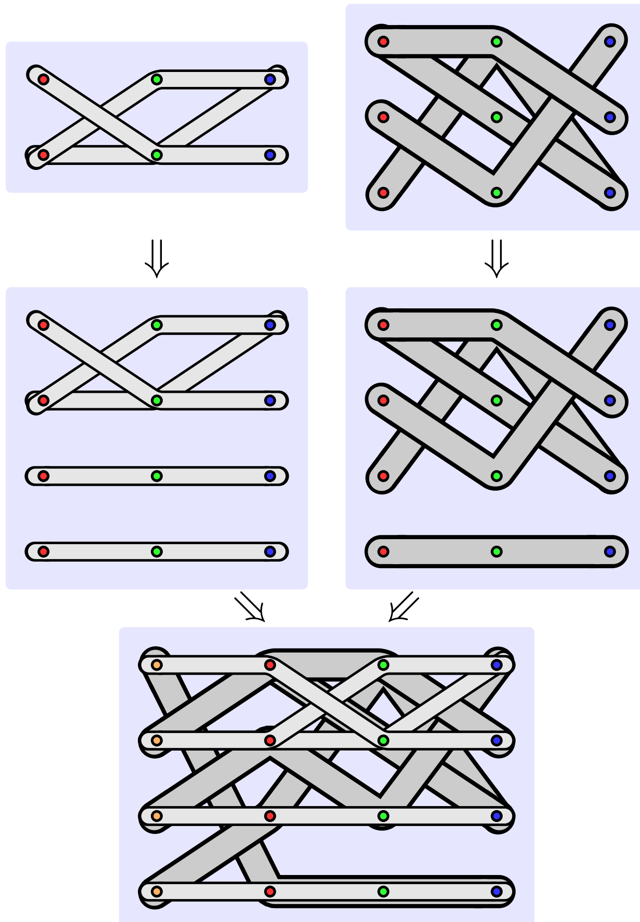

Figure 1. Two #3DM instances are joined by first padding them and then adding another dimension to each hyperedge and taking the union of both hypergraphs. The two special hyperedges H and H' are the ones containing the bottom right vertex. The different shades of gray for the hyperedges are just for illustration.

We start with some notation. For $\widetilde{H} \in \mathbb{Z}_u^4$ and $(i,j) \in \mathbb{Z}_u \times \mathbb{Z}_4$ we write $(i,j) \in \widetilde{H}$ if and only if $\widetilde{H}_j = i$. For each $(i,j) \in \mathbb{Z}_u \times \mathbb{Z}_4$ let $\mathrm{mult}(i,j)$ denote the number of $\widetilde{H} \in J$ with $(i,j) \in \widetilde{H}$. Let $r := 16 \cdot (\max\{4,u\} \cdot 5|J|+1)$, where the factor 16 will become clear in §4.4.

We use the notation $[a_1, a_2, a_3, \ldots] := a_1 r + a_2 r^2 + a_3 r^3 + \ldots$ We say that a_i is the *i-th coefficient*. Inside this notation we use the shorthand 0^j to denote a sequence of j many zeros. We also use the shorthand η_j^i to denote the sequence $(0^{j-1}, 1, 0^{i-j}) \in \{0,1\}^i$. We write $a \cdot \eta_j^i = (0^{j-1}, a, 0^{i-j}) \in \{0,a\}^i$. Let $\beta(j) := 3+1 = 4$ if $1 \le j \le 3$ and $\beta(4) := 3-3 = 0$.

Let there be $|J|$ many bins in this SetPartition instance, and let the bin size be given by $b_1 := [1,1,1,1,1,u,u,u,u,12]$, in other words $\mathbf{b} := (\underbrace{b_1, b_1, \ldots, b_1}_{|J| \text{ times}})$. The items are created as follows.

- For each $(i,j) \in \mathbb{Z}_u \times \mathbb{Z}_4$ we create an item of size $[\eta_j^4, 0, i \cdot \eta_j^4, 3]$. These are called *real vertex items*.
- For each $(i,j) \in \mathbb{Z}_u \times \mathbb{Z}_4$ we create $\mathrm{mult}(i,j) - 1$ many items of size $[\eta_j^4, 0, i \cdot \eta_j^4, \beta(j)]$. These are called *dummy vertex items*. Here we used that $\mathrm{mult}(i,j) \ge 1$, which is guaranteed, because J covers $\mathbb{Z}_u^4$.

