LOG-CONCAVE POSET INEQUALITIES

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Abstract. We study combinatorial inequalities for various classes of set systems: matroids, polymatroids, poset antimatroids, and interval greedoids. We prove log-concave inequalities for counting certain weighted feasible words, which generalize and extend several previous results establishing *Mason conjectures* for the numbers of independent sets of matroids. Notably, we prove matching equality conditions for both earlier inequalities and our extensions.

In contrast with much of the previous work, our proofs are combinatorial and employ nothing but linear algebra. We use the language formulation of greedoids which allows a linear algebraic setup, which in turn can be analyzed recursively. The underlying non-commutative nature of matrices associated with greedoids allows us to proceed beyond polymatroids and prove the equality conditions. As further application of our tools, we rederive both *Stanley's inequality* on the number of certain linear extensions, and its equality conditions, which we then also extend to the weighted case.

1. Introduction

1.1. **Foreword.** It is always remarkable and even a little suspicious, when a nontrivial property can be proved for a large class of objects. Indeed, this says that the result is "global", i.e. the property is a consequence of the underlying structure rather than individual objects. Such results are even more remarkable in combinatorics, where the structures are weak and the objects are plentiful. In fact, many reasonable conjectures in the area fail under experiments, while some are ruled out by theoretical considerations (cf. §16.1 and §17.1).

This paper is concerned with log-concavity results for counting problems in the general context of posets, and is motivated by a large body of amazing recent work in area, see a survey by Huh [Huh18]. Surprisingly, these results involve deep algebraic tools which go much beyond previous work on the subject, see earlier surveys [Brä15, Bre89, Bre94, Sta89]. This leads to several difficult questions, such as:

- How far do these inequalities generalize?
- How do we extend/develop new algebraic tools to prove these generalizations?

We aim to answer the first question in as many cases as we can, both generalizing the inequalities to larger classes of posets and strengthening these inequalities to match equality conditions which we also prove. We do this by sidestepping the second question, or avoiding it completely.

There is a very long and only partially justified tradition in combinatorics of looking for purely combinatorial proofs of combinatorial results. Although the very idea of using advanced algebraic tools to prove combinatorial inequalities is rather mesmerizing, one wonders if these tools are really necessary. Are they giving us a *true insight* into the nature of these inequalities that we were missing for so long? Or, perhaps, the absence of purely combinatorial proofs is a reflection of our continuing *lack of understanding*?

We posit that, in fact, all poset inequalities can be obtained by elementary means (cf. §1.21). We show how this can be done for a several large families of inequalities, and intend to continue this work in the future (see §17.17). There are certain tradeoffs, of course, as we need to introduce a technical linear algebraic setup (see §1.20), which allows us to quickly reprove both classical and recently established poset inequalities. The advantage of our approach is its flexibility and noncommutative nature, making it amenable to extend and generalize these inequalities in several directions.

Of course, none of what we did takes anything away from the algebraic proofs of poset inequalities which remained open for decades — the victors keep all the spoils (see Section 16). We do, however, hope the reader will appreciate that our combinatorial tools are indeed *more powerful* than the algebraic tools, at least in the cases we consider (cf. §§17.8–17.11).

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1.2. **What to expect now.** A long technical paper deserves a long technical introduction. Similarly, a friendly and accessible paper deserves a friendly and accessible introduction. Naturally, we aim to achieve both somewhat contradictory goals.

Below we present our main results and applications, all of which require definitions which are standard in the area, but not a common knowledge in the rest of mathematics. We make an effort to have the introduction thorough yet easily accessible, at the expense of brevity.[1]

In addition, rather than jump to the most general and thus most involved results, we begin slowly, and take time to introduce the reader to the world of poset inequalities. Essentially, the rest of the introduction can be viewed as an extensive survey of our own results interspersed with a few examples and some earlier results directly related to our work. The reader well versed in the greedoid literature can speed read a few early subsections.

We say very little about our tools at this stage, even though we consider them to be our main contribution (see §1.20 and §1.21). These are fully presented in the following sections, which in turn are followed by the proofs of all the results. As we mentioned above, our tools are elementary but technical, and are best enjoyed when the reader is convinced they are worth delving into.

Similarly, in the introduction, we say the bare minimum about the rich history of the subject and the previous work on poset inequalities. This is rather unfair to the many experts in the area whose names and contributions are mentioned only at the end of the paper. Our choice was governed by the effort to keep the introduction from exploding in size. We beg forgiveness on this point, and try to mitigate it by a lengthy historical discussion in Section 16, with quick pointer links sprinkled throughout the introduction.

1.3. **Matroids.** A (finite) *matroid* $\mathcal{M}$ is a pair $(X, \mathcal{I})$ of a *ground set* X, $|X| = n$, and a nonempty collection of *independent sets* $\mathcal{I} \subseteq 2^X$ that satisfies the following:

- (*hereditary property*) $S \subset T,\ T \in \mathcal{I} \ \Rightarrow\ S \in \mathcal{I}$, and
- (*exchange property*) $S, T \in \mathcal{I},\ |S| < |T| \ \Rightarrow\ \exists x \in T \setminus S$ s.t. $S + x \in \mathcal{I}$.

Rank of a matroid is the maximal size of the independent set: $\mathrm{rk}(\mathcal{M}) := \max_{S \in \mathcal{I}} |S|$. A *basis* of a matroid is an independent set of size $\mathrm{rk}(\mathcal{M})$. Finally, let $\mathcal{I}_k := \{S \in \mathcal{I},\ |S| = k\}$, and let $\mathrm{I}(k) = |\mathcal{I}_k|$ be the *number of independent sets* in $\mathcal{M}$ of size k, $0 \le k \le \mathrm{rk}(\mathcal{M})$.

Theorem 1.1 (*Log-concavity for matroids*, [AHK18, Thm 9.9 (3)], formerly *Welsh–Mason conjecture*). *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and integer $1 \le k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \ \ge\ \mathrm{I}(k-1) \cdot \mathrm{I}(k+1). \tag{1.1}$$

See §16.5 for the historical background. The *log-concavity* in (1.1) classically implies *unimodality* of the sequence $\{\mathrm{I}(k)\}$:

$$\mathrm{I}(0) \ \le\ \mathrm{I}(1) \ \le\ \ldots \ \le\ \mathrm{I}(k) \ \ge\ \mathrm{I}(k+1) \ \ge\ \ldots \ \ge\ \mathrm{I}(m), \quad \text{where} \quad m \,=\, \mathrm{rk}(\mathcal{M}).$$

It was noted in [Lenz11, Lem. 4.2] that other results in [AHK18] imply that the inequalities (1.1) are *always strict* (see §16.6). Further improvements to (1.1) have been long conjectured by Mason [Mas72] and were recently established in quick succession.

Theorem 1.2 (*One-sided ultra-log-concavity for matroids*, [HSW22, Cor. 9], formerly *weak Mason conjecture*). *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and integer $1 \le k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \ \ge\ \left(1 + \frac{1}{k}\right) \mathrm{I}(k-1)\ \mathrm{I}(k+1). \tag{1.2}$$

Theorem 1.3 (*Ultra-log-concavity for matroids*, [ALOV18, Thm 1.2] and [BH20, Thm 4.14], formerly *strong Mason conjecture*). *For a matroid $\mathcal{M} = (X, \mathcal{I})$, $|X| = n$, and integer $1 \le k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \ \ge\ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) \mathrm{I}(k-1)\ \mathrm{I}(k+1). \tag{1.3}$$

[1] In an effort to streamline the presentation, some basic notation is collected in a short Section 2, which we encourage the reader to consult whenever there is an apparent misunderstanding or ambiguity.

Equation (1.3) is a reformulation of *ultra-log-concavity* of the sequence $\{\mathrm{I}(k)\}$:

$$\mathrm{i}(k)^2 \,\geq\, \mathrm{i}(k-1)\cdot \mathrm{i}(k+1)\,, \quad \text{where} \quad \mathrm{i}(m) \,:=\, \frac{\mathrm{I}(m)}{\binom{n}{m}}$$

can be viewed as the probability that random m-subset of X is independent in $\mathcal{M}$.

1.4. **More matroids.** For an independent set $S \in \mathcal{I}$ of a matroid $\mathcal{M} = (X, \mathcal{I})$, denote by

$$\operatorname{Cont}(S) \,:=\, \big\{x \in X \setminus S \ : \ S + x \in \mathcal{I}\big\} \tag{1.4}$$

the set of *continuations* of S. For all $x, y \in \operatorname{Cont}(S)$, we write $x \sim_S y$ when $S+x+y \notin \mathcal{I}$ or when $x = y$. Note that "$\sim_S$" is an equivalence relations, see Proposition 4.1. We call an equivalence class of the relation $\sim_S$ a *parallel class* of S, and we denote by $\operatorname{Par}(S)$ the set of parallel classes of S.

For every $0 \leq k < \operatorname{rk}(\mathcal{M})$, define the *$k$-continuation number* of a matroid $\mathcal{M}$ as the maximal number of parallel classes of independent sets of size k:

$$\mathrm{p}(k) \ :=\ \max\big\{ \, \big|\operatorname{Par}(S)\big| \ : \ S \in \mathcal{I}_k \, \big\}. \tag{1.5}$$

Clearly, $\mathrm{p}(k) \leq n - k$.

Theorem 1.4 (*Refined log-concavity for matroids*)**.** *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and integer $1 \leq k < \operatorname{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \,\geq\, \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1)-1}\right)\, \mathrm{I}(k-1)\, \mathrm{I}(k+1)\,. \tag{1.6}$$

Clearly, Theorem 1.4 implies Theorem 1.3. This is our first result of the long series of generalizations that follow. Before we proceed, let us illustrate the power of this refinement in a special case.

Example 1.5 (*Graphical matroids*)**.** Let $G = (V, E)$ be a connected graph with $|V| = \textsc{n}$ edges. The corresponding *graphical matroid* $\mathcal{M}_G = (E, \mathcal{I})$ is defined to have independent sets to be all *spanning forests* in G, i.e. spanning subgraphs without cycles. Then $\mathrm{I}(k)$ is the number of spanning forests with k edges, bases are *spanning trees* in G, and $\operatorname{rk}\big(\mathcal{M}_G\big) = \textsc{n} - 1$.

Let $k = \textsc{n} - 2$ in Theorem 1.4. Observe that $\mathrm{p}(\textsc{n}-3) \leq 3$ since $T - e - e'$ can have at most three connected components, for every spanning tree T in G and edges $e, e' \in E$. Then (1.6) gives:

$$\frac{\mathrm{I}(\textsc{n}-2)^2}{\mathrm{I}(\textsc{n}-3)\,\cdot\,\mathrm{I}(\textsc{n}-1)} \ \geq\ \frac{3}{2}\left(1 + \frac{1}{\textsc{n}-2}\right) \ \to\ \frac{3}{2} \quad \text{as} \quad \textsc{n} \to \infty. \tag{1.7}$$

This is both numerically and asymptotically better than (1.3), cf. §17.12. For example, when $|E| - \textsc{n} \to \infty$, we have:

$$\frac{\mathrm{I}(\textsc{n}-2)^2}{\mathrm{I}(\textsc{n}-3)\,\cdot\,\mathrm{I}(\textsc{n}-1)} \ \geq_{(1.3)}\ \left(1 + \frac{1}{|E| - \textsc{n} + 2}\right)\left(1 + \frac{1}{\textsc{n}-2}\right) \ \to\ 1 \quad \text{as} \quad \textsc{n} \to \infty.$$

1.5. **Weighted matroid inequalities.** Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid, and let $\omega : X \to \mathbb{R}_{>0}$ be a positive *weight function* on the ground set X. We extend the weight function to every independent set $S \in \mathcal{I}$ as follows:

$$\omega(S) \ :=\ \prod_{x \in S} \omega(x).$$

For all $1 \leq k < \operatorname{rk}(\mathcal{M})$, define

$$\mathrm{I}_\omega(k) \ :=\ \sum_{S \in \mathcal{I}_k} \omega(S)\,.$$

Theorem 1.6 (*Refined weighted log-concavity for matroids*)**.** *Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid on $|X| = n$ elements, let $\omega : X \to \mathbb{R}_{>0}$ be a weight function, and let $1 \leq k < \operatorname{rk}(\mathcal{M})$. Then:*

$$\mathrm{I}_\omega(k)^2 \,\geq\, \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1)-1}\right)\, \mathrm{I}_\omega(k-1)\, \mathrm{I}_\omega(k+1)\,. \tag{1.8}$$

Remark 1.7. In this theorem, the setup is more important than the result as it can be easily reduced to Theorem 1.4. Indeed, note that one can take multiple copies of elements in a matroid $\mathcal{M}$. This implies the result for integer valued ω. The full version follows by homogeneity and continuity. This natural approach fails for the equality conditions as strict inequalities are not necessarily preserved in the limit, and for many generalizations below where we have constraints on the weight function. See §16.11 for some background.

1.6. **Equality conditions for matroids.** For a matroid $\mathcal{M} = (X, \mathcal{I})$ on $|X| = n$ elements, define $\operatorname{girth}(\mathcal{M}) := \min\{k : \mathrm{I}(k) < \binom{n}{k}\}$. By analogy with graph theory, *girth of a matroid* is the size of the smallest circuit in $\mathcal{M}$.

Theorem 1.8 (*Equality for matroids*, [MNY21, Cor. 1.2])**.** *Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid on $|X| = n$ elements, and let $1 \le k < \operatorname{rk}(\mathcal{M})$. Then:*

$$\mathrm{I}(k)^2 \;=\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right)\,\mathrm{I}(k-1)\,\mathrm{I}(k+1) \tag{1.9}$$

if and only if $\operatorname{girth}(\mathcal{M}) > (k+1)$.

See §16.12 for some background on equality conditions. The theorem says that in order to have equality (1.9), we must have probabilities $\mathrm{i}(k-1) = \mathrm{i}(k) = \mathrm{i}(k+1) = 1$. Now we present a weighted version of Theorem 1.8. We say that weight function $\omega : X \to \mathbb{R}_{>0}$ is *uniform* if $\omega(x) = \omega(y)$ for all $x, y \in X$.

Theorem 1.9 (*Weighted equality for matroids*)**.** *Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid on $|X| = n$ elements, let $1 \le k < \operatorname{rk}(\mathcal{M})$, and let $\omega : X \to \mathbb{R}_{>0}$ be a weight function. Then:*

$$\mathrm{I}_\omega(k)^2 \;=\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right)\,\mathrm{I}_\omega(k-1)\,\mathrm{I}_\omega(k+1) \tag{1.10}$$

if and only if $\operatorname{girth}(\mathcal{M}) > (k+1)$*, and the weight function ω is uniform.*

The uniform condition in the theorem is quite natural for integer weight functions, as it basically says that in order to have (1.10) all elements have to be repeated the same number of times. In other words, weighted inequalities do not have a substantially larger set of equality cases.

Theorem 1.10 (*Refined equality for matroids*)**.** *Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid, $1 \le k < \operatorname{rk}(\mathcal{M})$, and let $\omega : X \to \mathbb{R}_{>0}$ be a weight function. Then:*

$$\mathrm{I}_\omega(k)^2 \;=\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1) - 1}\right)\,\mathrm{I}_\omega(k-1)\,\mathrm{I}_\omega(k+1) \tag{1.11}$$

if and only if there exists $\mathrm{s}(k-1) > 0$, such that for every $S \in \mathcal{I}_{k-1}$ we have:

$$\big|\operatorname{Par}(S)\big| \;=\; \mathrm{p}(k-1)\,, \qquad \textit{and} \tag{ME1}$$

$$\sum_{x \in \mathcal{C}} \omega(x) \;=\; \mathrm{s}(k-1) \qquad \textit{for every} \quad \mathcal{C} \in \operatorname{Par}(S)\,. \tag{ME2}$$

Condition (ME1) says that the $(k-1)$-continuation number is achieved on all independent sets $S \in \mathcal{I}_{k-1}$. When the weight function is uniform, condition (ME2) is saying that all parallel classes $\mathcal{C} \in \operatorname{Par}(S)$ have the same size.

1.7. **Examples of matroids.** First, we prove that the equality conditions are rarely satisfied for graphical matroids, see Example 1.5. More precisely, we prove that the refined log-concavity inequality (1.7) is an equality only for cycles:

Proposition 1.11 (*Equality for graphical matroids*)**.** *Let $G = (V, E)$ be a simple connected graph on $|V| = \textsc{n}$ vertices, and let $\mathrm{I}(k)$ be the number of spanning forests with k edges. Then*

$$\frac{\mathrm{I}(\textsc{n}-2)^2}{\mathrm{I}(\textsc{n}-3)\,\cdot\,\mathrm{I}(\textsc{n}-1)} \;\ge\; \frac{3}{2}\left(1 + \frac{1}{\textsc{n}-2}\right), \tag{1.12}$$

and the equality holds if and only if G is an $\textsc{n}$-cycle.

We now show that the equality conditions in Theorem 1.10 have a rich family of examples (see §16.7 for more on these examples). The weight function is uniform in all these cases: $\omega(x) = 1$ for every $x \in X$.

Example 1.12 (*Finite field matroids*)**.** Let $\mathbb{F}_q$ be a finite field with q elements, let $m \geq 1$, and let $X = \mathbb{F}_q^m$. Let $\mathcal{I}$ be a set of subsets $S \subset \mathbb{F}_q^m$ which are linearly independent as vectors. Finally, let $\mathcal{M}(m, q) = (X, \mathcal{I})$ be a matroid of vectors in $\mathbb{F}_q^m$ of rank m.

Let $1 \leq k < m$ and let $S \in \mathcal{I}_{k-1}$, so we have $\dim_{\mathbb{F}_q}\langle S\rangle = k - 1$. For all parallel classes $\mathcal{C} \in \mathrm{Par}(S)$ we then have $|\,\mathcal{C}\,| = q^{k-1}$. Therefore,

$$\big|\mathrm{Par}(S)\big| \ = \ \frac{q^m \, - \, q^{k-1}}{q^{k-1}} \ = \ q^{m-k+1} \, - \, 1 \, . \tag{1.13}$$

The conditions (ME1) and (ME2) are then satisfied with $\mathrm{p}(k-1) = q^{m-k+1} - 1$ and $\mathrm{s}(k-1) = q^{k-1}$. We conclude that (1.6) is an equality for $\mathcal{M}(m, q)$, for all $1 \leq k < m$. Curiously, the equality (1.13) is optimal for matroids over $\mathbb{F}_q$, and we have the following result (see §10.6 for the proof).

Corollary 1.13. *Let $X \subseteq \mathbb{F}_q^m$ be a set of n vectors which span $\mathbb{F}_q^m$, and let $\mathcal{M} = (X, \mathcal{I})$ be the corresponding matroid of rank $m = \mathrm{rk}(\mathcal{M})$. Then, for all $1 \leq k < m$, we have:*

$$\mathrm{I}(k)^2 \ \geq \ \left(1 \, + \, \frac{1}{k}\right) \left(1 \, + \, \frac{1}{q^{m-k+1} \, - \, 2}\right) \ \mathrm{I}(k-1) \ \mathrm{I}(k+1) \, .$$

Example 1.14 (*Steiner system matroids*)**.** Fix integers $t < m < n$ and a ground set X, with $|X| = n$. A *Steiner system* $\mathrm{Stn}(t, m, n)$ is a collection $\mathcal{B}$ of m-subsets $B \subset X$ called *blocks*, such that each t-subset of X is contained in exactly one block $B \in \mathcal{B}$.

Let $\mathcal{M}(\mathcal{B}) = (X, \mathcal{I})$ be a matroid with $\mathrm{rk}(\mathcal{M}) = \mathrm{girth}(\mathcal{M}) = (t + 1)$, where the bases are $(t + 1)$-subsets of X that are not contained in any block of the Steiner system. It is easy to see that this indeed defines a matroid, cf. §16.7. Note that (1.8) is trivially an equality for all $1 \leq k < t$.

Let $S \in \mathcal{I}_{t-1}$ be an independent set of size $(t-1)$. The parallel classes of S are given by $B_1 \setminus S, \ldots, B_\ell \setminus S$, where $B_1, \ldots, B_\ell \in \mathcal{B}$ are blocks of the Steiner system that contain S, and $\ell = \frac{n-t+1}{m-t+1}$. Then we have:

$$|\,\mathrm{Par}(S)| \ = \ \ell \, , \qquad \text{and} \qquad |\,\mathcal{C}\,| \ = \ m - t + 1 \qquad \text{for every} \quad \mathcal{C} \in \mathrm{Par}(S).$$

Since the choice of S is arbitrary, the conditions (ME1) and (ME2) are satisfied with $\mathrm{p}(t-1) = \ell$ and $\mathrm{s}(t-1) = m - t + 1$. We conclude that (1.6) is also an equality for $k = t$.

1.8. Morphism of matroids.

For a matroid $\mathcal{M} = (X, \mathcal{I})$, the *rank function* $f : 2^X \to \mathbb{R}_{>0}$ is defined by

$$f(S) \ := \ \max \big\{ |A| \, : \, A \subseteq S, \, A \in \mathcal{I} \, \big\}.$$

Note that $\mathrm{rk}(\mathcal{M}) = f(X)$. There is an equivalent definition of a matroid in terms of monotonic submodular rank functions, see e.g. [Wel76].

Let $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$ be two matroids with rank functions f and g, respectively. Let $\Phi : X \to Y$ be a function that satisfies

$$g\big(\Phi(T)\big) \, - \, g\big(\Phi(S)\big) \ \leq \ f(T) \, - \, f(S) \qquad \text{for every} \quad S \subseteq T \subseteq X \, . \tag{1.14}$$

In this case we say that Φ is a *morphism* of matroids, write $\Phi : \mathcal{M} \to \mathcal{N}$. A subset $S \in \mathcal{I}$ is said to be a *basis* of Φ if $g(\Phi(S)) = \mathrm{rk}(\mathcal{N})$. In other words, S is contained in a basis of $\mathcal{M}$, and $\Phi(S)$ contains a basis of $\mathcal{N}$. Denote by $\mathcal{B}$ the set of bases of $\Phi : \mathcal{M} \to \mathcal{N}$, and let $\mathcal{B}_k := \mathcal{B} \cap \mathcal{I}_k$.

Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function on the ground set X. As before, for every $0 \leq k \leq \mathrm{rk}(\mathcal{M})$, let

$$\mathrm{B}_\omega(k) \ := \ \sum_{S \in \mathcal{B}_k} \omega(S) \, , \qquad \text{where} \qquad \omega(S) \ := \ \prod_{x \in S} \omega(x).$$

Theorem 1.15 (*Log-concavity for morphisms*, [EH20, Thm 1.3])**.** *Let $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$ be matroids, let $n := |X|$, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids. In addition, let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function, and let $1 \leq k < \mathrm{rk}(\mathcal{M})$. Then:*

$$\mathrm{B}_\omega(k)^2 \ \geq \ \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) \mathrm{B}_\omega(k-1) \, \mathrm{B}_\omega(k+1) \, . \tag{1.15}$$

Note that when $Y = \{y\}$ and $\mathcal{N} = (Y, \varnothing)$ is defined by $g(y) = 0$, we have condition (1.14) holds trivially and $\mathcal{B} = \mathcal{I}$. Thus, the theorem generalizes Theorem 1.3 to the morphism of matroids setting. We now give the corresponding generalization of Theorem 1.6.

Recall the equivalence relation "$\sim_S$" on the set $\mathrm{Cont}(S) \subseteq X \setminus S$ of continuations of $S \in \mathcal{I}$, see (1.4). Similarly, recall the set $\mathrm{Par}(S)$ of parallel classes of S, see (1.5). For every $1 \leq k \leq \mathrm{rk}(\mathcal{M})$, let

$$\mathrm{p}(k) \ := \ \max\left\{ \left|\mathrm{Par}(S)\right| \ : \ S \in \mathcal{B}_k \right\},$$

the maximum of the number of parallel classes of bases of morphism Φ of size k.

Theorem 1.16 (*Refined log-concavity for morphisms*)**.** *Let $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$ be matroids, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids. In addition, let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function, and let $1 \leq k < \mathrm{rk}(\mathcal{M})$. Then:*

$$\mathrm{B}_\omega(k)^2 \ \geq \ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1) \, - \, 1}\right) \mathrm{B}_\omega(k-1)\, \mathrm{B}_\omega(k+1). \tag{1.16}$$

As before, since $\mathrm{p}(k-1) \leq n - k + 1$, the theorem is an extension of Theorem 1.15.

Remark 1.17. The notion of morphism of matroids generalizes many classical notions in combinatorics such as *graph coloring*, *graph embeddings*, *graph homomorphism*, *matroid quotients*, and are a special case of the *induced matroids*. We refer to [EH20] for a detailed overview and further references (see also §16.8).

1.9. **Equality conditions for morphisms of matroids.** We start with the following characterization of equality in Theorem 1.15, which resolves an open problem in [MNY21, Question 5.7].

Theorem 1.18 (*Equality for morphisms*)**.** *Let $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$ be matroids, let $n := |X|$, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids. In addition, let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function, and let $1 \leq k < \mathrm{rk}(\mathcal{M})$. Suppose $\mathrm{B}_\omega(k) > 0$. Then:*

$$\mathrm{B}_\omega(k)^2 \ = \ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) \mathrm{B}_\omega(k-1)\, \mathrm{B}_\omega(k+1). \tag{1.17}$$

<u>if and only if</u> $\mathrm{girth}(\mathcal{M}) > k+1$, weight function ω is uniform, and $g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N})$ for all $S \in \mathcal{I}_{k-1}$.

Our next result is the following characterization of equality in Theorem 1.16.

Theorem 1.19 (*Refined equality for morphisms*)**.** *Let $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$ be matroids, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids. In addition, let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function, and let $1 \leq k < \mathrm{rk}(\mathcal{M})$. Suppose $\mathrm{B}_\omega(k) > 0$. Then:*

$$\mathrm{B}_\omega(k)^2 \ \geq \ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1) \, - \, 1}\right) \mathrm{B}_\omega(k-1)\, \mathrm{B}_\omega(k+1). \tag{1.18}$$

<u>if and only if</u> there exists $\mathrm{s}(k-1) > 0$, such that for every $S \in \mathcal{I}_{k-1}$ we have:

$$\left|\mathrm{Par}_S\right| \ = \ \mathrm{p}(k-1), \tag{MME1}$$

$$\sum_{x \in \mathcal{C}} \omega(x) \ = \ \mathrm{s}(k-1) \qquad \text{for every} \ \ \mathcal{C} \in \mathrm{Par}(S), \ \text{and} \tag{MME2}$$

$$g\big(\Phi(S)\big) \ = \ \mathrm{rk}(\mathcal{N}). \tag{MME3}$$

1.10. **Discrete polymatroids.** A *discrete polymatroid*[2] $\mathcal{D}$ is a pair $([n], \mathcal{J})$ of a ground set $[n] := \{1, \ldots, n\}$ and a nonempty finite collection $\mathcal{J}$ of integer points $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ that satisfy the following:

- (*hereditary property*) $\boldsymbol{a} \in \mathcal{I},\ \boldsymbol{b} \in \mathbb{N}^n$ s.t. $\boldsymbol{b} \leqslant \boldsymbol{a} \ \Rightarrow\ \boldsymbol{b} \in \mathcal{I}$, and
- (*exchange property*) $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{I},\ |\boldsymbol{a}| < |\boldsymbol{b}| \ \Rightarrow\ \exists i \in [n]$ s.t. $a_i < b_i$ and $\boldsymbol{a} + \boldsymbol{e}_i \in \mathcal{J}$.

Here $\boldsymbol{b} \leqslant \boldsymbol{a}$ is a componentwise inequality, $|\boldsymbol{a}| := a_1 + \ldots + a_n$, and $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is a standard linear basis in $\mathbb{R}^n$. When $\mathcal{J} \subseteq \{0,1\}^n$, discrete polymatroid $\mathcal{D}$ is a matroid. One can think of a discrete polymatroid as a set system where multisets are allowed, so we refer to $\mathcal{J}$ as *independent multisets* and to $|\boldsymbol{a}|$ as *size* of the multiset $\boldsymbol{a}$.

The role of bases in discrete polymatroids is played by *maximal elements* with respect to the order "$\leqslant$"; they are called *M-convex sets* in [BH20, §2]. Define $\mathrm{rk}(\mathcal{D}) := \max\{|\boldsymbol{a}| \,:\, \boldsymbol{a} \in \mathcal{J}\}$. For $0 \le k \le \mathrm{rk}(\mathcal{D})$, denote by $\mathcal{J}_k := \{\boldsymbol{a} \in \mathcal{J} \,:\, |\boldsymbol{a}| = k\}$ the subcollection of independent multisets of size k, and let $\mathrm{J}(k) := |\mathcal{J}_k|$.

Let $\omega : [n] \to \mathbb{R}_{>0}$ be a positive *weight function* on $[n]$. We extend weight function ω to all $\boldsymbol{a} \in \mathcal{J}$ as follows:

$$\omega(\boldsymbol{a}) \ := \ \omega(1)^{a_1} \cdots \omega(n)^{a_n}.$$

For every $0 \le k \le \mathrm{rk}(\mathcal{D})$, define

$$\mathrm{J}_\omega(k) \ := \ \sum_{\boldsymbol{a} \in \mathcal{J}_k} \frac{\omega(\boldsymbol{a})}{\boldsymbol{a}!}\,, \quad \text{where} \quad \boldsymbol{a}! := a_1! \cdots a_n!$$

Theorem 1.20 (*Log-concavity for polymatroids*, [BH20, Thm 3.10 (4) ⇔ (7)])**.** *Let $\mathcal{D} = ([n], \mathcal{J})$ be a discrete polymatroid, and let $\omega : [n] \to \mathbb{R}_{>0}$ be a positive weight function. For every $1 \le k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{J}_\omega(k)^2 \ \geq \left(1 + \frac{1}{k}\right) \mathrm{J}_\omega(k-1)\ \mathrm{J}_\omega(k+1). \tag{1.19}$$

We now give a common generalization of Theorem 1.6 and Theorem 1.20. Fix $t \in [0,1]$, and let

$$\pi(\boldsymbol{a}) \ := \ \sum_{i=1}^n \binom{a_i}{2}.$$

For every $0 \le k \le \mathrm{rk}(\mathcal{D})$, define

$$\mathrm{J}_{\omega,t}(k) \ := \ \sum_{\boldsymbol{a} \in \mathcal{J}_k} t^{\pi(\boldsymbol{a})}\, \frac{\omega(\boldsymbol{a})}{\boldsymbol{a}!}\,.$$

Note that $\binom{a}{2} = 0$ for $a \in \{0,1\}$, so $\pi(\boldsymbol{a}) = 0$ for all independent sets $\boldsymbol{a} \in \mathcal{I}$ in a matroid.

For an independent multiset $\boldsymbol{a} \in \mathcal{J}$ of a discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$, denote by

$$\mathrm{Cont}(\boldsymbol{a}) := \big\{ i \in [n] \ :\ \boldsymbol{a} + \boldsymbol{e}_i \in \mathcal{J} \big\}. \tag{1.20}$$

the set of *continuations* of $\boldsymbol{a}$. For all $i, j \in \mathrm{Cont}(\boldsymbol{a})$, we write $i \sim_{\boldsymbol{a}} j$ when $\boldsymbol{a} + \boldsymbol{e}_i + \boldsymbol{e}_j \notin \mathcal{J}$ or $i = j$. This is an equivalence relation again, see Proposition 4.2. We call an equivalence class of the relation $\sim_{\boldsymbol{a}}$ a *parallel class* of $\boldsymbol{a}$, and we denote by $\mathrm{Par}(\boldsymbol{a})$ the set of parallel classes of $\boldsymbol{a}$.

For every $0 \le k < \mathrm{rk}(\mathcal{D})$, define the *$k$-continuation number* of a discrete polymatroid $\mathcal{D}$ as the maximal number of parallel classes of independent multisets of size k:

$$\mathrm{p}(k) \ := \ \max\big\{ \, |\mathrm{Par}(\boldsymbol{a})| \ :\ \boldsymbol{a} \in \mathcal{J}_k \,\big\}. \tag{1.21}$$

For matroids, this is the same notion as defined above in §1.4.

Theorem 1.21 (*Refined log-concavity for polymatroids*)**.** *Let $\mathcal{D} = ([n], \mathcal{J})$ be a discrete polymatroid, and let $\omega : [n] \to \mathbb{R}_{>0}$ be a positive weight function. For every $t \in [0,1]$ and $1 \le k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{J}_{\omega,t}(k)^2 \ \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1-t}{\mathrm{p}(k-1) - 1 + t}\right) \mathrm{J}_{\omega,t}(k-1)\ \mathrm{J}_{\omega,t}(k+1). \tag{1.22}$$

[2] *Discrete polymatroids* are related but should not to be confused with *polymatroids*, which is a family of convex polytopes, see e.g. [Sch03, §44] and §16.9.

When $t = 1$, this gives Theorem 1.20. When $\mathcal{D}$ is a matroid and $t = 0$, this gives Theorem 1.6. For general discrete polymatroids $\mathcal{D}$ and $0 < t < 1$, this is a stronger result.

Example 1.22 (*Hypergraphical polymatroids*). Let $\mathcal{H} = (V, E)$ be a *hypergraph* on the finite set of vertices V, with *hyperedges* $E = \{e_1, \ldots, e_n\}$, where $e_i \subseteq V$, $e_i \neq \varnothing$. Let $W = \{w_1, \ldots, w_n\}$ be a collection of subsets of V, such that $w_i \subseteq e_i$, $w_i \neq \varnothing$, and every vertex $v \in V$ belongs to some w_i. A *hyperpath* is an alternating sequence $v \to w_i \to v' \to w_j \to v'' \to \ldots \to u$, where $v, v' \in w_i$, $v', v'' \in w_j$, etc., and the vertices $v, v', v'', \ldots, u \in V$ are not repeated.

A *spanning hypertree* in $\mathcal{H}$ is a collection W as above, such that every two vertices $v, u \in V$ are connected by exactly one such hyperpath. Similarly, a *spanning hyperforest* in $\mathcal{H}$ is a collection W as above, such that every two vertices are connected by at most one hyperpath. In the case all $|e_i| = 2$, we get the usual notions of (undirected) graphs, paths, spanning trees and spanning forests. We say that $\boldsymbol{d} = (d_1, \ldots, d_n)$, where $d_i = |w_i| - 1 \geq 0$, is a *degree sequence* of W. Note that in the graphical case, we have $d_i \in \{0, 1\}$, so a forest is determined by its degree sequence. In general hypergraphs this is no longer true.

Finally, a *hypergraphical polymatroid* corresponding to $\mathcal{H}$ is a discrete polymatroid $\mathcal{D}_{\mathcal{H}} = ([n], \mathcal{J})$, where $\mathcal{J}$ is a set of degree sequences of spanning hyperforests in $\mathcal{H}$. Similarly to graphical matroids (Example 1.29), the maximal elements are degree sequences of spanning hypertrees in $\mathcal{H}$. Therefore, Theorems 1.20 and 1.21 give log-concavity for the weighted sum $\mathrm{J}_{\omega,t}(k)$ over degree sequences with total degree $d_1 + \ldots + d_n = k$. See §16.10 for the background of this example.

1.11. **Equality conditions for polymatroids.** A discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$ is called *nondegenerate* if $\mathbf{e}_i \in \mathcal{J}$ for every $i \in [n]$. Define $\operatorname{polygirth}(\mathcal{D}) := \min\bigl\{k : \mathrm{J}(k) < \binom{n+k-1}{k-1}\bigr\}$. Observe that $\boldsymbol{a} \in \mathcal{J}$ for all $\boldsymbol{a} \in \mathbb{N}^k$, $|\boldsymbol{a}| < \operatorname{polygirth}(\mathcal{D})$. Note that the *polygirth* of a discrete polymatroid does not coincide with the girth of a matroid. In fact, $\operatorname{polygirth}(\mathcal{D}) = 2$ when $\mathcal{D}$ is a matroid with more than one element.

To get the equality conditions for (1.22), we separate the cases $t = 0$, $0 < t < 1$, and $t = 1$. The case $t = 0$ coincides with equality conditions for matroids given in Theorem 1.10. Examples in §1.7 show that this is a difficult condition with many nontrivial examples. The other two cases are in fact much less rich.

Theorem 1.23 (*Refined equality for polymatroids*, $t = 1$ case). *Let $\mathcal{D} = ([n], \mathcal{J})$ be a nondegenerate discrete polymatroid, let $\omega : [n] \to \mathbb{R}_{>0}$ be a positive weight function, and let $1 \leq k < \operatorname{rk}(\mathcal{M})$. Then:*

$$\mathrm{J}_\omega(k)^2 = \left(1 + \frac{1}{k}\right) \mathrm{J}_\omega(k-1)\, \mathrm{J}_\omega(k+1). \tag{1.23}$$

<u>if and only if</u> $\operatorname{polygirth}(\mathcal{D}) > (k+1)$.

We are giving the equality condition for (1.19) in place of (1.22), since $\mathrm{J}_{\omega,1}(k) = \mathrm{J}_\omega(k)$ for all k.

Theorem 1.24 (*Refined equality for polymatroids*, $0 < t < 1$ case). *Let $\mathcal{D} = ([n], \mathcal{J})$ be a nondegenerate discrete polymatroid, and let $\omega : [n] \to \mathbb{R}_{>0}$ be a positive weight function. Fix $1 \leq k < \operatorname{rk}(\mathcal{M})$ and $0 < t < 1$. Then:*

$$\mathrm{J}_{\omega,t}(k)^2 = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1-t}{\mathrm{p}(k-1) - 1 + t}\right) \mathrm{J}_{\omega,t}(k-1)\, \mathrm{J}_{\omega,t}(k+1). \tag{1.24}$$

<u>if and only if</u> $k = 1$, $\operatorname{polygirth}(\mathcal{D}) > 2$, and ω is uniform.

Remark 1.25. The reason the case $t = 0$ is substantially different, is because the combined weight function $t^{N(\boldsymbol{a})} \omega(\boldsymbol{a})$ is no longer strictly positive. Alternatively, one can view the dearth of nontrivial examples in these theorems as suggesting that the bound in Theorem 1.21 can be further improved for $t > 0$. This is based on the reasoning that Theorem 1.4 sharply improves over Theorem 1.3 because there are only trivial equality conditions for the latter (see Theorem 1.8), when compared with rich equality conditions for the former (see Theorem 1.9).

1.12. **Poset antimatroids.** Let X be finite set we call *letters*, let $n = |X|$, and let X^* be a set of *finite words* in the alphabet X. A *language over X* is a nonempty finite subset $\mathcal{L} \subset X^*$. A word is called *simple* if it contains each letter at most once; we consider only simple words from this point on. We write $x \in \alpha$ if word $\alpha \in \mathcal{L}$ contains letter x. Finally, let $|\alpha|$ be the length of the word, and denote $\mathcal{L}_k := \big\{\alpha \in \mathcal{L} \, : \, |\alpha| = k\big\}$.

A pair $\mathcal{A} = (X, \mathcal{L})$ is an *antimatroid*, if the language $\mathcal{L} \subset X^*$ satisfies:

- (*nondegenerate property*) every $x \in X$ is contained in at least one $\alpha \in \mathcal{L}$,
- (*normal property*) every $\alpha \in \mathcal{L}$ is simple,
- (*hereditary property*) $\alpha\beta \in \mathcal{L} \ \Rightarrow \ \alpha \in \mathcal{L}$, and
- (*exchange property*) $x \in \alpha, \ x \notin \beta$, and $\alpha, \beta \in \mathcal{L} \ \Rightarrow \ \exists y \in \alpha$ s.t. $\beta y \in \mathcal{L}$.