- For each hyperedge $(w, x, y, z) \in J$ we create an item of size $[0^4, 1, u-w, u-x, u-y, u-z, 0]$. These are called *hyperedge items.*

This defines a vector $\mathbf{a}$ of item sizes. The number of items is exactly $5|J|$, which can be seen for example by pairing each hyperedge item with 4 vertex items corresponding to that hyperedge. Moreover, $|\mathbf{a}| = |\mathbf{b}| = |J| \cdot b_1$, because the numbers of dummy vertex items for each $j \in \mathbb{Z}_4$ is equal to $|J| - u$, the number of hyperedges which are not part of the matching.

The item to $(u, 4)$ (the vertex in the bottom right in Figure 1) is called the *special vertex item.* By construction, it has size $[0, 0, 0, 1,\ 0,\ 0, 0, u, 0,\ 3]$ and is the unique item of this size. The item to hyperedge $H = (u, u, u, u)$ is called the *first special hyperedge item.* The item to hyperedge $H' = (1, u, u, u)$ is called the *second special hyperedge item.* These two items are also the unique items with their respective sizes.

Note that the value of every coefficient is nonnegative and at most $\max\{4, u\}$. Let

$$\delta := |J|! \cdot \prod_{(i,j)\in\mathbb{Z}_u\times\mathbb{Z}_4} \big(\mathrm{mult}(i, j) - 1\big)!$$

4.3.1. **Lemma.** *In every $\overrightarrow{K} \in \mathcal{P}(\mathbf{a}, \mathbf{b})$, the special vertex item is put in a bin with either the first special hyperedge item or the second special hyperedge item, but not with both at the same time. Let $\mathcal{P}(\mathbf{a}, \mathbf{b})_0$ be the subset of those $\overrightarrow{K}$ for which the special vertex item is put in a bin with the first special hyperedge item, and let $\mathcal{P}(\mathbf{a}, \mathbf{b})_1 = \mathcal{P}(\mathbf{a}, \mathbf{b}) \setminus \mathcal{P}(\mathbf{a}, \mathbf{b})_0$. Then*

$$\tfrac{1}{\delta} \cdot \big|\mathcal{P}(\mathbf{a}, \mathbf{b})_0\big| = \#4\mathrm{DM}\big(J \setminus \{H'\}\big) \quad \textit{and} \quad \tfrac{1}{\delta} \cdot \big|\mathcal{P}(\mathbf{a}, \mathbf{b})_1\big| = \#4\mathrm{DM}\big(J \setminus \{H\}\big).$$

Proof. Since r is large and the size of the bins is $[1, 1, 1, 1, 1, u, u, u, u, 12]$, a solution $\overrightarrow{K}$ to the instance must place exactly 5 items in every bin: One hyperedge item and four vertex items for some vertices $(i, j) \in \mathbb{Z}_u \times \mathbb{Z}_4$, one for each $j \in \mathbb{Z}_4$. Moreover, since r is large and since the 10th coefficient of the bin size is 12, in a solution we must have that in each bin the four vertex items are either all dummy vertex items or all real vertex items (by the construction of β).

Now, since the $\big(6\text{th}, 7\text{th}, 8\text{th}, 9\text{th}\big)$ coordinates of a hyperedge item are $(u-w, u-x, u-y, u-z)$, we conclude that each hyperedge must be placed together with its corresponding vertex items that constitute the hyperedge (real or dummy vertex items). From a placement $\overrightarrow{K}$ like this we can create a solution to $\#4\mathrm{DM}(J)$ by selecting exactly those hyperedges that are in a bin with real vertex items.

In fact, there are δ many different placements $\overrightarrow{K}$ that result in the same 4D-matching: The bins can be permuted, and for each vertex the dummy vertex items can be permuted. And vice versa: From a solution to $\#4\mathrm{DM}(J)$ we create δ many placements $\overrightarrow{K}$ of items by grouping the selected hyperedges together with their real vertex items, and grouping the unselected hyperedges together with their dummy vertex items.

These operations are inverses of each other, which gives a bijection between the set of 4D-matchings of J and the set of cardinality δ subsets of $\mathcal{P}(\mathbf{a}, \mathbf{b})$ in which all elements arise from each other by permuting the bins and the dummy vertices. Now Lemma 4.2.1 implies the result. □

4.4. **The Modified SetPartition Instance.** We modify the item vector $\mathbf{a}$ from the construction above, to obtain a vector $\mathbf{c}$ as follows.

- We add four items of sizes 1,2,4,5.
- We increase the size of the special vertex item by 1. We decrease the size of the first special hyperedge item by 5. We decrease the size of the second special hyperedge item by 2.

W.l.o.g. let the special vertex item, the first special hyperedge item, and the second special hyperedge item be the first three item sizes in $\mathbf{a}$. Then $\mathbf{c} := (1, 2, 4, 5, a_1+1, a_2-5, a_3-2, a_4, a_5, \ldots)$. Let $\mathbf{d} := \mathbf{b}$. We have $|\mathbf{c}| = |\mathbf{d}| + 6$. Let $\overline{\mathbf{d}} := (2, 4, d_1, d_2, \ldots)$ and $\overline{\mathbf{d}}' := (1, 5, d_1, d_2, \ldots)$. Finally, denote $\mathsf{setpartition}(J, H, H') := (\mathbf{c}, \mathbf{d}, \delta)$. This completes the construction process we started in §4.3.

4.4.1. **Lemma.** $\frac{1}{\delta}\,\mathrm{P}(\mathbf{c},\overline{\mathbf{d}}) = \#4\mathrm{DM}(J\setminus\{H'\})$ *and* $\frac{1}{\delta}\,\mathrm{P}(\mathbf{c},\overline{\mathbf{d}}') = \#4\mathrm{DM}(J\setminus\{H\})$.

Proof. The restrictions in the proof of Lemma 4.3.1 still directly apply, because we only made small changes to the item sizes and r is large. The new items and the changed sizes give additional constraints.

In $\mathcal{P}(\mathbf{c},\overline{\mathbf{d}})$, the bin of size 2 must be filled with the item of size 2, and thus the bin of size 4 must be filled with the item left of size 4. The special vertex item and the item of size 1 are placed with the second special hyperedge (because the item of size 2 has already been placed). The item of size 5 is placed with the first special hyperedge. The remaining placements of items can be done as in the proof of Lemma 4.3.1.

In $\mathcal{P}(\mathbf{c},\overline{\mathbf{d}}')$, the bin of size 1 must be filled with the item of size 1, and the bin of size 5 must contain a small (≤ 5) odd item, but the only such item left is the item of size 5. The parity now implies that the special vertex item is placed in a bin with the first special hyperedge item. The only two remaining small items of sizes 2 and 4 fill up the bins of the special hyperedge items. The remaining placements of items can be done as in the proof of Lemma 4.3.1. □

4.5. **Putting the Pieces Together.**

Proof of Theorem 1.1.4 and Theorem 1.1.5. Recall that $\mathsf{C_{=}P} = [\mathsf{GapP} = 0]$ and $\mathsf{PP} = [\mathsf{GapP} \geq 0]$. We prove both theorems simultaneously, so fix a comparison operator $\sim \in \{=, \geq\}$.

For every $L \in \mathsf{C_{=}P}$ there exist $F \in \#\mathsf{P}$ and $F' \in \#\mathsf{P}$ with $w \in L$ if and only if $F(w) \sim F'(w)$. By the Cook–Levin theorem, there exists a polynomial-time algorithm that on input w outputs a Boolean circuit C_w such that $F(w) = \#\textsc{CircuitSAT}(C_w)$. Analogously, there exists a polynomial-time algorithm that on input w outputs a Boolean circuit C'_w such that $F'(w) = \#\textsc{CircuitSAT}(C'_w)$.

Let $E := R(C_w)$ and $E' := R(C'_w)$, where R is defined as in §3.2. Let $(J, H, H') := \mathsf{join}(E, E')$. Let $(\mathbf{c}, \mathbf{d}, \delta) := \mathsf{setpartition}(J, H, H')$. Let $\overline{\mathbf{d}} := (2, 4, d_1, d_2, \ldots)$. Let $\overline{\mathbf{d}}' := (1, 5, d_1, d_2, \ldots)$. Let λ and α be from Proposition 4.1.5. We have:

$$\begin{aligned}
F(w) \sim F'(w) \quad &\Longleftrightarrow \quad \#\textsc{CircuitSAT}(C_w) \sim \#\textsc{CircuitSAT}(C'_w)\\
&\Longleftrightarrow \quad \#3\mathrm{DM}(E) \sim \#3\mathrm{DM}(E')\\
&\overset{\text{Lem. 4.2.1}}{\Longleftrightarrow} \quad \#4\mathrm{DM}(J\setminus\{H'\}) \sim \#4\mathrm{DM}(J\setminus\{H\})\\
&\overset{\text{Lem. 4.4.1}}{\Longleftrightarrow} \quad \tfrac{1}{\delta}\,\mathrm{P}(\mathbf{c},\overline{\mathbf{d}}) \sim \tfrac{1}{\delta}\,\mathrm{P}(\mathbf{c},\overline{\mathbf{d}}')\\
&\Longleftrightarrow \quad \mathrm{P}(\mathbf{c},\overline{\mathbf{d}}) \sim \mathrm{P}(\mathbf{c},\overline{\mathbf{d}}')\\
&\overset{\text{Pro. 4.1.5}}{\Longleftrightarrow} \quad \chi^\lambda(\alpha) \sim 0.
\end{aligned}$$