Note that for every antimatroid $\mathcal{A} = (X, \mathcal{L})$, it follows from the exchange property that

$$\operatorname{rk}(\mathcal{A}) \, := \, \max\{|\alpha| \, : \, \alpha \in \mathcal{L}\} \, = \, n.$$

Throughout the paper we use only one class of antimatroids which we now define (cf. §16.14).

Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements. A simple word $\alpha \in X^*$ is called *feasible* if α satisfies:

- (*poset property*) if α contains $x \in X$ and $y \prec x$, then letter y occurs before letter x in α.

A *poset antimatroid* $\mathcal{A}_{\mathcal{P}} = (X, \mathcal{L})$ is defined by the language $\mathcal{L}$ of all feasible words in X. The exchange property is satisfied because one can always take y to be the minimal letter (w.r.t. order $\prec$) that is not in β.

Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function on X. Denote by $\operatorname{Cov}(x) := \{y \in X \, : \, x \lessdot y\}$ the set of elements which cover x. We assume the weight function ω satisfies the following (*cover monotonicity property*):

$$\omega(x) \, \geq \sum_{y \in \operatorname{Cov}(x)} \omega(y)\,, \quad \text{for all} \ \ x \in X. \tag{CM}$$

Note that when (CM) is equality for all $x \in X$, we have:

$$\omega(x) \, = \, \text{number of maximal chains in } \mathcal{P} \text{ starting at } x. \tag{1.25}$$

For all $\alpha \in \mathcal{L}$ and $0 \leq k \leq n$, let

$$\mathrm{L}_\omega(k) \, := \sum_{\alpha \in \mathcal{L}_k} \omega(\alpha)\,, \quad \text{where} \quad \omega(\alpha) \, := \, \prod_{x \in \alpha} \omega(x)\,.$$

Theorem 1.26 (*Log-concavity for poset antimatroids*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\mathcal{A}_{\mathcal{P}} = (X, \mathcal{L})$ be the corresponding poset antimatroid. Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function which satisfies* (CM)*. Then, for every integer $1 \leq k < n$, we have:*

$$\mathrm{L}_\omega(k)^2 \, \geq \, \mathrm{L}_\omega(k-1) \cdot \mathrm{L}_\omega(k+1)\,. \tag{1.26}$$

Example 1.27 (*Standard Young tableaux of skew shape*)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, be a Young diagram, and let $\mathcal{P}_\lambda = (\lambda, \prec)$ be a poset on squares $\big\{(i-1, j-1) \, : \, 1 \leq i \leq \lambda_j, 1 \leq j \leq \ell\big\} \subset \mathbb{N}^2$, with $(i,j) \preccurlyeq (i', j')$ if $i \geq i'$ and $j \geq j'$. Following (1.25), let $\omega(i,j) = \binom{i+j}{i}$. Denote $a_\lambda(k) := \mathrm{L}_\omega(k)$, $0 \leq k \leq |\lambda|$, and we have:

$$a_\lambda(k) \, = \sum_{\mu \subset \lambda, \, |\lambda/\mu| = k} f^{\lambda/\mu} \prod_{(i,j) \in \lambda/\mu} \binom{i+j}{i},$$

where $f^{\lambda/\mu} = \big|\mathrm{SYT}(\lambda/\mu)\big|$ is the number of *standard Young tableaux* of shape λ/μ (see §16.15). Now Theorem 1.26 proves that the sequence $\big\{a_\lambda(k)\big\}$ is log-concave, for every λ.

This example also shows that the weight function condition (CM) is necessary. Indeed, let λ be a $m \times m$ square, $n = m^2$, and let $\omega(i,j) = 1$. Then, for all $k \leq m$, we have:

$$b(k) \, := \, \mathrm{L}_\omega(k) \, = \, \big|\mathcal{L}_k\big| \, = \sum_{\mu \vdash k} f^\mu\,.$$

The sequence $\{b_k\}$ is the *number of involutions in S_k*, see e.g. [OEIS, A000085], which satisfies $\log b_k = \frac{1}{2} n \log n + O(n)$, and is actually *log-convex*, see e.g. [Mező20, §4.5.2].

1.13. **Equality conditions for poset antimatroids.** Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\mathcal{A}_\mathcal{P} = (X, \mathcal{L})$ be the corresponding poset antimatroid.

For a word $\alpha \in \mathcal{L}$, denote by

$$\mathrm{Cont}(\alpha) := \{x \in X \,:\, \alpha x \in \mathcal{L}\}$$

the set of *continuations* of the word α. Define an equivalence relation "$\sim_\alpha$" on $\mathrm{Cont}(\alpha)$ by setting $x \sim_\alpha y$ if $\alpha x y \notin \mathcal{L}$, see Proposition 4.3. We call the equivalence classes of "$\sim_\alpha$" the *parallel classes* of α, and denote by $\mathrm{Par}(\alpha)$ the set of these parallel classes.

Let $\alpha \in \mathcal{L}$ and $x \in \mathrm{Cont}(\alpha)$. We say that $y \in X$ is a *descendent* of x with respect to α if $\alpha x y \in \mathcal{L}$ and $\alpha y \notin \mathcal{L}$. Denote by $\mathrm{Des}_\alpha(x)$ the set of descendants of x with respect to α. We omit α when the word is clear from the context.

Theorem 1.28 (*Equality for poset antimatroids*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\mathcal{A}_\mathcal{P} = (X, \mathcal{L})$ be the corresponding poset antimatroid. Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function which satisfies* (CM)*, and fix an integer $1 \le k < n$. Then:*

$$\mathrm{L}_\omega(k)^2 \;=\; \mathrm{L}_\omega(k-1) \cdot \mathrm{L}_\omega(k+1) \tag{1.27}$$

<u>if and only if</u> there exists $\mathrm{s}(k-1) > 0$, such that for every $\alpha \in \mathcal{L}_{k-1}$ and $x \in \mathrm{Cont}(\alpha)$, we have:

$$\sum_{x \in \mathrm{Cont}(\alpha)} \omega(x) \;=\; \mathrm{s}(k-1), \tag{AE1}$$

$$\mathrm{Des}_\alpha(x) \;=\; \mathrm{Cov}(x), \quad \textit{and} \tag{AE2}$$

$$\sum_{y \in \mathrm{Cov}(x)} \omega(y) \;=\; \omega(x). \tag{AE3}$$

The following is an example of a poset that satisfies conditions of Theorem 1.26.

Example 1.29 (*Tree posets*)**.** Let $\mathrm{T} = (V, E)$ be a finite *rooted tree* with root at $R \in V$, and the set of leaves $S \subset V$. Suppose further, that all leaves $v \in S$ are at distance h from R. Consider a poset $\mathcal{P}_\mathrm{T} = (V, \prec)$ with $v \prec v'$ if the shortest path $v' \to R$ goes through v, for all $v, v' \in V$. We call $\mathcal{P}_\mathrm{T}$ the *tree poset* corresponding to T. Denote by $S(v) := S \cap \{v' \in V \,:\, v' \succcurlyeq v\}$ the subset of leaves in the order ideal of v.

Let $\omega : X \to \mathbb{R}_{>0}$ be defined by (1.25). Observe that $\omega(v) = \big|S(v)\big|$, since maximal chains in $\mathcal{P}_\mathrm{T}$ are exactly the shortest paths in T towards one of the leaves, i.e. of the form $v \to w$ for some $w \in S$. Note that $S(v) \supseteq S(v')$ for all $v \prec v'$, $S(v) \cap S(v') = \varnothing$ for all v and v' that are incomparable, and $\sum_{x \in \mathrm{Cov}(v)} |S(x)| = |S(v)|$ for all $v \notin S$. These imply (AE1)–(AE3) for all $k \le h$, with $\mathrm{s}(k-1) = |S|$. By Theorem 1.26, we get an equality (1.27) in this case.

The following result shows the importance of tree posets for the equality conditions.

Theorem 1.30 (*Total equality for poset antimatroids*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\mathcal{A}_\mathcal{P} = (X, \mathcal{L})$ be the corresponding poset antimatroid. Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function which satisfies* (CM)*. Then:*

$$\mathrm{L}_\omega(k)^2 \;=\; \mathrm{L}_\omega(k-1) \cdot \mathrm{L}_\omega(k+1) \quad \textit{for all} \;\; 1 \le k < \mathrm{height}(\mathcal{P}) \tag{1.28}$$

<u>if and only if</u> $\mathcal{P} \cup \widehat{0}$ is a tree poset $\mathcal{P}_\mathrm{T}$ with a root at $\widehat{0}$, with all leaves at the same distance to the root, and such that $c\,\omega$ is defined by (1.25)*, for some constant multiple $c > 0$.*

1.14. **Interval greedoids.** Let X be finite set of letters, and let $\mathcal{L} \subset X^*$ be a language over X. A pair $\mathcal{G} = (X, \mathcal{L})$ is a *greedoid*, if the language $\mathcal{L}$ satisfies:

- (*nondegenerate property*) empty word $\varnothing$ is in $\mathcal{L}$,
- (*normal property*) every $\alpha \in \mathcal{L}$ is simple,
- (*hereditary property*) $\alpha\beta \in \mathcal{L} \;\Rightarrow\; \alpha \in \mathcal{L}$, and
- (*exchange property*) $\alpha, \beta \in \mathcal{L}$ s.t. $|\alpha| > |\beta| \;\Rightarrow\; \exists x \in \alpha$ s.t. $\beta x \in \mathcal{L}$.

Let $\operatorname{rk}(\mathcal{G}) := \max\{|\alpha| : \alpha \in \mathcal{L}\}$ be the *rank* of greedoid $\mathcal{G}$. Note that every maximal word in $\mathcal{L}$ has the same length by the exchange property. In the literature, greedoids are also defined via *feasible sets* of letters in $\alpha \in \mathcal{L}$, but we restrict ourselves to the language notation. We use [BZ92, §8.2.B] and [KLS91, §V.5] as our main references on interval greedoids; see also §16.13 for some background.

Greedoid $\mathcal{G} = (X, \mathcal{L})$ is called *interval* if the language $\mathcal{L}$ also satisfies:

- (*interval property*) $\alpha, \beta, \gamma \in X^*,\ x \in X$ s.t. $\alpha x, \alpha\beta\gamma x \in \mathcal{L} \quad \Rightarrow \quad \alpha\beta x \in \mathcal{L}$.

It is well known and easy to see that antimatroids are interval greedoids.

Let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a positive weight function. Let

$$\mathrm{L}_{\mathrm{q}}(k) \ = \ \sum_{\alpha \in \mathcal{L}_k} \mathrm{q}(\alpha).$$

In the next section, we define the notion of *k-admissible* weight function q, see Definition 3.2. This notion is much too technical to state here. We use it to formulate our first main result:

Theorem 1.31 (*Log-concavity for interval greedoids*, first main theorem)**.** *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid, let $1 \leq k < \operatorname{rk}(\mathcal{G})$, and let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a k-admissible weight function. Then:*

$$\mathrm{L}_{\mathrm{q}}(k)^2 \ \geq \ \mathrm{L}_{\mathrm{q}}(k-1) \cdot \mathrm{L}_{\mathrm{q}}(k+1). \tag{1.29}$$

This is the first main result of the paper, as it implies all previous inequalities for matroids, polymatroids and antimatroids.

Example 1.32 (*Directed branching greedoids*)**.** Let $G = (V, E)$ be a directed graph on $|V| = n$ vertices strongly connected towards the root $R \in V$. An *arborescence* is a tree in G strongly connected towards the root R. A word $\alpha = e_1 \cdots e_\ell \in E^*$ is called *pointed* if every prefix of α consists of edges which form an arborescence. One can think of pointed words as *increasing arborescences* in G (cf. §16.16).

The *directed branching greedoid* $\mathcal{G}_G = (E, \mathcal{L})$ is defined on the ground sets E by the language $\mathcal{L} \subset E^*$ of pointed words. It is well known and easy to see that $\mathcal{G}_G$ is an interval greedoid. When $G = T$ is a rooted tree, greedoid $\mathcal{G}_T$ is the poset antimatroid corresponding to the tree poset $\mathcal{P}_P$ (see Example 1.29). For general graphs, greedoid $\mathcal{G}_G$ is not necessarily a poset antimatroid. Theorem 1.31 in this case proves log-concavity for the numbers $\mathrm{L}_{\mathrm{q}}(k)$ of weighted increasing arborescences, cf. §16.16.

1.15. **Equality conditions for interval greedoids.** A word $\beta \in X^*$ is called a *continuation* of the word $\alpha \in \mathcal{L}$, if $\alpha\beta \in \mathcal{L}$. Denote by $\operatorname{Cont}_k(\alpha) \subset X^*$ the set of continuations of the word α with $\beta \in X^*$ of length $|\beta| = k$. Note that $\operatorname{Cont}(\alpha) = \operatorname{Cont}_1(\alpha)$. For notational convenience, we define $\operatorname{Cont}(\alpha) = \varnothing$ if $\alpha \notin \mathcal{L}$.

For every $\alpha \in \mathcal{L}$, let

$$\mathrm{L}_{\mathrm{q},\alpha}(k) \ := \sum_{\beta \in \operatorname{Cont}_k(\alpha)} \mathrm{q}(\alpha\beta).$$

Note that $\mathrm{L}_{\mathrm{q}}(k) = \mathrm{L}_{\mathrm{q},\varnothing}(k)$ and $\mathrm{L}_{\mathrm{q},\alpha}(0) = \mathrm{q}(\alpha)$.

Theorem 1.33 (*Equality for interval greedoids*, cf. Theorem 3.3)**.** *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid, let $1 \leq k < \operatorname{rk}(\mathcal{G})$, and let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a k-admissible weight function. Then:*

$$\mathrm{L}_{\mathrm{q}}(k)^2 \ = \ \mathrm{L}_{\mathrm{q}}(k-1) \cdot \mathrm{L}_{\mathrm{q}}(k+1)$$

if and only if there is $\mathrm{s}(k-1) > 0$, such that for every $\alpha \in \mathcal{L}_{k-1}$ we have:

$$\mathrm{L}_{\mathrm{q},\alpha}(2) \ = \ \mathrm{s}(k-1)\, \mathrm{L}_{\mathrm{q},\alpha}(1) \ = \ \mathrm{s}(k-1)^2\, \mathrm{L}_{\mathrm{q},\alpha}(0).$$

This is the second main result of the paper, giving an easy way to check the equality conditions. A more detailed and technical condition is given in Theorem 3.3, which we use to obtain the equality conditions for matroids, polymatroids and antimatroids.

1.16. **Linear extensions.** Let $\mathcal{P} := (X, \prec)$ be a poset on $n := |X|$ elements. A *linear extension* of $\mathcal{P}$ is a bijection $L : X \to \{1, \ldots, n\}$, such that $L(x) < L(y)$ for all $x \prec y$. Fix an element $z \in X$. Denote by $\mathcal{E} := \mathcal{E}(P)$ the set of linear extensions of $\mathcal{P}$, let $\mathcal{E}_k := \{L \in \mathcal{E} \, : \, L(z) = k\}$, and let $e(\mathcal{P}) := |\,\mathcal{E}\,|$. See §16.17 and §16.18 for some background.

Theorem 1.34 (*Stanley inequality* [Sta81, Thm 3.1])**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements, and let $z \in X$. Denote by $\mathrm{N}(k) := |\,\mathcal{E}_k\,|$ the number of linear extensions $L \in \mathcal{E}(P)$, such that $L(z) = k$. Then, for every $1 < k < n$, we have:*

$$\mathrm{N}(k)^2 \;\geq\; \mathrm{N}(k-1) \cdot \mathrm{N}(k+1). \tag{1.30}$$

We now give a weighted generalization of this result. Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function on X. We say that ω is *order-reversing* if it satisfies

$$x \preccurlyeq y \quad \Rightarrow \quad \omega(x) \,\geq\, \omega(y). \tag{Rev}$$

Fix $z \in X$, as above. Define $\omega : \mathcal{E} \to \mathbb{R}_{>0}$ by

$$\omega(L) \;:=\; \prod_{x \,:\, L(x) < L(z)} \omega(x), \tag{1.31}$$

and let

$$\mathrm{N}_\omega(k) \;:=\; \sum_{L \in \mathcal{E}_k} \omega(L)\,, \quad \text{for all } \ 1 \leq k \leq n. \tag{1.32}$$

Theorem 1.35 (*Weighted Stanley inequality*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements, and let $\omega : X \to \mathbb{R}_{>0}$ be a positive order-reversing weight function. Fix an element $z \in X$. Then, for every $1 < k < n$, we have:*

$$\mathrm{N}_\omega(k)^2 \;\geq\; \mathrm{N}_\omega(k-1) \cdot \mathrm{N}_\omega(k+1), \tag{1.33}$$

where $\mathrm{N}_\omega(k)$ is defined by (1.32).

Remark 1.36. In §14.8, we give further applications of our approach by extending the set of possible weights in Theorem 1.35 to a smaller class of *posets with belts*. We postpone this discussion to avoid cluttering, but the interested reader is encouraged to skip to that subsection which can be read separately from the rest of the paper.[3]

1.17. **Two permutation posets examples.** It is not immediately apparent that the numbers of linear extensions appear widely across mathematics. Below we present two notable examples from algebraic and enumerative combinatorics, see §16.19 for some background.

Example 1.37 (*Bruhat orders*)**.** Let $\sigma \in S_n$ and define the *permutation poset* $\mathcal{P}_\sigma = ([n], \prec)$ by letting

$$i \preccurlyeq j \quad \Leftrightarrow \quad i \leq j \ \text{ and } \ \sigma(i) \leq \sigma(j).$$

Fix $z \in [n]$. Viewing $\mathcal{E} = \mathcal{E}(\mathcal{P}_\sigma)$ as a subset of S_n, it is easy to see that $\mathcal{E}$ is the lower ideal of σ in the (weak) *Bruhat order* $\mathcal{B}_n = (S_n, \lhd)$. Thus, $\mathcal{E}_k = \big\{\nu \in S_n \, : \, \nu(z) = k, \, \nu \unlhd \sigma\big\}$.

Let $\omega(i) = q^i$, where $0 < q < 1$. Then ω is order-reversing. Now (1.31) gives $\omega(\nu) = q^{\beta(\nu)}$, where

$$\beta(\nu) \;:=\; \sum_{i=1}^{z-1} \, i \cdot \chi\big(k - \nu(i)\big) \quad \text{ and } \quad \chi(t) \;:=\; \begin{cases} 1 & \text{if } \ t > 0 \\ 0 & \text{if } \ t \leq 0 \end{cases}$$

Now Theorem 1.35 gives log-concavity $a_q(k)^2 \geq a_q(k-1) \cdot a_q(k+1)$, where $a_q(k) := \mathrm{N}_\omega(k) \geq 0$ is given by

$$a_q(k) \;=\; \sum_{\nu \in S_n \,:\, \nu \unlhd \sigma, \, \nu(z) = k} q^{\beta(\nu)}.$$

[3] In a followup investigation, we use the combinatorial atlas technology in [CP22b] to prove correlation inequalities for the numbers of linear extensions of posets.

Example 1.38 (*Euler–Bernoulli* and *Entringer numbers*)**.** Let $\mathcal{Q}_m = ([2m-1], \prec)$ be a height two poset corresponding to the skew Young diagram δ_m/δ_{m-2}, where $\delta_m := (m, \ldots, 2, 1)$. The linear extensions of $\mathcal{Q}_m$ are in natural bijection with *alternating permutations* $\sigma \in S_{2m-1}$ s.t. $\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) < \ldots$ Then the numbers $e(\mathcal{Q}_m)$ are the *Euler numbers*, which are closely related to the *Bernoulli numbers*, and have EGF

$$\sum_{m=1}^{\infty} (-1)^{m-1}\, e(\mathcal{Q}_m)\, \frac{t^{2m-1}}{(2m-1)!} \, = \, \tan(t)\,,$$

see e.g. [OEIS, A000111]. Fix $z=1$. It is easy to see that triangle of numbers $a(m,k) = \big|\mathcal{E}_k(\mathcal{Q}_m)\big|$ are *Entringer numbers* [OEIS, A008282], and Stanley's Theorem 1.34 proves their log-concavity:

$$a(m,k)^2 \, \geq \, a(m,k-1)\, a(m,k+1) \quad \text{for} \ \ 1 \leq k \leq 2m-2.$$

Now, let $\omega(2) = \omega(4) = \ldots = 1$, $\omega(1) = \omega(3) = \ldots = q$, where $0 < q < 1$. Similarly to the previous example, we have $\omega(\sigma) = q^{\gamma(\sigma)}$, where $\gamma(\sigma)$ is the number of permutation entries in the odd positions which are $< k$. Theorem 1.35 then proves log-concavity for the corresponding q-deformation of the Entringer numbers.

1.18. **Equality conditions for linear extensions.** Let $\mathcal{P} := (X, \prec)$ be a poset on $|X| = n$ elements. Denote by $f(x) := \big|\{y \in X \,:\, y \prec x\}\big|$ and $g(x) := \big|\{y \in X \,:\, y \succ x\}\big|$ the sizes of lower and upper ideals of $x \in X$, respectively, excluding the element x.

Theorem 1.39 (*Equality condition for Stanley inequality* [SvH20, Thm 15.3])**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements. Let $z \in X$ and let $\mathrm{N}(k)$ be the number of linear extensions $L \in \mathcal{E}(P)$, such that $L(z) = k$. Suppose that $\mathrm{N}(k) > 0$. Then the following are equivalent:*

- (a) $\mathrm{N}(k)^2 \, = \, \mathrm{N}(k-1) \cdot \mathrm{N}(k+1)$,
- (b) $\mathrm{N}(k+1) \, = \, \mathrm{N}(k) \, = \, \mathrm{N}(k-1)$,
- (c) *we have $f(x) > k$ for all $x \succ z$, and $g(x) > n-k+1$ for all $x \prec z$.*

See §16.22 for some background. The weighted version of this theorem is a little more subtle and needs the following *(s,k)-cohesiveness property*:

(Coh) $$\omega\big(L^{-1}(k-1)\big) \, = \, \omega\big(L^{-1}(k+1)\big) \, = \, \mathrm{s}, \quad \text{for all} \ \ L \in \mathcal{E}_k\ .$$

Note that (Coh) can hold for non-uniform weight functions ω, for example for $\mathcal{P} = A_{k+1} \oplus C_{n-k-1}$, i.e. the *linear sum* of an antichain on which ω is uniform and a chain on which ω can be non-uniform. In fact, if z is an element in A_{k+1}, we can have $\omega(z)$ different from the rest of the antichain.

Theorem 1.40 (*Equality condition for weighted Stanley inequality*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\omega : X \to \mathbb{R}_{>0}$ be a positive order-reversing weight function. Fix element $z \in X$ and let $\mathrm{N}_\omega(k)$ be defined as in (1.32). Suppose that $\mathrm{N}_\omega(k) > 0$. Then the following are equivalent:*

- (a) $\mathrm{N}_\omega(k)^2 \, = \, \mathrm{N}_\omega(k-1) \cdot \mathrm{N}_\omega(k+1)$,
- (b) *there exists $\mathrm{s} = \mathrm{s}(k,z) > 0$, s.t.*
$$\mathrm{N}_\omega(k+1) \, = \, \mathrm{s}\, \mathrm{N}_\omega(k) \, = \, \mathrm{s}^2\, \mathrm{N}_\omega(k-1),$$
- (c) *there exists $\mathrm{s} = \mathrm{s}(k,z) > 0$, s.t. $f(x) > k$ for all $x \succ z$, $g(x) > n-k+1$ for all $x \prec z$, and* (Coh).

1.19. **Summary of results and implications.** Here is a chain of matroid results from new to known:

$$\text{Thm 1.6} \ \Rightarrow \ \text{Thm 1.4} \ \Rightarrow \ \text{Thm 1.3} \ \Rightarrow \ \text{Thm 1.2} \ \Rightarrow \ \text{Thm 1.1}.$$

The first two of these introduce the *refined log-concave inequalities*, both weighted and unweighted, and they imply the last three known theorems. For morphisms of matroids and for polymatroids, we have two new results which extend two earlier results:

$$\text{Thm 1.16} \ \Rightarrow \ \text{Thm 1.15} \qquad \text{and} \qquad \text{Thm 1.21} \ \Rightarrow \ \text{Thm 1.20}.$$

Here is a family of implications of log-concave inequalities across matroid generalizations, from interval greedoids to polymatroids to matroids, and from interval greedoids to poset antimatroids:

$$\text{Thm 1.31} \ \Rightarrow_{\S 4.4} \ \text{Thm 1.21} \ \Rightarrow_{\S 1.10} \ \text{Thm 1.6} \qquad \text{and} \qquad \text{Thm 1.31} \ \Rightarrow_{\S 4.2} \ \text{Thm 1.26}.$$

All these results are new. Note that both polymatroids and poset antimatroids are different special cases of interval greedoids, while our results on morphisms of matroids are separate and do not generalize.

For the equality conditions, we have a similar chain of implications across matroid generalizations:

Thm 3.3 $\Rightarrow$ Thm 1.33 $\Rightarrow$ Thm 1.24 $\cup$ Thm 1.23 $\Rightarrow$ Thm 1.10 $\Rightarrow$ Thm 1.9 $\Rightarrow$ Thm 1.8,
Thm 1.19 $\Rightarrow$ Thm 1.18 and Thm 3.3 $\Rightarrow$ Thm 1.28 $\Rightarrow$ Thm 1.30.

Of these, only Theorem 1.8 was previously known. The most general of these, Theorem 3.3, is too technical to be stated in the introduction. The same holds for Definition 3.2 needed in Theorem 1.31. We postpone both the definition and the general theorem until Section 3.

Finally, for the Stanley inequality and its equality conditions, we have:

Thm 1.35 $\Rightarrow$ Thm 1.34 and Thm 15.1 $\Rightarrow$ Thm 1.40 $\Rightarrow$ Thm 1.39.

In both cases, more general results are new and correspond to the case of weighted linear extensions.

Let us emphasize that while some of these implications are trivial or follow immediately from definitions, others are more involved and require a critical change of notation and some effort to verify certain poset and weight function properties. These implications are discussed in Section 4.

1.20. **Proof ideas.** Although we prove multiple results, the proof of each log-concavity inequality uses the same approach and technology, so we refer to it as "the proof".

At the first level, the proof is an inductive argument proving a stronger claim about eigenvalues of certain matrices associated with the posets. The induction is not over posets of smaller size, but over other matrices which can in fact be larger, but correspond to certain parameters decreasing as we go along. The claim then reduces to the base of induction, which is the only part of the proof requiring a computation. The latter involves checking eigenvalues of explicitly written small matrices, making the proof fully elementary.

Delving a little deeper, we set up a new type of structure which we call a *combinatorial atlas*. In the special case of greedoids, a combinatorial atlas $\mathbb{A}$ associated with a greedoid $\mathcal{G} = (X, \mathcal{L})$, $|X| = n$, is comprised of:

- acyclic digraph $\Gamma_{\mathcal{G}} = (\mathcal{L}, \Theta)$, with the unique source at the empty word $\varnothing \in \mathcal{L}$, and edges corresponding to multiplications by a letter: $\Theta = \big\{(\alpha, \alpha x) \, : \, \alpha, \alpha x \in \mathcal{L}, \, x \in X\big\}$,
- each vertex $\alpha \in \mathcal{L}$ is associated with a pair $(\mathbf{M}_\alpha, \mathbf{h}_\alpha)$, where $\mathbf{M}_\alpha = \big(\mathrm{M}_{ij}\big)$ is a nonnegative symmetric $d \times d$ matrix, $\mathbf{h}_\alpha = (\mathrm{h}_1, \ldots, \mathrm{h}_d)$ is a nonnegative vector, and $d = n+1$,
- each edge $(\alpha, \alpha x) \in \Theta$ is associated with a linear transformation $\mathbf{T}_\alpha^{\langle x \rangle} : \mathbb{R}^d \to \mathbb{R}^d$.

The key technical observation is that under certain conditions on the atlas, we have every matrix $\mathbf{M} := \mathbf{M}_\alpha$, $\alpha \in \mathcal{L}$, is *hyperbolic*:

$$\text{(Hyp)} \qquad \langle \mathbf{v}, \mathbf{M}\mathbf{w}\rangle^2 \ \geq \ \langle \mathbf{v}, \mathbf{M}\mathbf{v}\rangle\langle \mathbf{w}, \mathbf{M}\mathbf{w}\rangle \quad \text{for every} \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^d, \quad \text{such that} \quad \langle \mathbf{w}, \mathbf{M}\mathbf{w}\rangle > 0.$$

Log-concavity inequalities now follow from (Hyp) for the matrix $\mathbf{M}_\varnothing$, by interpreting the inner products as numbers $\mathrm{L_q}(k)$, $\mathrm{L_q}(k-1)$ and $\mathrm{L_q}(k+1)$, respectively.

We prove (Hyp) by induction, reducing the claim for $\mathbf{M}_\alpha$ to that of $\mathbf{M}_{\alpha x}$, for all $x \in \mathrm{Cont}(\alpha)$. Proving (Hyp) for the base of induction required the eigenvalue interlacing argument, cf. §17.5. This is where our conditions for the weight function ω appear in the calculation. We also need a few other properties of the atlas. Notably, we require every matrix $\mathbf{M}_\alpha$ to be *irreducible* with respect to its support, but that is proved by a direct combinatorial argument.

For other log-concavity inequalities in the paper, we consider similar atlas constructions and similar claims. For the equalities, we works backwards and observe that we need equations (Hyp) to be equalities. These imply the local properties which must hold for certain edges $(\alpha, \alpha x) \in \Theta$. Analyzing these properties gives the equality conditions we present.

1.21. **Discussion.** Skipping over the history of the subject (see Section 16), in recent years a great deal of progress on the subject was made by Huh and his coauthors. In fact, until the celebrated Adiprasito–Huh–Katz paper [AHK18], even the log-concavity for the number of k-forests (Welsh–Mason conjecture for graphical matroids), remained open. That paper was partially based on the earlier work [Huh12, Huh15, HK12], and paved a way to a number of further developments, most notably [ADH20, BES19, BST20, B+20a, B+20b, HSW22, HW17].

From the traditional order theory point of view, the level of algebra used in these works overwhelms the senses. The inherent rigidity of the original algebraic approach required either to extend the algebra as in the papers above, or to downshift in the technology. The *Lorentzian polynomials* approach developed by Brändén–Huh [BH18, BH20] and by Anari et. al [ALOV18] allowed stronger results such as Theorem 1.3 and led to further results and applications such as [ALOV19, BLP20, HSW22, MNY21]. This paper represented the first major downshift in the technology.

($\circ$) A casual reader can be forgiven in thinking of this paper as a successful deconstruction of the Lorentzian polynomials into the terminology of linear algebra. This is the opposite of what happens both mathematically and philosophically. Our approach does in fact contain much of the Lorentzian polynomials approach as a special case (cf. §17.9). This can be made precise, but we postpone that discussion until [CP22a].

However, viewing greedoids and its special cases as *languages* allows us to reach far beyond what the Lorentzian polynomials possibly can.[4] To put this precisely, our maps $\mathbf{T}_\alpha^{\langle x\rangle}$ have a complete flexibility in their definition. In the world of Lorentzian polynomials, the corresponding maps are trivial. We trade the elegance of that approach to more complexity, flexibility and strength.

($\circ$) The true origin of our "combinatorial atlas" technology lies in our deconstruction of the Stanley inequality (1.30). This is both one of oldest and the most mysterious results in the area, and our proof is elementary but highly technical, more so than our proof of greedoid results.

To understand the conundrum Stanley's inequality represents, consider the original proof in [Sta81] which is barely a page long via a simple reduction to the classical *Alexandrov–Fenchel inequality*. The latter is a fundamental result on the subject, with many different proofs across the fields, all of them difficult (see §16.20). This difficulty represented the main obstacle in obtaining an elementary proof of Stanley's inequality.

($\circ$) Most recently, the new proof of the Alexandrov–Fenchel inequality by Shenfeld and van Handel [SvH19] using "Bochner formulas", renewed our hopes for the elementary proof of Stanley's inequality. Their proof exploits the finiteness of the set of normals to polytope facets in a very different way from Alexandrov's original approach in [Ale38], see discussion in [SvH19, §6.1]. Our next point of inspiration was a most recent paper [SvH20] by Shenfeld and van Handel, where the authors obtain the equality conditions for Stanley's inequality (see Theorem 1.39) with applications to Stanley's inequality (cf. §17.11).

Deconstruction of [SvH19, SvH20] combined with ideas from [BH20, Sta81] and our earlier work [CPP22a, CPP21], led to our "combinatorial atlas" approach. Both the Stanley inequality and the conditions for equality followed from our linear algebra setting and became amenable to generalizations. Part of the reason for this is the explicit construction of maps $\mathbf{T}_\alpha^{\langle x\rangle}$, which for convex polytopes are shown in [SvH19] to exist only indirectly albeit in greater generality, see also §17.6.

($\circ$) Now, once we climbed the mountain of Stanley's inequality by means of the new technology, going down to poset antimatroids, polymatroids and matroids became easier. Our ultimate extension to interval greedoids required additional effort, as evidenced in the technical definitions in Section 3. Furthermore, our approach retained the flexibility of allowing us to match the results with equality conditions.

($\circ$) In conclusion, let us mention that the ultimate goal we set out in [Pak19], remains unresolved. There, we observed that the Adiprasito–Huh–Katz inequalities for graphs and Stanley inequalities for numbers of linear extensions correspond to nonnegative integer functions in GapP = #P − #P. We asked whether these functions are themselves in #P. This amounts to finding a *combinatorial interpretation* for the difference of the LHS and the RHS of these inequalities. While we use only elementary tools, the eigenvalue based argument is not direct enough to imply a positive answer. See §17.17 for more on this problem.

[4] Lest one think to use a straightforward generalization to noncommutative polynomials, try imagining the right notion of a *partial derivative* which plays a crucial role in [ALOV18, BH20].

1.22. **Paper structure.** We start with basic definitions and notions in Section 2. In the next Section 3 we present the main results of the paper on log-concave inequalities and the matching equality conditions for interval greedoids. We follow in Section 4 with a chain of combinatorial reductions explaining how our greedoids results imply poset antimatroid, polymatroid and matroid results.

In Section 5 we introduce the notion of *combinatorial atlas*, which is the main technical structure of this paper. We then show how to derive log-concave inequalities in this general setting. The key combinatorial properties of the atlases are given in Section 6. In the next Section 7, we show that under additional conditions on the atlas, we can characterize the equality conditions.

From this point on, much of the paper occupy proofs of the results:

- Thm 1.31 (interval greedoids inequality) is proved in Section 8,
- Thm 3.3 (interval greedoids equality conditions) is proved in Section 9,
- Thm 1.6, Thm 1.9, Thm 1.10 (matroid inequality and equality conditions) are proved in Section 10; in addition, this section includes proof of Prop. 1.11, further results on log-concavity for graphs (§10.5), and examples of combinatorial atlases (§10.7),
- Thm 1.21, Thm 1.23 and Thm 1.24 (discrete polymatroid inequality and equality conditions) are proved in Section 11,
- Thm 1.26, Thm 1.28 and Thm 1.30 (poset antimatroid inequality and equality conditions) are proved in Section 12,
- Thm 1.16, Thm 1.18 and Thm 1.19 (morphism of matroids inequality and equality conditions) are proved in Section 13,
- Thm 1.35 (weighted Stanley's inequality) is proved in Section 14; in addition, this section includes §14.8 on posets with belts and an example §14.7 of a combinatorial atlas in this case,
- Thm 1.40 (equality condition for weighted Stanley's inequality) is proved in Section 15.

These last two sections are the most technically involved parts of this paper. Note that although Sections 10–13 are somewhat independent, we do recommend the reader start with the matroid proofs in Section 10 because of the examples and as a starting point of generalizations, and antimatroid proofs in Section 12 because it has the shortest and cleanest reduction to the earlier greedoid results.

We conclude the paper with a lengthy historical Section 16 which cover to some degree various background behind results int he introduction. Since the material is so vast, we are somewhat biased towards most recent and general results. We present final remarks and open problems in Section 17.

2. Definitions and notations

2.1. **Basic notation.** We use $[n] = \{1, \ldots, n\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_+ = \{1, 2, \ldots\}$, $\mathbb{R}_{\geq 0} = \{x \geq 0\}$ and $\mathbb{R}_{>0} = \{x > 0\}$. For a subset $S \subseteq X$ and element $x \in X$, we write $S + x := S \cup \{x\}$ and $S - x := S \smallsetminus \{x\}$.

2.2. **Matrices and vectors.** Throughout the paper we denote matrices with bold capitalized letter and the entries by roman capitalized letters: $\mathbf{M} = (\mathrm{M}_{ij})$. We also keep conventional index notations, so, e.g., $\big(\mathbf{M}^3 + \mathbf{M}^2\big)_{ij}$ is the (i, j)-th matrix entry of $\mathbf{M}^3 + \mathbf{M}^2$. We denote vectors by bold small letters, while vector entries by either unbolded uncapitalized letters or vector components, e.g. $\mathbf{h} = (\mathrm{h}_1, \mathrm{h}_2, \ldots)$ and $\mathrm{h}_i = (\mathbf{h})_i$.

A real matrix (resp., a real vector) is *nonnegative* if all its entries are nonnegative real numbers, and is *strictly positive* if all of its entries are positive real numbers. The *support* of a real $d \times d$ symmetric matrix $\mathbf{M}$ is defined as:

$$\operatorname{supp}(\mathbf{M}) \ := \ \big\{\, i \in [d] \ : \ \mathrm{M}_{ij} \neq 0 \ \text{ for some } j \in [d] \,\big\}.$$

In other words, $\operatorname{supp}(\mathbf{M})$ is the set of indexes for which the corresponding row and column of $\mathbf{M}$ are nonzero vectors. Similarly, the *support* of a real d-dimensional vector $\mathbf{h}$ is defined as:

$$\operatorname{supp}(\mathbf{h}) \ := \ \{\, i \in [d] \ : \ \mathrm{h}_i \neq 0 \,\}.$$

For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$, we write $\mathbf{v} \leqslant \mathbf{w}$ to mean the componentwise inequality, i.e. $\mathrm{v}_i \leq \mathrm{w}_i$ for all $i \in [d]$. We write $|\,\mathbf{v}\,| := \mathrm{v}_1 + \ldots + \mathrm{v}_d$. We also use $\mathbf{e}_1, \ldots, \mathbf{e}_d$ to denote the standard basis of $\mathbb{R}^d$.

Finally, for a subset $S \subseteq [d]$, the *characteristic vector* of S is the vector $\mathbf{v} \in \mathbb{R}^d$ such that $\mathrm{v}_i = 1$ if $i \in S$ and $\mathrm{v}_i = 0$ if $i \notin S$. We use $\mathbf{0} \in \mathbb{R}^d$ to denote the zero vector.

2.3. **Words.** For a finite ground set X, we denote by X^* the set of all sequences $x_1 \cdots x_\ell$ ($\ell \geq 0$) of elements $x_i \in X$ for $i \in [\ell]$. We call an element of X^* a *word* in the *alphabet* X. By a slight abuse of notation we use x_i to also denote the i-th letter in the word α. The *length* of a word $\alpha = x_1 \cdots x_\ell$ is the number of letters ℓ in the word, and is denoted by $|\alpha|$. The *concatenation* $\alpha\beta$ of two words α and β is the string α followed by the string β. In this case α is called a *prefix* of $\alpha\beta$. For every $\alpha = x_1 \cdots x_\ell \in X^*$, we write $z \in \alpha$ if $x_i = z$ for some $i \in [\ell]$.