This completes the proof of both theorems. □

Proof of Theorem 1.1.3. Combine Theorem 1.1.4 and Proposition 3.1.1. □

5. Final remarks and open problems

5.1. **Combinatorial interpretations.** Finding positive combinatorial interpretations for the Kronecker, plethysm and Schubert coefficients remains a central open problem in Algebraic Combinatorics. Special cases for the Kronecker coefficients have been studied, e.g., in [BO05, BOR09, Bla17, IMW17, PP13, RW94], among many others. Combinatorial interpretations for plethysm coefficients have been even harder to find, see [BBP22, DIP20, FI20] for some special cases.

For the Schubert coefficients, see [Knu16, KZ17, Man01, MPP14] for positive combinatorial interpretations in several special cases, and [ARY21] for complexity of a related problem. For the row character sums a_λ defined in (1.2.1), Frumkin [Fru86] proved that $a_\lambda \geq 1$ for all $|\lambda| > 1$. See also [Sol61] for a generalization to all finite groups. We refer to [KW01] for a combinatorial interpretation of an ingredient in the sum in (1.2.1), and to [Sun18, p. 323] for a connection to plethysm coefficients. For the column character sums b_λ, see [Sta99, Exc 7.69] and references therein.

Finally, let us note that the notion of a "positive combinatorial interpretation" is informal, so the apparent lack of such should not be viewed as a strong indicator of not being in #P. For example, it could be that the Kronecker coefficients are in #P, but a technical polynomial time witness is not accepted as a combinatorial interpretation by the Algebraic Combinatorics community. We refer to §§4.5-7 in [Pak18] for several examples of this phenomenon in the context of Enumerative Combinatorics.

5.2. **Unary vs binary input.** Our results are independent of the input encoding in the following sense: the description size of (λ, π) and (λ, μ) can differ exponentially if μ is provided as a list of integers that are encoded in binary. Our results hold in both of these settings. It is noteworthy that such results do not exist for other quantities of interest, for example the Kostka numbers, Littlewood–Richardson and Kronecker coefficients, and the Schubert structure constants.

Narayanan [Nar06] proved that computing the *Kostka coefficients* $K_{\lambda\mu}$ and the LR-coefficients $c^\lambda_{\mu\nu}$ are #P-complete when the inputs λ, μ, ν are encoded in binary. It was conjectured in [PP17, Conj. 8.1] that the LR-coefficients are #P-complete in unary.[2]

We should note however, that the decision problems $[K_{\lambda\mu} = 0]$ and $[c^\lambda_{\mu\nu} = 0]$ are in P even when the input is binary. The first one reduces to checking the linear inequalities whether $\lambda \trianglerighteq \mu$ in the dominance order. By the Knutson–Tao *saturation theorem* [KT99], the vanishing of LR-coefficients reduces to checking if the *Gelfand–Tsetlin polytope* is empty, see [BI13, MNS12, DM06].

The unary hardness of the counting problems would imply that the Schubert coefficients are also #P-hard to compute. Indeed, the natural encoding for the Schubert coefficients, when the inputs are permutations, is in unary. On the other hand, the LR-coefficients are special cases of the Schubert coefficients, but so far #P-completeness is only known when λ, μ, ν are encoded in binary. Thus, we cannot yet conclude the computational hardness result.[3]

By contrast, the Kronecker coefficients of the symmetric group $g(\lambda, \mu, \nu)$ are #P-hard with input in unary; this follows form the proof in [IMW17] that vanishing of $g(\lambda, \mu, \nu)$ is NP-hard in unary.

5.3. **GapP-completeness and parsimonious reductions.** To emphasize the difference, consider the following two problems:

Problem ComputeCharUnary:
- Input: An integer n, and partitions $\lambda, \mu \vdash n$, as lists of numbers encoded in unary
- Output: $\chi^\lambda(\mu)$

Problem ComputeCharBinary:
- Input: An integer n, and partitions $\lambda, \mu \vdash n$, as lists of numbers encoded in binary
- Output: $\chi^\lambda(\mu)$

As we mentioned in the introduction, Hepler [Hep94] proved that computing $\chi^\lambda(\mu)$ is #P-hard in unary, an thus in binary.[4] The following result has not been observed before, but follows directly from Proposition 4.1.2:

5.3.1. **Theorem.** *The problem* ComputeCharBinary *is* GapP*-complete under Turing reductions.*

We note that we cannot at this point strengthen the result to parsimonious many-one reductions, because the reduction from matchings to counting ordered set partitions is itself not parsimonious, having the factor of δ.