2.4. **Posets.** A *poset* $\mathcal{P} = (X, \prec)$ is a pair of *ground set* X and a *partial order* "$\prec$" on X. For $x, y \in X$, we say that y *covers* x in $\mathcal{P}$, write $x \lessdot y$, if $x \prec y$, and there exists no $z \in X$ such that $x \prec z \prec y$. For $x, y \in X$, we write $x || y$ if x and y are *incomparable* in $\mathcal{P}$. Denote by $\operatorname{inc}(x) \subset X$ the subset of elements $y \in X$ incomparable with x.

A *lower ideal* of $\mathcal{P}$ is a subset $S \subseteq X$ such that, if $x \in S$ and $y \prec x$, then $y \in S$. Similarly, an *upper ideal* of $\mathcal{P}$ is a subset $S \subseteq X$ such that, if $x \in S$ and $y \succ x$, then $y \in X$. The *Hasse diagram* $\mathcal{H} := \mathcal{H}_{\mathcal{P}}$ of $\mathcal{P}$ is the acyclic digraph with X as the vertex set, and with (x, y) as an edge if $x \lessdot y$.

A *chain* of $\mathcal{P}$ is a subset of X that is totally ordered: $x_1 \prec x_2 \prec \ldots \prec x_\ell$. An antichain is a subset $S \subset X$, such that every two elements in S are incomparable. *Height* of a poset $\operatorname{height}(\mathcal{P})$ is the length of the maximal chain in $\mathcal{P}$. Similarly, *width* of a poset $\operatorname{width}(\mathcal{P})$ is the size of the maximal antichain in $\mathcal{P}$. Element $x \in X$ is called *minimal* if there is no $y \in X$, s.t. $y \prec x$. Define *maximal* elements similarly.

3. Combinatorics of interval greedoids

3.1. **Preliminaries.** Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid of rank $m := \operatorname{rk}(\mathcal{G})$. Recall the definitions of $\operatorname{Par}(\alpha)$ and $\operatorname{Des}_\alpha(x)$ given in §1.13 above, and note that "$\sim_\alpha$" remains an equivalence relation, see Proposition 4.3.

For all $\alpha \in \mathcal{L}$ and $x, y \in X$, define *passive* and *active non-continuations* as follows:

$$\begin{aligned} \operatorname{Pas}_\alpha(x,y) \ &:= \ \big\{ z \in X \ : \ \alpha z \notin \mathcal{L}, \ \alpha x z, \alpha y z \notin \mathcal{L}, \ \alpha x y z \in \mathcal{L} \big\}, \\ \operatorname{Act}_\alpha(x,y) \ &:= \ \big\{ z \in X \ : \ \alpha z \notin \mathcal{L}, \ \alpha x z, \alpha y z \in \mathcal{L}, \ \alpha x y z \in \mathcal{L} \big\}. \end{aligned}$$

Let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a positive *weight function*, which we extend to $\mathrm{q} : X^* \to \mathbb{R}$ by setting $\mathrm{q}(\alpha) = 0$ for all $\alpha \notin \mathcal{L}$. Let $\boldsymbol{c} = (c_0, \ldots, c_m) \in \mathbb{R}_{>0}^{m+1}$, where $m = \operatorname{rk}(\mathcal{G})$, be a fixed positive sequence, which we call the *scale sequence*. Consider another weight function $\omega : X^* \to \mathbb{R}$:

$$\omega(\alpha) \ := \ \frac{\mathrm{q}(\alpha)}{c_\ell}\,, \quad \text{where } \ \ell = |\alpha| \ \text{ and } \ \alpha \in X^*, \tag{3.1}$$

which we call the *scaled weight function*.

3.2. **Properties.** Fix weight function $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ and scale sequence $\boldsymbol{c} \in \mathbb{R}_{>0}^{m+1}$. For every word $\alpha \in \mathcal{L}$ of length $\ell := |\alpha|$, consider the following properties.

1. ***Continuation invariance property:***

(ContInv)
$$\mathrm{q}(\alpha x y \beta) \ = \ \mathrm{q}(\alpha y x \beta) \quad \text{for all } \ x, y \in \operatorname{Cont}(\alpha) \ \text{ and } \ \beta \in X^*.$$

Note that by the exchange property, we have $\alpha x y \beta \in \mathcal{L}$ if and only if $\alpha y x \beta \in \mathcal{L}$.

2. ***Passive-active monotonicity property:***

(PAMon)
$$\sum_{z \in \operatorname{Pas}_\alpha(x,y)} \ \sum_{\beta \in \operatorname{Cont}_k(\alpha x y z)} \mathrm{q}\big(\alpha x y z \beta\big) \ \geq \ \sum_{z \in \operatorname{Act}_\alpha(x,y)} \ \sum_{\beta \in \operatorname{Cont}_k(\alpha x y z)} \mathrm{q}\big(\alpha x y z \beta\big)\,,$$

for all distinct $x, y \in \operatorname{Cont}(\alpha)$, and $k \geq 0$. We also have a stronger property stated in terms of $\mathcal{L}$.

2′. ***Weak local property:***

(WeakLoc)
$$x, y, z \in X \quad \text{s.t.} \quad \alpha x z, \alpha y z, \alpha x y z \in \mathcal{L} \quad \Rightarrow \quad \alpha z \in \mathcal{L}\,.$$

Observe that (WeakLoc) implies that $\mathrm{Act}_\alpha(x,y) = \varnothing$ for all distinct $x, y \in \mathrm{Cont}(\alpha)$, which in turn trivially implies (PAMon). Note also that (WeakLoc) is a property of a greedoid rather than the weight function. Greedoids that satisfy (WeakLoc) are called *weak local greedoids*.[5]

3. ***Log-modularity property:***

$$\text{(LogMod)} \qquad \omega(\alpha x)\,\omega(\alpha y) \;=\; \omega(\alpha)\,\omega(\alpha x y) \quad \text{for all} \;\; x, y \in \mathrm{Cont}(\alpha) \;\; \text{s.t.} \;\; \alpha x y \in \mathcal{L}.$$

4. ***Few descendants property:***

$$\text{(FewDes)} \qquad |\,\mathcal{C}\,| \geq 2 \quad \Rightarrow \quad \mathrm{Des}_\alpha(x) \;=\; \varnothing\,, \qquad \text{for every} \;\; x \in \mathcal{C} \;\; \text{and} \;\; \mathcal{C} \in \mathrm{Par}(\alpha).$$

Note that (FewDes) is satisfied if $|\,\mathcal{C}\,| \leq 1$, or if $\mathrm{Des}_\alpha(x) = \varnothing$.

5. ***Syntactic monotonicity property:***

$$\text{(SynMon)} \qquad \omega(\alpha x)^2 \;\geq \sum_{y \,\in\, \mathrm{Des}_\alpha(x)} \omega(\alpha)\,\omega(\alpha x y)\,, \quad \text{for all} \;\; x \in \mathrm{Cont}(\alpha).$$

For all $\mathcal{C} \in \mathrm{Par}(\alpha)$, define

$$\mathrm{b}_\alpha(\mathcal{C}) \;:=\; \begin{cases} \displaystyle\sum_{y \,\in\, \mathrm{Des}_\alpha(x)} \frac{\omega(\alpha)\,\omega(\alpha x y)}{\omega(\alpha x)^2} & \text{if} \;\; \mathcal{C} = \{x\}, \\ 0 & \text{if} \;\; |\,\mathcal{C}\,| \geq 2. \end{cases} \tag{3.2}$$

Note that properties (FewDes) and (SynMon) imply that $\mathrm{b}_\alpha(\mathcal{C}) \leq 1$ for all $\mathcal{C} \in \mathrm{Par}(\alpha)$. This sets up our final

6. ***Scale monotonicity property:***

$$\text{(ScaleMon)} \qquad \left(1 - \frac{c_{\ell+1}^2}{c_\ell\, c_{\ell+2}}\right) \sum_{\mathcal{C} \,\in\, \mathrm{Par}(\alpha)} \frac{1}{1 - \mathrm{b}_\alpha(\mathcal{C})} \;\leq\; 1\,, \quad \text{for all} \;\; \mathcal{C} \in \mathrm{Par}(\alpha).$$

We adopt the convention that (ScaleMon) is always satisfied whenever $c_{\ell+1}^2 \geq c_\ell\, c_{\ell+2}$ (because then the LHS is considered nonpositive), and that $\mathrm{b}_\alpha(\mathcal{C}) < 1$ for all $\mathcal{C} \in \mathrm{Par}(\alpha)$ whenever $c_{\ell+1}^2 < c_\ell\, c_{\ell+2}$ (as otherwise the LHS is considered to be ∞) . In particular, note that (ScaleMon) is satisfied for the uniform scale sequence $\boldsymbol{c} = (1, \ldots, 1)$.

Remark 3.1. The last four properties (LogMod), (FewDes), (SynMon) and (ScaleMon) have a linear algebraic interpretation as certain matrix being *hyperbolic*. We postpone a discussion of this until the next section.

3.3. **Admissible weight functions.** We can now give the main definition used in the first main result of the paper (Theorem 1.31).

Definition 3.2 (*k-admissible weight functions*)**.** Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid of rank $m := \mathrm{rk}(\mathcal{G})$, and let $1 \leq k < m$. Weight function $\omega : \mathcal{L} \to \mathbb{R}_{>0}$ is called *k-admissible*, if there is a scale sequence $\boldsymbol{c} = (c_0, \ldots, c_m) \in \mathbb{R}_{>0}^{m+1}$, such that properties (ContInv), (PAMon), (LogMod), (FewDes), (SynMon) and (ScaleMon) are satisfied for all $\alpha \in \mathcal{L}$ of length $|\alpha| < k$.

We can also state our second main result of the paper, which gives the third equivalent condition in Theorem 1.33 that is both more detailed and useful in applications.

[5]This is a new class of greedoids which is similar but more general than the *local poset greedoids*. See Section 4 for the properties of weak local greedoids, relationships to other classes, and §17.14 for further background.

Theorem 3.3 (*Equality for interval greedoids*, second main theorem)**.** *Let* $\mathcal{G} = (X, \mathcal{L})$ *be an interval greedoid of rank* $m := \mathrm{rk}(\mathcal{G})$*, let* $1 \leq k < m$*, and let* $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ *be a* k*-admissible weight function with a scale sequence* $\boldsymbol{c} = (c_0, \ldots, c_m) \in \mathbb{R}^{m+1}_{>0}$*. Then, the following are equivalent:*

a. *We have:*

$$\mathrm{L}_{\mathrm{q}}(k)^2 \ = \ \mathrm{L}_{\mathrm{q}}(k-1) \cdot \mathrm{L}_{\mathrm{q}}(k+1). \tag{GE-a}$$

b. *There is* $\mathrm{s}(k-1) > 0$*, such that for every* $\alpha \in \mathcal{L}_{k-1}$ *we have:*

$$\mathrm{L}_{\mathrm{q},\alpha}(2) \ = \ \mathrm{s}(k-1)\, \mathrm{L}_{\mathrm{q},\alpha}(1) \ = \ \mathrm{s}(k-1)^2\, \mathrm{L}_{\mathrm{q},\alpha}(0). \tag{GE-b}$$

c. *There is* $\mathrm{s}(k-1) > 0$*, such that for every* $\alpha \in \mathcal{L}_{k-1}$ *we have:*

$$\sum_{x \,\in\, \mathrm{Cont}(\alpha)} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} \ = \ \mathrm{s}(k-1), \quad \textit{and} \tag{GE-$c1$}$$

$$\big(1 - \mathrm{b}_\alpha(\mathcal{C})\big) \sum_{x \,\in\, \mathcal{C}} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} \ = \ \mathrm{s}(k-1) \left(1 - \frac{c_k^2}{c_{k-1}\, c_{k+1}}\right) \qquad \textit{for all} \quad \mathcal{C} \in \mathrm{Par}(\alpha), \tag{GE-$c2$}$$

where $\mathrm{b}_\alpha(\mathcal{C})$ *is defined in* (3.2).

Note that (GE-$c1$) and (GE-$c2$) imply that (ScaleMon) is always an equality for $\alpha \in \mathcal{L}_{k-1}$.

Remark 3.4. Note that the k-admissible property of weight functions q is quite constraining and there are interval greedoid for which there are no such q. Given the abundance of examples where such weight functions are natural, we do not investigate the structural properties they constrain (cf. §16.11).

4. Combinatorial preliminaries

In this section we present basic properties of matroids, polymatroids, poset antimatroids, local poset greedoids and interval greedoids. We include the relations between these classes which will be important in the proofs. Most of these are relatively straightforward, but stated in a different way and often dispersed across the literature. We include the short proofs for completeness and as a way to help the reader get more familiar with the notions. The reader well versed with greedoids can skip this section and come back whenever proofs call for the specific results.

4.1. **Equivalence relations.** Here we prove that equivalence relations given in the introduction are well defined. We include short proofs both for completeness.

Proposition 4.1. *Let* $\mathcal{M} = (X, \mathcal{I})$ *be a matroid, and let* $S \in \mathcal{I}$ *be an independent set. Then the relation "*$\sim_S$*" defined in* §1.4 *is an equivalence relation.*

Proof. Observe that $x \sim_S y$ if and only if x and y are *parallel* in the matroid $\mathcal{M}/S$ obtained from $\mathcal{M}$ by contracting over S. □

Proposition 4.2. *Let* $\mathcal{D} = ([n], \mathcal{J})$ *be a discrete polymatroid, and let* $\boldsymbol{a} \in \mathcal{J}$ *be an independent multiset. Then the relation "*$\sim_{\boldsymbol{a}}$*" defined in* §1.10 *is an equivalence relation.*

Proof. It suffices to prove transitivity of "$\sim_{\boldsymbol{a}}$", as reflexivity and symmetry follow immediately from the definition. Let $i \sim_{\boldsymbol{a}}$ and $j \sim_{\boldsymbol{a}} k$. Suppose to the contrary that $i \not\sim_{\boldsymbol{a}} k$, so $\boldsymbol{a} + \mathbf{e}_i + \mathbf{e}_k \in \mathcal{J}$. On the other hand, $\boldsymbol{a} + \mathbf{e}_j \in \mathcal{J}$ since $j \in \mathrm{Cont}(\boldsymbol{a})$. It then follows from applying the exchange property to $\boldsymbol{a} + \mathbf{e}_j$ and $\boldsymbol{a} + \mathbf{e}_i + \mathbf{e}_k$, that either $\boldsymbol{a} + \mathbf{e}_j + \mathbf{e}_i \in \mathcal{J}$ or $\boldsymbol{a} + \mathbf{e}_j + \mathbf{e}_k \in \mathcal{J}$, both of which give us a contradiction. □

Proposition 4.3. *Let* $\mathcal{G} = (X, \mathcal{L})$ *be an interval greedoid, and let* $\alpha \in \mathcal{L}$ *be a fixed word. Then the relation "*$\sim_\alpha$*" defined in* §1.12 *is an equivalence relation.*

Proof. Reflexivity follows immediately from the definition. For the symmetry, let $x \sim_\alpha y$ and suppose to the contrary that $y \not\sim_\alpha x$. This is equivalent to $\alpha yx \in \mathcal{L}$. On the other hand, $\alpha x \in \mathcal{L}$ since $x \in \mathrm{Cont}(\alpha)$. It then follows from applying the exchange property to αx and αyx that $\alpha xy \in \mathcal{L}$, which contradicts $x \sim_\alpha y$.

For transitivity, let $x \sim_\alpha y$ and $y \sim_\alpha z$. Suppose to the contrary, that $x \not\sim_\alpha z$, so $\alpha xz \in \mathcal{L}$. On the other hand, $\alpha y \in \mathcal{L}$ since $y \in \mathrm{Cont}(\alpha)$. It then follows from applying the exchange property to αy and αxz, that either $\alpha yx \in \mathcal{L}$ or $\alpha yz \in \mathcal{L}$, both of which gives us a contradiction. □

We conclude with another equivalence relation, which will prove important in §13.2. Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids, let f be the rank function for $\mathcal{M} = (X, \mathcal{I})$, and let g be the rank function for $\mathcal{N} = (Y, \mathcal{J})$. For an independent set $S \in \mathcal{I}$, let $H \subseteq X$ be given by

$$H \ := \ \big\{ x \in X \setminus S \ : \ g\big(\Phi(S+x)\big) = \mathrm{rk}(\mathcal{N}) - 1 \big\}. \tag{4.1}$$

Denote by "$\sim_H$" the equivalence relation on H, defined by

$$x \sim_H y \quad \Longleftrightarrow \quad g\big(\Phi(S+x+y)\big) \ = \ \mathrm{rk}(\mathcal{N}) - 1. \tag{4.2}$$

Proposition 4.4. *The relation "$\sim_H$" defined in (4.2) is an equivalence relation.*

Proof. Reflexivity and symmetry follows directly from definition, so it suffices to prove transitivity. Suppose that $x, y, z \in H$ are distinct elements, such that $x \sim_H y$ and $y \sim_H z$. Assume to the contrary, that $x \not\sim_H z$. This implies that $g\big(\Phi(S+x+z)\big) = \mathrm{rk}(\mathcal{N})$. Applying the exchange property for matroid $\mathcal{N}$ to $\Phi(S+y)$ and $\Phi(S+x+z)$, we have that either $g\big(\Phi(S+y+x)\big) = \mathrm{rk}(\mathcal{N})$ or $g\big(\Phi(S+y+z)\big) = \mathrm{rk}(\mathcal{N})$. This contradicts the assumption, and completes the proof. □

4.2. **Antimatroids $\subset$ interval greedoids.** Note that (*nondegenerate property*) defining the language of a greedoid is vacuously true for poset antimatroids. Also note that two properties defining the language of a greedoid are identical to those defining antimatroids: (*normal property*) and (*hereditary property*). Similarly, the (*exchange property*) for antimatroids is more restrictive than the (*exchange property*) for greedoids.

It remains to show that the (*interval property*) holds for antimatroids. Let $\mathcal{A} = (X, \mathcal{L})$ be an antimatroid. Suppose $\alpha, \beta, \gamma \in X^*$ and $x \in X$, s.t. αx, $\alpha\beta\gamma x \in \mathcal{L}$. Write $\alpha' := \alpha x$ and $\beta' := \alpha\beta$. Then note that $x \in \alpha'$ and $x \notin \beta'$, as otherwise $w := \alpha\beta\gamma x \notin \mathcal{L}$ since w is not a simple word, and $\alpha', \beta' \in \mathcal{L}$. Also note that x is the only letter in α' that is not contained in β'. It then follows from the (*exchange property*) for $\mathcal{A}$, that $\alpha\beta x = \beta' x \in \mathcal{L}$, as desired. □

Proposition 4.5. *Let $\mathcal{P} = (X, \prec)$ be a poset, and let $\mathcal{A} = (X, \mathcal{L})$ be the corresponding antimatroid. Then $\mathcal{A}$ satisfies the* (interval property)*,* (FewDes) *and* (WeakLoc)*.*[6]

Proof. The (*interval property*) is proved above for all antimatroids. For (WeakLoc), let $x, y, z \in X$, s.t. αxz, αyz, $\alpha xyz \in \mathcal{L}$. Since $\alpha xz \in \mathcal{L}$ and $y \notin \alpha xz$, this implies z is incomparable to y in $\mathcal{P}$. Together with $\alpha yz \in \mathcal{L}$, this implies that $\alpha z \in \mathcal{L}$, as desired.

For (FewDes), note that $\mathcal{A}$ satisfies

$$\alpha x,\, \alpha y \in \mathcal{L}, \quad x, y \in X \quad \Longrightarrow \quad \alpha xy \in \mathcal{L}. \tag{4.3}$$

Indeed, this is because $\alpha y \in \mathcal{L}$ implies that every element in $\mathcal{P}$ that is less than y is contained in α, so they are also contained in αx. This in turn implies that $\alpha xy \in \mathcal{L}$. Now note that (4.3) implies that $|\,\mathcal{C}\,| = 1$ for every parallel class $\mathcal{C} \in \mathrm{Par}(\alpha)$ of $\alpha \in \mathcal{L}$, and thus (FewDes) is satisfied trivially. □

[6] Weak local property does not hold for all antimatroids, but holds for all poset antimatroids.

4.3. **Matroids $\subset$ greedoids.** Given a matroid $\mathcal{M} = (X, \mathcal{I})$, we construct the corresponding greedoid $\mathcal{G} = (X, \mathcal{L})$, where $\mathcal{L}$ is defined as follows:

$$\alpha \, = \, x_1 \cdots x_\ell \in \mathcal{L} \quad \Longleftrightarrow \quad \alpha \ \text{ is simple and } \{x_1, \ldots, x_\ell\} \in \mathcal{I}\,.$$

Observe that (*nondegenerate property*) for $\mathcal{G}$ follows from matroid $\mathcal{M}$ being nonempty, (*normal property*) follows from definition, (*hereditary property*) for $\mathcal{G}$ follows from the (*hereditary property*) for $\mathcal{M}$, and the (*exchange property*) for $\mathcal{G}$ follows from (*exchange property*) for $\mathcal{M}$.

Proposition 4.6. *Given a matroid $\mathcal{M} = (X, \mathcal{I})$, the greedoid $\mathcal{G} = (X, \mathcal{L})$ constructed above satisfies the* (interval property)*,* (FewDes) *and* (WeakLoc)*.*

Proof. Now note that, the greedoid $\mathcal{G}$ satisfies

$$\alpha x y \in \mathcal{L}, \quad x, y \in X \quad \Longrightarrow \quad \alpha y \in \mathcal{L}\,. \tag{4.4}$$

This follows from commutativity of $\mathcal{L}$ and the (*hereditary property*) of $\mathcal{M}$. The (*interval property*) for $\mathcal{G}$ follows immediately from (4.4).

Now, it follows from (4.4) that $\mathrm{Des}_\alpha(x) = \varnothing$ for every $\alpha \in \mathcal{L}$ and $x \in X$, and (FewDes) then follows trivially. Finally, let $x, y, z \in X$, s.t. αxz, αyz, $\alpha xyz \in \mathcal{L}$. Applying (4.4) to $\alpha xz \in \mathcal{L}$, it then follows that $\alpha z \in \mathcal{L}$. This proves (WeakLoc), and completes the proof. $\square$

4.4. **Discrete polymatroids $\subset$ greedoid.** Given a discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$, we construct the corresponding greedoid $\mathcal{G} = (X, \mathcal{L})$ as follows. Let $X := \big\{x_{i\,j} \,:\, 1 \le i, j \le n\big\}$ be the alphabet.[7]

For every word $\alpha \in X^*$, denote by $\boldsymbol{a}_\alpha = (\mathrm{a}_1, \ldots, \mathrm{a}_n) \in \mathbb{N}^n$ the vector counting the number of occurrences of $x_{i,*}$'s in α, i.e. $\mathrm{a}_i := \big|\big\{ j \in [n] \,:\, x_{i\,j} \in \alpha \big\}\big|$. The word $\alpha \in X^*$ is called *well-ordered* if for every letter $x_{i\,j}$ in α, letter $x_{i\,j-1}$ is also in α *before* $x_{i\,j}$.

Define $\mathcal{L}$ to be the set of simple well-ordered words $\alpha \in X^*$, such that $\boldsymbol{a}_\alpha \in \mathcal{J}$. Note that, each vector $\boldsymbol{a} \in \mathcal{J}$ corresponds to $\binom{|\boldsymbol{a}|}{\mathrm{a}_1, \ldots, \mathrm{a}_n}$ many feasible words $\alpha \in \mathcal{L}$ for which $\boldsymbol{a}_\alpha = \boldsymbol{a}$. Namely, these are all permutations of the word $x_{1\,1} \cdots x_{1\,\mathrm{a}_1} \cdots x_{n\,1} \cdots x_{n\,\mathrm{a}_n}$ preserving the relative order of letters $x_{i\,1}, \ldots, x_{i\,\mathrm{a}_i}$.

For the greedoid $\mathcal{G} = (X, \mathcal{L})$, the (*nondegenerate property*) and the (*normal property*) follow from definition. On the other hand, the (*hereditary property*) and the (*exchange property*) for $\mathcal{G}$ follows from the corresponding properties for $\mathcal{D}$. This completes the proof. $\square$

Proposition 4.7. *Given a discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$, the greedoid $\mathcal{G} = (X, \mathcal{L})$ constructed above satisfies the* (interval property)*,* (FewDes) *and* (WeakLoc)*.*

Proof. First, let us show that (*interval property*) holds for $\mathcal{G}$. Let $\alpha, \beta, \gamma \in X^*$, and let $z = x_{i\,j} \in X$ s.t. $\alpha z, \alpha\beta\gamma z \in \mathcal{L}$. Since $\alpha\beta\gamma z \in \mathcal{L}$, this implies that $x_{i\,j+1}, \ldots, x_{i\,n} \notin \beta$. Since $\alpha z \in \mathcal{L}$, this implies that $\alpha\beta z$ is well-ordered. On the other hand, by applying the (*hereditary property*) of $\mathcal{D}$ to the word $\alpha\beta\gamma z$, it then follows that $\boldsymbol{a}_{\alpha\beta z} \in \mathcal{J}$. Hence, the word $\alpha\beta z \in \mathcal{L}$, which proves the (*interval property*).

Now, note that $\mathcal{G}$ satisfies

$$\mathrm{Des}_\alpha(x_{ij}) \ \subseteq \ \{x_{i\,j+1}\} \qquad \text{for every} \quad \alpha \in \mathcal{L} \quad \text{and} \quad x_{ij} \in \mathrm{Cont}(\alpha). \tag{4.5}$$

For (WeakLoc), let $x, y, z \in X$, s.t. αxz, αyz, $\alpha xyz \in \mathcal{L}$. Suppose to the contrary, that $\alpha z \notin \mathcal{L}$. Since $\alpha xz \in \mathcal{L}$ and $\alpha yz \in \mathcal{L}$, this implies that $z \in \mathrm{Des}_\alpha(x)$ and $z \in \mathrm{Des}_\alpha(y)$. On the other hand, this intersection is empty by (4.5). This gives a contradiction, and proves (WeakLoc).

For (FewDes), let $\boldsymbol{a} = \boldsymbol{a}_\alpha$ where $\alpha \in \mathcal{L}$, and let $x, y \in \mathrm{Cont}(\alpha)$ be distinct elements s.t. $x \sim_\alpha y$. Let $i, j \in [n]$ be such that $\boldsymbol{a}_{\alpha x} = \boldsymbol{a} + \mathbf{e}_i$ and $\boldsymbol{a}_{\alpha y} = \boldsymbol{a} + \mathbf{e}_j$. Note that $i \neq j$ and $\boldsymbol{a} + \mathbf{e}_i, \boldsymbol{a} + \mathbf{e}_j \in \mathcal{J}$. Suppose to the contrary, that (FewDes) is not satisfied, so we can assume that $\mathrm{Des}_\alpha(x) \neq \varnothing$. By (4.5), this implies that $\boldsymbol{a} + 2\,\mathbf{e}_i \in \mathcal{J}$. Now, by applying the polymatroid exchange property to $\boldsymbol{a} + \mathbf{e}_j$ and $\boldsymbol{a} + 2\,\mathbf{e}_i$, we then have $\boldsymbol{a} + \mathbf{e}_i + \mathbf{e}_j \in \mathcal{J}$. This contradicts the assumption that $x \sim_\alpha y$, and proves (FewDes). $\square$

[7]Unlike the rest of the paper, here $|X| = n^2$.

4.5. **Exchange property for morphism of matroids.** We will also need the following basic result.

Proposition 4.8. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$. Let $S, T \subset X$, $|S| = |T|$ be two distinct bases of Φ. Then there exists $z \in S \setminus T$ and $w \in T \setminus S$ such that $S - z + w$ is also a basis of Φ.*

Proof. Fix an arbitrary $z \in S \setminus T$. We split the proof into two cases. First, suppose that $\Phi(S - z)$ contains a basis of $\mathcal{N}$. Applying the exchange property of $\mathcal{M}$ to the independent sets $S - z$ and T, there exists $w \in T \setminus S$ such that $S' := S - z + w$ is an independent set of $\mathcal{M}$. Note that $\Phi(S') \supset \Phi(S - z)$ contains a basis of $\mathcal{N}$ by assumption, so S' is a basis of Φ, as desired.

Second, suppose that $\Phi(S - z)$ does not contain a basis of $\mathcal{N}$. Applying the exchange property of $\mathcal{N}$ to $\Phi(S - z)$ and $\Phi(T)$, there exists $w \in T \setminus S$ such that $\Phi(S - z + w)$ contains a basis of $\mathcal{N}$. Since Φ is a morphism of matroid, we have

$$f(S - z + w) \, - \, f(S - z) \; \geq \; g\big(\Phi(S - z + w)\big) \, - \, g\big(\Phi(S - z)\big) \; = \; 1,$$

where f and g are rank functions in $\mathcal{M}$ and $\mathcal{N}$, respectively. This implies that $S - z + w$ is an independent set of $\mathcal{M}$, and therefore $S - z + w$ is a basis of the morphism Φ. This completes the proof. $\square$

5. Combinatorial atlases and hyperbolic matrices

In this section we introduce combinatorial atlases and present the local–global principle which allows one to recursively establish hyperbolicity of vertices. See §17.4 for some background.

5.1. **Combinatorial atlas.** Let $\mathcal{P} = (\Omega, \prec)$ be a locally finite poset of bounded height.[8] Denote by $\Gamma = (\Omega, \Theta) = \mathcal{H}_{\mathcal{P}}$ be the acyclic digraph given by the Hasse diagram of $\mathcal{P}$. Let $\Omega^0 \subseteq \Omega$ be the set of maximal elements in $\mathcal{P}$, so these are *sink vertices* in Γ. Similarly, denote by $\Omega^+ := \Omega \smallsetminus \Omega^0$ the *non-sink vertices*. We write v^* for the set of out-neighbor vertices $v' \in \Omega$, such that $(v, v') \in \Theta$.

Definition 5.1. A *combinatorial atlas* $\mathbb{A} = \mathbb{A}_{\mathcal{P}}$ of dimension d is an acyclic digraph $\Gamma := (\Omega, \Theta) = \mathcal{H}_{\mathcal{P}}$ with an additional structure:

- Each vertex $v \in \Omega$ is associated with a pair $(\mathbf{M}_v, \mathbf{h}_v)$, where $\mathbf{M}_v$ is a nonnegative symmetric $d \times d$ matrix, and $\mathbf{h}_v \in \mathbb{R}^d_{\geq 0}$ is a nonnegative vector.
- Every vertex $v \in \Omega^+$ has outdegree d, and the outgoing edges of each vertex $v \in \Omega^+$ are labeled with indices $i \in [d]$. We denote the edge labeled i as $e^{\langle i \rangle} = (v, v^{\langle i \rangle})$, where $1 \leq i \leq d$.
- Each edge $e^{\langle i \rangle}$ is associated to a linear transformation $\mathbf{T}^{\langle i \rangle}_v : \mathbb{R}^d \to \mathbb{R}^d$.

Whenever clear, we drop the subscript v to avoid cluttering. We call $\mathbf{M} = (\mathrm{M}_{ij})_{i,j \in [d]}$ the *associated matrix* of v, and $\mathbf{h} = (\mathrm{h}_i)_{i \in [d]}$ the *associated vector* of v. In notation above, we have $v^{\langle i \rangle} \in v^*$, for all $1 \leq i \leq d$.

5.2. **Local-global principle.** As in the introduction (see §1.20), matrix $\mathbf{M}$ is called *hyperbolic*, if

$$\text{(Hyp)} \qquad \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle^2 \; \geq \; \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \quad \text{for every} \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^d, \quad \text{such that} \quad \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle > 0.$$

For the atlas $\mathbb{A}$, we say that $v \in \Omega$ is *hyperbolic*, if the associated matrix $\mathbf{M}_v$ is hyperbolic, i.e. satisfies (Hyp). We say that atlas $\mathbb{A}$ satisfies *hyperbolic property* if every $v \in \Omega$ is hyperbolic.

Note that property (Hyp) depends only on the support of $\mathbf{M}$, i.e. it continues to hold after adding or removing zero rows or columns. This simple observation will be used repeatedly through the paper.

We say that atlas $\mathbb{A}$ satisfies *inheritance property* if for every non-sink vertex $v \in \Omega^+$, we have:

$$\text{(Inh)} \qquad (\mathbf{M}\mathbf{v})_i \; = \; \big\langle \mathbf{T}^{\langle i \rangle} \mathbf{v}, \, \mathbf{M}^{\langle i \rangle} \, \mathbf{T}^{\langle i \rangle} \, \mathbf{h} \, \big\rangle \quad \text{for every} \quad i \in \operatorname{supp}(\mathbf{M}) \quad \text{and} \quad \mathbf{v} \in \mathbb{R}^d,$$

where $\mathbf{T}^{\langle i \rangle} = \mathbf{T}^{\langle i \rangle}_v$, $\mathbf{h} = \mathbf{h}_v$ and $\mathbf{M}^{\langle i \rangle} := \mathbf{M}_{v^{\langle i \rangle}}$ is the matrix associated with $v^{\langle i \rangle}$.

Similarly, we say that atlas $\mathbb{A}$ satisfies *pullback property* if for every non-sink vertex $v \in \Omega^+$, we have:

$$\text{(Pull)} \qquad \sum_{i \in \operatorname{supp}(\mathbf{M})} \mathrm{h}_i \, \big\langle \mathbf{T}^{\langle i \rangle} \mathbf{v}, \, \mathbf{M}^{\langle i \rangle} \, \mathbf{T}^{\langle i \rangle} \, \mathbf{v} \, \big\rangle \; \geq \; \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \qquad \text{for every } \mathbf{v} \in \mathbb{R}^d.$$

[8] In our examples, the poset $\mathcal{P}$ can be both finite and infinite.

We say that a non-sink vertex $v \in \Omega^+$ is *regular* if the following positivity conditions are satisfied:

(Irr) The associated matrix $\mathbf{M}_v$ restricted to its support is irreducible.

(h-Pos) The associated vector $\mathbf{h}_v$ restricted to the support of $\mathbf{M}_v$ is strictly positive.

Note that a matrix is *irreducible* if if it is not similar via a permutation to a block upper triangular matrix that has more than one block of positive size.

We now present the first main result of this section, which is a local-global principle for (Hyp).

Theorem 5.2 (*local–global principle*). *Let $\mathbb{A}$ be a combinatorial atlas that satisfies properties* (Inh) *and* (Pull)*, and let $v \in \Omega^+$ be a non-sink regular vertex of Γ. Suppose every out-neighbor of v is hyperbolic. Then v is also hyperbolic.*

Theorem 5.2 reduces checking the property (Hyp) to sink vertices $v \in \Omega^0$. In our applications, the pullback property (Pull) is more complicated condition to check than the inheritance property (Inh). In the next Section 6, we present conditions implying (Pull) that are easier to check.

5.3. **Eigenvalue interpretation of hyperbolicity.** The following lemma that gives an equivalent condition to (Hyp) that is often easier to check. A symmetric matrix $\mathbf{M}$ satisfies (OPE) if

(OPE) $\mathbf{M}$ has at most *one positive eigenvalue* (counting multiplicity).

The equivalence between (Hyp) and (OPE) is well-known in the literature, see e.g., [Gre81], [COSW04, Thm 5.3], [SvH19, Lem. 2.9] and [BH20, Lem. 2.5]. We present a short proof for completeness.

Lemma 5.3. *Let $\mathbf{M}$ be a self-adjoint operator on $\mathbb{R}^d$ for an inner product $\langle \cdot, \cdot \rangle$. Then $\mathbf{M}$ satisfies* (Hyp) *if and only if $\mathbf{M}$ satisfies* (OPE).

Proof. For the (Hyp) $\Rightarrow$ (OPE) direction, suppose to the contrary that $\mathbf{M}$ has eigenvalues $\lambda_1, \lambda_2 > 0$ (not necessarily distinct). Let $\mathbf{v}$ and $\mathbf{w}$ be orthonormal eigenvectors of $\mathbf{M}$ for λ_1 and λ_2, respectively. It then follows that

$$0 \ = \ \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle \quad \text{and} \quad \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \, \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \ = \ \lambda_1 \lambda_2 \,,$$

which contradicts (Hyp).

For the (OPE) $\Rightarrow$ (Hyp) direction, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ be such that $\langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle > 0$. Let λ be the largest eigenvalue of $\mathbf{M}$, and let $\mathbf{h}$ be a corresponding eigenvector. Since $\langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle > 0$, this implies that λ is a positive eigenvalue. Since $\mathbf{M}$ has at most one positive eigenvalue (counting multiplicity), it follows that λ is the unique positive eigenvalue of $\mathbf{M}$, and is a simple eigenvalue. In particular, this implies that

$$\langle \mathbf{w}, \mathbf{M}\mathbf{h} \rangle \ \neq \ 0,$$

as otherwise, we would have $\langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \leq 0$. Let $\mathbf{z} \in \mathbb{R}^d$ be the vector

$$\mathbf{z} \ := \ \mathbf{v} \ - \ \frac{\langle \mathbf{v}, \mathbf{M}\mathbf{h} \rangle}{\langle \mathbf{w}, \mathbf{M}\mathbf{h} \rangle} \, \mathbf{w} \,.$$

It follows that $\langle \mathbf{z}, \mathbf{M}\mathbf{h} \rangle = 0$. Since λ is the only positive eigenvalue of $\mathbf{M}$, we then have

$$\langle \mathbf{z}, \mathbf{M}\mathbf{z} \rangle \ \leq \ 0. \tag{5.1}$$

On the other hand, we have

$$\begin{aligned} \langle \mathbf{z}, \mathbf{M}\mathbf{z} \rangle \ &= \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \ - \ 2 \, \frac{\langle \mathbf{v}, \mathbf{M}\mathbf{h} \rangle \, \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{M}\mathbf{h} \rangle} \ + \ \frac{\langle \mathbf{v}, \mathbf{M}\mathbf{h} \rangle^2 \, \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{M}\mathbf{h} \rangle^2} \\ &\geq \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \ - \ \frac{\langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle^2}{\langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle} \,, \end{aligned}$$

where the last inequality is due to the AM–GM inequality. Combining this inequality with (5.1), we get

$$\langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle^2 \ \geq \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \, \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \,,$$

which proves (Hyp). ∎

5.4. **Proof of Theorem 5.2.** Let $\mathbf{M} := \mathbf{M}_v$ and $\mathbf{h} := \mathbf{h}_v$ be the associated matrix and the associated vector of v, respectively. Since (Hyp) is a property that is invariant under restricting to the support of $\mathbf{M}$, it follows from (Irr) that we can assume that $\mathbf{M}$ is irreducible.

Let $\mathbf{D} := (\mathrm{D}_{ij})$ be the $d \times d$ diagonal matrix given by

$$\mathrm{D}_{ii} \ := \ \frac{(\mathbf{M}\mathbf{h})_i}{\mathrm{h}_i} \qquad \text{for every} \ \ 1 \leq i \leq d.$$

Note that $\mathbf{D}$ is well defined and $\mathrm{D}_{ii} > 0$, by (h-Pos) and the assumption that $\mathbf{M}$ is irreducible. Define a new inner product $\langle \cdot, \cdot \rangle_{\mathbf{D}}$ on $\mathbb{R}^d$ by $\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{D}} \ := \ \langle \mathbf{v}, \mathbf{D}\mathbf{w} \rangle$.

Let $\mathbf{N} := \mathbf{D}^{-1}\mathbf{M}$. Note that $\langle \mathbf{v}, \mathbf{N}\mathbf{w} \rangle_{\mathbf{D}} = \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle$ for every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$. Since $\mathbf{M}$ is a symmetric matrix, this implies that $\mathbf{N}$ is a self-adjoint operator on $\mathbb{R}^d$ for the inner product $\langle \cdot, \cdot \rangle_{\mathbf{D}}$. A direct calculation shows that $\mathbf{h}$ is an eigenvector of $\mathbf{N}$ for eigenvalue $\lambda = 1$. Since $\mathbf{M}$ is irreducible matrix and $\mathbf{h}$ is a strictly positive vector, it then follows from the Perron–Frobenius theorem that $\lambda = 1$ is the largest real eigenvalue of $\mathbf{N}$, and that it has multiplicity one.

Claim: $\lambda = 1$ *is the only positive eigenvalue of* $\mathbf{N}$ (*counting multiplicity*).

By applying Lemma 5.3 to the matrix $\mathbf{N}$ and the inner product $\langle \cdot, \cdot \rangle_{\mathbf{D}}$, it then follows that

$$\langle \mathbf{v}, \mathbf{N}\mathbf{w} \rangle_{\mathbf{D}}^2 \ \geq \ \langle \mathbf{v}, \mathbf{N}\mathbf{v} \rangle_{\mathbf{D}} \, \langle \mathbf{w}, \mathbf{N}\mathbf{w} \rangle_{\mathbf{D}} \quad \text{ for every } \ \mathbf{v}, \mathbf{w} \in \mathbb{R}^d.$$

Since $\langle \mathbf{v}, \mathbf{N}\mathbf{w} \rangle_{\mathbf{D}} = \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle$, this implies (Hyp) for v, and completes the proof of the theorem. □

Proof of the Claim. Let $i \in [d]$ and $\mathbf{v} \in \mathbb{R}^d$. It follows from (Inh) that

$$\big((\mathbf{M}\mathbf{v})_i\big)^2 \ = \ \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{v}, \, \mathbf{M}^{\langle i \rangle} \, \mathbf{T}^{\langle i \rangle}\mathbf{h} \big\rangle^2. \tag{5.2}$$

Since $\mathbf{M}^{\langle i \rangle}$ satisfies (Hyp) by the assumption of the theorem, applying (Hyp) to the RHS of (5.2) gives:

$$\big((\mathbf{M}\mathbf{v})_i\big)^2 \ \geq \ \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{v}, \, \mathbf{M}^{\langle i \rangle} \, \mathbf{T}^{\langle i \rangle}\mathbf{v} \big\rangle \, \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{h}, \, \mathbf{M}^{\langle i \rangle} \, \mathbf{T}^{\langle i \rangle}\mathbf{h} \big\rangle, \tag{5.3}$$

Here (Hyp) can be applied since $\big\langle \mathbf{T}^{\langle i \rangle}\mathbf{h}, \, \mathbf{M}^{\langle i \rangle} \, \mathbf{T}^{\langle i \rangle}\mathbf{h} \big\rangle \ = \ (\mathbf{M}\mathbf{h})_i > 0$. Now note that

$$\begin{aligned}
\big((\mathbf{N}\mathbf{v})_i\big)^2 \, \mathrm{D}_{ii} \ &= \ \big((\mathbf{M}\mathbf{v})_i\big)^2 \, \frac{\mathrm{h}_i}{(\mathbf{M}\mathbf{h})_i} \ \ =_{(\text{Inh})} \ \ \big((\mathbf{M}\mathbf{v})_i\big)^2 \, \frac{\mathrm{h}_i}{\big\langle \mathbf{T}^{\langle i \rangle}\mathbf{h}, \, \mathbf{M}^{\langle i \rangle} \, \mathbf{T}^{\langle i \rangle}\mathbf{h} \big\rangle} \\
&\geq_{(5.3)} \ \mathrm{h}_i \, \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{v}, \, \mathbf{M}^{\langle i \rangle} \, \mathbf{T}^{\langle i \rangle}\mathbf{v} \big\rangle.
\end{aligned}$$

Summing this inequality over all $i \in [d]$, gives:

$$\langle \mathbf{N}\mathbf{v}, \mathbf{N}\mathbf{v} \rangle_{\mathbf{D}} \ \geq \ \sum_{i=1}^{r} \mathrm{h}_i \, \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{v}, \, \mathbf{M}^{\langle i \rangle} \, \mathbf{T}^{\langle i \rangle}\mathbf{v} \big\rangle \ \geq_{(\text{Pull})} \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \ = \ \langle \mathbf{v}, \mathbf{N}\mathbf{v} \rangle_{\mathbf{D}}. \tag{5.4}$$

Now, let λ be an arbitrary eigenvalue of $\mathbf{N}$, and let $\mathbf{g}$ be an eigenvector of λ. We have:

$$\lambda^2 \langle \mathbf{g}, \mathbf{g} \rangle_{\mathbf{D}} \ = \ \langle \mathbf{N}\mathbf{g}, \mathbf{N}\mathbf{g} \rangle_{\mathbf{D}} \ \geq_{(5.4)} \ \langle \mathbf{g}, \mathbf{N}\mathbf{g} \rangle_{\mathbf{D}} \ = \ \lambda \, \langle \mathbf{g}, \mathbf{g} \rangle_{\mathbf{D}}.$$

This implies that $\lambda \geq 1$ or $\lambda \leq 0$. Since $\lambda = 1$ is the largest eigenvalue of $\mathbf{N}$ and is simple, we obtain the result. □

Remark 5.4. In the proof above, neither the Claim nor the proof of the Claim are new, but a minor revision of Theorem 5.2 in [SvH19]. We include the proof for completeness and to help the reader get through our somewhat cumbersome notation.

6. Pullback property

In this section we present sufficient conditions for (Pull) that are easier to verify, together with a construction of the maps $\mathbf{T}^{\langle i \rangle}$.

6.1. **Three new properties.** Let $\mathbb{A}$ be a combinatorial atlas. We say that $\mathbb{A}$ satisfies the *projective property*, if for every non-sink vertex $v \in \Omega^+$ and every $i \in \operatorname{supp}(\mathbf{M})$, we have:

$$\text{(Proj)} \qquad \big(\mathbf{T}^{\langle i\rangle}\mathbf{v}\big)_j \ = \ \begin{cases} \mathbf{v}_j & \text{if} \quad j \in \operatorname{supp}\big(\mathbf{M}^{\langle i\rangle}\big) \cap \operatorname{supp}(\mathbf{M}), \\ \mathbf{v}_i & \text{if} \quad j \in \operatorname{supp}\big(\mathbf{M}^{\langle i\rangle}\big) \setminus \operatorname{supp}(\mathbf{M}). \end{cases}$$

We say that $\mathbb{A}$ satisfies the *transposition-invariant property*, if for every non-sink vertex $v \in \Omega^+$, we have:

$$\text{(T-Inv)} \qquad \mathrm{M}_{jk}^{\langle i\rangle} \ = \ \mathrm{M}_{ki}^{\langle j\rangle} \ = \ \mathrm{M}_{ij}^{\langle k\rangle} \qquad \text{for every distinct } i,j,k \in \operatorname{supp}(\mathbf{M}).$$

Now, let $v \in \Omega^+$ be a non-sink vertex of Γ, and let $i \in \operatorname{supp}(\mathbf{M})$. We partition the support of matrix $\mathbf{M}^{\langle i\rangle}$ associated with vertex $v^{\langle i\rangle}$, into two parts:

$$(6.1) \qquad \mathrm{Aunt}^{\langle i\rangle} \ := \ \operatorname{supp}\big(\mathbf{M}^{\langle i\rangle}\big) \cap \big(\operatorname{supp}(\mathbf{M}) - i\big), \qquad \mathrm{Fam}^{\langle i\rangle} \ := \ \operatorname{supp}\big(\mathbf{M}^{\langle i\rangle}\big) \setminus \big(\operatorname{supp}(\mathbf{M}) - i\big).$$

In other words, $\mathrm{Aunt}^{\langle i\rangle}$ consists of elements in the support of $\mathbf{M}$ that do not include i,[9] while $\mathrm{Fam}^{\langle i\rangle}$ consists of i together with elements that initially are not in the support of $\mathbf{M}$, but is then included in the support of $\mathbf{M}^{\langle i\rangle}$.[10] For every distinct $i,j \in \operatorname{supp}(\mathbf{M})$, let

$$(6.2) \qquad \mathrm{K}_{ij} \ := \ \mathrm{h}_j\, \mathrm{M}_{jj}^{\langle i\rangle} \ - \ \mathrm{h}_j \sum_{k \in \mathrm{Fam}^{\langle j\rangle}} \mathrm{M}_{ik}^{\langle j\rangle}.$$

Let us emphasize that $\mathrm{Aunt}^{\langle i\rangle}$, $\mathrm{Fam}^{\langle i\rangle}$, and K_{ij} all depend on non-sink vertex v of Γ, even though v does not appear in these notation.

We say that $\mathbb{A}$ satisfies the K*-nonnegative property*, if for every non-sink vertex $v \in \Omega^+$,

$$\text{(K-Non)} \qquad \mathrm{K}_{ij} \ \geq \ 0 \quad \text{for every distinct} \quad i,j \ \in \ \operatorname{supp}(\mathbf{M}).$$

The main result of this subsection is the following sufficient condition for (Pull).

Theorem 6.1. *Let $\mathbb{A}$ be a combinatorial atlas that satisfies* (Inh)*,* (Proj)*,* (T-Inv) *and* (K-Non)*. Then $\mathbb{A}$ also satisfies* (Pull)*.*

6.2. **Symmetry lemma.** To prove Theorem 6.1, we need the following:

Lemma 6.2. *Let $\mathbb{A}$ be a combinatorial atlas that satisfies* (Inh)*,* (Proj)*, and* (T-Inv)*. Then, for every non-sink vertex $v \in \Omega^+$, we have:*

$$\mathrm{K}_{ij} \ = \ \mathrm{K}_{ji} \quad \textit{for every distinct} \quad i,j \ \in \ \operatorname{supp}(\mathbf{M}).$$

Proof. Let $\mathbf{e}_1,\ldots,\mathbf{e}_d$ be the standard basis for $\mathbb{R}^d$. It follows from (Inh) that:

$$\begin{aligned} \mathrm{M}_{ij} \ &= \ \big(\mathbf{M}\mathbf{e}_j\big)_i \ =_{(\mathrm{Inh})} \ \big\langle \mathbf{T}^{\langle i\rangle}\,\mathbf{e}_j, \mathbf{M}^{\langle i\rangle}\,\mathbf{T}^{\langle i\rangle}\mathbf{h}\big\rangle \ = \ \sum_{k=1}^{d} \mathrm{M}_{jk}^{\langle i\rangle}\,\big(\mathbf{T}^{\langle i\rangle}\mathbf{h}\big)_k \\ &= \ \sum_{k \in \mathrm{Fam}^{\langle i\rangle}} \mathrm{M}_{jk}^{\langle i\rangle}\,\big(\mathbf{T}^{\langle i\rangle}\mathbf{h}\big)_k \ + \ \sum_{k \in \mathrm{Aunt}^{\langle i\rangle}} \mathrm{M}_{jk}^{\langle i\rangle}\,\big(\mathbf{T}^{\langle i\rangle}\mathbf{h}\big)_k \\ &= \ \mathrm{M}_{jj}^{\langle i\rangle}\,\big(\mathbf{T}^{\langle i\rangle}\mathbf{h}\big)_j \ + \ \sum_{k \in \mathrm{Fam}^{\langle i\rangle}} \mathrm{M}_{jk}^{\langle i\rangle}\,\big(\mathbf{T}^{\langle i\rangle}\mathbf{h}\big)_k \ + \ \sum_{k \in \operatorname{supp}(\mathbf{M})\setminus\{i,j\}} \mathrm{M}_{jk}^{\langle i\rangle}\,\big(\mathbf{T}^{\langle i\rangle}\mathbf{h}\big)_k\,. \end{aligned}$$

Applying (Proj) to the equation above, we get:

$$(6.3) \qquad \mathrm{M}_{ij} \ = \ \mathrm{M}_{jj}^{\langle i\rangle}\,\mathrm{h}_j \ + \ \sum_{k \in \mathrm{Fam}^{\langle i\rangle}} \mathrm{M}_{jk}^{\langle i\rangle}\,\mathrm{h}_i \ + \ \sum_{k \in \operatorname{supp}(\mathbf{M})\setminus\{i,j\}} \mathrm{M}_{jk}^{\langle i\rangle}\,\mathrm{h}_k\,.$$

By the same reasoning, we also get:

$$(6.4) \qquad \mathrm{M}_{ji} \ = \ \mathrm{M}_{ii}^{\langle j\rangle}\,\mathrm{h}_i \ + \ \sum_{k \in \mathrm{Fam}^{\langle j\rangle}} \mathrm{M}_{ik}^{\langle j\rangle}\,\mathrm{h}_j \ + \ \sum_{k \in \operatorname{supp}(\mathbf{M})\setminus\{i,j\}} \mathrm{M}_{ik}^{\langle j\rangle}\,\mathrm{h}_k\,.$$

[9] The name "aunt" here is referring to the siblings of the parent.

[10] The name "family" here is referring to both the parents and their children.

By (T-Inv), the rightmost sums in (6.3) and (6.4) are equal. On the other hand, the left side of (6.3) and (6.4) are equal since $\mathbf{M}$ is a symmetric matrix. Equating (6.3) and (6.4), we obtain:

$$\mathrm{M}_{jj}^{\langle i\rangle}\,\mathrm{h}_j \;+\; \sum_{k\,\in\,\mathrm{Fam}^{\langle i\rangle}} \mathrm{M}_{jk}^{\langle i\rangle}\,\mathrm{h}_i \;=\; \mathrm{M}_{ii}^{\langle j\rangle}\,\mathrm{h}_i \;+\; \sum_{k\,\in\,\mathrm{Fam}^{\langle j\rangle}} \mathrm{M}_{ik}^{\langle j\rangle}\,\mathrm{h}_j\,,$$

which is equivalent to

$$\mathrm{M}_{jj}^{\langle i\rangle}\,\mathrm{h}_j \;-\; \sum_{k\,\in\,\mathrm{Fam}^{\langle j\rangle}} \mathrm{M}_{ik}^{\langle j\rangle}\,\mathrm{h}_j \;=\; \mathrm{M}_{ii}^{\langle j\rangle}\,\mathrm{h}_i \;-\; \sum_{k\,\in\,\mathrm{Fam}^{\langle i\rangle}} \mathrm{M}_{jk}^{\langle i\rangle}\,\mathrm{h}_i\,.$$

The lemma now follows by noting that the LHS of the equation above is equal to K_{ij}, while the RHS is equal to K_{ji}. $\square$

6.3. **Proof of Theorem 6.1.** Let v be a non-sink vertex of Γ, and let $\mathbf{v} \in \mathbb{R}^d$. The left side of (Pull) is equal to

$$\sum_{i\,\in\,\mathrm{supp}(\mathbf{M})} \mathrm{h}_i \left\langle \mathbf{T}^{\langle i\rangle}\mathbf{v}, \mathbf{M}^{\langle i\rangle}\mathbf{T}^{\langle i\rangle}\mathbf{v}\right\rangle \;=\; \sum_{i\,\in\,\mathrm{supp}(\mathbf{M})}\;\sum_{j,k\,\in\,\mathrm{supp}(\mathbf{M}^{\langle i\rangle})} \mathrm{h}_i \left(\mathbf{T}^{\langle i\rangle}\mathbf{v}\right)_j \left(\mathbf{T}^{\langle i\rangle}\mathbf{v}\right)_k \mathrm{M}_{jk}^{\langle i\rangle}\,. \tag{6.5}$$

<u>First</u>, this sum can be partitioned into the sum over the following five families:

(1) The triples (i,j,k), where $i \in \mathrm{supp}(\mathbf{M})$, and $j,k \in \mathrm{Aunt}^{\langle i\rangle}$ are distinct. By (Proj), the term in (6.5) is equal to
$$\mathrm{h}_i\,\mathrm{v}_j\,\mathrm{v}_k\,\mathrm{M}_{jk}^{\langle i\rangle}\,.$$
(2) The triples (i,j,k), where $i \in \mathrm{supp}(\mathbf{M})$, and $j,k \in \mathrm{Fam}^{\langle i\rangle}$ (not necessarily distinct). By (Proj), the term in (6.5) is equal to
$$\mathrm{h}_i\,\mathrm{v}_i^2\,\mathrm{M}_{jk}^{\langle i\rangle}\,.$$
(3) The triples (i,j,k), where $i \in \mathrm{supp}(\mathbf{M})$, $j \in \mathrm{Aunt}^{\langle i\rangle}$, and $k \in \mathrm{Fam}^{\langle i\rangle}$. By (Proj), the term in (6.5) is equal to
$$\mathrm{h}_i\,\mathrm{v}_i\,\mathrm{v}_j\,\mathrm{M}_{jk}^{\langle i\rangle}\,.$$
(4) The triples (i,j,k), where $i \in \mathrm{supp}(\mathbf{M})$, $j \in \mathrm{Fam}^{\langle i\rangle}$, and $k \in \mathrm{Aunt}^{\langle i\rangle}$. By (Proj), the term in (6.5) is equal to
$$\mathrm{h}_i\,\mathrm{v}_i\,\mathrm{v}_k\,\mathrm{M}_{jk}^{\langle i\rangle}\,.$$
(5) The triples (i,j,k), where $i \in \mathrm{supp}(\mathbf{M})$, and $j = k \in \mathrm{Aunt}^{\langle i\rangle}$. By (Proj), the term in (6.5) is equal to
$$\mathrm{h}_i\,\mathrm{v}_j^2\,\mathrm{M}_{jj}^{\langle i\rangle} \;=\; \frac{\mathrm{h}_i}{\mathrm{h}_j}\,\mathrm{v}_j^2\,\mathrm{K}_{ij} \;+\; \sum_{k\,\in\,\mathrm{Fam}^{\langle j\rangle}} \mathrm{h}_i\,\mathrm{v}_j^2\,\mathrm{M}_{ik}^{\langle j\rangle}\,.$$
Thus the sum over this family can be partitioned further into the sum over the following two families:
(5a) The pair (i,j), where $i,j \in \mathrm{supp}(\mathbf{M})$ are distinct, with the term
$$\frac{\mathrm{h}_i}{\mathrm{h}_j}\,\mathrm{v}_j^2\,\mathrm{K}_{ij}\,.$$
(5b) The triples (i,j,k), where $i,j \in \mathrm{supp}(\mathbf{M})$ are distinct, and $k \in \mathrm{Fam}^{\langle j\rangle}$, with the term
$$\mathrm{h}_i\,\mathrm{v}_j^2\,\mathrm{M}_{ik}^{\langle j\rangle}\,.$$

<u>Second</u>, the right side of (Pull) is equal to

$$\begin{aligned}\langle \mathbf{v}, \mathbf{M}\,\mathbf{v}\rangle \;&=\; \sum_{i'\,\in\,\mathrm{supp}(\mathbf{M})} \mathrm{v}_{i'}\,(\mathbf{M}\,\mathbf{v})_{i'} \;=_{(\mathrm{Inh})}\; \sum_{i'\,\in\,\mathrm{supp}(\mathbf{M})} \mathrm{v}_{i'} \left\langle \mathbf{T}^{\langle i'\rangle}\mathbf{v},\, \mathbf{M}^{\langle i'\rangle}\,\mathbf{T}^{\langle i'\rangle}\mathbf{h}\right\rangle \\ &=\; \sum_{i'\,\in\,\mathrm{supp}(\mathbf{M})}\;\sum_{j',k'\,\in\,\mathrm{supp}(\mathbf{M}^{\langle i'\rangle})} \mathrm{v}_{i'} \left(\mathbf{T}^{\langle i'\rangle}\mathbf{v}\right)_{j'} \left(\mathbf{T}^{\langle i'\rangle}\mathbf{h}\right)_{k'} \mathrm{M}_{j'k'}^{\langle i'\rangle}. \end{aligned} \tag{6.6}$$

This sum can be partitioned into the sum over the following five families:

(1′) The triples (i', j', k'), where $i' \in \operatorname{supp}(\mathbf{M})$, and $j', k' \in \mathrm{Aunt}^{\langle i' \rangle}$ are distinct. By (Proj), the term in (6.6) is equal to

$$\mathrm{h}_{k'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle i' \rangle}_{j'k'} .$$

(2′) The triples (i', j', k'), where $i' \in \operatorname{supp}(\mathbf{M})$, and $j', k' \in \mathrm{Fam}^{\langle i' \rangle}$ (not necessarily distinct). By (Proj), the term in (6.6) is equal to

$$\mathrm{h}_{i'}\, \mathrm{v}^2_{i'}\, \mathrm{M}^{\langle i' \rangle}_{j'k'} .$$

(3′) The triples (i', j', k'), where $i' \in \operatorname{supp}(\mathbf{M})$, $j' \in \mathrm{Aunt}^{\langle i' \rangle}$, and $k' \in \mathrm{Fam}^{\langle i' \rangle}$. By (Proj), the term in (6.6) is equal to

$$\mathrm{h}_{i'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle i' \rangle}_{j'k'} .$$

(4′) The triples (i', j', k'), where $i' \in \operatorname{supp}(\mathbf{M})$, $j' \in \mathrm{Fam}^{\langle i' \rangle}$, and $k' \in \mathrm{Aunt}^{\langle i' \rangle}$. By (Proj), the term in (6.6) is equal to

$$\mathrm{h}_{k'}\, \mathrm{v}^2_{i'}\, \mathrm{M}^{\langle i' \rangle}_{j'k'} .$$

(5′) The triples (i', j', k'), where $i' \in \operatorname{supp}(\mathbf{M})$, and $j' = k' \in \mathrm{Aunt}^{\langle i' \rangle}$. By (Proj), the term in (6.6) is equal to

$$\mathrm{h}_{j'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle i' \rangle}_{j'j'} \;=\; \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{K}_{i'j'} \;+\; \sum_{k' \in \mathrm{Fam}^{\langle j' \rangle}} \mathrm{h}_{j'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle j' \rangle}_{i'k'} .$$

Thus the sum over this family can be partitioned further into the sum over the following two families:

(5a′) The pair (i', j'), where $i', j' \in \operatorname{supp}(\mathbf{M})$ are distinct, with the term

$$\mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{K}_{i'j'} .$$

(5b′) The triples (i', j', k'), where $i', j' \in \operatorname{supp}(\mathbf{M})$ are distinct, and $k' \in \mathrm{Fam}^{\langle j' \rangle}$, with the term

$$\mathrm{h}_{j'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle j' \rangle}_{i'k'} .$$

Third, we show that the RHS of (6.5) is at least as large as the RHS of (6.6). We have the following six cases:

(i) The term in (1) is equal to that of (1′) by substituting $i' \leftarrow j$, $j' \leftarrow k$, $k' \leftarrow i$ (counterclockwise substitution) to (1):

$$\mathrm{h}_i\, \mathrm{v}_j\, \mathrm{v}_k\, \mathrm{M}^{\langle i \rangle}_{jk} \;=_{(\text{T-Inv})}\; \mathrm{h}_i\, \mathrm{v}_j\, \mathrm{v}_k\, \mathrm{M}^{\langle j \rangle}_{ki} \;=\; \mathrm{h}_{k'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle i' \rangle}_{j'k'} .$$

(ii) The term in (2) is equal to that of (2′) by substituting $i' \leftarrow i$, $j' \leftarrow j$, $k' \leftarrow k$ (identity substitution) to (2):

$$\mathrm{h}_i\, \mathrm{v}^2_i\, \mathrm{M}^{\langle i \rangle}_{jk} \;=\; \mathrm{h}_{i'}\, \mathrm{v}^2_{i'}\, \mathrm{M}^{\langle i' \rangle}_{j'k'} .$$

(iii) The term in (3) is equal to that of (3′) by substituting $i' \leftarrow i$, $j' \leftarrow j$, $k' \leftarrow k$ (identity substitution) to (3):

$$\mathrm{h}_i\, \mathrm{v}_i\, \mathrm{v}_j\, \mathrm{M}^{\langle i \rangle}_{jk} \;=\; \mathrm{h}_{i'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle i' \rangle}_{j'k'} .$$

(iv) The term in (4) is equal to that of (5b′) by substituting $i' \leftarrow k$, $j' \leftarrow i$, $k' \leftarrow j$ (clockwise substitution) to (4):

$$\mathrm{h}_i\, \mathrm{v}_i\, \mathrm{v}_k\, \mathrm{M}^{\langle i \rangle}_{jk} \;=\; \mathrm{h}_{j'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle j' \rangle}_{i'k'} .$$

(v) The term in (5a) is equal to that of (5a′) by substituting $i' \leftarrow i$, $j' \leftarrow j$ (identity substitution) to (5a):

$$\begin{aligned} \frac{\mathrm{h}_i}{\mathrm{h}_j}\, \mathrm{v}^2_j\, \mathrm{K}_{ij} \;+\; \frac{\mathrm{h}_j}{\mathrm{h}_i}\, \mathrm{v}^2_i\, \mathrm{K}_{ji} \;&\geq\; 2\, \mathrm{v}_i\, \mathrm{v}_j \sqrt{\mathrm{K}_{ij}\, \mathrm{K}_{ji}} \;=_{\text{Lem 6.2}}\; \mathrm{v}_i\, \mathrm{v}_j\, \mathrm{K}_{ij} \;+\; \mathrm{v}_j\, \mathrm{v}_i\, \mathrm{K}_{ji} \\ &=\; \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{K}_{i'j'} \;+\; \mathrm{v}_{j'}\, \mathrm{v}_{i'}\, \mathrm{K}_{j'i'} , \end{aligned}$$

where the first inequality follows from (K-Non) and the AM-GM inequality.[11]

[11]Note that this is only instance of inequality in this proof.

(vi) The term in (5b) is equal to that of (4′) by substituting $i' \leftarrow j$, $j' \leftarrow k$, $k' \leftarrow i$ (clockwise substitution) to (5b):

$$\mathrm{h}_i \, \mathrm{v}_j^2 \, \mathrm{M}_{ik}^{\langle j \rangle} \ = \ \mathrm{h}_{k'} \, \mathrm{v}_{i'}^2 \, \mathrm{M}_{j'k'}^{\langle i' \rangle} .$$

This completes the proof of the theorem. □

Remark 6.3. The condition (K-Non) in Theorem 6.1 can be weakened as follows. Let $v \in \Omega^+$ be a non-sink vertex, and let $\mathbf{K} := (\mathrm{K}_{ij})_{i,j \in \mathrm{supp}(\mathbf{M})}$ be the matrix defined by

$$\mathrm{K}_{ij} \ := \ \begin{cases} \mathrm{K}_{ij} \text{ as in (6.2)} & \text{if} \quad i,j \in \mathrm{supp}(\mathbf{M}), \quad i \neq j, \\ -\displaystyle\sum_{\ell \in \mathrm{supp}(\mathbf{M}) \setminus \{i\}} \frac{\mathrm{h}_i}{\mathrm{h}_\ell} \, \mathrm{K}_{i,\ell} & \text{if} \quad i = j \in \mathrm{supp}(\mathbf{M}). \end{cases}$$

We claim that the condition (K-Non) in Theorem 6.1 can be replaced with

(K-PSD) $$\text{The matrix} \ \ -\mathbf{K} \ \ \text{is positive semidefinite,}$$

for every non-sink vertex v of Γ. This generalization follows from the same proof as Theorem 6.1 by a straightforward modification to step (v). Note that in this paper we never apply this (slightly more general) version of Theorem 6.1, as all interesting applications that we found satisfy the stronger condition (K-Non), which is also easier to check.

7. Hyperbolic equality for combinatorial atlases

In this section we characterize when the equality conditions in (Hyp) hold for all non-sink vertices in a combinatorial atlas. For that, we obtain the equality variation of the local-global principle (Theorem 5.2). See §17.4 for some background.

7.1. **Statement.** Let $\mathbb{A}$ be a combinatorial atlas of dimension d. Recall that, for a non-sink vertex v of Γ, we denote by $\mathbf{M} = \mathbf{M}_v$ the associated matrix of v, by $\mathbf{h} = \mathbf{h}_v$ the associated vector of v, by $\mathbf{T}^{\langle i \rangle} = \mathbf{T}_v^{\langle i \rangle}$ the associated linear transformation of the edge $e^{\langle i \rangle} = (v, v^{\langle i \rangle})$, and by $\mathbf{M}^{\langle i \rangle}$ the associated matrix of the vertex $v^{\langle i \rangle}$.

A *global pair* $\mathbf{f}, \mathbf{g} \in \mathbb{R}^d$ is a pair of nonnegative vectors, such that

(Glob-Pos) $$\mathbf{f} + \mathbf{g} \ \ \text{is a strictly positive vector}.$$

Here $\mathbf{f}$ and $\mathbf{g}$ are global in a sense that they are the same for all vertices $v \in \Omega$.

Fix a number $\mathrm{s} > 0$. We say that a vertex $v \in \Omega$ satisfies (s-Equ), if

(s-Equ) $$\langle \mathbf{f}, \mathbf{M}\mathbf{f} \rangle \ = \ \mathrm{s} \, \langle \mathbf{g}, \mathbf{M}\mathbf{f} \rangle \ = \ \mathrm{s}^2 \, \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle,$$

where $\mathbf{M} = \mathbf{M}_v$ as above. Observe that (s-Equ) implies that equality occurs in (Hyp) for substitutions $\mathbf{v} \leftarrow \mathbf{g}$ and $\mathbf{w} \leftarrow \mathbf{f}$, since

$$\langle \mathbf{g}, \mathbf{M}\mathbf{f} \rangle^2 \ = \ \mathrm{s} \, \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle \, \mathrm{s}^{-1} \, \langle \mathbf{f}, \mathbf{M}\mathbf{f} \rangle \ = \ \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle \, \langle \mathbf{f}, \mathbf{M}\mathbf{f} \rangle \, . \tag{7.1}$$

We say that the atlas $\mathbb{A}$ satisfies s-*equality property* if (s-Equ) holds for every $v \in \Omega$.

We now present the first main result of this section, which is a *local-global principle* for (s-Equ). A vertex $v \in \Omega^+$ is called *functional source* if the following conditions are satisfied:

(Glob-Proj) $$\mathrm{f}_j \ = \ \big(\mathbf{T}^{\langle i \rangle} \, \mathbf{f}\,\big)_j \quad \text{and} \quad \mathrm{g}_j \ = \ \big(\mathbf{T}^{\langle i \rangle} \, \mathbf{g}\,\big)_j \quad \text{for every} \ \ i \in \mathrm{supp}(\mathbf{M}), \ j \in \mathrm{supp}(\mathbf{M}^{\langle i \rangle}),$$

(h-Glob) $$\mathbf{f} \ = \ \mathbf{h}_v \, .$$

Here condition (Glob-Proj) means that $\mathbf{f}, \mathbf{g}$ are fixed points of the projection $\mathbf{T}^{\langle i \rangle}$ when restricted to the support.

We say that an edge $e^{\langle i \rangle} = (v, v^{\langle i \rangle}) \in \Theta$ is *functional* if v is a functional source and $i \in \mathrm{supp}(\mathbf{M}) \cap \mathrm{supp}(\mathbf{h})$. A vertex $w \in \Omega$ is a *functional target* of v, if there exists a directed path $v \to w$ in Γ consisting of only functional edges. Note that a functional target is not necessarily a functional source.

Theorem 7.1 (*local-global equality principle*)**.** *Let $\mathbb{A}$ be a combinatorial atlas that satisfies properties* (Inh), (Pull)*. Suppose also $\mathbb{A}$ satisfies property* (Hyp) *for every vertex $v \in \Omega$. Let $\mathbf{f}, \mathbf{g}$ be a global pair of $\mathbb{A}$. Suppose a non-sink vertex $v \in \Omega^+$ satisfies* (s-Equ) *with constant $\mathrm{s} > 0$. Then every functional target of v also satisfies* (s-Equ) *with the same constant* s.

7.2. **Algebraic lemma.** We start with the following general algebraic result. Recall that a matrix is *hyperbolic* if it satisfies (Hyp).

Lemma 7.2. *Let $\mathbf{M}$ be a nonnegative symmetric hyperbolic $\mathrm{r} \times \mathrm{r}$ matrix. Let $\mathbf{f}, \mathbf{g} \in \mathbb{R}^{\mathrm{r}}$ be nonnegative vectors, let $\mathrm{s} > 0$, and let $\mathbf{z} := \mathbf{f} - \mathrm{s}\,\mathbf{g}$. Then* (s-Equ) *holds* <u>*if and only if*</u> $\mathbf{M}\mathbf{z} = 0$.

Proof. The $\Leftarrow$ direction follows from the fact that

$$\langle \mathbf{f}, \mathbf{M}\mathbf{f}\rangle - \mathrm{s}\,\langle \mathbf{g}, \mathbf{M}\mathbf{f}\rangle = \langle \mathbf{z}, \mathbf{M}\mathbf{f}\rangle = \langle \mathbf{M}\mathbf{z}, \mathbf{f}\rangle, \quad \text{and} \quad \mathrm{s}\,\langle \mathbf{g}, \mathbf{M}\mathbf{f}\rangle - \mathrm{s}^2\,\langle \mathbf{g}, \mathbf{M}\mathbf{g}\rangle = \mathrm{s}\,\langle \mathbf{g}, \mathbf{M}\mathbf{z}\rangle. \tag{7.2}$$

Thus it suffices to prove the $\Rightarrow$ direction. We will assume that $\mathbf{M}$ is nonzero when restricted to the support of $\mathbf{g}+\mathbf{f}$, as otherwise every term in (s-Equ) is equal to 0 and the lemma follows immediately. Let $\mathbf{w} := \mathbf{g}+\mathbf{f}$, and the assumption implies that $\langle \mathbf{w}, \mathbf{M}\mathbf{w}\rangle > 0$. By (Hyp), we then have that the matrix $\mathbf{M}$ is negative semidefinite on $(\mathbf{M}\mathbf{w})^{\perp}$. Now note that $\mathbf{z} \in (\mathbf{M}\mathbf{w})^{\perp}$, since $\langle \mathbf{z}, \mathbf{M}\mathbf{w}\rangle = 0$ by (7.2) and (s-Equ). Also note that

$$\langle \mathbf{z}, \mathbf{M}\mathbf{z}\rangle = \langle \mathbf{f}, \mathbf{M}\mathbf{f}\rangle - 2\,\mathrm{s}\,\langle \mathbf{g}, \mathbf{M}\mathbf{f}\rangle + \mathrm{s}^2\,\langle \mathbf{g}, \mathbf{M}\mathbf{g}\rangle =_{\text{(s-Equ)}} 0. \tag{7.3}$$

It then follows from these three observations that $\mathbf{M}\mathbf{z} = 0$, as desired. ☐

7.3. **Proof of Theorem 7.1.** By induction, it suffices to show that, for every functional edge $(v, v^{\langle i\rangle}) \in v^*$, we have that $v^{\langle i\rangle}$ satisfies (s-Equ) with the same constant $\mathrm{s} > 0$.

It follows from (Inh), that for every $i \in \operatorname{supp}(\mathbf{M})$ we have:

$$\big(\mathbf{M}\mathbf{g}\big)_i = \langle \mathbf{T}^{\langle i\rangle}\,\mathbf{g}, \mathbf{M}^{\langle i\rangle}\mathbf{T}^{\langle i\rangle}\,\mathbf{h}\rangle \qquad \text{and} \qquad \big(\mathbf{M}\mathbf{h}\big)_i = \langle \mathbf{T}^{\langle i\rangle}\,\mathbf{h}, \mathbf{M}^{\langle i\rangle}\,\mathbf{T}^{\langle i\rangle}\,\mathbf{h}\rangle\,.$$

It then follows from (Glob-Proj) and the fact that $\mathbf{f} = \mathbf{h} = \mathbf{h}_v$ by (h-Glob) that

$$\big(\mathbf{M}\mathbf{g}\big)_i = \langle \mathbf{g}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle \qquad \text{and} \qquad \big(\mathbf{M}\mathbf{f}\big)_i = \langle \mathbf{f}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle. \tag{7.4}$$

Let $\mathbf{z} := \mathbf{f} - s\,\mathbf{g}$. It then follows from (s-Equ) and (7.3) that $\langle \mathbf{z}, \mathbf{M}\mathbf{z}\rangle = 0$. By Lemma 7.2, (s-Equ) implies that $\mathbf{M}\mathbf{z} = \mathbf{0}$, which is equivalent to $\mathrm{s}\,\mathbf{M}\mathbf{g} = \mathbf{M}\mathbf{f}$. Together with (7.4), this implies that

$$\mathrm{s}\,\langle \mathbf{g}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle =_{\text{(Inh)}} \mathrm{s}\,\big(\mathbf{M}\mathbf{g}\big)_i = \big(\mathbf{M}\mathbf{f}\big)_i = \langle \mathbf{f}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle. \tag{7.5}$$

On the other hand, we have

$$\langle \mathbf{f}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle =_{(7.4)} \big(\mathbf{M}\mathbf{f}\big)_i =_{(7.5)} \frac{\mathrm{s}+1}{\mathrm{s}}\,\big(\mathbf{M}(\mathbf{f}+\mathbf{g})\big)_i > 0,$$

where the positivity follows by (Glob-Pos) and the assumption that $i \in \operatorname{supp}(\mathbf{M})$. Now note that,

$$\begin{aligned} \langle \mathbf{z}, \mathbf{M}^{\langle i\rangle}\,\mathbf{z}\rangle \quad &= \mathrm{s}^2\,\langle \mathbf{g}, \mathbf{M}^{\langle i\rangle}\,\mathbf{g}\rangle - 2\mathrm{s}\,\langle \mathbf{g}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle + \langle \mathbf{f}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle \\ &=_{(7.5)} \mathrm{s}^2\left(\langle \mathbf{g}, \mathbf{M}^{\langle i\rangle}\,\mathbf{g}\rangle - \frac{\langle \mathbf{g}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle^2}{\langle \mathbf{f}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle}\right), \end{aligned} \tag{7.6}$$

which is nonpositive as $v^{\langle i\rangle}$ satisfies (Hyp). On the other hand, we have

$$\sum_{i\in\operatorname{supp}(\mathbf{M})} \mathrm{h}_i\,\langle \mathbf{z}, \mathbf{M}^{\langle i\rangle}\,\mathbf{z}\rangle =_{\text{(Glob-Proj)}} \sum_{i\in\operatorname{supp}(\mathbf{M})} \mathrm{h}_i\,\langle \mathbf{T}^{\langle i\rangle}\,\mathbf{z}, \mathbf{M}^{\langle i\rangle}\mathbf{T}^{\langle i\rangle}\,\mathbf{z}\rangle \geq_{\text{(Pull)}} \langle \mathbf{z}, \mathbf{M}\mathbf{z}\rangle =_{(7.3)} 0.$$

So the RHS of this inequality is equal to 0, while the LHS is a sum of nonpositive terms by (7.6). This implies that every term in the first sum is equal to 0, and thus $\mathrm{h}_i\langle \mathbf{z}, \mathbf{M}^{\langle i\rangle}\,\mathbf{z}\rangle = 0$ for every $i \in \operatorname{supp}(\mathbf{M})$. This in turn implies that $\langle \mathbf{z}, \mathbf{M}^{\langle i\rangle}\,\mathbf{z}\rangle = 0$ whenever $(v, v^{\langle i\rangle})$ is a functional edge. This is equivalent to saying that the left side of (7.6) is zero, and we have:

$$\langle \mathbf{g}, \mathbf{M}^{\langle i\rangle}\,\mathbf{g}\rangle = \frac{\langle \mathbf{g}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle^2}{\langle \mathbf{f}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle} =_{(7.5)} \frac{1}{\mathrm{s}}\,\langle \mathbf{g}, \mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle. \tag{7.7}$$

It then follows from (7.5) and (7.7) that $v^{\langle i\rangle}$ satisfies (s-Equ) whenever $(v, v^{\langle i\rangle})$ is a functional edge, which completes the proof. $\square$

8. Log-concave inequalities for interval greedoids

In this section, we prove Theorem 1.31 by constructing a combinatorial atlas corresponding to a greedoid, and applying both local-global principle in Theorem 5.2 and sufficient conditions for hyperbolicity given in Theorem 6.1.