[2] The distinction between unary and binary input was underscored in [GJ78]. Unfortunately, the original naming of "strong" vs. (the usual) "weak" NP-completeness added to the confusion, and is best to be avoided.

[3] This argument points out the error in [MQ17, p. 885] which concludes that Schubert coefficients are #P-hard.

[4] In [PP17], the second and third authors made erroneous claims on this point.

5.3.2. **Conjecture.** *The problem* ComputeCharBinary *is* GapP*-complete under many-one reductions.*

We should note though that the reduction from #SetPartition to ComputeCharUnary is parsimonious from the following:

5.3.3. **Proposition.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two positive sequences with equal sums, and let* $\mathbf{b}$ *have* $p-1$ *many parts. Let* $\lambda = p\,\mathrm{sort}(\mathbf{b})$ *and* $\alpha = p\mathbf{a}$. *Then*

$$\chi^\lambda(\alpha) = \mathrm{P}(\mathbf{a}, \mathbf{b}).$$

The proof follows directly from applying Proposition 4.1.2 and observing that since all sizes α are divisible by p, we must have $p \,|\, (\lambda_i + \sigma_i - i) = (pb_i + \sigma_i - i)$ for all bin sizes. Then $\sigma_i = i$, and the only nonzero term which survives is $\mathrm{P}(\alpha, \lambda) = \mathrm{P}(\mathbf{a}, \mathbf{b})$.

5.4. **Combinatorial identities.** The irreducible characters of a finite group G are orthonormal with respect to the inner product

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\, \psi(g^{-1}),$$

see e.g. [Ser77, §2.3]. Thus, equation (1.1.2) gives the squared norm of a character χ^λ.

Equation (1.1.1) for general finite groups is called *Burnside's identity* [Bur11, §208] and can be generalized as follows. For every partition $\mu = (1^{m_1} \ldots \ell^{m_\ell}) \vdash n$ with m_i parts of size i, we have:

$$1^{m_1} m_1! \, \cdots \, \ell^{m_\ell} \ell! \,=\, \sum_{\lambda \vdash n} \bigl(\chi^\lambda(\mu)\bigr)^2, \tag{5.4.1}$$

see e.g. [Sag01, Thm 1.10.3]. When $\mu = (1^n)$ we get (1.1.1), but in this case finding a natural combinatorial partition of the objects from the LHS to sets of sizes given by the character squares is unlikely for the same reason as for (1.1.2).

It would be interesting to see if (5.4.1) has a combinatorial interpretation for *some* classes of μ. For example, when $\mu = (n)$, the characters $\chi^\lambda(\mu) \in \{0, \pm 1\}$ and there is an easy combinatorial interpretation for the character squares $\bigl(\chi^\lambda(\mu)\bigr)^2$. More generally, for $\mu = (k^{n/k})$, all rim hook tableaux in the Murnaghan–Nakayama rule for $\chi^\lambda(\mu)$ have the same sign, see e.g. [JK81, §2.7] and [SW85], so again character squares have a combinatorial interpretation. These "equal cycles" characters also appear in the mysterious identities in [KK98, Thm 3.3]. We note that they do not have a combinatorial proof except for the first identity which coincides with (5.4.1).

5.5. **Other values.** As discussed e.g. in [PPV16, §8] and [Mil19, Pel20], other values of the character table are of interest as well, notably the uniqueness and parity of the characters. The corresponding complexity problems $\bigl[\chi^\lambda(\mu) = 1\bigr]$ and $\bigl[\chi^\lambda(\mu) = 0 \bmod 2\bigr]$ are also very interesting and worth studying.

5.6. **Implications of our results.** Since our main results are written in the language of computational complexity, let us give a few quick and very informal implications in a purely combinatorial language.

We study the vanishing and positivity problems for characters of the symmetric group, in our notation $\bigl[\chi^\lambda(\mu) = 0\bigr]$ and $\bigl[\chi^\lambda(\mu) \geq 0\bigr]$, respectively. In both cases, we determine the exact complexity classes of each problem. Combined with classical results in the area, we make the following conclusions:

(1) If $\bigl[\chi^\lambda(\mu) \geq 0\bigr]$ can be decided in polynomial time, then $\mathsf{P} = \mathsf{NP}$ (Theorem 1.1.5). For example, the existence of Hamiltonian cycles in graphs can be decided in polynomial time.

(2) If either of $\bigl(\chi^\lambda(\mu)\bigr)^2$, or $|\chi^\lambda(\mu)|$, has a positive combinatorial interpretation, then there is a polynomial time certificate for *nonequality* of every two #P functions (Theorem 1.1.3). For example,

whenever two graphs have a different number of Hamiltonian cycles, this property can be proved by a certificate which can be verified in polynomial time.[5] In particular, by choosing one graph with a unique Hamiltonian cycle, one should be able to find such certificate that another graph has either zero or at least two Hamiltonian cycles — while the latter is easy to do by explicitly showing two Hamiltonian cycles, the former is very unlikely and hard to fathom.

Acknowledgements. We are grateful to Swee Hong Chan, Josh Grochow, Sam Hopkins, Greg Kuperberg and Alejandro Morales for fruitful discussions. The first author was partially supported by the DFG grant IK 116/2-1 and the EPSRC grant EP/W014882/1. The second author was partially supported by the NSF grant CCF-2007891. The third author was partially supported by the NSF grant CCF-2007652.

References

[ARY21] A. Adve, C. Robichaux and A. Yong, An efficient algorithm for deciding vanishing of Schubert polynomial coefficients, *Adv. Math.* **383** (2021), Paper No. 107669, 38 pp.

[And76] G. E. Andrews, *The theory of partitions*, Addison-Wesley, Reading, MA, 1976, 255 pp.

[BO05] C. M. Ballantine and R. C. Orellana, On the Kronecker product $s_{(n-p,p)} * s_\lambda$, *Electron. J. Combin.* **12** (2005), RP 28, 26 pp.

[BBP22] C. Bessenrodt, C. Bowman and R. Paget, The classification of multiplicity-free plethysms of Schur functions, *Trans. AMS* **375** (2022), 5151–5194.

[Bla17] J. Blasiak, Kronecker coefficients for one hook shape, *Sém. Lothar. Combin* **77** (2017), Art. B77c, 40 pp.

[BOR09] E. Briand, R. Orellana and M. Rosas, Reduced Kronecker coefficients and counter-examples to Mulmuley's strong saturation conjecture SH, *Comput. Complexity* **18** (2009), 577–600.

[BS17] D. Bump and A. Schilling, *Crystal bases*, World Sci., Hackensack, NJ, 2017, 279 pp.

[Bur11] W. Burnside, *Theory of groups of finite order* (Second ed.), Cambridge Univ. Press, Cambride, UK, 1911, 512 pp.

[BI08] P. Bürgisser and C. Ikenmeyer, The complexity of computing Kronecker coefficients, in *Proc. 20th FPSAC* (2008), DMTCS, Nancy, 2008, 357–368.

[BI13] P. Bürgisser and C. Ikenmeyer, Deciding positivity of Littlewood–Richardson coefficients, *SIAM J. Discrete Math.* **27** (2013), 1639–1681.

[CDW12] M. Christandl, B. Doran and M. Walter, Computing multiplicities of Lie group riepresentations, in *Proc. 53rd FOCS* (2012), IEEE, 639–648.

[DNV22] D. Dauvergne, M. Nica and B. Virág, RSK in last passage percolation: a unified approach, *Probab. Surv.* **19** (2022), 65–112.

[DM06] J. A. De Loera and T. B. McAllister, On the computation of Clebsch–Gordan coefficients and the dilation effect, *Experimental Math.* **15** (2006), 7–19.

[Dia88] P. Diaconis, *Group representations in probability and statistics*, IMS, Hayward, CA, 1988, 198 pp.

[DIP20] J. Dörfler, C. Ikenmeyer and G. Panova, On geometric complexity theory: multiplicity obstructions are stronger than occurrence obstructions, *SIAM J. Appl. Algebra Geom.* **4** (2020), 354–376.

[EFP11] D. Ellis, E. Friedgut and H. Pilpel, Intersecting families of permutations, *J. AMS* **24** (2011), 649–682.

[FGHP99] S. Fenner, F. Green, S. Homer and R. Pruim, Determining acceptance possibility for a quantum computation is hard for the polynomial hierarchy, *R. Soc. Lond. Proc. Ser. A, Math. Phys. Eng. Sci.* **455** (1999), no. 1991, 3953–3966.

[FI20] N. Fischer and C. Ikenmeyer, The computational complexity of plethysm coefficients, *Comput. Complexity* **29** (2020), no. 2, Paper 8, 43 pp.

[Fru86] A. Frumkin, Theorem about the conjugacy representation of S_n, *Israel J. Math.* **55** (1986), 121–128.