8.1. **Combinatorial atlas for interval greedoids.** Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid on $|X| = n$ elements, let $1 \leq k < \mathrm{rk}(\mathcal{G})$, and let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be the weight function in Theorem 1.31. We define a combinatorial atlas $\mathbb{A}$ corresponding to $(\mathcal{G}, k, \mathrm{q})$ as follows.

Define an acyclic graph $\Gamma := (\Omega, \Theta)$, where the set of vertices $\Omega := \Omega^0 \cup \Omega^1 \cup \ldots \cup \Omega^{k-1}$ is given by[12]

$$\begin{aligned} \Omega^m \ &:= \ \big\{ (\alpha, m, t) \ \mid \ \alpha \in X^* \text{ with } |\alpha| \leq k-1-m, \ t \in [0,1] \big\} \quad \text{for} \quad m \geq 1, \\ \Omega^0 \ &:= \ \big\{ (\alpha, 0, 1) \ \mid \ \alpha \in X^* \text{ with } |\alpha| \leq k-1 \big\}. \end{aligned}$$

Here the restriction $t = 1$ in Ω^0 is crucial for a technical reason that will be apparent later in the section.

Let $\widehat{X} := X \cup \{\mathsf{null}\}$ be the set of letters X with one special element null added. The reader should think of element null as the empty letter. Let $d := |\widehat{X}| = (n+1)$ be the dimension of the atlas, so each vertex $v \in \Omega^m$, $m \geq 1$, has exactly $(n+1)$ outgoing edges we label $\big(v, v^{\langle x\rangle}\big) \in \Theta$, where $x \in \widehat{X}$ and $v^{\langle x\rangle} \in \Omega^{m-1}$ is defined as follows:

$$v^{\langle x\rangle} \ := \ \begin{cases} (\alpha x, m-1, 1) & \text{if} \quad x \in X, \\ (\alpha, m-1, 1) & \text{if} \quad x = \mathsf{null}. \end{cases}$$

Let us emphasize that this is not a typo and we indeed have the last parameter $t = 1$, for all $v^{\langle x\rangle}$ (see Figure 8.1).

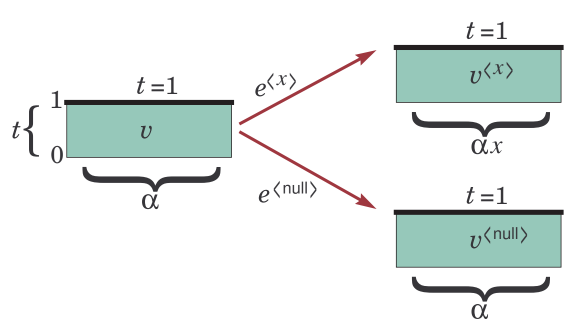


Figure 8.1. Edges of two type: $e^{\langle x\rangle} = \big(v, v^{\langle x\rangle}\big)$, $v = (\alpha, m, t)$, $v^{\langle x\rangle} = (\alpha x, m-1, 1)$, and $e^{\langle \mathsf{null}\rangle} = \big(v, v^{\langle \mathsf{null}\rangle}\big)$, $v = (\alpha, m, t)$, $v^{\langle \mathsf{null}\rangle} = (\alpha, m-1, 1)$.

[12]Yes, graph Γ has uncountably many vertices.

For every $\alpha \in X^*$ and every $m \in \{1, \ldots, \mathrm{rk}(\mathcal{G}) - |\alpha| - 1\}$, we denote by $\mathbf{A}(\alpha, m) := (\mathrm{A}_{xy})_{x,y \in \widehat{X}}$ the symmetric $d \times d$ matrix defined as follows:[13]

$$\begin{aligned}
\mathrm{A}_{xy} &:= 0 \qquad \text{for} \quad x \notin \mathrm{Cont}(\alpha) + \mathsf{null} \quad \text{or} \quad y \notin \mathrm{Cont}(\alpha) + \mathsf{null}, \\
\mathrm{A}_{xy} &:= \sum_{\beta \in \mathrm{Cont}_{m-1}(\alpha xy)} \mathrm{q}(\alpha xy\beta) \qquad \text{for} \quad x \neq y, \quad x, y \in \mathrm{Cont}(\alpha), \\
\mathrm{A}_{xx} &:= \sum_{y \in \mathrm{Des}_\alpha(x)} \ \sum_{\beta \in \mathrm{Cont}_{m-1}(\alpha xy)} \mathrm{q}(\alpha xy\beta) \qquad \text{for} \quad x \in \mathrm{Cont}(\alpha), \\
\mathrm{A}_{x\,\mathsf{null}} &= \mathrm{A}_{\mathsf{null}\,x} := \sum_{\beta \in \mathrm{Cont}_{m-1}(\alpha x)} \mathrm{q}(\alpha x\beta) \qquad \text{for} \quad x \in \mathrm{Cont}(\alpha) \quad \text{and} \quad y = \mathsf{null}, \\
\mathrm{A}_{\mathsf{null}\,\mathsf{null}} &:= \sum_{\beta \in \mathrm{Cont}_{m-1}(\alpha)} \mathrm{q}(\alpha\beta).
\end{aligned}$$

For the second line, note that (ContInv) implies $\mathrm{A}_{xy} = \mathrm{A}_{yx}$. Note also that $\mathrm{A}_{x\,\mathsf{null}} > 0$, since by the exchange property the word $\alpha x \in \mathcal{L}$ can be extended to $\alpha x\beta \in \mathcal{L}$ for some $\beta \in X^*$ with $|\beta| \leq \mathrm{rk}(\mathcal{G}) - |\alpha| - 1$.

For each vertex $v = (\alpha, m, t) \in \Omega$, define the associated matrix as follows:

$$\mathbf{M} = \mathbf{M}_{(\alpha,m,t)} := t\,\mathbf{A}(\alpha, m+1) + (1-t)\,\mathbf{A}(\alpha, m).$$

Similarly, define the associated vector $\mathbf{h} = \mathbf{h}_{(\alpha,m,t)} \in \mathbb{R}^d$ with coordinates

$$\mathrm{h}_x := \begin{cases} t & \text{if } x \in X, \\ 1-t & \text{if } x = \mathsf{null}. \end{cases}$$

Finally, define the linear transformation $\mathbf{T}^{\langle x \rangle} : \mathbb{R}^d \to \mathbb{R}^d$ associated to the edge $(v, v^{\langle x \rangle})$, as follows:

$$\left(\mathbf{T}^{\langle x \rangle}\,\mathbf{v}\right)_y := \begin{cases} \mathrm{v}_y & \text{if } y \in \mathrm{supp}(\mathbf{M}), \\ \mathrm{v}_x & \text{if } y \in \widehat{X} \setminus \mathrm{supp}(\mathbf{M}). \end{cases}$$

8.2. **Properties of the atlas.** We now show that our combinatorial atlas $\mathbb{A}$ satisfies the conditions in Theorem 5.2, in the following series of lemmas.

Lemma 8.1. *For every vertex $v = (\alpha, m, t) \in \Omega$, we have:*

(i) *the support of the associated matrix $\mathbf{M}_v$ is given by*

$$\mathrm{supp}(\mathbf{M}_v) = \mathrm{supp}(\mathbf{A}(\alpha, m+1)) = \mathrm{supp}(\mathbf{A}(\alpha, m)) = \begin{cases} \mathrm{Cont}(\alpha) + \mathsf{null} & \text{if } \alpha \in \mathcal{L}, \\ \varnothing & \text{if } \alpha \notin \mathcal{L}. \end{cases}$$

(ii) *vertex v satisfies (Irr), and*

(iii) *vertex v satisfies (h-Pos) for $t \in (0,1)$.*

Proof. Part (i) follows directly from the definition of matrices $\mathbf{M}$, $\mathbf{A}(\alpha, m+1)$, and $\mathbf{A}(\alpha, m)$. Part (iii) follows from the fact that $\mathbf{h}_v$ is a strictly positive vector when $t \in (0,1)$.

We now prove part (ii). If $\alpha \notin \mathcal{L}$, then $\mathbf{M}$ is a zero matrix and v trivially satisfies (Irr). If $\alpha \in \mathcal{L}$, then it follows from the definition of $\mathbf{M} = \left(\mathrm{M}_{xy}\right)$, that $\mathrm{M}_{x\,\mathsf{null}} > 0$ for every $x \in \mathrm{Cont}(\alpha)$. Since the support of $\mathbf{M}$ is $\mathrm{Cont}(\alpha) + \mathsf{null}$, this proves (Irr), as desired. □

Lemma 8.2. *For every greedoid $\mathcal{G} = (X, \mathcal{L})$, the atlas $\mathbb{A}$ satisfies (Proj).*

Proof. Let $v = (\alpha, m, t) \in \Omega^m$, $m \geq 1$, be a non-sink vertex of Γ. The condition (Proj) follows directly from the definition of $\mathbf{T}^{\langle x \rangle}$. □

Lemma 8.3. *For every greedoid $\mathcal{G} = (X, \mathcal{L})$, the atlas $\mathbb{A}$ satisfies (Inh).*

[13] When $m = 1$ and $\alpha \in \mathcal{L}$, we have $\mathrm{Cont}_{m-1}(\alpha)$ consists of exactly one element, namely the empty word.

Proof. Let $v = (\alpha, m, t) \in \Omega^m$, $m \geq 1$, be a non-sink vertex of Γ. Let $x \in \operatorname{supp}(\mathbf{M}) = \operatorname{Cont}(\alpha) \cup \{\textsf{null}\}$. By the linearity of $\mathbf{T}^{\langle x \rangle}$, it suffices to show that for every $y \in \operatorname{Cont}(\alpha) \cup \{\textsf{null}\}$, we have:

$$\mathrm{M}_{xy} \ = \ \big\langle \mathbf{T}^{\langle x \rangle} \, \mathbf{e}_y \, , \mathbf{M}^{\langle x \rangle} \, \mathbf{T}^{\langle x \rangle} \, \mathbf{h} \, \big\rangle,$$

where $\big\{ \mathbf{e}_y, \, y \in \widehat{X} \big\}$ is the standard basis for $\mathbb{R}^d$. We present only the proof for the case $x, y \in \operatorname{Cont}(\alpha)$, as the proof of the other cases are analogous.

First suppose that $x, y \in \operatorname{Cont}(\alpha)$ are distinct. Then:

$$\begin{aligned}
\big\langle \mathbf{T}^{\langle x \rangle} \, \mathbf{e}_y \, , \mathbf{M}^{\langle x \rangle} \, \mathbf{T}^{\langle x \rangle} \, \mathbf{h} \, \big\rangle \quad &= \sum_{z \,\in\, \operatorname{supp}(\mathbf{M}^{\langle x \rangle})} \mathrm{M}^{\langle x \rangle}_{yz} \, \big(\mathbf{T}^{\langle x \rangle} \, \mathbf{h} \big)_z \\
= \sum_{z \,\in\, \operatorname{supp}(\mathbf{M}^{\langle x \rangle}), \, z \neq \textsf{null}} & t \, \mathbf{A}(\alpha x, m)_{yz} \ + \ (1-t) \, \mathbf{A}(\alpha x, m)_{y \, \textsf{null}} \\
= \sum_{z \,\in\, X} \ \sum_{\beta \in \operatorname{Cont}_{m-1}(\alpha x y z)} & t \, \mathrm{q}(\alpha x y z \beta) \ + \sum_{\beta \,\in\, \operatorname{Cont}_{m-1}(\alpha x y)} (1-t) \, \mathrm{q}(\alpha x y \beta) \\
= \sum_{\gamma \,\in\, \operatorname{Cont}_m(\alpha x y)} & t \, \mathrm{q}(\alpha x y \gamma) \ + \sum_{\beta \in \operatorname{Cont}_{m-1}(\alpha x y)} (1-t) \, \mathrm{q}(\alpha x y \beta),
\end{aligned}$$

where we substitute $\gamma \leftarrow z\beta$ in the first term of the last equality. This implies that

$$\big\langle \mathbf{T}^{\langle x \rangle} \, \mathbf{e}_y \, , \mathbf{M}^{\langle x \rangle} \, \mathbf{T}^{\langle x \rangle} \, \mathbf{h} \, \big\rangle \ = \ t \, \mathbf{A}(\alpha, m+1)_{xy} \ + \ (1-t) \, \mathbf{A}(\alpha, m)_{xy} \ = \ \mathrm{M}_{xy} \, ,$$

which proves (Inh) for this case.

Now suppose that $x = y \in \operatorname{Cont}(\alpha)$. Then:

$$\big\langle \mathbf{T}^{\langle x \rangle} \, \mathbf{e}_x, \mathbf{M}^{\langle x \rangle} \, \mathbf{T}^{\langle x \rangle} \, \mathbf{h} \, \big\rangle \ = \sum_{y \,\in\, \operatorname{supp}(\mathbf{M}^{\langle x \rangle}) \setminus \operatorname{supp}(\mathbf{M})} \ \sum_{z \,\in\, \operatorname{supp}(\mathbf{M}^{\langle x \rangle})} \mathrm{M}^{\langle x \rangle}_{yz} \, \big(\mathbf{T}^{\langle x \rangle} \, \mathbf{h} \big)_z \, .$$

By the same argument as above, this equation becomes

$$\begin{aligned}
& \sum_{y \,\in\, \operatorname{supp}(\mathbf{M}^{\langle x \rangle}) \setminus \operatorname{supp}(\mathbf{M})} \ \sum_{\gamma \,\in\, \operatorname{Cont}_m(\alpha x y)} t \, \mathrm{q}(\alpha x y \gamma) \ + \sum_{\beta \,\in\, \operatorname{Cont}_{m-1}(\alpha x y)} (1-t) \, \mathrm{q}(\alpha x y \beta) \\
& = \sum_{y \,\in\, \operatorname{Des}_\alpha(x)} \ \sum_{\gamma \,\in\, \operatorname{Cont}_{\alpha x y}(m)} t \, \mathrm{q}(\alpha x y \gamma) \ + \sum_{y \,\in\, \operatorname{Des}_\alpha(x)} \ \sum_{\beta \,\in\, \operatorname{Cont}_{\alpha x y}(m-1)} (1-t) \, \mathrm{q}(\alpha x y \beta) \\
& = \ t \, \mathbf{A}(\alpha, m+1)_{xx} \ + \ (1-t) \, \mathbf{A}(\alpha, m)_{xx} \ = \ \mathrm{M}_{xx} \, ,
\end{aligned}$$

which proves (Inh) for this case. This completes the proof. $\square$

Lemma 8.4. *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid, and suppose the weight function $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ satisfies* (ContInv)*. Then the atlas $\mathbb{A}$ satisfies* (T-Inv)*.*

Proof. Let $v = (\alpha, m, t) \in \Omega^m$, $m \geq 1$, be a non-sink vertex of Γ, and let x, y, z be distinct elements of $\operatorname{supp}(\mathbf{M}) = \operatorname{Cont}(\alpha) + \textsf{null}$. We present only the proof for the case when $x, y, z \in \operatorname{Cont}(\alpha)$, as other cases follow analogously.

First suppose that $\alpha x' y' z' \notin \mathcal{L}$ for every permutation (x', y', z') of $\{x, y, z\}$. Then

$$\mathrm{M}^{\langle x \rangle}_{yz} \ = \ \mathrm{M}^{\langle y \rangle}_{zx} \ = \ \mathrm{M}^{\langle z \rangle}_{xy} \ = \ 0,$$

and (T-Inv) is satisfied. So, without loss of generality, we assume that $\alpha x y z \in \mathcal{L}$. It then follows from the interval exchange property that $\alpha x' y' z' \in \mathcal{L}$ for every permutation (x', y', z') of $\{x, y, z\}$. This allows us to apply (ContInv) for $\alpha \in \mathcal{L}$ and any two elements from $\{x, y, z\}$.

We now have

$$\begin{aligned}
\mathrm{M}^{\langle x \rangle}_{yz} \ = \ \mathbf{A}(\alpha x, m)_{yz} \ &= \sum_{\beta \,\in\, \operatorname{Cont}_{m-1}(\alpha x y z)} \mathrm{q}(\alpha x y z \beta) \\
&=_{(\text{ContInv})} \sum_{\beta \,\in\, \operatorname{Cont}_{m-1}(\alpha y x z)} \mathrm{q}(\alpha y x z \beta) \ = \ \mathbf{A}(\alpha y, m)_{xz} \ = \ \mathrm{M}^{\langle y \rangle}_{xz} \, .
\end{aligned}$$

By an analogous argument, it follows that $\mathrm{M}^{\langle x \rangle}_{yz} = \mathrm{M}^{\langle z \rangle}_{xy}$, and thus (T-Inv) is satisfied, as desired. $\square$

Lemma 8.5. *Let $\mathcal{G} = (X, \mathcal{L})$ be a greedoid, let $1 \leq k < \operatorname{rk}(\mathcal{G})$, and suppose the weight function $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ satisfies* (ContInv) *and* (PAMon)*. Then the atlas $\mathbb{A}$ satisfies* (K-Non)*.*

Proof. Let $v = (\alpha, m, t) \in \Omega^m$, $m \geq 1$, be a non-sink vertex of Γ. We need to check the condition (K-Non) for distinct $x, y \in \operatorname{supp}(\mathbf{M}) = \operatorname{Cont}(\alpha) + \mathsf{null}$.

First suppose that x, y are distinct elements of $\operatorname{Cont}(\alpha)$. We have:

$$\mathrm{M}_{yy}^{\langle x \rangle} \ = \ \mathbf{A}(\alpha x, m)_{yy} \ = \sum_{z \in \operatorname{Des}_{\alpha x}(y)} \ \sum_{\beta \in \operatorname{Cont}_{m-1}(\alpha x y z)} \mathrm{q}(\alpha x y z \beta).$$

Note that $z \in X$ in the equation above is summed over the set

$$\big\{ z \in X \ : \ \alpha x z \notin \mathcal{L}, \, \alpha x y z \in \mathcal{L} \big\}.$$

By the interval exchange property, every element z in the set above also satisfies $\alpha z \notin \mathcal{L}$. We can then partition the set above into

$$\begin{aligned} &\big\{ z \ : \ \alpha z \notin \mathcal{L}, \, \alpha x z \notin \mathcal{L}, \, \alpha y z \notin \mathcal{L}, \, \alpha x y z \in \mathcal{L} \big\} \ \cup \ \big\{ z \ : \ \alpha z \notin \mathcal{L}, \, \alpha x z \notin \mathcal{L}, \, \alpha y z \in \mathcal{L}, \, \alpha x y z \in \mathcal{L} \big\} \\ &\quad = \ \operatorname{Pas}_\alpha(x,y) \ \cup \ \big\{ z \ : \ \alpha z \notin \mathcal{L}, \, \alpha x z \notin \mathcal{L}, \, \alpha y z \in \mathcal{L}, \, \alpha x y z \in \mathcal{L} \big\}. \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} \sum_{z \in \operatorname{Fam}^{\langle y \rangle}} \mathrm{M}_{xz}^{\langle y \rangle} \ &= \sum_{z \in \operatorname{Fam}^{\langle y \rangle}} \mathbf{A}(\alpha y, m)_{xz} \ = \sum_{z \in \operatorname{Des}_\alpha(y)} \ \sum_{\beta \in \operatorname{Cont}_{m-1}(\alpha y x z)} \mathrm{q}(\alpha y x z \beta) \\ &=_{(\text{ContInv})} \sum_{z \in \operatorname{Des}_\alpha(y)} \ \sum_{\beta \in \operatorname{Cont}_{m-1}(\alpha x y z)} \mathrm{q}(\alpha x y z \beta), \end{aligned}$$

where in the last equality we apply (ContInv) to swap x and y. Note that $z \in X$ in the equation above is summed over the set

$$\{ z \in X \ : \ \alpha z \notin \mathcal{L}, \, \alpha y z \in \mathcal{L}, \, \alpha x y z \in \mathcal{L} \},$$

which can be partitioned into

$$\begin{aligned} &\big\{ z \ : \ \alpha z \notin \mathcal{L}, \, \alpha x z \in \mathcal{L}, \, \alpha y z \in \mathcal{L}, \, \alpha x y z \in \mathcal{L} \big\} \ \cup \ \big\{ z \ : \ \alpha z \notin \mathcal{L}, \, \alpha x z \notin \mathcal{L}, \, \alpha y z \in \mathcal{L}, \, \alpha x y z \in \mathcal{L} \big\} \\ &\quad = \ \operatorname{Act}_\alpha(x,y) \ \cup \ \big\{ z \ : \ \alpha z \notin \mathcal{L}, \, \alpha x z \notin \mathcal{L}, \, \alpha y z \in \mathcal{L}, \, \alpha x y z \in \mathcal{L} \big\}. \end{aligned}$$

It follows from the calculations above that

$$\begin{aligned} \mathrm{M}_{yy}^{\langle x \rangle} \ - \sum_{z \in \operatorname{Fam}^{\langle y \rangle}} \mathrm{M}_{xz}^{\langle y \rangle} \ &= \sum_{z \in \operatorname{Pas}_\alpha(x,y)} \ \sum_{\beta \in \operatorname{Cont}_{m-1}(\alpha x y z)} \mathrm{q}(\alpha x y z \beta) \\ &\quad - \sum_{z \in \operatorname{Act}_\alpha(x,y)} \ \sum_{\beta \in \operatorname{Cont}_{m-1}(\alpha x y z)} \mathrm{q}(\alpha x y z \beta), \end{aligned}$$

which is nonnegative by (PAMon). This proves (K-Non) in this case.

Now suppose that $x \in \operatorname{Cont}(\alpha)$ and $y = \mathsf{null}$. Without loss of generality, we assume that $\alpha \in \mathcal{L}$, as otherwise $\mathrm{M}_{yz}^{\langle x \rangle}$ is always equal to the zero matrix and (K-Non) is trivially satisfied. Then we have:

$$\mathrm{M}_{\mathsf{null}\,\mathsf{null}}^{\langle x \rangle} \ = \ \mathbf{A}(\alpha x, m)_{\mathsf{null}\,\mathsf{null}} \ = \sum_{\beta \in \operatorname{Cont}_{m-1}(\alpha x)} \mathrm{q}(\alpha x \beta).$$

On the other hand, we have $\operatorname{supp}(\mathbf{M}) = \operatorname{supp}\big(\mathbf{M}^{\langle \mathsf{null} \rangle}\big) = \operatorname{Cont}(\alpha) + \mathsf{null}$, which implies that

$$\operatorname{Fam}^{\langle \mathsf{null} \rangle} \ = \ \operatorname{supp}\big(\mathbf{M}^{\langle \mathsf{null} \rangle}\big) \setminus \big(\operatorname{supp}(\mathbf{M}) \ - \ \mathsf{null}\big) \ = \ \{\mathsf{null}\}.$$

Therefore, we have:

$$\sum_{z \in \operatorname{Fam}^{\langle \mathsf{null} \rangle}} \mathrm{M}_{xz}^{\langle \mathsf{null} \rangle} \ = \ \mathrm{M}_{x\,\mathsf{null}}^{\langle \mathsf{null} \rangle} \ = \ \mathbf{A}(\alpha, m)_{x\,\mathsf{null}} \ = \sum_{\beta \in \operatorname{Cont}_{m-1}(\alpha x)} \mathrm{q}(\alpha x \beta).$$

It thus follows from the calculations above that $\mathrm{K}_{x\,\mathsf{null}} = 0$. This completes the proof of (K-Non). □

8.3. **Basic hyperbolicity.** To prove hyperbolicity of vertices in Ω^0, we need the following straightforward linear algebra lemma. We include the proof for completeness.

Lemma 8.6. *Let $\mathbf{N} = \big(\mathrm{N}_{ij}\big)$ be a nonnegative symmetric $(n+1)\times(n+1)$ matrix, such that its nondiagonal entries are equal to 1. Suppose that*

$$(*) \qquad \mathrm{N}_{11}, \ldots, \mathrm{N}_{nn} \, \leq \, 1 \qquad \text{and} \qquad \mathrm{N}_{n+1\,n+1} \, \geq \, \sum_{i=1}^{n} \frac{\mathrm{N}_{n+1\,n+1}-1}{1-\mathrm{N}_{ii}} \quad \text{if } \ \mathrm{N}_{ii} < 1 \text{ for all } \ i \in [n].$$

Then $\mathbf{N}$ satisfies (Hyp).

Proof. Fix $\epsilon > 0$. Substituting $\mathrm{N}_{ii} \leftarrow \mathrm{N}_{ii} - \epsilon$ for every $1 \leq i \leq n$ if necessary, we can assume that all inequalities in $(*)$ are strict. Note that (Hyp) is preserved under taking the limit $\epsilon \to 0$, so it suffices to prove the result in this case.

We prove that $\mathbf{N}$ satisfies (OPE) by induction on n. By Lemma 5.3 this implies (Hyp). The base case $n = 0$ is trivial. Assume that the claim is true for $(n-1)$. Let $\lambda_1 \geq \ldots \geq \lambda_{n+1}$ be the eigenvalues of $\mathbf{N}$, and let $\lambda'_1 \geq \ldots \geq \lambda'_n$ be the eigenvalues of the matrix obtained by removing the first row and column of $\mathbf{N}$. By the *Cauchy interlacing theorem*, we have

$$\lambda_1 \, \geq \, \lambda'_1 \, \geq \, \lambda_2 \, \geq \, \ldots \, \geq \, \lambda'_n \, \geq \, \lambda_{n+1} \, .$$

Note that $\lambda'_2, \ldots, \lambda'_n$ are nonpositive by induction. It then follows that $\lambda_3, \ldots, \lambda_{n+1}$ are nonpositive. By the Perron–Frobenius theorem, we also have $\lambda_1 > 0$. It thus suffices to show that $\lambda_2 \leq 0$, which will follow from showing that $\det(\mathbf{N})$ has sign $(-1)^n$. Observe that $\det(\mathbf{N})$ is equal to

$$\begin{vmatrix} \mathrm{N}_{11}-1 & 0 & \cdots & 0 & 1-\mathrm{N}_{n+1\,n+1} \\ 0 & \mathrm{N}_{22}-1 & & 0 & 1-\mathrm{N}_{n+1\,n+1} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{N}_{nn}-1 & 1-\mathrm{N}_{n+1\,n+1} \\ 1 & 1 & \ldots & 1 & \mathrm{N}_{n+1\,n+1} \end{vmatrix} = \begin{vmatrix} \mathrm{N}_{11}-1 & 0 & \cdots & 0 & 1-\mathrm{N}_{n+1\,n+1} \\ 0 & \mathrm{N}_{22}-1 & & 0 & 1-\mathrm{N}_{n+1\,n+1} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{N}_{nn}-1 & 1-\mathrm{N}_{n+1\,n+1} \\ 0 & 0 & \ldots & 0 & \mathrm{J} \end{vmatrix},$$

$$\text{where} \qquad \mathrm{J} \, := \, \mathrm{N}_{n+1+1} \, - \, \sum_{i=1}^{n} \frac{\mathrm{N}_{n+1\,n+1} \, - \, 1}{1 \, - \, \mathrm{N}_{ii}} \, > \, 0 \, ,$$

by the assumption $(*)$. Therefore, we have

$$\det(\mathbf{N}) \, = \, \mathrm{J} \cdot \prod_{i=1}^{n} \big(\mathrm{N}_{ii} \, - \, 1\big),$$

and by the assumptions on N_{ii} this determinant has sign $(-1)^n$. This completes the proof. $\square$

8.4. **Proof of Theorem 1.31.** We first show that every sink vertex in the combinatorial atlas $\mathbb{A}$ is hyperbolic.

Lemma 8.7. *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid on $|X| = n$ elements, let $1 \leq k < \mathrm{rk}(\mathcal{G})$, and let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a k-admissible weight function. Then every vertex in Ω^0 satisfies (Hyp).*

Proof. Let $v = (\alpha, 0, 1) \in \Omega^0$ be a sink vertex. It suffices to show that $\mathbf{A}(\alpha, 1)$ satisfies (Hyp). First note that if $\alpha \notin \mathcal{L}$, then $\mathbf{A}(\alpha, 1)$ is a zero matrix, and (Hyp) is trivially true. Thus, we can assume that $\alpha \in \mathcal{L}$. We write $\mathrm{A}_{x,y} := \mathbf{A}(\alpha, 1)_{xy}$ for every $x, y \in X$.

Let $\mathcal{C} \in \mathrm{Par}(\alpha)$ be a parallel class. Suppose that $|\mathcal{C}| \geq 2$, and let x, y be distinct elements of $\mathcal{C}$.

Claim: *For every $z \in \widehat{X}$, we have $\ \omega(\alpha y)\, \mathrm{A}_{xz} \, = \, \omega(\alpha x)\, \mathrm{A}_{yz}$.*

Proof of Claim. First suppose that $z \in \{x, y\}$. It then follows from (FewDes) and the fact that $\alpha x y \notin \mathcal{L}$ that $\mathrm{A}_{x,z} \, = \, \mathrm{A}_{y,z} \, = \, 0$, which implies the claim in this case.

Now suppose that $z \in X \setminus \{x, y\}$. It follows from the exchange property that $\alpha xz \in \mathcal{L}$ if and only if $\alpha yz \in \mathcal{L}$. There are now two cases. If $\alpha xz \notin \mathcal{L}$ and $\alpha yz \notin \mathcal{L}$, then again we have $\mathrm{A}_{x,z} = \mathrm{A}_{y,z} = 0$, which implies the claim. If $\alpha xz \in \mathcal{L}$ and $\alpha yz \in \mathcal{L}$, we then have:

$$\mathrm{A}_{xz} \ = \ \mathrm{q}(\alpha xz) =_{(\text{LogMod})} c_{\ell+2} \, \frac{\omega(\alpha x)\, \omega(\alpha z)}{\omega(\alpha)}\,, \qquad \mathrm{A}_{yz} \ = \ \mathrm{q}(\alpha yz) =_{(\text{LogMod})} c_{\ell+2} \, \frac{\omega(\alpha y)\, \omega(\alpha z)}{\omega(\alpha)}\,,$$

where $\ell := |\alpha|$. This implies the claim in this case. Finally, let $z = \mathsf{null}$. Then we have $\mathrm{A}_{xz} = c_{\ell+1}\, \omega(\alpha x)$ and $\mathrm{A}_{yz} = c_{\ell+1}\, \omega(\alpha y)$, which implies the claim. □

Deduct the y-row and y-column of $\mathbf{A}(\alpha, 1)$ by $\frac{\omega(\alpha y)}{\omega(\alpha x)}$ of the x-row and x-column of $\mathbf{A}(\alpha, 1)$. It then follows from the claim that the resulting matrix has y-row and y-column is equal to zero. Also, note that (Hyp) is preserved under this transformation. Applying this linear transformation repeatedly, and by restricting to the support of resulting matrix which preserves (Hyp), without loss of generality we can assume that $|\,\mathcal{C}\,| = 1$ for every parallel class $\mathcal{C} \in \mathrm{Par}(\alpha)$. Then the matrix $\mathbf{A}(\alpha, 1)$ is equal to

$$\begin{pmatrix} c_{\ell+2}\, \mathrm{b}_\alpha(\mathcal{C}_1)\, \frac{\omega(\alpha x_1)^2}{\omega(\alpha)} & \mathrm{q}(\alpha x_1 x_2) & \cdots & \mathrm{q}(\alpha x_1 x_n) & \mathrm{q}(\alpha x_1) \\ \mathrm{q}(\alpha x_2 x_1) & c_{\ell+2}\, \mathrm{b}_\alpha(\mathcal{C}_2)\, \frac{\omega(\alpha x_1)^2}{\omega(\alpha)} & & \vdots & \vdots \\ \vdots & & \ddots & \mathrm{q}(\alpha x_{n-1} x_n) & \mathrm{q}(\alpha x_{n-1}) \\ \mathrm{q}(\alpha x_n x_1) & \cdots & \mathrm{q}(\alpha x_n x_{n-1}) & c_{\ell+2}\, \mathrm{b}_\alpha(\mathcal{C}_d)\, \frac{\omega(\alpha x_1)^2}{\omega(\alpha)} & \mathrm{q}(\alpha x_n) \\ \mathrm{q}(\alpha x_1) & \cdots & \mathrm{q}(\alpha x_{n-1}) & \mathrm{q}(\alpha x_n) & \mathrm{q}(\alpha) \end{pmatrix},$$

where $\mathcal{C}_i = \{x_i\}$ for $i \in [n]$, the rows and columns are indexed by $\widehat{X} = \big\{x_1, \ldots, x_n, \mathsf{null}\big\}$, and $\mathrm{b}_\alpha(\mathcal{C})$ is as defined in (3.2). We now rescale the x_i-row and x_i-column by $\frac{\sqrt{\omega(\alpha)}}{\sqrt{c_{\ell+2}}\,\omega(\alpha x_i)}$, and the null-row and null-column by $\frac{\sqrt{c_{\ell+2}}}{c_{\ell+1}\,\sqrt{\omega(\alpha)}}$. Again, note that (Hyp) is preserved under this transformation. It then follows from (LogMod) that the matrix becomes

$$\begin{pmatrix} \mathrm{b}_\alpha(\mathcal{C}_1) & 1 & \cdots & 1 & 1 \\ 1 & \mathrm{b}_\alpha(\mathcal{C}_2) & & \vdots & \vdots \\ \vdots & & \ddots & 1 & 1 \\ 1 & \cdots & 1 & \mathrm{b}_\alpha(\mathcal{C}_n) & 1 \\ 1 & \cdots & 1 & 1 & \frac{c_{\ell+2}\, c_\ell}{c_{\ell+1}^2} \end{pmatrix}.$$

It follows from (SynMon) and (ScaleMon) that this matrix satisfies conditions $(*)$ in Lemma 8.6. Hence, by the lemma, this matrix satisfies (Hyp). We conclude that v satisfies (Hyp), as desired. □

We can now prove that every vertex in Γ is hyperbolic.

Proposition 8.8. *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid on $|X| = n$ elements, let $1 \le k < \mathrm{rk}(\mathcal{G})$, and let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a k-admissible weight function. Then every vertex in Ω satisfies* (Hyp).

Proof. We will show that every vertex in Ω^m for $m \le k-1$ satisfies (Hyp) by induction on m. The claim is true for $m = 0$ by Lemma 8.7. Suppose that the claim is true for Ω^{m-1}. Note that the atlas $\mathbb{A}$ satisfies the assumptions of Theorem 5.2 by Lemmas 8.2, 8.3, 8.4, and 8.5. It then follows that every regular vertex in Ω^m satisfies (Hyp).

On the other hand, by Lemma 8.1, the regular vertices of Ω^m are those of the form $v = (\alpha, m, t)$ with $t \in (0, 1)$. Since (Hyp) is preserved under taking the limits $t \to 0$ and $t \to 1$, it then follows that every vertex in Ω^m satisfies (Hyp), and the proof is complete. □

Proof of Theorem 1.31. Let $\mathbf{M} = \mathbf{M}_v$ be the matrix associated with the vertex $v = (\varnothing, k-1, 1)$. Let $\mathbf{v}$ and $\mathbf{w}$ be the characteristic vectors of X and $\{\mathsf{null}\}$, respectively. Then:

$$\mathrm{L}_\mathrm{q}(k+1) \ = \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle, \qquad \mathrm{L}_\mathrm{q}(k) \ = \ \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle, \qquad \mathrm{L}_\mathrm{q}(k-1) \ = \ \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle. \tag{8.1}$$

By Proposition 8.8, vertex v satisfies (Hyp). Substituting (8.1) into (Hyp), gives the log-concave inequality (1.29) in the theorem. $\square$

9. Proof of equality conditions for interval greedoids

In this section we prove Theorem 3.3. The implication (GE-b) $\Rightarrow$ (GE-a) is obvious. We now prove the other implications.

9.1. **Proof of (GE-a) $\Rightarrow$ (GE-$c1$) & (GE-$c2$).** Let $\mathbb{A}$ be the combinatorial atlas defined in §8.1, that corresponds to $(\mathcal{G}, k, \mathrm{q})$. Recall that every vertex of Γ satisfies (Hyp) by Proposition 8.8.

As at the end of previous section, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ be the characteristic vectors of X and $\{\mathsf{null}\}$, respectively. It is straightforward to verify that $\mathbf{v}, \mathbf{w}$ is a global pair of Γ, i.e. they satisfy (Glob-Pos).

Let $v = (\varnothing, k-1, 1) \in \Omega$ and let $\mathbf{M} = \mathbf{M}_v$ be the matrix associated with v. Recall that $\mathbf{M} = \mathbf{A}(\varnothing, k)$ and we have equalities (8.1) again:

$$\mathrm{L}_{\mathrm{q}}(k+1) \ = \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle\,, \qquad \mathrm{L}_{\mathrm{q}}(k) \ = \ \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle\,, \qquad \mathrm{L}_{\mathrm{q}}(k-1) \ = \ \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle\,. \tag{9.1}$$

Note also that $\mathrm{L}_{\mathrm{q}}(k+1), \mathrm{L}_{\mathrm{q}}(k), \mathrm{L}_{\mathrm{q}}(k-1) > 0$ since $k < \mathrm{rk}(\mathcal{G})$. It then follows from (GE-a), that v satisfies (s-Equ) for some $\mathrm{s} > 0$.

Let us show that, for every $\alpha \in \mathcal{L}$ of length $(k-1)$, we have:

$$\langle \mathbf{v}, \mathbf{A}(\alpha,1)\,\mathbf{v} \rangle \ = \ \mathrm{s}\,\langle \mathbf{w}, \mathbf{A}(\alpha,1)\,\mathbf{v} \rangle \ = \ \mathrm{s}^2\,\langle \mathbf{w}, \mathbf{A}(\alpha,1)\,\mathbf{w} \rangle. \tag{9.2}$$

First, suppose that $k = 1$. It then follows that $\alpha = \varnothing$ and $v = (\varnothing, 0, 1)$. Equation (9.2) now follows from the fact that v satisfies (s-Equ).

Now suppose that $k > 1$. Then it is straightforward to verify that v is a functional source, i.e. satisfies (Glob-Proj) and (h-Glob), where we apply the substitution $\mathbf{f} \leftarrow \mathbf{v}$ for (h-Glob). By Theorem 7.1, every functional target of v also satisfies (s-Equ) with the same $\mathrm{s} > 0$. On the other hand, it is straightforward to verify that the functional targets of v are those of the form $(\alpha, 0, 1)$. Combining these observations, we conclude (9.2).

Let $\mathbf{z} := \mathbf{v} - \mathrm{s}\,\mathbf{w}$. It follows from (9.2) that $\langle \mathbf{z}, \mathbf{A}(\alpha,1)\,\mathbf{z} \rangle = 0$. It then follows from Lemma 7.2 that $\mathbf{A}(\alpha,1)\,\mathbf{z} = \mathbf{0}$, which is equivalent to $\mathrm{s}\,\mathbf{A}(\alpha,1)\,\mathbf{w} = \mathbf{A}(\alpha,1)\,\mathbf{v}$. This implies that

$$\mathrm{s}\,\mathrm{q}(\alpha) \ = \ \mathrm{s}\,\big(\mathbf{A}(\alpha,1)\,\mathbf{w}\big)_{\mathsf{null}} \ = \ \big(\mathbf{A}(\alpha,1)\,\mathbf{v}\big)_{\mathsf{null}} \ = \ \sum_{x \in \mathrm{Cont}(\alpha)} \mathrm{q}(\alpha x),$$

which proves (GE-$c1$) for $\mathrm{s}(k-1) = \mathrm{s}$.

Let $x \in \mathrm{Cont}(\alpha)$ be an arbitrary continuation. By the same reasoning as above, we have:

$$\mathrm{s}\,\mathrm{q}(\alpha x) \ = \ \mathrm{s}\,\big(\mathbf{A}(\alpha,1)\,\mathbf{w}\big)_x \ = \ \big(\mathbf{A}(\alpha,1)\,\mathbf{v}\big)_x\,.$$

On the other hand, we also have:

$$\big(\mathbf{A}(\alpha,1)\,\mathbf{v}\big)_x \ = \ \sum_{y \in \mathrm{Des}_\alpha(x)} \mathrm{q}(\alpha x y) \ + \sum_{\substack{y \in \mathrm{Cont}(\alpha) \\ y \not\sim_\alpha x}} \mathrm{q}(\alpha x y)\,.$$

It then follows that:

$$\mathrm{s}\,\mathrm{q}(\alpha x) \ = \ \sum_{y \in \mathrm{Des}_\alpha(x)} \mathrm{q}(\alpha x y) \ + \sum_{\substack{y \in \mathrm{Cont}(\alpha) \\ y \not\sim_\alpha x}} \mathrm{q}(\alpha x y)\,. \tag{9.3}$$

Let $\mathcal{C}$ be the parallel class in Par_α containing x. We now show that (9.3) is equivalent to (GE-$c2$).

Applying (LogMod) to (9.3) and dividing both sides by $\omega(\alpha)$, we get:

$$\mathrm{s}\,c_k\,\frac{\omega(\alpha x)}{\omega(\alpha)} \ = \ \sum_{y \in \mathrm{Des}_\alpha(x)} c_{k+1}\,\frac{\omega(\alpha x y)}{\omega(\alpha)} \ + \sum_{\substack{y \in \mathrm{Cont}(\alpha) \\ y \not\sim_\alpha x}} c_{k+1}\,\frac{\omega(\alpha x)\,\omega(\alpha y)}{\omega(\alpha)^2}\,. \tag{9.4}$$

Now note that, (GE-c1) gives:

$$\sum_{\substack{y \in \mathrm{Cont}(\alpha) \\ y \not\succ_\alpha x}} \frac{\omega(\alpha y)}{\omega(\alpha)} \;=\; \mathrm{s}\, \frac{c_{k-1}}{c_k} \;-\; \mathrm{a}_\alpha(\mathcal{C}), \tag{9.5}$$

where

$$\mathrm{a}_\alpha(\mathcal{C}) \;:=\; \sum_{y \in \mathcal{C}} \frac{\omega(\alpha y)}{\omega(\alpha)}.$$

Now note that, when $|\,\mathcal{C}\,| \geq 2$,

$$\sum_{y \in \mathrm{Des}_\alpha(x)} \frac{\omega(\alpha x y)}{\omega(\alpha)} \;=_{(\text{FewDes})}\; 0 \;=\; \frac{\omega(\alpha x)}{\omega(\alpha)}\, \mathrm{a}_\alpha(\mathcal{C})\, \mathrm{b}_\alpha(\mathcal{C}),$$

where the last equality is because $\mathrm{b}_\alpha(\mathcal{C}) = 0$ when $|\,\mathcal{C}\,| \geq 2$. On the other hand, when $\mathcal{C} = \{x\}$,

$$\sum_{y \in \mathrm{Des}_\alpha(x)} \frac{\omega(\alpha x y)}{\omega(\alpha)} \;=\; \frac{\omega(\alpha x)^2}{\omega(\alpha)^2}\, \mathrm{b}_\alpha(\mathcal{C}) \;=\; \frac{\omega(\alpha x)}{\omega(\alpha)}\, \mathrm{a}_\alpha(\mathcal{C})\, \mathrm{b}_\alpha(\mathcal{C}),$$

where the last equality is because $\mathrm{a}_\alpha(\mathcal{C}) = \frac{\omega(\alpha x)}{\omega(\alpha)}$ when $\mathcal{C} = \{x\}$. This allows us to conclude that

$$\sum_{y \in \mathrm{Des}_\alpha(x)} \frac{\omega(\alpha x y)}{\omega(\alpha)} \;=\; \frac{\omega(\alpha x)}{\omega(\alpha)}\, \mathrm{a}_\alpha(\mathcal{C})\, \mathrm{b}_\alpha(\mathcal{C}), \tag{9.6}$$

Substituting (9.5) and (9.6) into (9.4), we obtain:

$$\mathrm{s}\, c_k\, \frac{\omega(\alpha x)}{\omega(\alpha)} \;=\; c_{k+1}\, \frac{\omega(\alpha x)}{\omega(\alpha)}\, \mathrm{a}_\alpha(\mathcal{C})\, \mathrm{b}_\alpha(\mathcal{C}) \;+\; c_{k+1}\, \frac{\omega(\alpha x)}{\omega(\alpha)} \left(\mathrm{s}\, \frac{c_{k-1}}{c_k} \;-\; \mathrm{a}_\alpha(\mathcal{C}) \right).$$

This is equivalent to

$$\mathrm{s}\left(\frac{c_{k-1}}{c_k} - \frac{c_k}{c_{k+1}} \right) \;=\; \mathrm{a}_\alpha(\mathcal{C})\, \big(1 - \mathrm{b}_\alpha(\mathcal{C})\big),$$

which is (GE-c2). This completes the proof. □

9.2. **Proof of** (GE-c1) & (GE-c2) $\Rightarrow$ (GE-b)**.** Write $\mathrm{s} := \mathrm{s}(k-1)$. We have $\mathrm{L}_{\mathrm{q},\alpha}(0) = \mathrm{q}(\alpha)$ by definition, and

$$\mathrm{L}_{\mathrm{q},\alpha}(1) \;=\; \sum_{x \in \mathrm{Cont}(\alpha)} \mathrm{q}(\alpha x) \;=_{(\text{GE-}c1)}\; \mathrm{s}\, \mathrm{q}(\alpha) \;=\; \mathrm{s}\, \mathrm{L}_{\mathrm{q},\alpha}(0),$$

which proves the first part of (GE-b). For the second part of (GE-b), we have:

$$\mathrm{L}_{\mathrm{q},\alpha}(2) \;=\; \sum_{x \in \mathrm{Cont}(\alpha)} \left(\sum_{y \in \mathrm{Des}_\alpha(x)} \mathrm{q}(\alpha x y) \;+\; \sum_{\substack{y \in \mathrm{Cont}(\alpha) \\ y \not\succ_\alpha x}} \mathrm{q}(\alpha x y) \right). \tag{9.7}$$

On the other hand, we showed in the proof above (see §9.1), that (GE-c2) is equivalent to (9.3). Therefore, for every $x \in \mathrm{Cont}(\alpha)$, we have:

$$\mathrm{s}\, \mathrm{q}(\alpha x) \;=\; \sum_{y \in \mathrm{Des}_\alpha(x)} \mathrm{q}(\alpha x y) \;+\; \sum_{\substack{y \in \mathrm{Cont}(\alpha) \\ y \not\succ_\alpha x}} \mathrm{q}(\alpha x y).$$

Substituting this equation into (9.7), we conclude:

$$\mathrm{L}_{\mathrm{q},\alpha}(2) \;=\; \sum_{x \in \mathrm{Cont}(\alpha)} \mathrm{s}\, \mathrm{q}(\alpha x) \;=_{(\text{GE-}c1)}\; \mathrm{s}^2\, \mathrm{q}(\alpha) \;=\; \mathrm{s}^2\, \mathrm{L}_{\mathrm{q},\alpha}(0).$$

This proves the second part of (GE-b), and completes the proof. □

10. Proof of matroid inequalities and equality conditions

In this section we give proofs of Theorem 1.6, Theorem 1.9, Theorem 1.10, Proposition 1.11 and give further extension of graphical matroid results. We conclude with two explicit examples of small combinatorial atlases.

10.1. **Proof of Theorem 1.6.** We deduce the result from Theorem 1.31. Let $\mathcal{G} = (X, \mathcal{L})$ be the interval greedoid constructed in §4.3, and corresponding to matroid $\mathcal{M} = (X, \mathcal{I})$. Let $1 \leq k < \mathrm{rk}(\mathcal{M})$ and let $\omega : X \to \mathbb{R}_{>0}$ be as in the theorem. Define the weight function $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ by the product formula:

$$\mathrm{q}(\alpha) \ := \ c_\ell \prod_{x \in \alpha} \omega(x), \tag{10.1}$$

where $\ell := |\alpha|$, and c_ℓ is given by

$$c_\ell \ := \ \begin{cases} 1 & \text{for } \ \ell \neq k+1, \\ 1 + \dfrac{1}{\mathrm{p}(k-1) - 1} & \text{for } \ \ \ell = k+1. \end{cases} \tag{10.2}$$

Since every permutation of an independent set gives rise to a feasible word, we then have:

$$\mathrm{L}_{\mathrm{q}}(k-1) \ = \ (k-1)! \cdot \mathrm{I}_\omega(k-1), \qquad \mathrm{L}_{\mathrm{q}}(k) \ = \ k! \cdot \mathrm{I}_\omega(k), \quad \text{and}$$
$$\mathrm{L}_{\mathrm{q}}(k+1) \ = \ (k+1)! \left(1 + \frac{1}{\mathrm{p}(k-1) - 1}\right) \cdot \mathrm{I}_\omega(k+1).$$

This reduces (1.8) to (1.29).

By Theorem 1.31, it remains to show that q is a k-admissible weight function. First note that the weight function q is multiplicative and thus satisfies (ContInv) and (LogMod). By Proposition 4.6, greedoid $\mathcal{G}$ satisfies (WeakLoc), which in turn implies (PAMon). By the same proposition, greedoid $\mathcal{G}$ is interval and satisfies (FewDes). Further, property (4.4) implies that $\mathrm{Des}_\alpha(x) = \varnothing$ for every $\alpha \in \mathcal{L}$ and $x \in X$, which in turn trivially implies (SynMon).

To verify (ScaleMon), first suppose that $\ell < k - 1$. Then $c_\ell = c_{\ell+1} = c_{\ell+2} = 1$, which implies that the LHS of (ScaleMon) is equal to 0 while the RHS of (ScaleMon) is equal to 1, as desired. Now suppose that $\ell = k - 1$. Note that $\mathrm{b}_\alpha(\mathcal{C}) = 0$ for every $\alpha \in \mathcal{L}$ and $\mathcal{C} \in \mathrm{Par}(\alpha)$, since $\mathrm{Des}_\alpha(x) = \varnothing$. Then, for every $\alpha \in \mathcal{L}$ of length $k - 1$, we have:

$$\left(1 \ - \ \frac{c_k^2}{c_{k-1}\, c_{k+1}}\right) \sum_{\mathcal{C} \in \mathrm{Par}(\alpha)} \frac{1}{1 - \mathrm{b}_\alpha(\mathcal{C})} \ = \ \left(1 \ - \ \frac{c_k^2}{c_{k-1}\, c_{k+1}}\right) \big|\, \mathrm{Par}_\alpha \,\big| \ = \ \frac{\big|\, \mathrm{Par}_\alpha \,\big|}{\mathrm{p}(k-1)} \ \leq \ 1.$$

This finishes the proof of (ScaleMon).

In summary, greedoid $\mathcal{G} = (X, \mathcal{L})$ satisfies (ContInv), (PAMon), (LogMod), (FewDes), (SynMon) and (ScaleMon). By Definition 3.2, we conclude that weight function q is k-admissible, which completes the proof of the theorem. □

10.2. **Proof of Theorem 1.10.** We will prove the theorem as a consequence of Theorem 3.3. From the proof of Theorem 1.6 given above, it suffices to show that (GE-c1) and (GE-c2) are equivalent to (ME1) and (ME2) for the greedoid $\mathcal{G}$.

Let $\alpha \in \mathcal{L}$ of length $|\alpha| = k - 1$. We denote by $\mathrm{s}_{\mathcal{M}}(k-1)$ the constant that appears in (ME2), and $\mathrm{s}_{\mathcal{G}}(k-1)$ the constant that appears in (GE-c2). Recall that $\mathrm{b}_\alpha(\mathcal{C}) \ = \ 0$ for every $\alpha \in \mathcal{L}$ and $\mathcal{C} \in \mathrm{Par}(\alpha)$. Note that

$$\sum_{x \in \mathcal{C}} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} \ = \ \sum_{x \in \mathcal{C}} \omega(x) \quad \text{and} \quad 1 \ - \ \frac{c_k^2}{c_{k-1}\, c_{k+1}} \ = \ \frac{1}{\mathrm{p}(k-1)}\,,$$

where the first equality follows from the product formula (10.1) and the second equality is because of the choice of constants c_ℓ in (10.2). This implies that (GE-c2) and (ME2) are equivalent under the substitution $\mathrm{s}_{\mathcal{M}}(k-1) := \mathrm{s}_{\mathcal{G}}(k-1)/\mathrm{p}(k-1)$.

Now, let $S = \{x_1, \ldots, x_{k-1}\}$ be an arbitrary independent set of size $(k-1)$, and let $\alpha := x_1 \cdots x_{k-1}$. We have:

$$\sum_{x \in \operatorname{Cont}(\alpha)} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} = \sum_{x \in \operatorname{Cont}(\alpha)} \omega(x) =_{\text{(ME2)}} \left| \operatorname{Par}_S \right| \cdot \mathrm{s}_{\mathcal{M}}(k-1).$$

This implies that (GE-$c1$) and (ME1) are equivalent, and completes the proof of the theorem. $\square$

10.3. **Proof of Theorem 1.9.** The direction $\Leftarrow$ is trivial, so it suffices to prove the $\Rightarrow$ direction.

Let S be an arbitrary independent set of size $k-1$. Recall that $\mathrm{p}(k-1) \leq n-k+1$. From the equality (1.10) and inequality (1.8), it follows that $\mathrm{p}(k-1) = n-k+1$. On the other hand, it follows from equation (ME1) in Theorem 1.10, that $|\operatorname{Par}_S| = \mathrm{p}(k-1)$. Combining these two observations, we obtain:

$$S \cup \{x, y\} \text{ is an independent set for every distinct } x, y \in X \setminus S. \tag{10.3}$$

Let us show that every $(k+1)$-subset of X is independent. Fix an independent set U of size $k+1$, and take an arbitrary $(k+1)$-subset T of X. If $T = U$ then we are done, so suppose that $T \neq U$. Let $x \in T \setminus U$ and let $y \in U \setminus T$. Let U' be the $(k+1)$-subset given by $U' := U + x - y$. It follows from (10.3) that U' is an independent set. Observe that the size of the intersection has increased: $|T \cap U'| > |T \cap U|$, Letting $U \leftarrow U'$, we can iterate this argument until we eventually get $U' = T$, as desired.

We can now prove that the weight function $\omega : X \to \mathbb{R}_{>0}$ is uniform. Let $x, y \in X$ be distinct elements. Let S be a $(k-1)$-subset of X that contains neither x nor y. It follows from the argument in the previous paragraph that S is an independent set of the matroid $\mathcal{M}$, and every parallel class of S has cardinality 1. By applying (ME2) to the parallel class $\mathcal{C}_1 = \{x\}$ and $\mathcal{C}_2 = \{y\}$, we conclude that $\omega(x) = \omega(y)$. This completes the proof. $\square$

10.4. **Proof of Proposition 1.11.** The inequality (1.12) in the proposition is a restatement of (1.7). Thus, we need to show that equality in (1.7) holds if and only if G is an $\textsc{n}$-cycle. The $\Leftarrow$ direction follows from a direct calculation, so it suffices to prove the $\Rightarrow$ direction.

We first show that $\deg(v) \geq 2$ for every $v \in V$. Suppose to the contrary, that there exists $v \in V$ such that $\deg(v) = 1$. Let e be the unique edge adjacent to v, and let $S \subset E$ be a forest with $\textsc{n} - 3$ edges not containing e. Then v is a leaf vertex in the contraction graph G/S. On the other hand, the graph G/S is the complete graph K_3 by (ME1), a contradiction.

We now show that $\deg(v) \leq 2$ for every $v \in V$. Suppose to the contrary, that $\deg(v) \geq 3$ for some $v \in V$. Let $e, f, g \in E$ be three distinct edges adjacent to v. Then there exists a spanning tree T in G that contains e, f, g. Let $S = T - \{e, f\}$, and let x and y be the other endpoint of e and f, respectively. Note that S is a forest with $\textsc{n} - 3$ edges, so it follows from (ME2) that there exists $\mathrm{s}(\textsc{n} - 3)$ many edges connecting the component of G/S containing x, and the component of G/S containing y. Now let $U := T - \{f, g\} = S + e - g$, which is another forest with $\textsc{n} - 3$ edges. Note that there are at least $\mathrm{s}(\textsc{n}-3)+1$ edges connecting the component of G/U containing $\{v, x\}$, and the component of G/S containing y, namely the edge f and the other $\mathrm{s}(\textsc{n} - 3)$ many edges connecting the component of G/S containing x and the component of G/S containing y. This contradicts (ME2).

Finally, observe that the $\textsc{n}$-cycle is the only connected graph for which every vertex has degree two. This completes the proof. $\square$

10.5. **More graphical matroids.** The following result is a counterpart to the Proposition 1.11 proved above.

Theorem 10.1. *Let $G = (V, E)$ be a simple connected graph on $|V| = \textsc{n}$ vertices, and let $\mathrm{I}(k)$ be the number of spanning forests with k edges. Then*

$$\frac{\mathrm{I}(k)^2}{\mathrm{I}(k+1) \cdot \mathrm{I}(k-1)} \geq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\binom{\textsc{n}-k+1}{2} - 1}\right), \tag{10.4}$$

and the inequality is always strict if $1 < k < \textsc{n} - 2$.

Proof. The inequality (10.4) follows immediately from (1.6) and the fact that $\mathrm{p}(k-1) \leq \binom{\mathrm{N}-k+1}{2}$.

For the second part, suppose to the contrary, that we have equality in (10.4) for some simple connected graph G. It then follows from (ME1) and (ME2), that there exists $\mathrm{s} > 0$ such that G satisfies the following *clique-partition property*:

> Let $A_1, \ldots, A_{\mathrm{N}-k+1}$ be a partition of V, such that each $A_i \subset V$ spans a connected subgraph of G. Then the graph obtained by contracting each A_i to one vertex (loops are removed but multiple edges remain) is the complete graph $K_{\mathrm{N}-k+1}$, with the multiplicity of every edge equal to s.

Now, start with an arbitrary partition $A_1, \ldots, A_{\mathrm{N}-k+1}$ of V such that each $A_i \subset V$ is nonempty and spans a connected subgraph of G. We get our contradiction if this partition does not satisfy the clique-partition property above. Since $k > 1$, without loss of generality, we assume that A_1 has at least two vertices.

Let x be a vertex in A_1 that is adjacent to a vertex in A_2. If x is adjacent to any other vertex in A_i, $i \geq 3$, then by moving x to A_2 we create a new partition $A'_1, \ldots, A'_{\mathrm{N}-k+1}$ and note that now there are $\mathrm{s}+1$ edges connecting A'_2 and A'_i, contradicting the clique-partition property. Thus, x is not adjacent to $A_3, \ldots, A_{\mathrm{N}-k+1}$, and we can then move x to A_2 to create a new partition. By iteratively moving elements to A_2 until only one element y remains in A_1, and applying the clique-partition property to the resulting partition, we conclude that y is adjacent to $A_3, \ldots, A_{\mathrm{N}-k+1}$.

We now return to the original partition $A_1, \ldots, A_{\mathrm{N}-k+1}$, and we move y to A_3 to obtain a new partition $A''_1, \ldots A''_{\mathrm{N}-k+1}$. In this new partition, there are $\mathrm{s}+1$ edges connecting A''_3 and A''_4, a contradiction. Note that here that part A''_4 is nonempty since $k < \mathrm{N}-2$. This completes the proof. $\square$

Remark 10.2. The inequality (10.4) is incomparable with (1.3), and is stronger only for very dense graphs:

$$|E| \ \geq \ \binom{\mathrm{N}-k+1}{2} + k - 1.$$

To explain this, note that $\mathrm{p}(k-1)$ is usually smaller than the binomial coefficient above. This is why the inequality (10.4) is strict for $1 < k < \mathrm{N}-2$. This also underscores the power of our main matroid inequality (1.6).

10.6. **Proof of Corollary 1.13.** The result follows immediately from Theorem 1.4 and the fact that

$$\mathrm{Par}(S) \ \leq \ q^{m-k+1} - 1 \quad \text{for every} \quad S \in \mathcal{I}_{k-1}\,.$$

This is because the contraction $\mathcal{M}/S$ with parallel elements removed is a realizable matroid over $\mathbb{F}_q$ of rank $m-k+1$, which can have at most $q^{m-k+1}-1$ nonzero vectors. $\square$

10.7. **Examples of combinatorial atlases.** The numbers of independent sets can grow rather large, so we give two rather small matroid examples to help the reader navigate the definitions. We assume that the weight function ω is uniform in both examples.

Example 10.3 (Free matroid)**.** Let $\mathcal{M} = (X, \mathcal{I})$ be a *free matroid* on $n = 4$ elements: $X = \{x_1, \ldots x_4\}$ and $\mathcal{I} = 2^X$. In this case, we have $\mathrm{I}(1) = \mathrm{I}(3) = 4$, $\mathrm{I}(2) = 6$, and the inequality (1.3) is an equality.

Following §4.3 and the proof of Theorem 1.6, the corresponding greedoid $\mathcal{G} = (X, \mathcal{L})$ has all simple words in X^*. Let $k = 2$ and $\alpha = \varnothing$. Then $\mathbf{A}(\alpha, k-1)$ and $\mathbf{A}(\alpha, k)$ are $(n+1)\times(n+1)$ matrices given by

$$\mathbf{A}(\varnothing, 1) \ = \ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A}(\varnothing, 2) \ = \ \begin{pmatrix} 0 & 3 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{pmatrix},$$

where the rows and columns are labeled by $\{x_1, x_2, x_3, x_4, \mathsf{null}\}$. Recall that each entry of the matrices is counting the number of certain feasible words, and only words of length $k+1 = 3$ are weighted by $1 + \frac{1}{\mathrm{p}(k-1)-1} = \frac{3}{2}$.

As in the proof of Theorem 1.31 (see §8.4), let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^5$ be the vectors given by

$$\mathbf{v} \ := \ (1,1,1,1,0)^{\intercal} \quad \text{and} \quad \mathbf{w} \ := \ (0,0,0,0,1)^{\intercal}.$$

Inequality (1.3) in this case is equivalent to (1.29), which in turn can be rewritten as:

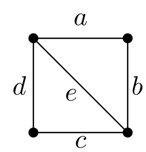
$$\langle \mathbf{v}, \mathbf{A}(\alpha,k)\,\mathbf{w}\rangle^2 \;\geq\; \langle \mathbf{v}, \mathbf{A}(\alpha,k)\,\mathbf{v}\rangle\,\langle \mathbf{w}, \mathbf{A}(\alpha,k)\,\mathbf{w}\rangle. \tag{10.5}$$

In this case the equality holds, since

$$\langle \mathbf{v}, \mathbf{A}(\varnothing,2)\,\mathbf{w}\rangle \;=\; 12, \qquad \langle \mathbf{v}, \mathbf{A}(\varnothing,2)\,\mathbf{v}\rangle \;=\; 36, \qquad \langle \mathbf{v}, \mathbf{A}(\varnothing,2)\,\mathbf{v}\rangle \;=\; 4,$$

as implied by Theorem 1.8.

Example 10.4 (Graphical matroid)**.** Let $G = (V, E)$ be a graph as in the figure below, where $\textsc{n} := |V| = 4$ and $E = \{a, b, c, d, e\}$. Let $\mathcal{M} = (E, \mathcal{L})$ be the corresponding graphical matroid (see Example 1.5). In this case $n = |E| = 5$ and $\mathrm{rk}(\mathcal{M}) = \textsc{n} - 1 = 3$.



Let $\alpha = \varnothing$ and $k = 2$. Then $\mathbf{A}(\alpha, k-1)$ and $\mathbf{A}(\alpha, k)$ are $(n+1)\times(n+1)$ matrices given by

$$\mathbf{A}(\varnothing,1) \;=\; \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A}(\varnothing,2) \;=\; \begin{pmatrix} 0 & 3 & 4.5 & 4.5 & 3 & 4 \\ 3 & 0 & 4.5 & 4.5 & 3 & 4 \\ 4.5 & 4.5 & 0 & 3 & 3 & 4 \\ 4.5 & 4.5 & 3 & 0 & 3 & 4 \\ 3 & 3 & 3 & 3 & 0 & 4 \\ 4 & 4 & 4 & 4 & 4 & 5 \end{pmatrix},$$

where the rows and columns are labeled by $\{a, b, c, d, e, \textsf{null}\}$. As in the previous example, each entry of the matrices is counting the number of certain feasible words, and only words of length $k+1 = 3$ is weighted by $1 + \frac{1}{\mathrm{p}(k-1)-1} = \frac{3}{2}$.

As above, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^6$ be the vectors given by

$$\mathbf{v} \;:=\; (1,1,1,1,1,0)^{\intercal} \quad \text{and} \quad \mathbf{w} \;:=\; (0,0,0,0,0,1)^{\intercal}.$$

Inequality (1.3) in this case is equivalent to (1.29), which in turn can be rewritten as (10.5). Note that in this case we have

$$\langle \mathbf{v}, \mathbf{A}(\alpha,k)\,\mathbf{w}\rangle \;=\; 72, \qquad \langle \mathbf{v}, \mathbf{A}(\alpha,k)\,\mathbf{v}\rangle \;=\; 20, \qquad \langle \mathbf{v}, \mathbf{A}(\alpha,k)\,\mathbf{v}\rangle \;=\; 5,$$

and indeed we have a strict inequality $20^2 > 72 \times 5$, as implied by Theorem 1.8.

11. Proof of discrete polymatroid inequalities and equality conditions

In this section we give proofs of Theorem 1.21, Theorem 1.23 and Theorem 1.24.

11.1. **Proof of Theorem 1.21.** We deduce the result from Theorem 1.31. This proof is similar to the argument in the proof of Theorem 1.6 in the previous section, so we will emphasize the differences.

Let $\mathcal{G} = (X, \mathcal{L})$ be the interval greedoid constructed in §4.4, and corresponding to discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$. Let $1 \leq k < \mathrm{rk}(\mathcal{D})$, let $0 < t \leq 1$, and let $\omega : X \to \mathbb{R}_{>0}$ be as in the theorem.

Let $\boldsymbol{a}_\alpha = (\mathrm{a}_1, \ldots, \mathrm{a}_n) \in \mathbb{N}^n$ be the vector corresponding to the word $\alpha \in \mathcal{L}$. We define the weight function $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ by the product formula

$$\mathrm{q}(\alpha) \;:=\; c_\ell\, t^{\pi(\boldsymbol{a}_\alpha)}\, \omega(\boldsymbol{a}_\alpha),$$

where $\ell := |\alpha| = |\mathrm{b}_\alpha|$, and c_ℓ is given by

$$c_\ell \;:=\; \begin{cases} 1 & \text{for } \;\ell \neq k+1, \\ 1 + \dfrac{1-t}{\mathrm{p}(k-1) - 1 + t} & \text{for } \;\ell = k+1. \end{cases} \tag{11.1}$$

Using this weight function, we obtain:

$$\mathrm{L_q}(k) \;=\; \sum_{\alpha\in\mathcal{L}_k} t^{\pi(\boldsymbol{a}_\alpha)}\,\omega(\boldsymbol{a}_\alpha) \;=\; \sum_{\mathrm{b}\in\mathcal{J}_k} t^{\pi(\boldsymbol{a}_\alpha)}\,\omega(\boldsymbol{a})\,\frac{k!}{\mathrm{a}_1!\cdots\mathrm{a}_n!} \;=\; k!\,\cdot\,\mathrm{J}_{\omega,t}(k),$$

where the third equality follows from every permutation of a feasible word that is well-ordered is again a feasible word. By the same calculation, we have

$$\mathrm{L_q}(k-1) \;=\; (k-1)!\,\cdot\,\mathrm{J}_{\omega,t}(k-1)\,, \quad \mathrm{L}(k+1) \;=\; (k+1)!\left(1+\frac{1-t}{\mathrm{p}(k-1)-1+t}\right)\cdot\,\mathrm{J}_{\omega,t}(k+1).$$

This reduces (1.22) to (1.29).

By Theorem 1.31, it remains to show that q is a k-admissible weight function. First note that the weight function q is multiplicative and thus satisfies (ContInv) and (LogMod). By Proposition 4.7, greedoid $\mathcal{G}$ satisfies (WeakLoc), which in turn implies (PAMon). By the same proposition, greedoid $\mathcal{G}$ is interval and satisfies (FewDes).

We now verify (SynMon), which is no longer similar to the matroid case. It follows from (4.5) that the right side of (SynMon) is 0, unless $x = x_{ij}$ and $\mathrm{Des}_\alpha(x) = x_{i\,j+1}$. In the latter case, we have

$$\frac{\omega(\alpha x)}{\omega(\alpha)} \;=\; t^{j-1}\,\omega(i)\,, \qquad \sum_{y\in\mathrm{Des}_\alpha(x)}\frac{\omega(\alpha x y)}{\omega(\alpha x)} \;=\; t^j\,\omega(i), \tag{11.2}$$

and (SynMon) follows from $t \le 1$.

For (ScaleMon), the same argument as for matroids works for $\ell < k-1$. Now suppose that $\ell = k-1$. Note that, for every $\alpha \in \mathcal{L}$ and $\mathcal{C} \in \mathrm{Par}(\alpha)$, we have $\mathrm{b}_\alpha(\mathcal{C}) \le t$ by (11.2). Hence, for every $\alpha \in \mathcal{L}_{k-1}$, we have:

$$\begin{aligned}\left(1-\frac{c_k^2}{c_{k-1}\,c_{k+1}}\right)\sum_{\mathcal{C}\in\mathrm{Par}(\alpha)}\frac{1}{1-\mathrm{b}_\alpha(\mathcal{C})} \;&\le\; \left(1-\frac{c_k^2}{c_{k-1}\,c_{k+1}}\right)\sum_{\mathcal{C}\in\mathrm{Par}(\alpha)}\frac{1}{1-t}\\ &=\; \left(1-\frac{c_k^2}{c_{k-1}\,c_{k+1}}\right)\frac{\left|\,\mathrm{Par}_\alpha\,\right|}{1-t} \;=\; \frac{\left|\,\mathrm{Par}_\alpha\,\right|}{\mathrm{p}(k-1)} \;\le\; 1\,,\end{aligned}$$

which proves (ScaleMon).

In summary, greedoid $\mathcal{G} = (X,\mathcal{L})$ satisfies (ContInv), (PAMon), (LogMod), (FewDes), (SynMon) and (ScaleMon). By Definition 3.2, we conclude that weight function q is k-admissible, which completes the proof of the theorem. $\square$

11.2. **Proof of Theorem 1.23.** We deduce the result from Theorem 3.3. The proof below only assumes that $0 < t \le 1$. For the $\Rightarrow$ direction, let $\alpha \in \mathcal{L}$ with $|\alpha| = k-1$. Note that, since $c_k = c_{k-1} = 1$, we have

$$\sum_{x\in\mathcal{C}}\frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} \;=\; \sum_{x\in\mathcal{C}}\frac{\omega(\alpha x)}{\omega(\alpha)} \;=\; \mathrm{a}_\alpha(\mathcal{C}).$$

By (GE-c2), there exists $\mathrm{s} > 0$, s.t. for every $\mathcal{C} \in \mathrm{Par}(\alpha)$ we have:

$$\mathrm{a}_\alpha(\mathcal{C})\,\big(1-\mathrm{b}_\alpha(\mathcal{C})\big) \;=\; \mathrm{s}\left(1-\frac{c_k^2}{c_{k-1}\,c_{k+1}}\right) \;=\; \mathrm{s}\,\frac{1-t}{\mathrm{p}(k-1)}\,. \tag{11.3}$$

Summing over all $\mathcal{C} \in \mathrm{Par}(\alpha)$, we get:

$$\sum_{\mathcal{C}\in\mathrm{Par}_\alpha}\mathrm{a}_\alpha(\mathcal{C})\,\big(1-\mathrm{b}_\alpha(\mathcal{C})\big) \;=\; \mathrm{s}\;(1-t)\,\frac{\left|\,\mathrm{Par}_\alpha\,\right|}{\mathrm{p}(k-1)}\,.$$

On the other hand, the equality (GE-c1) gives:

$$\mathrm{s} \;=\; \sum_{\mathcal{C}\in\mathrm{Par}_\alpha}\mathrm{a}_\alpha(\mathcal{C}).$$

Combining these equations, we obtain:

$$\sum_{\mathcal{C}\in\mathrm{Par}_\alpha}\mathrm{a}_\alpha(\mathcal{C})\,\big(1-\mathrm{b}_\alpha(\mathcal{C})\big) \;=\; \sum_{\mathcal{C}\in\mathrm{Par}_\alpha}\mathrm{a}_\alpha(\mathcal{C})\,(1-t)\,\frac{\left|\,\mathrm{Par}_\alpha\,\right|}{\mathrm{p}(k-1)}\,,$$

which is equivalent to

$$\sum_{\mathcal{C}\in\mathrm{Par}_\alpha} \mathrm{a}_\alpha(\mathcal{C}) \left(1 \,-\, \mathrm{b}_\alpha(\mathcal{C}) \,-\, (1-t)\,\frac{\left|\,\mathrm{Par}_\alpha\,\right|}{\mathrm{p}(k-1)}\right) \;=\; 0.$$

Now note that, the LHS of the equation above is always nonnegative since $\mathrm{b}_\alpha(\mathcal{C}) \leq t$ by (11.2), and $\left|\,\mathrm{Par}_\alpha\,\right| \leq \mathrm{p}(k-1)$ by definition. Therefore, the equality hold for both inequalities, so in particular we have:

$$\mathrm{b}_\alpha(\mathcal{C}) \,=\, t \quad \text{for every} \quad \alpha\in\mathcal{L}_{k-1} \quad \text{and} \quad \mathcal{C}\in\mathrm{Par}(\alpha).$$

Since $t>0$ by assumption, it follows from (FewDes) that

(11.4) $$|\,\mathcal{C}\,| \,=\, 1 \quad \text{and} \quad \mathrm{b}_\alpha(\mathcal{C})=t>0 \quad \text{for every} \quad \alpha\in\mathcal{L} \quad \text{with} \quad |\alpha|=k-1 \quad \text{and} \quad \mathcal{C}\in\mathrm{Par}(\alpha).$$

Restating this equation in the language of polymatroids, we conclude: for every $\boldsymbol{a}\in\mathcal{J}_{k-1}$, and every $i,j\in[n]$ (not necessarily distinct), we have:

(11.5) $$\boldsymbol{a}+\mathbf{e}_i,\, \boldsymbol{a}+\mathbf{e}_j \in \mathcal{J} \quad \Longrightarrow \quad \boldsymbol{a}+\mathbf{e}_i+\mathbf{e}_j\in\mathcal{J}.$$

We can now show that every $\mathbf{n}=(\mathrm{n}_1,\ldots,\mathrm{n}_n)\in\mathbb{N}^n$ with $|\,\mathbf{n}\,|=k+1$ is contained in $\mathcal{J}$. We follow the corresponding argument int the matroid case. Let $\boldsymbol{a}\in\mathcal{J}$ with $|\boldsymbol{a}|=k+1$. If $\boldsymbol{a}=\mathbf{n}$, we are done, so suppose that $\boldsymbol{a}\neq\mathbf{n}$. Then there exists $i,j\in[n]$, such that $\mathrm{a}_i>\mathrm{n}_i$ and $\mathrm{a}_j<\mathrm{n}_j$. By the polymatroid hereditary property, we have $\boldsymbol{a}-\mathbf{e}_i\in\mathcal{J}$. Since $\mathbf{e}_j\in\mathcal{J}$ by the assumption that the polymatroid is normal, we can then apply the exchange property to $\mathbf{e}_j$ and $\boldsymbol{a}-\mathbf{e}_i$ to conclude that $\boldsymbol{a}-\mathbf{e}_i+\mathbf{e}_j-\mathbf{e}_h\in\mathcal{J}$ for some $h\in[n]$. Let $\mathbf{u}:=\boldsymbol{a}-\mathbf{e}_i-\mathbf{e}_h$. Note that $\mathbf{u}\in\mathcal{J}_{k-1}$ by hereditary property, and

$$\mathbf{u}+\mathbf{e}_j \;=\; \boldsymbol{a}-\mathbf{e}_i+\mathbf{e}_j-\mathbf{e}_h \,\in\, \mathcal{J}, \quad \text{and} \quad \mathbf{u}+\mathbf{e}_h \;=\; \boldsymbol{a}-\mathbf{e}_i\in\mathcal{J}.$$

It then follows from (11.5) that

$$\boldsymbol{a}-\mathbf{e}_i+\mathbf{e}_j \;=\; \mathbf{u}+\mathbf{e}_j+\mathbf{e}_h\in\mathcal{J}\,.$$

Make substitution $\boldsymbol{a}\leftarrow\boldsymbol{a}-\mathbf{e}_i+\mathbf{e}_j$ and iterate this argument until eventually $\mathbf{n}=\boldsymbol{a}$, as desired. This proves the $\Rightarrow$ direction.

For the $\Leftarrow$ direction, assume now that $t=1$. The equality now follows from a direct calculation, since that

$$1\,+\,\frac{1-t}{\mathrm{p}(k-1)-1+t} \,=\, 1, \quad \text{and} \quad \mathrm{J}_\omega(\ell) \;=\; \frac{\big(\omega(1)\,+\,\ldots\,+\,\omega(n)\big)^\ell}{\ell!} \quad \text{for every} \;\; \ell\leq k+1.$$

This completes the proof. □

11.3. **Proof of Theorem 1.24.** Assume now that $0<t<1$. From the proof above, it remains to show that $k=1$ and that the weight function ω is uniform.

Let $i,j\in[n]$ be distinct elements, let $\alpha:=x_{i\,1}\cdots x_{i\,k-1}$, let $x:=x_{i\,k}$, and let $y:=x_{j\,1}$. By (11.4), we have $\mathcal{C}_1=\{x\}$ and $\mathcal{C}_2=\{y\}$ are both parallel classes of α. It then follows from (11.3) and (11.4), that

$$\mathrm{a}_\alpha(\mathcal{C}_1) \;=\; \mathrm{a}_\alpha(\mathcal{C}_2).$$

On the other hand, we have

$$\mathrm{a}_\alpha(\mathcal{C}_1) \;=\; t^k\,\omega(i), \qquad \mathrm{a}_\alpha(\mathcal{C}_2) \;=\; t\,\omega(j),$$

so $t^{k-1}=\omega(j)/\omega(i)$. Since the choice of i and j was arbitrary, we can switch i and j to obtain $t^{k-1}=\omega(j)/\omega(i)$. This implies that $\omega(i)=\omega(j)$ and $k=1$, which proves the $\Rightarrow$ direction.

For the $\Leftarrow$ direction, assume now that $k=1$. From the proof above, $\omega(i)=C$ for every $i\in[n]$ and some $C>0$. It then follows from a direct calculation that

$$1+\frac{1-t}{\mathrm{p}(k-1)-1+t} \;=\; \frac{n}{n-1+t}\,,$$

and

$$\mathrm{J}_{\omega,t}(0) \;=\; 1\,, \qquad \mathrm{J}_{\omega,t}(1) \;=\; C\,t\,n\,, \qquad \mathrm{J}_{\omega,t}(2) \;=\; C^2\left(\frac{t^3\,n}{2}\,+\,\frac{t^2\,n\,(n-1)}{2}\right).$$

Thus, the equality (1.24) holds in this. This completes the proof. □

12. Proof of poset antimatroid inequalities and equality conditions

In this section we prove Theorem 1.26 and Theorem 1.28.