[GJ78] M. R. Garey and D. S. Johnson, "Strong" NP-completeness results: motivation, examples, and implications, *J. Assoc. Comput. Mach.* **25** (1978), 499–508.

[GJ79] M. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of* NP*-completeness*, Freeman, San Francisco, CA, 1979.

[Gre93] F. Green, On the power of deterministic reductions to $C_=P$, *Math. Systems Theory* **26** (1993), 215–233.

[5]Here the prover has every time in the world to construct the certificate, i.e. by examining and listing all possible subgraphs, but the verifier is constrained and cannot possible analyze this exponentially long list in polynomial time.

[Hep94] C. T. Hepler, *On the complexity of computing characters of finite groups*, thesis, University of Calgary, 1994, 117 pp.; available at `https://dspace.ucalgary.ca/handle/1880/45530`

[HVW95] U. Hertrampf, H. Vollmer and K. Wagner, On the power of number-theoretic operations with respect to counting, in *Proc. 10th CCC* (1995), 299–314.

[IMW17] C. Ikenmeyer, K. D. Mulmuley and M. Walter, On vanishing of Kronecker coefficients, *Comput. Complexity* **26** (2017), 949–992.

[IP22] C. Ikenmeyer and I. Pak, What is in #P and what is not?, preprint (2022), 82 pp.; `arXiv:2204.13149`; extended abstract to appear in *Proc. 63rd FOCS* (2022).

[JK81] G. James and A. Kerber, *The representation theory of the symmetric group*, Addison-Wesley, Reading, MA, 1981, 510 pp.

[KM18] T. Kahle and M. Michałek, Obstructions to combinatorial formulas for plethysm, *Electron. J. Combin.* **25** (2018), no. 1, Paper 1.41, 9 pp.

[KP21] D. Kim and P. Pylyavskyy, Robinson–Schensted correspondence for unit interval orders, *Selecta Math.* **27** (2021), no. 5, Paper No. 97, 66 pp.

[KK98] A. Klyachko and E. Kurtaran, Some identities and asymptotics for characters of the symmetric group, *J. Algebra* **206** (1998), 413–437.

[Knu16] A. Knutson, Schubert calculus and puzzles, in *Adv. Stud. Pure Math.* **71**, Math. Soc. Japan, Tokyo, 2016, 185–209.

[KT99] A. Knutson and T. Tao, The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture, *J. AMS* **12** (1999), 1055–1090.

[KZ17] A. Knutson and P. Zinn-Justin, Schubert puzzles and integrability I: invariant trilinear forms, preprint (2017), 51 pp.; `arXiv:1706.10019`.

[KW01] W. Kraśkiewicz and J. Weyman, Algebra of coinvariants and the action of a Coxeter element, in *Bayreuth. Math. Schr.* **63** (2001), 265–284.

[Man01] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracyloci*, SMF/AMS, Providence, RI, 2001.

[MPP14] K. Mészáros, G. Panova and A. Postnikov, Schur times Schubert via the Fomin–Kirillov algebra, *Electron. J. Combin.* **21** (2014), no. 1, Paper 1.39, 22 pp.

[Mil19] A. R. Miller, On parity and characters of symmetric groups, *J. Combin. Theory, Ser. A* **162** (2019), 231–240.

[MRS08] C. Moore, A. Russell and L. J. Schulman, The symmetric group defies strong Fourier sampling, *SIAM J. Comput.* **37** (2008), 1842–1864.

[MQ17] P. Mukhopadhyay and Y. Qiao, Sparse multivariate polynomial interpolation on the basis of Schubert polynomials, *Comput. Complexity* **26** (2017), 881–909.

[Mul09] K. D. Mulmuley, Geometric Complexity Theory VI: The flip via saturated and positive integer programming in representation theory and algebraic geometry, preprint (2009), 139 pp.; `arXiv:0704.0229v4`.

[MNS12] K. D. Mulmuley, H. Narayanan and M. Sohoni, Geometric complexity theory III. On deciding nonvanishing of a Littlewood–Richardson coefficient, *J. Algebraic Combin.* **36** (2012), 103–110.

[Nar06] H. Narayanan, On the complexity of computing Kostka numbers and Littlewood–Richardson coefficients, *J. Algebraic Combin.* **24** (2006), 347–354.

[O'Con03] N. O'Connell, Conditioned random walks and the RSK correspondence. Random matrix theory, *J. Phys. A* **36** (2003), 3049–3066.

[OW03] R. O'Donnell and J. Wright, Efficient quantum tomography II, in *Proc. 49th STOC* (2017), 962–974.