12.1. Proof of Theorem 1.26. As in the previous sections, we deduce the result from Theorem 1.31. Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements and let $\mathcal{A} = (X, \mathcal{L})$ be the corresponding poset antimatroid which is an interval greedoid by the argument in §4.2. In the notation of Section 3, let $c_\ell = 1$ for all $\ell \geq 1$, and let $\mathrm{q}(\alpha) := \omega(\alpha)$, so that (1.26) coincides with (1.29) in this case.

It remains to show that ω is a k-admissible weight function. First note that ω satisfies (ContInv) and (LogMod) since the weight function is multiplicative. The condition (ScaleMon) is also trivially satisfied. By Proposition 4.5, both (WeakLoc) and (FewDes) are satisfied, and the former implies (PAMon).

For (SynMon), mote that the poset ideal greedoid $\mathcal{G}$ satisfies

$$\mathrm{Des}_\alpha(x) \ \subseteq \ \big\{ y \in X \, : \, x \leftarrow\!\!\prec y \big\}, \tag{12.1}$$

for every $\alpha \in \mathcal{L}$ and every $x \in \mathrm{Cont}(\alpha)$. It then follows that

$$\frac{\omega(\alpha x)}{\omega(\alpha)} \ = \ \omega(x) \ \geq_{\text{(CM)}} \ \sum_{y \, : \, x \leftarrow\!\!\prec y} \omega(y) \ \geq_{\text{(12.1)}} \ \sum_{y \in \mathrm{Des}_\alpha(x)} \omega(y) \ = \ \sum_{y \in \mathrm{Des}_\alpha(x)} \frac{\omega(\alpha x y)}{\omega(\alpha x)}, \tag{12.2}$$

which proves (SynMon). Hence q is indeed a k-admissible weight function, which completes the proof. $\square$

12.2. Proof of Theorem 1.28. We deduce the result from Theorem 3.3. From the argument above, it suffices to show that (GE-c1) and (GE-c2) are equivalent to properties (AE1)–(AE3). First note that,

$$\sum_{x \in \mathrm{Cont}(\alpha)} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} \ = \ \sum_{x \in \mathrm{Cont}(\alpha)} \omega(x),$$

for every $\alpha \in \mathcal{L}$. This implies that (GE-c1) is equivalent to (AE1).

Let $\alpha \in \mathcal{L}$, let $x \in \mathrm{Cont}(\alpha)$, and let $\mathcal{C} = \{x\}$ be the parallel class in $\mathrm{Par}(\alpha)$ containing x. Since $c_{k+1} = c_k = c_{k-1} = 1$, it then follows that the RHS of (GE-c2) is equal to 0, so (GE-c2) is equivalent to

$$\sum_{y \in \mathrm{Des}_\alpha(x)} \frac{\omega(\alpha)\, \omega(\alpha x y)}{\omega(\alpha x)^2} \ = \ \mathrm{b}_\alpha(\mathcal{C}) \ = \ 1.$$

This implies that (GE-c2) is equivalent to equality in (12.2), which in turn is equivalent to equality in both (CM) and (12.1). The latter is equivalent to (AE2) and (AE3), which completes the proof. $\square$

13. Proof of morphism of matroids inequalities and equality conditions

In this section we give proofs of Theorem 1.16, Theorem 1.18 and Theorem 1.19.

13.1. Combinatorial atlas construction. Let $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$ be two matroids, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids. Let $1 \leq k < \mathrm{rk}(\mathcal{M})$ and let $\omega : X \to \mathbb{R}_{>0}$ be the weight function as in Theorem 1.16. We now define a combinatorial atlas $\mathbb{A}$ that corresponds to (Φ, k, ω).

Let $\mathcal{G} = (X, \mathcal{L})$ be the greedoid which corresponds to matroid $\mathcal{M}$, see §4.3. We extend ω to a *nonnegative weight function* $\mathrm{q} : \mathcal{L}_{\mathcal{M}} \to \mathbb{R}_{\geq 0}$ by the product formula:

$$\mathrm{q}(x_1 \cdots x_\ell) \ := \ \begin{cases} c_\ell \, \omega(x_1) \, \cdots \, \omega(x_\ell) & \text{if } \ \{x_1, \ldots, x_\ell\} \in \mathcal{B}_\ell \, , \\ 0 & \text{otherwise}, \end{cases}$$

where c_ℓ is defined in (10.2). Let $\Gamma := (\Omega, \Theta)$ be the acyclic graph and let $\mathbb{A}$ be the combinatorial atlas defined in §10.1 that corresponds to the greedoid $\mathcal{G}$. Note that Γ depends only on the matroid $\mathcal{M}$, but $\mathbb{A}$ depends also on the morphism Φ. Note also that, unlike the weight function in §1.14 and §10.1, here the weight q is *not* strictly positive, so Theorem 1.31 does not apply in this case.

In this section, we rework the combined proofs of Theorem 1.31 and Theorem 1.6 to apply for morphisms of matroids. Recall the properties we need to establish as summarized in the key results:

$$\text{Theorem 5.2:} \quad \left\{ \begin{array}{c} \text{(Inh), (Pull) hold for } \mathbb{A} \\ v \in \Omega^+ \text{ satisfies (Irr), (h-Pos)} \\ \text{(Hyp) holds for all } v^{\langle i \rangle} \in v^* \end{array} \right\} \quad \Rightarrow \quad \text{(Hyp) holds for } v.$$

$$\text{Theorem 6.1:} \quad \Big\{ \text{(Inh), (Proj), (T-Inv), (K-Non)} \Big\} \quad \Rightarrow \quad \text{(Pull)}.$$

Now, observe that (Inh), (Proj), (T-Inv), and (K-Non) are closed properties, i.e. preserved under taking limits. Thus, they follow from the arguments in §8.2. On the other hand, properties (Irr) and (h-Pos) need to be verified separately, a the arguments in §8.2 use the strict positivity of q.

Lemma 13.1. *Let $v = (\alpha, m, t) \in \Omega^m$ be a non-sink vertex of the acyclic graph Γ defined above, where $\alpha \in X^*$, $|\alpha| \le k-1-m$, $0 < m \le k-1$, and $0 < t < 1$. Then v satisfies* (Irr) *and* (h-Pos)*.*

Proof. For the second part, it follows from the definition of $\mathbf{h}_v$ that the vector is strictly positive for all $t \in (0,1)$. Thus, vertex v satisfies (h-Pos), for all $t \in (0,1)$.

For the first part, let $\mathbf{M}_v$ be the associated matrix of v. Without loss of generality, we can assume that $\alpha \in \mathcal{L}$, as otherwise $\mathbf{M}_v = 0$ and (Irr) holds trivially.

Claim: *Every $x, y \in X$ in the support of $\mathbf{M}_v$ belong to the same irreducible component of $\mathbf{M}_v$.*

When null is not in the support of $\mathbf{M}_v$, property (Irr) follows from the claim. Now assume that null is in the support of $\mathbf{M}_v$. By the Claim, it remains to show that null belong to the same irreducible component of *some* $x \in X$ in the support of $\mathbf{M}_v$.

Let $\alpha = x_1 \cdots x_\ell$, where $\ell \le k - m - 1$. By the assumption, there exits a subset $S \in \mathcal{B}_\Phi$ such that $\{x_1, \ldots, x_\ell\} \subset S$ and $|S| \in \{\ell+m-1, \ell+m, \ell+m+1\}$. To see this, observe that if null is in the support of $\mathbf{M}_v$, then either $\mathrm{A}(\alpha, m)_{\mathsf{null}\,\mathsf{null}} \neq 0$, or $\mathrm{A}(\alpha, m)_{\mathsf{null}\,x} \neq 0$, or $\mathrm{A}(\alpha, m+1)_{\mathsf{null}\,x} \neq 0$, for some $x \in X$.

By adding extra elements to S if necessary, without loss of generality we can assume that $|S| = \ell+m+1$. Let $x \in S \setminus \{x_1, \ldots, x_\ell\}$. Then we have $\mathrm{A}(\alpha, m+1)_{\mathsf{null}\,x} \neq 0$. This implies that x and null belong to the same irreducible component of $\mathbf{A}(\alpha, m+1)$. Since $0 < t < 1$, this implies that x and null belong to the same irreducible component of $\mathbf{M}_v$, and completes the proof of the lemma. □

Proof of Claim. Let $\ell = |\alpha|$, as above. Since x is contained in the support of $\mathbf{M}_v$, this implies that there exits $S \in \mathcal{B}_\Phi$ such that $\{x_1, \ldots, x_\ell, x\} \subset S$ and $|S| \in \{\ell+m, \ell+m+1, \ell+m+2\}$. Similarly, there exists $T \in \mathcal{B}_\Phi$ such that $\{x_1, \ldots, x_\ell, y\} \subset T$ and $|T| \in \{\ell+m, \ell+m+1, \ell+m+2\}$. By adding extra elements to S and T if necessary, without loss of generality we can assume that $|S| = |T| = \ell+m+2$.

For $S = T$, the claim follows immediately from the definition of $\mathbf{M}_v$ and q, since $\mathrm{A}(\alpha, m+1)_{xy} \neq 0$ in this case. So assume that $S \neq T$. By the exchange property for morphism of matroids (Proposition 4.8), there exists $z \in S \setminus T$ and $w \in T \setminus S$ such that $S - z + w \in \mathcal{B}_\Phi$.

Let $S' := S - z + w$. Note that $|S' \setminus \{x_1, \ldots, x_\ell, x, w\}| = m \ge 1$, and let $x' \in S' \setminus \{x_1, \ldots, x_\ell, x, w\}$. Note that $x' \in S \setminus \{x_1, \ldots, x_\ell\}$, which implies that $\mathrm{A}(\alpha, m+1)_{xx'} \neq 0$ in this case. Therefore, elements x and x' belongs to the same irreducible component of $\mathbf{A}(\alpha, m+1)$, and thus the same irreducible component of $\mathbf{M}_v$ since $0 < t < 1$. Note also that we have $|S' \cap T| > |S \cap T|$ by the construction of S'. Substitute $x \leftarrow x'$ and $S \leftarrow S'$, and iteratively apply the same argument, until the set S will eventually becomes T. This implies that x and y are contained in the same irreducible component of $\mathbf{M}_v$, as desired. □

13.2. **All atlas vertices are hyperbolic.** We first show that every sink vertex in $\mathbb{A}$ satisfies (Hyp). We then use Theorem 5.2 to obtain the result.

Let $\mathcal{G} = (X, \mathcal{L})$ be the greedoid corresponding to matroid $\mathcal{M} = (X, \mathcal{I})$. Let $\alpha = x_1 \cdots x_\ell \in \mathcal{L}$ of length $\ell := |\alpha| \le k-1$, let $S := \{x_1, \ldots, x_\ell\}$, and let $\mathbf{A}(\alpha, 1)$ be the matrix defined in §8.1 for $\mathcal{G}$. Recall that

$$\omega(S) \; = \; \omega(x_1) \cdots \omega(x_\ell) \; = \; \frac{\mathrm{q}(x_1 \ldots x_\ell)}{c_\ell}.$$

For each $x \in X$, divide the x-row and x-column of $\mathbf{A}(\alpha, 1)$ by $\sqrt{c_{\ell+2}\ \omega(S)}\,\omega(x)$. Multiply the null-row and the null-column by $\frac{1}{c_{\ell+1}}\sqrt{\frac{c_{\ell+2}}{\omega(S)}}$. Denote by $\mathbf{B}$ the resulting matrix. Note that (Hyp) is preserved under this transformation, so it suffices to show that $\mathbf{B}$ satisfies (Hyp). Observe that $\mathbf{B} = \big(\mathrm{B}_{xy}\big)_{x,\, y \in \widehat{X}}$ is given by

$$
\begin{aligned}
\mathrm{B}_{xy} \ &= \ \begin{cases} 1 & \text{if } \ S+x+y \in \mathcal{B} \\ 0 & \text{if } \ S+x+y \notin \mathcal{B} \end{cases} \qquad \text{for distinct} \quad x,y \in X, \\
\mathrm{B}_{x\,\mathsf{null}} \ &= \ \begin{cases} 1 & \text{if } \ S+x \in \mathcal{B} \\ 0 & \text{if } \ S+x \notin \mathcal{B} \end{cases} \qquad \text{for} \quad x \in X\,, \\
\mathrm{B}_{\mathsf{null}\,\mathsf{null}} \ &= \ \begin{cases} \dfrac{c_\ell\, c_{\ell+2}}{c_{\ell+1}^2} & \text{if } \ S \in \mathcal{B} \\ \qquad 0 & \text{if } \ S \notin \mathcal{B} \end{cases} \\
\mathrm{B}_{xx} \ &= \ 0 \qquad \text{for} \quad x \in X.
\end{aligned}
$$

We now split the proof into three cases, each discussed as a separate lemma.

Lemma 13.2. *Suppose that $g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N})$. Then the matrix $\mathbf{B}$ satisfies* (Hyp).

Proof. By the assumption of the lemma, every independent set of $\mathcal{M}$ containing $\{x_1, \ldots, x_\ell\}$ is also a basis of Φ. It then follows that $\mathbf{B} = \big(\mathrm{B}_{xy}\big)$ is equal to

$$
\mathrm{B}_{xy} \ = \ \mathrm{B}_{yx} \ = \ \begin{cases} 1 & \text{if } \ x, y \in \mathrm{Cont}(S) \ \text{ and } \ x \not\sim_S y, \\ 1 & \text{if } \ x \in \mathrm{Cont}(S) \ \text{ and } \ y = \mathsf{null}, \\ \frac{c_\ell\, c_{\ell+2}}{c_{\ell+1}^2} & \text{if } \ x = y = \mathsf{null}, \\ 0 & \text{otherwise.} \end{cases}
$$

In particular, if $x \sim_\alpha y$, then x-row and x-column of $\mathbf{B}$ is equal to y-row and y-column of $\mathbf{B}$. Now, choose a representative element x_i for each equivalence class $\mathcal{C}_i$ in $\mathrm{Par}(\alpha)$. For every other y in $\mathcal{C}_i$, we subtract from the y-row and y-column of $\mathbf{A}$ the x_i-row and x_i-column of $\mathbf{A}$, respectively. Note that (Hyp) is preserved under these transformations. Restricting to the support, we obtain:

$$
\mathbf{N} \ := \ \begin{pmatrix} 0 & 1 & \ldots & 1 & 1 \\ 1 & 0 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 1 \\ 1 & 1 & \ldots & 1 & \frac{c_\ell c_{\ell+2}}{c_{\ell+1}^2} \end{pmatrix},
$$

where the rows and columns are indexed by $\{x_1, \ldots, x_m, \mathsf{null}\}$, with $m := |\,\mathrm{Par}(\alpha)|$. Now note that

$$
\frac{c_\ell c_{\ell+2}}{c_{\ell+1}^2} \ = \ \begin{cases} 1 & \text{for } \ \ell < k-1, \\ 1 \ + \ \dfrac{1}{\mathrm{p}(k-1) \ - \ 1} & \text{for } \ \ell = k-1\,. \end{cases}
$$

In both cases, we have:

$$
1 \ \leq \ \frac{c_\ell c_{\ell+2}}{c_{\ell+1}^2} \ \leq \ 1 + \frac{1}{|\,\mathrm{Par}(\alpha)| - 1} \ = \ 1 + \frac{1}{m-1}\,. \tag{13.1}
$$

This implies that matrix $\mathbf{N}$ satisfies the conditions in Lemma 8.6. Thus $\mathbf{N}$ is hyperbolic, as desired. $\square$

Lemma 13.3. *Suppose that $g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N}) - 1$. Then the matrix $\mathbf{B}$ satisfies* (Hyp).

Proof. By assumptions of the lemma, we can partition $\mathrm{Cont}(\alpha) := X_1 \cup X_2$ into two subsets:

$$
\begin{aligned}
X_1 \ &:= \ \big\{ x \in \mathrm{Cont}(\alpha) \ : \ g\big(\Phi(S+x)\big) \ = \ \mathrm{rk}(\mathcal{N}) \big\}, \\
X_2 \ &:= \ \big\{ x \in \mathrm{Cont}(\alpha) \ : \ g\big(\Phi(S+x)\big) \ = \ \mathrm{rk}(\mathcal{N}) - 1 \big\}.
\end{aligned}
$$

We now make the observations in three possible cases of $x, y \in X$:

(1) For every $x, y \in X_1$, we have $S+x+y \in \mathcal{B}$ if and only if $x \not\sim_S y$. This is because $\Phi(S+x+y) \supseteq \Phi(S+x)$, which implies that $\Phi(S+x+y)$ contains a basis of $\mathcal{N}$, and because $S+x+y \in \mathcal{I}$ if and only if $x \not\sim_S y$.

(2) For every $x \in X_1$ and $y \in X_2$, we have $S+x+y$ is a basis of Φ. This is because $\Phi(S+x+y) \supseteq \Phi(S+x)$, which implies that $\Phi(S+x+y)$ contains a basis of $\mathcal{N}$, and because

$$f(S+x+y) - f(S+y) \geq g\big(\Phi(S+x+y)\big) - g\big(\Phi(S+y)\big) = \mathrm{rk}(\mathcal{N}) - \big(\mathrm{rk}(\mathcal{N})-1\big) = 1,$$

which implies that $S+x+y \in \mathcal{I}$.

(3) For every $x, y \in X_2$, we have $S+x+y$ is not a basis of Φ. This is because $g\big(\Phi(S+x)\big) = g\big(\Phi(S+y)\big) = g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N}) - 1$, which implies that $g\big(\Phi(S+x+y)\big) = \mathrm{rk}(\mathcal{N}) - 1$.

It follows from the observations above that

$$\mathrm{B}_{xy} = \mathrm{B}_{yx} = \begin{cases} 1 & \text{if } x, y \in X_1 \text{ and } x \not\sim_S y, \\ 1 & \text{if } x \in X_1 \text{ and } y \in X_2, \\ 1 & \text{if } x \in X_1 \text{ and } y = \textsf{null}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for $x, y \in X_1$ and $x \sim_S y$, we have x-row (x-column) of $\mathbf{B}$ equal to y-row (y-column) of $\mathbf{B}$. Similarly, for $x, y \in X_2$, we have x-row (x-column) of $\mathbf{B}$ is equal to y-row (y-column) of $\mathbf{B}$.

Now let $x_1, \ldots, x_m$ be representatives of the equivalence classes under the relation "$\sim_S$" on X_1, and let y be a representative element of X_2. For every other $z \in X_1$ in the same equivalence class of x_i, we subtract from the z-row (z-column) of $\mathbf{B}$ the x_i-row (x_i-column) of $\mathbf{B}$. For every other $w \in X_2$, subtract from the w-row (w-column) of $\mathbf{B}$ the y-row (y-column) of $\mathbf{B}$. Recall that (Hyp) is preserved under these transformations.

By applying these transformations and restricting to the support, we obtain the following matrix:

$$\begin{pmatrix} 0 & 1 & \ldots & 1 & 1 \\ 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & 1 \\ 1 & \ldots & 1 & 0 & 0 \\ 1 & \ldots & 1 & 0 & 0 \end{pmatrix},$$

where the rows and columns are indexed by $\{x_1, \ldots, x_m, y, \textsf{null}\}$. The eigenvalues of this matrix are $\lambda_1 = m$, $\lambda_2 = 0$, $\lambda_3 = \ldots = \lambda_{m+2} = -1$. This implies that the matrix satisfies (OPE), and by Lemma 5.3 also (Hyp), as desired. □

Lemma 13.4. *Suppose that* $g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N}) - 2$. *Then the matrix* $\mathbf{B}$ *satisfies* (Hyp).

Proof. Let $H \subseteq X$ be given in (4.1), and let "$\sim_H$" be an equivalence relation defined by (4.2). Let us show that for every $x, y \in H$, we have:

$$S+x+y \in \mathcal{B} \iff x \not\sim_H y. \tag{13.2}$$

The $\Rightarrow$ direction is clear, so it suffices to prove the $\Leftarrow$ direction. Let $x, y \in H$ such that $x \not\sim_H y$. Then we have:

$$f(S+x+y) - f(S) \geq g\big(\Phi(S+x+y)\big) - g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N}) - \big(\mathrm{rk}(\mathcal{N}) - 2\big) = 2,$$

which implies that $S+x+y \in \mathcal{I}$. Since $\Phi(S+x+y)$ is a basis of $\mathcal{N}$ by assumption, it then follows that $S+x+y$ is a basis of Φ, as desired.

It then follows from the claim above that

$$\mathrm{B}_{xy} = \mathrm{B}_{yx} = \begin{cases} 1 & \text{if } x, y \in H \text{ and } x \not\sim_H y, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, if $x, y \in H$ and $x \sim_H y$, then x-row (x-column) of $\mathbf{B}$ is equal to y-row (y-column) of $\mathbf{B}$. Also note that, the support of $\mathbf{B}$ is contained in H.

Let $x_1, \ldots, x_m$ be the representatives of the equivalence classes $\mathcal{C}_1, \ldots, \mathcal{C}_m$ of the relation "$\sim_H$". For every other $y \in \mathcal{C}_i$, we subtract from the y-row (y-column) of $\mathbf{B}$ the x_i-row (x_i-column) of $\mathbf{B}$. By applying this transformation and restricting to the support of the resulting matrix, we obtain the following matrix:

$$\begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \ldots & 1 & 0 \end{pmatrix},$$

where the rows and columns are indexed by $\{x_1, \ldots, x_m\}$. The eigenvalues of this matrix are $\lambda_1 = m-1$, $\lambda_2 = \ldots = \lambda_m = -1$. This implies that the matrix satisfies (OPE), and by Lemma 5.3 also (Hyp), as desired. $\square$

Lemma 13.5. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroid, let $1 \leq k < \operatorname{rk}(\mathcal{M})$, and let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function. Let $\mathbb{A}$ be a combinatorial atlas corresponding to Φ. Then every sink vertex $v = (\alpha, 0, 1) \in \Omega^0$ satisfies* (Hyp).

Proof. Let $\alpha = x_1 \cdots x_\ell$ and $S = \{x_1, \ldots, x_\ell\}$. It suffices to show that $\mathbf{A}(\alpha, 1)$ satisfies (Hyp). If $\alpha \notin \mathcal{L}$ or $g(S) < \operatorname{rk}(\mathcal{N}) - 2$, then $\mathbf{A}(\alpha, 1)$ is equal to a zero matrix, so (Hyp) is trivially satisfied. Now suppose that $\alpha \in \mathcal{L}_{\mathcal{M}}$ and $g(S) \geq \operatorname{rk}(\mathcal{N}) - 2$. Then it follows from Lemma 13.2, Lemma 13.3 and Lemma 13.4, that $\mathbf{A}(\alpha, 1)$ satisfies (Hyp). $\square$

Lemma 13.6. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroid, let $1 \leq k < \operatorname{rk}(\mathcal{M})$, and let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function. Let $\mathbb{A}$ be a combinatorial atlas corresponding to Φ. Then every vertex $v \in \Omega$ satisfies* (Hyp).

Proof. Let $v = (\alpha, m, t) \in \Omega^m$. We prove that v satisfies (Hyp) by induction on m. The claim is true for $m = 0$ by Lemma 13.5. Suppose that the claim is true for Ω^{m-1}. It then follows from Theorem 5.2 that every regular vertex in Ω^m satisfies (Hyp). On the other hand, by Lemma 13.1, the regular vertices of Ω^m contain those of the form $v = (\alpha, m, t)$, where $t \in (0,1)$. Since (Hyp) is a property that is preserved under taking limits $t \to 0$ and $t \to 1$, we conclude that every vertex in Ω^m satisfies (Hyp). This completes the proof. $\square$

13.3. **Proof of Theorem 1.16.** Let $\mathbf{M}_v$ be the associated matrix of the vertex $v := (\varnothing, k-1, 1) \in \Omega$. Let $\mathbf{v}$ and $\mathbf{w}$ be the characteristic vector of X and $\{\mathsf{null}\}$, respectively. Then

$$\begin{aligned} \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \ &= \ (k-1)! \cdot \mathrm{B}_\omega(k-1), \qquad \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle \ = \ k! \cdot \mathrm{B}_\omega(k), \\ \text{and} \quad \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \ &= \ (k+1)! \left(1 + \frac{1}{\mathrm{p}(k-1)-1}\right) \cdot \mathrm{B}_\omega(k+1). \end{aligned} \tag{13.3}$$

Since v satisfies (Hyp) by Lemma 13.6, it then follows from the equations above that

$$\mathrm{B}_\omega(k)^2 \ \geq \ \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\mathrm{p}(k-1)-1}\right) \cdot \mathrm{B}_\omega(k+1)\, \mathrm{B}_\omega(k-1),$$

as desired. $\square$

13.4. **Proof of Theorem 1.19.** We first prove the $\Leftarrow$ direction. It follows from (MME3) that

$$\mathrm{B}_\omega(k+1) \,=\, \mathrm{I}_\omega(k+1), \quad \mathrm{B}_\omega(k) \,=\, \mathrm{I}_\omega(k), \quad \text{and} \quad \mathrm{B}_\omega(k-1) \,=\, \mathrm{I}_\omega(k-1), \tag{13.4}$$

where

$$\mathrm{I}_\omega(r) \,:=\, \sum_{S\in\mathcal{I}_r} \omega(S).$$

Similarly, $\mathrm{p}(k-1)$ coincide for Φ and $\mathcal{M}$. Thus (MME1) is equivalent to (ME1) for $\mathcal{M}$, and (MME2) is equivalent to (ME2) for $\mathcal{M}$. It then follows from Theorem 1.6 that

$$\mathrm{I}_\omega(k)^2 \;=\; \left(1+\frac{1}{k}\right)\left(1+\frac{1}{\mathrm{p}(k-1)-1}\right)\mathrm{I}_\omega(k+1)\,\mathrm{I}_\omega(k-1),$$

which together with (13.4) proves the $\Leftarrow$ direction.

We now prove the $\Rightarrow$ direction. It follows from the same argument as in the $\Leftarrow$ direction, that it suffices to show that (MME3) is satisfied. Let $\mathbb{A}$ be the combinatorial atlas that corresponds to (Φ, k, ω) from §13.1. In particular, every vertex of Γ satisfies (Hyp) by Lemma 13.6.

As in §13.3, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ be the characteristic vector of X and $\{\mathsf{null}\}$, respectively. It is straightforward to verify that $\mathbf{v}, \mathbf{w}$ is a global pair for Γ, i.e. they satisfy (Glob-Pos).

Let $v = (\varnothing, k-1, 1) \in \Omega$ and let $\mathbf{M} = \mathbf{M}_v$ be the associated matrix. Note that $\mathrm{B}_\omega(k+1)$, $\mathrm{B}_\omega(k)$ and $\mathrm{B}_\omega(k-1) > 0$ by the assumption of the theorem. It then follows from (13.4) that v satisfies (s-Equ) for some $\mathrm{s} > 0$.

We now show that, for every $\alpha \in \mathcal{L}_{k-1}$ such that $\langle \mathbf{v}, \mathbf{A}(\alpha,1)\,\mathbf{v}\rangle > 0$, we have:

$$\langle \mathbf{v}, \mathbf{A}(\alpha,1)\,\mathbf{v}\rangle \;=\; \mathrm{s}\,\langle \mathbf{w}, \mathbf{A}(\alpha,1)\,\mathbf{v}\rangle \;=\; \mathrm{s}^2\,\langle \mathbf{w}, \mathbf{A}(\alpha,1)\,\mathbf{w}\rangle \;>\; 0. \tag{13.5}$$

First suppose that $k=1$. This implies that $\alpha = \varnothing$ and $v = (\varnothing, 0, 1)$. Thus, (13.5) follows from the fact that v satisfies (s-Equ).

Suppose now that $k>1$. It is easy to see that v is a functional source in this case, i.e. it satisfies (Glob-Proj) and (h-Glob), where we apply the substitution $\mathbf{f} \leftarrow \mathbf{v}$ for (h-Glob). By Theorem 7.1, every functional target of v in Γ also satisfies (s-Equ) with the same $\mathrm{s}>0$. On the other hand, observe that the functional targets of v in Ω^0 contain those of the form $(\alpha, 0, 1)$, with $\alpha \in \mathcal{L}_{k-1}$ satisfying $\langle \mathbf{v}, \mathbf{A}(\alpha,1)\,\mathbf{v}\rangle > 0$. Combining these two observations, we obtain (13.5).

Claim: *For every $T \in \mathcal{I}_{k-1}$, we have $g\big(\Phi(T)\big) \neq \mathrm{rk}(\mathcal{N}) - 1$.*

Proof. Let $T = \{y_1, \ldots, y_{k-1}\}$ and let $\beta = y_1 \,\cdots\, y_{k-1} \in \mathcal{L}$. For every $\ell \geq 0$, let

$$\mathrm{B}_{\omega,T}(\ell) \,:=\, \sum_{\substack{S\in\mathcal{B}_{|T|+\ell}\\ S\supseteq T}} \omega(S).$$

Then:

$$\begin{aligned} \langle \mathbf{w}, \mathbf{A}(\beta,1)\,\mathbf{w}\rangle \;&=\; \mathrm{B}_{\omega,T}(0), \qquad \langle \mathbf{w}, \mathbf{A}(\beta,1)\,\mathbf{v}\rangle \;=\; \mathrm{B}_{\omega,T}(1),\\ \langle \mathbf{v}, \mathbf{A}(\beta,1)\,\mathbf{v}\rangle \;&=\; 2\left(1+\frac{1}{\mathrm{p}(0)-1}\right)\mathrm{B}_{\omega,T}(2). \end{aligned} \tag{13.6}$$

Now suppose to the contrary that $g\big(\Phi(T)\big) = \mathrm{rk}(\mathcal{N}) - 1$. Since $\mathrm{B}_\omega(k+1) > 0$, there is a basis $S \in \mathcal{B}_{k+1}$. Applying the exchange property for Φ, it follows that there exist $x, y \in S \setminus T$, such that $T \cup \{x,y\} \in \mathcal{B}_{k+1}$. This implies that $\mathrm{B}_{\omega,T}(2) > 0$, which in turn implies that $\langle \mathbf{v}, \mathbf{A}(\beta,1)\,\mathbf{v}\rangle > 0$ by (13.6). Hence (13.5) applies to β, which implies that $\langle \mathbf{w}, \mathbf{A}(\beta,1)\,\mathbf{w}\rangle > 0$. Again, by (13.6) we conclude that $\mathrm{B}_{\omega,T}(0) > 0$. This contradicts the assumption that $g\big(\Phi(T)\big) = \mathrm{rk}(\mathcal{N}) - 1$. This completes the proof of the claim. □

It remains to prove (MME3), i.e. that every $T \in \mathcal{I}_{k-1}$ satisfies $g\big(\Phi(T)\big) = \mathrm{rk}(\mathcal{N})$. Suppose to the contrary that $g\big(\Phi(T)\big) < \mathrm{rk}(\mathcal{N})$. Since $\mathrm{B}_\omega(k-1) > 0$, there is at least one basis $S \in \mathcal{B}_{k-1}$. By the exchange property of the matroid $\mathcal{M}$ the basis exchange graph is connected, i.e. there exist a sequence of bases $T_1, \ldots, T_m \in \mathcal{I}_{k-1}$, such that $\big|T_{i+1} \smallsetminus T_i\big| = 1$, $T_1 = T$, and $T_m = S$. Since $g\big(\Phi(T)\big) < \mathrm{rk}(\mathcal{N})$ and $g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N})$, there exists $i \in [m]$ such that $g\big(\Phi(T_i)\big) = \mathrm{rk}(\mathcal{N}) - 1$. This contradicts the claim above, and completes the proof of (MME3). □

13.5. **Proof of Theorem 1.18.** The $\Leftarrow$ direction is straightforward. For the $\Rightarrow$ direction, it follows from (MME3) in Theorem 1.19, that for every $S \subseteq X$, $|S| = k-1$, the image $\Phi(S)$ contains a basis of $\mathcal{N}$. This implies that (13.4) holds. It then follows from Theorem 1.8, that every subset of X of size $k+1$ is independent, and the weight $\omega : X \to \mathbb{R}_{>0}$ is uniform. This completes the proof. $\square$

14. Proof of log-concavity for linear extensions

In this section we give proofs of Theorem 1.35 and some variations of the results for posets with belts (§14.8). We also give an example of a combinatorial atlas in this case (§14.7).

14.1. **New notation.** In the next two sections we fix a ground set X and an element $z \in X$. Let $\mathcal{P} = (X_P, \prec)$ be a poset for which the ground set X_P is a subset of X. Let $k \in \{2, \ldots, |X_P| - 1\}$, and let $\omega : X \to \mathbb{R}_{>0}$ be the order-reversing weight function, see §1.16. We define a combinatorial atlas $\mathbb{A}(\mathcal{P}, k) := \mathbb{A}(\mathcal{P}, z, k, \omega)$ as follows.

Recall that $\mathcal{E}(\mathcal{P})$ denotes the set of linear extensions of $\mathcal{P}$. By a slight abuse of notation, in the next two sections a linear extension α of $\mathcal{P}$ is a simple word $\alpha := x_1 \ldots x_{|\alpha|} \in X_P^*$ of length $|X_P|$ such that $x_i \prec x_j$ in $\mathcal{P}$ implies that $i \leq j$. Denote by $\mathcal{E}(P, k)$ the set of linear extensions $\alpha \in \mathcal{E}(\mathcal{P})$ such that $\alpha_k = z$.

For a simple word $\alpha \in X^*$, we write $x \lhd z$ if x appears to the left of z in α. Following (1.31), for a word $\alpha \in X^*$, let

$$\omega(\alpha) \ := \ \prod_{x \lhd z} \omega(x),$$

and $\omega(S) := \sum_{\alpha \in S} \omega(\alpha)$ for every $S \subseteq X^*$.

Let $Z_{\text{down}} := X - z$ and denote every element in Z_{down} as x_{down} instead of x. Similarly, let $Z_{\text{up}} := X - z$, and denote every element in Z_{up} as x_{up} instead of x. Since Z_{down} Z_{up} are two copies of the same set, labels "down" and "up" are used to distinguish them. We write $Z := Z_{\text{down}} \cup Z_{\text{up}}$. Note that $d := |Z| = 2n - 2$ since Z_{down} and Z_{up} do not intersect because of the labeling. We will sometimes drop the "down" and "up" labels from x_{down} and x_{up} when the labels are either clear from the context or are irrelevant to the discussion. We denote by $\min(\mathcal{P}, \text{down}) \subseteq Z_{\text{down}}$ the set of elements of Z_{down} that correspond to minimal elements of $\mathcal{P}$, and by $\max(\mathcal{P}, \text{up}) \subseteq Z_{\text{up}}$ the set of elements of Z_{up} that correspond to maximal elements of $\mathcal{P}$. More generally, for a subset $S \subseteq X - z$, we denote by $S_{\text{down}} \subseteq Z_{\text{down}}$ the subset in Z_{down} that corresponds to S, and by $S_{\text{up}} \subseteq Z_{\text{up}}$ the subset in Z_{up} that corresponds to S.

Let $\mathcal{P}^{\text{op}} := (X, \prec^{\text{op}})$ denote the *opposite poset* of $\mathcal{P}$, defined by $x \prec^{\text{op}} y$ if and only if $y \prec x$.[14] For every $\alpha = x_1 \ldots x_\ell \in X^*$, we denote by $\alpha^{\text{op}} := x_\ell \ldots x_1$. Let $\mathcal{E}^{\text{op}}$ denote the set of linear extensions of $\mathcal{P}^{\text{op}}$, and note that $|\,\mathcal{E}^{\text{op}}\,| = |\,\mathcal{E}\,| = e(\mathcal{P})$. Denote by $\omega^{\text{op}} : X \to \mathbb{R}_{>0}$ the weight function defined by $\omega^{\text{op}}(x) := \omega(x)^{-1}$. Note that ω^{op} is an order-reversing weight function for $\mathcal{P}^{\text{op}}$. It then follows that

$$\mathrm{N}_\omega(\mathcal{P}, k) \ = \ \mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}}, |X_P| - k + 1) \prod_{x \in X - z} \omega(x). \tag{14.1}$$

In the subsequent two sections, we shall frequently utilize this technique of interchanging between $\mathcal{P}$ and $\mathcal{P}^{\text{op}}$ to streamline certain parts of the proofs.

[14] Sometimes, $\mathcal{P}^{\text{op}}$ is also called *dual* or *reverse poset*.

14.2. **Combinatorial atlas construction.** We denote by $\mathbf{C}(\mathcal{P},k) := \mathbf{C}(\mathcal{P},k,\omega) := \big(\mathrm{C}_{xy}\big)_{x,y\in Z}$ the symmetric $d\times d$ matrix where,[15]

$$\text{(DefC-1)}\qquad \begin{aligned}
\mathrm{C}_{xy} \ &:= \ \begin{cases} \omega(x)\,\omega(\mathcal{E}(\mathcal{P}-x-y,k-1)) & \text{if } x\in\min(\mathcal{P},\mathrm{down}),\ y\in\max(\mathcal{P},\mathrm{up}),\\ \omega(x)\,\omega(y)\,\omega(\mathcal{E}(\mathcal{P}-x-y,k-1)) & \text{if } x,y\in\min(\mathcal{P},\mathrm{down}),\ x\neq y,\\ \omega(\mathcal{E}(\mathcal{P}-x-y,k-1)) & \text{if } x,y\in\max(\mathcal{P},\mathrm{up}),\ x\neq y,\\ 0 & \text{otherwise,}\end{cases}\\
\mathrm{C}_{xx} \ &:= \ \sum_{y\in\min(\mathcal{P}-x,\mathrm{down}),\ y\succ x} \omega(x)\omega(y)\,\omega(\mathcal{E}(\mathcal{P}-x-y,k-1)) \quad \text{for } x\in\min(\mathcal{P},\mathrm{down}),\\
\mathrm{C}_{xx} \ &:= \ \sum_{y\in\max(\mathcal{P}-x,\mathrm{up}),\ y\prec x} \omega(\mathcal{E}(\mathcal{P}-x-y,k-1)) \quad \text{for } x\in\max(\mathcal{P},\mathrm{up}),\\
\mathrm{C}_{xx} \ &:= \ 0 \quad \text{for } x\notin\min(\mathcal{P},\mathrm{down})\cup\max(\mathcal{P},\mathrm{up}).
\end{aligned}$$

Equivalently, $\mathbf{C}(\mathcal{P},k)$ is given by[16]

$$\text{(DefC-2)}\qquad \begin{aligned}
\mathrm{C}_{xy} \ := \ \mathrm{C}_{yx} \ &:= \ \begin{cases} \omega\big(\{x\beta y\in\mathcal{E}(P,k)\}\big) & \text{if } x\in\min(\mathcal{P},\mathrm{down}),\ y\in\min(\mathcal{P},\mathrm{up}),\\ \omega\big(\{xy\beta\in\mathcal{E}(P,k+1)\}\big) & \text{if } x,y\in\min(\mathcal{P},\mathrm{down}),\ x\neq y,\\ \omega\big(\{\beta xy\in\mathcal{E}(P,k-1)\}\big) & \text{if } x,y\in\max(\mathcal{P},\mathrm{up}),\ x\neq y,\\ 0 & \text{if } \{x,y\}\nsubseteq\min(\mathcal{P},\mathrm{down})\cup\max(\mathcal{P},\mathrm{up}),\end{cases}\\
\mathrm{C}_{xx} \ &:= \ \begin{cases} \omega\big(\{xy\beta\in\mathcal{E}(\mathcal{P},k+1)\mid y\succ x\}\big) & \text{if } x\in\min(\mathcal{P},\mathrm{down}),\\ \omega\big(\{\beta yx\in\mathcal{E}(\mathcal{P},k-1)\mid y\prec x\}\big) & \text{if } x\in\max(\mathcal{P},\mathrm{up}).\end{cases}
\end{aligned}$$

Note that both definitions will be frequently employed throughout the next two sections, chosen based on their suitability. Also note that it follows from the definition that $\mathbf{C}$ is a nonnegative symmetric matrix.