[Pak18] I. Pak, Complexity problems in enumerative combinatorics, in *Proc. ICM Rio de Janeiro*, Vol. IV, World Sci., Hackensack, NJ, 2018, 3153–3180; an expanded version of the paper is at `arXiv:1803.06636`.

[Pak19] I. Pak, Combinatorial inequalities, *Notices AMS* **66** (2019), 1109–1112; an expanded version of the paper is available at `https://www.math.ucla.edu/~pak/papers/full-story1.pdf`

[PP13] I. Pak and G. Panova, Strict unimodality of q-binomial coefficients, *C.R. Math. Acad. Sci. Paris* **351** (2013), 415–418.

[PP17] I. Pak and G. Panova, On the complexity of computing Kronecker coefficients, *Comput. Complexity* **26** (2017), 1–36.

[PPV16] I. Pak, G. Panova and E. Vallejo, Kronecker products, characters, partitions, and the tensor square conjectures, *Adv. Math.* **288** (2016), 702–731.

[Pap94] C. H. Papadimitriou, *Computational Complexity*, Addison-Wesley, Reading, MA, 1994.

[Pau95] R. Pauncz, The symmetric group in quantum chemistry, CRC Press, Boca Raton, FL, 1995, 330 pp.

[Pel20] S. Peluse, On even entries in the character table of the symmetric group, preprint (2020), 12 pp.; `arXiv:2007.06652`.

[PS09] A. Postnikov and R. P. Stanley, Chains in the Bruhat order, *J. Algebraic Combin.* **29** (2009), 133–174.

[RW94] J. B. Remmel and T. Whitehead, On the Kronecker product of Schur functions of two row shapes, *Bull. Belg. Math. Soc. Simon Stevin* **1** (1994), 649–683.

[Rom15] D. Romik, *The surprising mathematics of longest increasing subsequences*, Cambridge Univ. Press, New York, 2015, 353 pp.

[Sag01] B. E. Sagan, *The symmetric group*, Springer, New York, 2001.

[Sch78] T. J. Schaefer, The complexity of satisfiability problems, in *Proc. 10th STOC* (1978), 216–226.

[Ser77] J.-P. Serre, Linear representations of finite groups, Springer, New York, 1977, 170 pp.

[Sol61] L. Solomon, On the sum of the elements in the character table of a finite group, *Proc. AMS* **12** (1961), 962–963.

[Sta84] R. P. Stanley, The stable behavior of some characters of $\mathrm{SL}(n, \mathbb{C})$, *Linear and Multilinear Algebra* **16** (1984), 3–27.

[Sta99] R. P. Stanley, *Enumerative Combinatorics*, vol. 1 (second ed.) and vol. 2, Cambridge Univ. Press, 2012 and 1999.

[Sta00] R. P. Stanley, Positivity problems and conjectures in algebraic combinatorics, in *Mathematics: Frontiers and Perspectives*, AMS, Providence, RI, 2000, 295–319.

[SW85] D. W. Stanton and D. E. White, A Schensted algorithm for rim hook tableaux, *J. Combin. Theory, Ser. A* **40** (1985), 211–247.

[Ste94] S. Sternberg, *Group theory and physics*, Cambridge Univ. Press, Cambridge, 1994, 429 pp.

[Sun18] S. Sundaram, The conjugacy action of S_n and modules induced from centralisers, *J. Algebraic Combin.* **48** (2018), 179–225.

[Tar91] J. Tarui, Randomized polynomials, threshold circuits, and the polynomial hierarchy, in *Proc. 8th STACS* (1991), 238–250.

[TY08] H. Thomas and A. Yong, An S_3-symmetric Littlewood–Richardson rule, *Math. Res. Lett.* **15** (2008), 1027–1037.

[Toda89] S. Toda, On the computational power of PP and $\oplus$P, in *Proc. 30th FOCS* (1989), 514–519.

[Toda91] S. Toda, PP is as hard as the polynomial-time hierarchy, *SIAM J. Comput.* **20** (1991), 865–877.

[Wilf82] H. S. Wilf, What is an answer?, *Amer. Math. Monthly* **89** (1982), no. 5, 289–292.

[Whi83] D. White, A bijection proving orthogonality of the characters of S_n, *Adv. Math.* **50** (1983), 160–186.

[Whi85] D. White, Orthogonality of the characters of S_n, *J. Combin. Theory, Ser. A* **40** (1985), 265–275.

[You20] N. Young, Why is the reduction from 3-SAT to 3-dimensional Matching Parsimonious?, *CS Theory Stack Exchange* answer (Aug. 31, 2020); `https://cstheory.stackexchange.com/q/47491/16704`