Note that, it follows from (DefC-2) that, for every $x\in\min(\mathcal{P},\mathrm{down})$,

$$\text{(14.2)}\qquad \begin{aligned}
\sum_{y\in Z_{\mathrm{down}}} \mathrm{C}_{xy} \ &= \ \omega\big(\{x\beta\mid\mathcal{E}(\mathcal{P},k+1)\}\big) \ = \ \omega(x)\,\omega\big(\mathcal{P}-x,k\big),\\
\sum_{y\in Z_{\mathrm{up}}} \mathrm{C}_{xy} \ &= \ \omega\big(\{x\beta\mid\mathcal{E}(\mathcal{P},k)\}\big) \ = \ \omega(x)\,\omega\big(\mathcal{P}-x,k-1\big).
\end{aligned}$$

Similarly, for every $x\in\max(\mathcal{P},\mathrm{up})$,

$$\text{(14.3)}\qquad \begin{aligned}
\sum_{y\in Z_{\mathrm{down}}} \mathrm{C}_{xy} \ &= \ \omega\big(\{\beta\,x\mid\mathcal{E}(\mathcal{P},k)\}\big) \ = \ \omega\big(\mathcal{P}-x,k\big),\\
\sum_{y\in Z_{\mathrm{up}}} \mathrm{C}_{xy} \ &= \ \omega\big(\{\beta\,x\mid\mathcal{E}(\mathcal{P},k-1)\}\big) \ = \ \omega\big(\mathcal{P}-x,k-1\big).
\end{aligned}$$

Let $\mathbf{f},\mathbf{g}\in\mathbb{R}^d$ be the indicator vector of Z_{down} and Z_{up}, respectively. It follows from (14.2) and (14.3) that

$$\text{(Cfg)}\qquad \begin{aligned}
\langle\mathbf{f},\mathbf{C}(\mathcal{P},k)\,\mathbf{g}\rangle \ &= \ \mathrm{N}_\omega(\mathcal{P},k), \qquad \langle\mathbf{f},\mathbf{C}(\mathcal{P},k)\,\mathbf{f}\rangle \ = \ \mathrm{N}_\omega(\mathcal{P},k+1),\\
\langle\mathbf{g},\mathbf{C}(\mathcal{P},k)\,\mathbf{g}\rangle \ &= \ \mathrm{N}_\omega(\mathcal{P},k-1),
\end{aligned}$$

where recall that $\mathrm{N}_\omega(\mathcal{P},k)$ is the sum of ω-weight of linear extensions of $\mathcal{P}$ such that z is the k-th smallest element.

Let $\Gamma := \Gamma(\mathcal{P},k) := (\Omega,\Theta)$ be the acyclic graph with $\Omega=\Omega^0\cup\Omega^1$, where

$$\Omega^1 \ := \ \{t\in\mathbb{R}\mid 0\le t\le 1\}, \qquad \Omega^0 \ := \ Z.$$

For a non-sink vertex $v=t\in\Omega^1$ and $x\in Z$, the corresponding outneighbor in Ω^0 is $v^{\langle x\rangle}:=x$.

[15]Here $\omega(\mathcal{E}(\mathcal{P}-x-y,k-1))$ is the sum of ω-weight of all linear extensions α of $\mathcal{P}-x-y$ for which $\alpha_{k-1}=z$.

[16]Here $\{x\beta y\in\mathcal{E}(P,k)\}$ is the set of linear extensions $\alpha\in\mathcal{E}(P,k)$ such that $\alpha_1=x$ and $\alpha_{|X_{\mathcal{P}}|}=y$. The word $\beta\in X^*$ here denotes $\alpha_2\cdots\alpha_{|X_{\mathcal{P}}|-1}$.

Define the combinatorial atlas $\mathbb{A}(\mathcal{P}, k)$ of dimension d corresponding to poset $\mathcal{P}$, and $k \in \{3, \ldots, |X_P| - 1\}$ by the acyclic graph Γ and the linear algebraic data defined as follows. For each vertex $v = x \in \Omega^0$, the associated matrix is

$$\mathbf{M}_v := \begin{cases} \omega(x)\,\mathbf{C}(\mathcal{P} - x, k-1) & \text{if } x \in \min(\mathcal{P}, \text{down}), \\ \mathbf{C}(\mathcal{P} - x, k-1) & \text{if } x \in \max(\mathcal{P}, \text{up}), \end{cases}$$

and is equal to the zero matrix otherwise. For each vertex $v = t \in \Omega^1$, the associated matrix is

$$\mathbf{M} := \mathbf{M}_v := t\,\mathbf{C}(\mathcal{P}, k) + (1-t)\,\mathbf{C}(\mathcal{P}, k-1),$$

and the associated vector $\mathbf{h} := \mathbf{h}_v \in \mathbb{R}^d$ is defined to have coordinates

$$\mathrm{h}_x := \begin{cases} t & \text{if } \; x \in Z_{\text{down}}\,, \\ 1-t & \text{if } \; x \in Z_{\text{up}}\,. \end{cases}$$

Finally, let the linear transformation $\mathbf{T}^{\langle x \rangle} : \mathbb{R}^d \to \mathbb{R}^d$ associated to the edge $(v, v^{\langle x \rangle})$, be

$$\big(\mathbf{T}^{\langle x \rangle}\,\mathbf{v}\big)_y := \begin{cases} \mathrm{v}_y & \text{if } \; y \in \operatorname{supp}(\mathbf{M}), \\ \mathrm{v}_x & \text{if } \; y \in Z \setminus \operatorname{supp}(\mathbf{M}). \end{cases}$$

14.3. **Properties of the matrix $\mathbf{C}(\mathcal{P}, k)$.** In this subsection we gather properties of the matrix $\mathbf{C}(\mathcal{P}, k)$ that will be used in this paper.

Lemma 14.1. *Let $\mathcal{P}$ be a poset, and let and let $k \in \{2, \ldots, |X_P| - 1\}$ such that $\mathrm{N}(\mathcal{P}, k) > 0$. Then*

- *The support of $\boldsymbol{C}(\mathcal{P}, k)$ is equal to $\min(\mathcal{P}, \text{down}) \cup \max(\mathcal{P}, \text{up})$, and*
- *The matrix $\boldsymbol{C}(\mathcal{P}, k)$ is irreducible when restricted to the support.*

Proof. Let $n := |X_P|$. It follows from (DefC-1) that the support of $\mathbf{C}(\mathcal{P}, k)$ is a subset of $\min(\mathcal{P}, \text{down}) \cup \max(\mathcal{P}, \text{up})$. Now note that, since $\mathrm{N}(\mathcal{P}, k) > 0$, there exists a linear extension $\alpha = x_1 \cdots x_n \in \mathcal{E}(P, k)$, and note that $x_1 = (x_1)_{\text{down}} \in \min(\mathcal{P}, \text{down})$ and $x_n = (x_n)_{\text{up}} \in \max(\mathcal{P}, \text{up})$. Now, let y be an arbitrary element of $\min(\mathcal{P}, \text{down}) \cup \max(\mathcal{P}, \text{up})$. For the first claim it suffices to show that $y \in \operatorname{supp}(\mathbf{C}(\mathcal{P}, k))$, and for the second claim it suffices to show that that y is contained in the same irreducible component (of the matrix $\mathbf{C}(\mathcal{P}, k)$) as $(x_1)_{\text{down}}$ and $(x_n)_{\text{up}}$.

By switching to the dual poset in (14.1) if necessary, we will without loss of generality assume that $y = y_{\text{down}} \in \min(\mathcal{P}, \text{down})$. Let α' be the linear extension obtained from α by *demoting* y to be the smallest element, i.e.

$$\alpha' := y\, x_1 \cdots x_{i-1} x_{i+1} \cdots x_n, \qquad \text{where} \quad \alpha =: x_1 \cdots x_{i-1}\, y\, x_{i+1} \cdots x_n.$$

Note that α' is still a linear extension of $\mathcal{P}$ since y is a minimal element of $\mathcal{P}$. Now note that either $\alpha' \in \mathcal{E}(\mathcal{P}, k)$ or $\alpha' \in \mathcal{E}(\mathcal{P}, k+1)$. In the first case we then have $(\mathbf{C}(\mathcal{P}, k))_{y x_n} > 0$, so y is contained in the support of $\mathbf{C}(\mathcal{P}, k)$ and is in the same irreducible component as x_n. In the second case we then have $(\mathbf{C}(\mathcal{P}, k))_{y x_1} > 0$, so y is contained in the support of $\mathbf{C}(\mathcal{P}, k)$ and is in the same irreducible component at x_1. This completes the proof. □

Lemma 14.2. *Let $\mathcal{P}$ be a poset, and let $k \in \{3, \ldots, |X_P| - 1\}$ such that $\mathrm{N}(\mathcal{P}, k) > 0$ and $\mathrm{N}(\mathcal{P}, k-1) > 0$. Then, for every $x \in \min(\mathcal{P}, \text{down}) \cup \max(\mathcal{P}, \text{up})$,*

$$\mathrm{N}(\mathcal{P} - x, k-1) > 0.$$

Proof. By switching to the dual poset in (14.1) if necessary, we will without loss of generality assume that $x = x_{\text{down}} \in \min(\mathcal{P}, \text{down})$. By assumption there exists linear extensions $\alpha \in \mathcal{E}(\mathcal{P}, k-1)$ and $\beta \in \mathcal{E}(\mathcal{P}, k)$. Let α' and β' be the linear extension of $\mathcal{P}$ obtained from α and β by demoting x to be the smallest element, respectively. It then follows that $\alpha' \in \mathcal{E}(\mathcal{P}, k-1) \cup \mathcal{E}(\mathcal{P}, k)$ and $\beta' \in \mathcal{E}(\mathcal{P}, k) \cup \mathcal{E}(\mathcal{P}, k+1)$. If $\alpha' \in \mathcal{E}(\mathcal{P}, k)$ then we are done, as removing the smallest element from α' (which is x) will give us a linear extension in $\mathcal{E}(\mathcal{P} - x, k-1)$. If $\beta' \in \mathcal{E}(\mathcal{P}, k)$ then we are also done, as removing the smallest element from β' (which is x) will give us a linear extension in $\mathcal{E}(\mathcal{P} - x, k-1)$. So we assume that $\alpha' \in \mathcal{E}(\mathcal{P}, k-1)$ and $\beta' \in \mathcal{E}(\mathcal{P}, k+1)$.

This assumption implies that there exists $y \in X_P$ which appears to the right of z in α', but appears to the left of z in β'. This in turn implies that y is incomparable to z in $\mathcal{P}$. Now, let j be the smallest integer in the set

$$\{i \,:\, x'_i \,\|\, z \ \text{ in } \mathcal{P},\, k \leq i \leq |X_P|\,\},$$

where $x'_1 \cdots x'_n := \alpha'$. Note that this set is non-empty by the preceding argument. Let γ be the linear extension of $\mathcal{P}$ obtained from α' by demoting x'_j to the $k-1$-th position. Then $\gamma \in \mathcal{E}(\mathcal{P}, k)$ and furthermore x is the smallest element in γ. Then, removing the smallest element of γ gives us a linear extension in $\mathcal{E}(\mathcal{P} - x, k-1)$, and the proof is complete. □

Remark 14.3. The arguments in Lemma 14.1 and Lemma 14.2 are variations of the maximality argument that appears in the proof of Thm 8.9 in [CPP21]. We refer to [CP23, §12.3, §14.2] for a detailed survey.

14.4. **Properties of the combinatorial atlas.** We now show that the atlas $\mathbb{A}(\mathcal{P}, k)$ defined above, satisfies all four conditions in Theorem 6.1, namely properties (Inh), (Proj), (T-Inv) and (K-Non). We prove these properties one by one, in the following series of lemmas. For every lemma in this subsection we assume that $\mathcal{P} = (X_P, \prec)$ is a poset, and $k \in \{3, \ldots, |X_P|-1\}$ such that $\mathrm{N}(\mathcal{P}, k) > 0$ and $\mathrm{N}(\mathcal{P}, k-1) > 0$.

Lemma 14.4. *The atlas $\mathbb{A}(\mathcal{P}, k)$ satisfies* (Inh) *and* (Proj)*.*

Proof. Let $v = t \in \Omega^+$ be a non-sink vertex of Γ. The property (Proj) follows directly from the definition of $\mathbf{T}^{\langle x \rangle}$. For (Inh), let $x \in \operatorname{supp}(\mathbf{M})$. By linearity of $\mathbf{T}^{\langle x \rangle}$, it suffices to show that, for every $y \in Z$, we have:

$$\mathrm{M}_{xy} \ = \ \big\langle \mathbf{T}^{\langle x \rangle}\, \mathbf{e}_y, \mathbf{M}^{\langle x \rangle}\, \mathbf{T}^{\langle x \rangle}\, \mathbf{h} \,\big\rangle, \tag{14.4}$$

where $(\mathbf{e}_y)_{y \in Z}$ is the standard basis for $\mathbb{R}^d$. Note that we can assume $y \in \operatorname{supp}(\mathbf{M})$, as otherwise $\mathbf{M}\mathbf{e}_y = \mathbf{T}^{\langle x \rangle}\, \mathbf{e}_y = \mathbf{0}$, and (14.4) then follows trivially. It then follows from Lemma 14.1 that $x, y \in \min(\mathcal{P}, \mathrm{down}) \cup \max(\mathcal{P}, \mathrm{up})$. Without loss of generality, assume that $x = x_{\mathrm{down}} \in Z_{\mathrm{down}}$ and $y = y_{\mathrm{down}} \in Z_{\mathrm{down}}$, as the proofs of the other cases are analogous.

We split the proof of (14.4) into two cases. First suppose that x and y are distinct. It then follows that $\mathbf{T}^{\langle x \rangle}\, \mathbf{e}_y \ = \ \mathbf{e}_y$, and

$$\begin{aligned}
\big\langle \mathbf{T}^{\langle x \rangle}\, \mathbf{e}_y, \mathbf{M}^{\langle x \rangle}\, \mathbf{T}^{\langle x \rangle}\, \mathbf{h} \,\big\rangle \ &= \ \sum_{u \in Z} \mathrm{M}^{\langle x \rangle}_{uy} \, \big(\mathbf{T}^{\langle x \rangle}\, \mathbf{h}\big)_u \\
&= \sum_{u \in Z_{\mathrm{down}}} \big(\mathbf{C}(\mathcal{P} - x, k-1)\big)_{uy}\, t \ + \sum_{u \in Z_{\mathrm{up}}} \big(\mathbf{C}(\mathcal{P} - x, k-1)\big)_{uy}\, (1-t) \\
&=_{(14.2)} \ \omega(x)\, \omega(y)\, \omega(\mathcal{E}(\mathcal{P} - x - y, k-1))\, t \ + \ \omega(x)\, \omega(y)\, \omega(\mathcal{E}(\mathcal{P} - x - y, k-2))\, (1-t) \\
&=_{(\mathrm{DefC}\text{-}1)} (\mathbf{C}(\mathcal{P}, k))_{xy}\, t \ + \ (\mathbf{C}(\mathcal{P}, k-1))_{xy}\, (1-t) \quad = \quad \mathrm{M}_{xy},
\end{aligned}$$

as desired.

Now suppose that $x = y$. Then

$$\begin{aligned}
\big\langle \mathbf{T}^{\langle x \rangle}\, \mathbf{e}_x, \mathbf{M}^{\langle x \rangle}\, \mathbf{T}^{\langle x \rangle}\, \mathbf{h} \,\big\rangle \ &= \ \sum_{\substack{w \in Z_{\mathrm{down}} \\ w \succ x}} \ \sum_{u \in Z} \mathrm{M}^{\langle x \rangle}_{uw} \, \big(\mathbf{T}^{\langle x \rangle}\, \mathbf{h}\big)_u \\
&= \sum_{\substack{w \in Z_{\mathrm{down}} \\ w \succ x}} \left(\sum_{u \in Z_{\mathrm{down}}} \big(\mathbf{C}(\mathcal{P} - x, k-1)\big)_{uw}\, t + \sum_{u \in Z_{\mathrm{up}}} \big(\mathbf{C}(\mathcal{P} - x, k-1)\big)_{uw}\, (1-t) \right) \\
&=_{(14.2)} \sum_{\substack{w \in Z_{\mathrm{down}} \\ w \succ x}} \omega(x)\, \omega(w)\, \omega(\mathcal{E}(\mathcal{P} - x - w, k-1))\, t \ + \ \omega(x)\, \omega(w)\, \omega(\mathcal{E}(\mathcal{P} - x - w, k-2))\, (1-t) \\
&=_{(\mathrm{DefC}\text{-}1)} \ \big(\mathbf{C}(\mathcal{P}, k)\big)_{xx}\, t \ + \ \big(\mathbf{C}(\mathcal{P}, k-1)\big)_{x,x}\, (1-t) \quad = \quad \mathrm{M}_{xx}\,,
\end{aligned}$$

which completes the proof. □

Lemma 14.5. *The atlas $\mathbb{A}(\mathcal{P}, k)$ satisfies* (T-Inv)*.*

Proof. Let $v = t \in \Omega^+$ be a non-sink vertex, and let a, b, c be distinct elements of $\operatorname{supp}(\mathbf{M})$. It follows from Lemma 14.1 that $a, b, c \in \min(\mathcal{P}, \mathrm{down}) \cup \max(\mathcal{P}, \mathrm{up})$. Without loss of generaltiy assume that $a, b, c \in \min(\mathcal{P}, low)$. It then follows from (DefC-1) that

$$\mathrm{M}_{bc}^{\langle a\rangle} \;=\; \mathrm{M}_{ca}^{\langle b\rangle} \;=\; \mathrm{M}_{ab}^{\langle c\rangle} \;=\; \omega(a)\,\omega(b)\,\omega(c)\,\omega(\mathcal{E}(\mathcal{P} - a - b - c, k-2)),$$

and the lemma follows. $\square$

Lemma 14.6. *The atlas $\mathbb{A}(\mathcal{P}, k)$ satisfies* (K-Non).

Proof. Let $v = t \in \Omega^+$ be a non-sink vertex. We need to check the condition (K-Non) for distinct $x, y \in \operatorname{supp}(\mathbf{M})$. It follows from Lemma 14.1 that $x, y \in \min(\mathcal{P}, \mathrm{down}) \cup \max(\mathcal{P}, \mathrm{up})$. We will without loss of generality assume that $x = x_{\mathrm{down}} \in \min(\mathcal{P}, \mathrm{down})$ and $y = y_{\mathrm{down}} \in \min(\mathcal{P}, \mathrm{down})$, as the proof of other cases are analogous.

It follows from (DefC-1) that

$$\mathrm{M}_{yy}^{\langle x\rangle} \;=\; \sum_{w \in \min(\mathcal{P}-x-y, \mathrm{down}),\; w \succ y} \omega(x)\,\omega(y)\,\omega(w)\,\omega(\mathcal{E}(\mathcal{P} - x - y - w, k-2)).$$

Now note that, it follows from Lemma 14.1 that

$$\begin{aligned} \operatorname{supp}(\mathbf{M}) - y \;&=\; (\min(\mathcal{P}, \mathrm{down}) - y) \cup \max(\mathcal{P}, \mathrm{up}), \\ \operatorname{supp}\big(\mathbf{M}^{\langle y\rangle}\big) \;&=\; \min(\mathcal{P} - y, \mathrm{down}) \cup \max(\mathcal{P} - y, \mathrm{up}), \end{aligned}$$

Then, the set $\mathrm{Fam}^{\langle y\rangle}$ defined in (6.1), in this case is equal to

$$\begin{aligned} \mathrm{Fam}^{\langle y\rangle} \;&=\; \operatorname{supp}\big(\mathbf{M}^{\langle y\rangle}\big) \setminus \big(\operatorname{supp}(\mathbf{M}) - y\big) \;=\; \min(\mathcal{P} - y, \mathrm{down}) \setminus \min(\mathcal{P}, \mathrm{down}) \\ &=\; \{w \in \min(\mathcal{P} - y, \mathrm{down}) \mid w \succ y\}. \end{aligned}$$

This implies that

$$\begin{aligned} \sum_{w \in \mathrm{Fam}^{\langle y\rangle}} \mathrm{M}_{xw}^{\langle y\rangle} \;&=\; \sum_{w \in \mathrm{Fam}^{\langle y\rangle}} \omega(x)\,\omega(y)\,\omega(w)\,\omega(\mathcal{E}(\mathcal{P} - x - -yw, k-2)) \\ &=\; \sum_{w \in \min(\mathcal{P}-y, \mathrm{down}),\; w \succ y} \omega(x)\,\omega(y)\,\omega(w)\omega(\mathcal{E}(\mathcal{P} - x - y - w, k-2)). \end{aligned}$$

Taking the difference of the two equations above, we get:

$$\mathrm{M}_{yy}^{\langle x\rangle} - \sum_{w \in \mathrm{Fam}^{\langle y\rangle}} \mathrm{M}_{xw}^{\langle y\rangle} \;=\; \sum_{w \in \min(\mathcal{P}-x-y, \mathrm{down}),\; w \succ y,\; w \succ x} \omega(x)\,\omega(y)\,\omega(w)\,\omega(\mathcal{E}(\mathcal{P} - x - y - w, k-2)).$$

This is clearly nonnegative, and thus (K-Non) holds, as desired. $\square$

Lemma 14.7. *Every $v \in \Omega^+$ satisfies* (Irr). *Furthermore, every $v = t \in \Omega^+$ satisfies* (h-Pos)*, for all $0 < t < 1$.*

Proof. Property (Irr) follows directly from Lemma 14.1, and Property (h-Pos) follows from the observation that $\mathbf{h}_v$ is a positive vector when $t \in (0, 1)$. $\square$

14.5. **Sink vertices are hyperbolic.** Before we can apply the local-global principle, we need the following result:

Lemma 14.8. *Let $\mathcal{P} = (X_P, \prec)$ be a finite poset with $|X_P| = 3$, let $\omega : X \to \mathbb{R}_{>0}$ be an order-reversing weight function. Then the matrix $\boldsymbol{C}(\mathcal{P}, 2)$ satisfies* (Hyp).

Proof. Let $\{x, y, z\} := X_P$. We index the rows and columns of $\mathbf{C}(\mathcal{P}, 2)$ with $\{x_{\text{down}}, y_{\text{down}}, x_{\text{up}}, y_{\text{up}}\}$.

We now split the proof of the lemma into seven cases, depending on the relative order of $\{x, y, z\}$. First, suppose that x, y, z are incomparable to each other. Then

$$\mathbf{C}(\mathcal{P}, 2) \quad = \quad \begin{pmatrix} 0 & \omega(x)\,\omega(y) & 0 & \omega(x) \\ \omega(x)\,\omega(y) & 0 & \omega(y) & 0 \\ 0 & \omega(y) & 0 & 1 \\ \omega(x) & 0 & 1 & 0 \end{pmatrix}. \tag{C1}$$

We now divide x_{down}-row and x_{down}-column by $\omega(x)$, and the y_{down}-row and the y_{down}-column by $\omega(y)$. Recall that (Hyp) is preserved under this transformation. Then the matrix becomes

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\{2, 0, 0, -2\}$. This implies that the matrix satisfies (OPE). By Lemma 5.3 we also have (Hyp), as desired.

Second, suppose that $x \prec y$, and z are incomparable to both elements. Then

$$\mathbf{C}(\mathcal{P}, 2) \quad = \quad \begin{pmatrix} \omega(x)\,\omega(y) & 0 & 0 & \omega(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega(x) & 0 & 0 & 1 \end{pmatrix}. \tag{C2}$$

Restricting the rows and columns to the support $\{x_{\text{down}}, y_{\text{up}}\}$, we get

$$\begin{pmatrix} \omega(x)\,\omega(y) & \omega(x) \\ \omega(x) & 1 \end{pmatrix}.$$

This matrix has determinant

$$\omega(x)\,\big(\omega(y) - \omega(x)\big) \ \leq\ 0,$$

where the inequality follows from ω being order-reversing. This implies that the matrix satisfies (OPE), and thus also (Hyp), as desired.

In the remaining cases, element z is comparable to either x or y, or both. By symmetry, without loss of generality, we assume that $x \prec z$. Third, suppose that $x \prec z$, $x \prec y$, and $y \,||\, z$. Then:

$$\mathbf{C}(\mathcal{P}, 2) \quad = \quad \begin{pmatrix} \omega(x)\,\omega(y) & 0 & 0 & \omega(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega(x) & 0 & 0 & 0 \end{pmatrix}. \tag{C3}$$

Restricting the rows and columns to the support $\{x_{\text{down}}, y_{\text{up}}\}$, we get

$$\begin{pmatrix} \omega(x)\,\omega(y) & \omega(x) \\ \omega(x) & 0 \end{pmatrix}.$$

This matrix has a negative determinant, so it satisfies (OPE). Thus, it also satisfies (Hyp), as desired.

Fourth, suppose that $x \prec z$, $x \,||\, y$, and $y \,||\, z$. Then:

$$\mathbf{C}(\mathcal{P}, 2) \quad = \quad \begin{pmatrix} 0 & \omega(x)\,\omega(y) & 0 & \omega(x) \\ \omega(x)\,\omega(y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega(x) & 0 & 0 & 0 \end{pmatrix}. \tag{C4}$$

By restricting the rows and columns to the support $\{x_{\text{down}}, y_{\text{down}}, y_{\text{up}}\}$, followed by dividing the x_{down}-row and x_{down}-column by $\omega(x)$, and the y_{down}-row and the y_{down}-column by $\omega(y)$, we get

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\{\sqrt{2}, 0, -\sqrt{2}\}$, so it satisfies (OPE). Thus it also satisfies (Hyp), as desired.

Fifth, suppose that $x \prec z$, $y \prec z$, and $x \,||\, y$. Then:

$$\mathbf{C}(\mathcal{P},2) \quad = \quad \begin{pmatrix} 0 & \omega(x)\,\omega(y) & 0 & 0 \\ \omega(x)\,\omega(y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C5}$$

The eigenvalues of this matrix are $\{\,\omega(x)\,\omega(y), 0, 0, -\,\omega(x)\,\omega(y)\}$, so it satisfies (OPE). Thus, it also satisfies (Hyp), as desired.

For the sixth case, suppose that $x \prec z \prec y$. Then:

$$\mathbf{C}(\mathcal{P},2) \quad = \quad \begin{pmatrix} 0 & 0 & \omega(x) & 0 \\ 0 & 0 & 0 & 0 \\ \omega(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C6}$$

The eigenvalues of this matrix are $\{\,\omega(x), 0, 0, -\,\omega(x)\}$, so it satisfies (OPE). Thus it also satisfies (Hyp), as desired.

Seventh and final case, suppose that $x \prec y \prec z$. Then:

$$\mathbf{C}(\mathcal{P},2) \quad = \quad \begin{pmatrix} \omega(x)\,\omega(y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C7}$$

The eigenvalues of this matrix are $\{\,\omega(x)\,\omega(y), 0, 0, 0\}$, so it satisfies (OPE). Thus it also satisfies (Hyp), as desired. This completes the proof. □

14.6. **Proof of Theorem 1.34.** We can now prove that the matrix $\mathbf{C}(\mathcal{P},k)$ is always hyperbolic.

Proposition 14.9. *Let $\mathcal{P} = (X_P, \prec)$ be a finite poset, let $k \in \{2, \ldots, |X_P| - 1\}$, and let $\omega : X \to \mathbb{R}_{>0}$ be an order-reversing weight function. Then the matrix $\boldsymbol{C}(\mathcal{P},k)$ satisfies* (Hyp).

Proof. We will prove the proposition by induction on $|X_P|$. The base case $|X_P| = 3$ follows from Lemma 14.8. Suppose that the claim is true for $|X_P| - 1$.

First note that, if $\mathrm{N}(\mathcal{P}, k-1) = \mathrm{N}(\mathcal{P}, k) = \mathrm{N}(\mathcal{P}, k+1) = 0$, then $\mathbf{C}(\mathcal{P},k)$ is the zero matrix, and (Hyp) immediately follows. So we will assume that either one of $\mathrm{N}(\mathcal{P}, k-1), \mathrm{N}(\mathcal{P}, k), \mathrm{N}(\mathcal{P}, k+1)$ is nonzero.

We split the proof into case (1) and case (2): For case(1), suppose that at least two of the three numbers are nonzero. Since the sequence $\mathrm{N}(\mathcal{P}, k-1), \mathrm{N}(\mathcal{P}, k), \mathrm{N}(\mathcal{P}, k+1)$ cannot have internal zeroes (this follows from the demotion argument in the proof of Lemma 14.2), this reduces to either $\mathrm{N}(\mathcal{P}, k-1), \mathrm{N}(\mathcal{P}, k) > 0$ or $\mathrm{N}(\mathcal{P}, k+1), \mathrm{N}(\mathcal{P}, k) > 0$. By switching to the dual poset in (14.1) if necessary, we can without loss of generality assume that $\mathrm{N}(\mathcal{P}, k-1), \mathrm{N}(\mathcal{P}, k) > 0$. We split the proof further into case (1a), case (1b), and case (1c).

For case (1a), assume that $k \geq 3$. Let $\mathbb{A}(\mathcal{P}, k)$ be the atlas defined in §14.2. It follows from Lemma 14.4, 14.5, 14.6 that this atlas satisfies the assumptions of Theorem 5.2 (note that these lemmas require $k \geq 3$). Also note that every sink vertex in Ω^0 satisfies (Hyp) by the induction assumption, as they correspond to posets with cardinality $|X_P| - 1$. It then follows from Theorem 5.2 that every regular vertex in Ω^1 satisfies (Hyp). On the other hand, it follows from Lemma 14.7 that every $v = t \in \Omega^+$ with $0 < t < 1$ is a regular vertex. This implies that, for $0 < t < 1$, the matrix $t\,\mathbf{C}(\mathcal{P},k) + (1-t)\,\mathbf{C}(\mathcal{P},k-1)$ satisfies (Hyp). By taking the limit $t \to 0$ and $t \to 1$, we then conclude that both $\mathbf{C}(\mathcal{P},k)$ and $\mathbf{C}(\mathcal{P},k-1)$ satisfies (Hyp), as desired.

For case (1b), assume that $k = 2$ and $\mathrm{N}(\mathcal{P}, k+1) > 0$. Then by applying the same argument as in case (1a) to the atlas $\mathbb{A}(\mathcal{P}, k+1)$, it follows that both $\mathbf{C}(\mathcal{P},k+1)$ and $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp), as desired.

For case (1c), assume that $k = 2$ and $\mathrm{N}(\mathcal{P}, k+1) = 0$. The assumptions imply that X_P can be partitioned into $\{x\} \cup \{z\} \cup \{T\}$, where x is the only element in X_P incomparable to z, and T is the upper ideal of z in $\mathcal{P}$. Also note that the support of $\mathbf{C}(\mathcal{P},k)$ is contained in $\{x_{\text{down}}\} \cup T_{\text{up}}$. Now suppose that there exists $y \in \min(T)$ such that $x \,||\, y$. Let $\mathcal{P}' := (X_P, \prec')$ be the poset with the same ground set as $\mathcal{P}$ and with $\prec'$ being obtained from $\prec$ by removing the relation $z \prec y$. Now note that $\mathrm{N}(\mathcal{P}, k+1) > 0$ by construction, so it follows from case (1b) that $\mathbf{C}(\mathcal{P}',k)$ satisfies (Hyp). On the other hand, it follows from the construction that $\mathbf{C}(\mathcal{P},k)$ is equal to $\mathbf{C}(\mathcal{P}',k)$ when restricted to rows and columns indexed by $\{x_{\text{down}}\} \cup T_{\text{up}}$. Since (Hyp) is preserved under restricting to principal submatrices, we have $\mathbf{C}(\mathcal{P},k)$ also satisfies (Hyp), as desired. Now suppose that every element T is ordered to be greater than x by $\prec$. This implies that, for every $y \in \max(\mathcal{P}, up) \cap T_{\text{up}}$,

$$\omega(\mathcal{E}(\mathcal{P}-x-y,k-1)) \ = \ \omega(\mathcal{E}(\mathcal{P}-y,k-1)),$$

because x is the second smallest element in every linear extension counted by $\mathcal{E}(\mathcal{P}-y,k-1)=\mathcal{E}(\mathcal{P}-y,1)$. This implies that

$$\begin{aligned}(\mathbf{C}(\mathcal{P},k))_{x_{\text{down}},y_{\text{up}}} \ &=_{(\text{DefC-1})} \ \omega(x)\,\omega(\mathcal{E}(\mathcal{P}-x-y,k-1)) \ = \ \omega(x)\,\omega(\mathcal{E}(\mathcal{P}-y,k-1)) \\ &=_{(14.3)} \omega(x) \sum_{w\in Z_{\text{up}}} (\mathbf{C}(\mathcal{P},k))_{w,y_{\text{up}}}.\end{aligned}$$

Let $\mathbf{D}$ be the matrix obtained by deducting the x_{down}-row by the $\omega(x)$ times the sum of the other rows, followed by deducting the the x_{down}-column by the $\omega(x)$ times sum of the other columns (note that this operation preserves (Hyp)). Then it follows from the previous equation that the entries of $\mathbf{D}$ are given by

$$(\mathbf{D})_{uv} \ = \ \begin{cases}(\mathbf{C}(\mathcal{P},k))_{uv} & \text{if } u,v\in Z-x,\\ 0 & \text{if } u\in Z-x \text{ and } v=x,\\ -\omega(x)\,\omega(\mathcal{E}(\mathcal{P}-x,k-1)) & \text{if } u=v=x.\end{cases}$$

It then follows that $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp) if and only if, the restriction of $\mathbf{D}$ to rows and columns indexed by T_{up}, satisfies (Hyp). Now let $\mathcal{P}'$ be the induced subposet of $\mathcal{P}$ on $X_P - x$. Note that $\mathbf{C}(\mathcal{P}',k)$ satisfies (Hyp) by the induction assumption. Also note that $\mathbf{C}(\mathcal{P}',k)$ is equal to $\mathbf{D}$ when restricted to rows and columns indexed by T_{up}. Since (Hyp) is preserved under restricting to principal submatrices, it follows that this submatrix of $\mathbf{D}$ also satisfies (Hyp). Combined with previous observations, we then conclude that $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp), as desired. This completes the proof of case (1).

For case (2), suppose that exactly one of $\mathrm{N}(\mathcal{P},k-1), \mathrm{N}(\mathcal{P},k), \mathrm{N}(\mathcal{P},k+1)$ is nonzero. The proof then splits into three subcases. For case (2a), let $\mathrm{N}(\mathcal{P},k)=0$ while $\mathrm{N}(\mathcal{P},k-1)=\mathrm{N}(\mathcal{P},k+1)=0$. This implies that X_P can be partitioned into $S\cup T\cup\{x\}$, where S is the lower ideal of z in $\mathcal{P}$ and $|S|=k-1$, and T is the upper ideal of x in $\mathcal{P}$. Then the entries of $\mathbf{C}(\mathcal{P},k)$ is given by

$$(\mathbf{C}(\mathcal{P},k))_{x,y} \ = \ \begin{cases}\omega(x) & \text{if } x\in\min(\mathcal{P},\text{down})\cap S_{\text{down}},\ y\in\max(\mathcal{P},\text{up})\cap T_{\text{up}},\\ 0 & \text{otherwise.}\end{cases}$$

By a direct computation, the eigenvalues of this matrix are $\lambda,-\lambda,0,\ldots,0$, where

$$\lambda \ := \ |\max(\mathcal{P},\text{up})\cap T_{\text{up}}| \sum_{x\in\min(\mathcal{P},\text{down})\cap S_{\text{down}}} \omega(x).$$

Thus this matrix satisfies (OPE), and so it satisfies (Hyp).

For case (2b), let $\mathrm{N}(\mathcal{P},k+1)>0$ while $\mathrm{N}(\mathcal{P},k-1)=\mathrm{N}(\mathcal{P},k)=0$. Let S be the lower ideal of z in $\mathcal{P}$. This implies that $|S|=k$, and the support of $\mathbf{C}(\mathcal{P},k)$ is contained in S_{down}. Now let $\mathcal{P}' := (X_{\mathcal{P}'},\prec')$ be the poset with ground set $X_{\mathcal{P}'} := S\cup\{z\}\subseteq X$, and with relations $\prec'$ given by

$$\begin{aligned}&\forall\, x,y\in S, \qquad x \prec' y \quad \Longleftrightarrow \quad x\prec y,\\ &\forall\, x\in S, \qquad x\,||\,z.\end{aligned}$$

It follows from the construction that, for all $x,y\in S_{\text{down}}$,

$$\big(\mathbf{C}(\mathcal{P},k)\big)_{x_{\text{down}},y_{\text{down}}} \ = \ \mathcal{E}(\mathcal{P}-S-x)\big(\mathbf{C}(\mathcal{P}',k)\big)_{x_{\text{down}},y_{\text{down}}}.$$

Since (Hyp) is a property that is preserved by restricting to principal submatrices, we have that $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp) if $\mathbf{C}(\mathcal{P}',k)$ also satisfies (Hyp). Also note that $\mathrm{N}(\mathcal{P}',k-1)>0$, $\mathrm{N}(\mathcal{P}',k)>0$. Thus by the same argument as in case (1), it follows that $\mathbf{C}(\mathcal{P}',k)$ satisfies (Hyp), which in turn implies that $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp).

Finally, for case (2c), let $\mathrm{N}(\mathcal{P},k-1)>0$ while $\mathrm{N}(\mathcal{P},k+1)=\mathrm{N}(\mathcal{P},k)=0$. This case follows by applying the same argument as in case (2b) to the dual poset $\mathcal{P}^{\text{op}}$ in (14.1). This completes the proof. $\square$

Proof of Theorem 1.35. It follows from Proposition 14.9 that the matrix $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp), and it follows from (Cfg) that the theorem is a special case of $\mathbf{C}(\mathcal{P},k)$ satisfying (Hyp). This completes the proof. $\square$

14.7. **Example of a combinatorial atlas.** Let $\mathcal{P}$ be the poset on $X = \{a,b,c,d,z\}$, with the order given by $a \prec b \prec c$, $a \prec z$, $d \prec c$. Fix $z \in \mathcal{P}$ as in Stanley's inequality, and with uniform weight on all linear extensions. Let $k = 3$. Then the matrices $\mathbf{C}(\mathcal{P},k)$ and $\mathbf{C}(\mathcal{P},k+1)$ are given by

$$\mathbf{C}(\mathcal{P},3) \ = \ \begin{pmatrix} 1&0&0&1&0&0&2&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 2&0&1&0&0&0&2&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \qquad \mathbf{C}(\mathcal{P},4) \ = \ \begin{pmatrix} 1&1&0&0&0&0&2&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 2&0&0&1&0&0&3&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

where the rows and columns are labeled by $\{a_{\min}, b_{\min}, c_{\min}, d_{\min}, a_{\max}, b_{\max}, c_{\max}, d_{\max}\}$. In this notation, we have:

$$\mathbf{f} \ = \ (1,1,1,1,0,0,0,0)^{\intercal}, \qquad \mathbf{g} \ = \ (0,0,0,0,1,1,1,1)^{\intercal}.$$

Recall that the inner products of these two vectors with the matrix $\mathbf{C}(\mathcal{P},k)$ has the following combinatorial interpretation by (Cfg):

$$\langle \mathbf{f}, \mathbf{C}(\mathcal{P},3)\,\mathbf{f}\rangle \ = \ \mathrm{N}(4), \qquad \langle \mathbf{f}, \mathbf{C}(\mathcal{P},3)\,\mathbf{g}\rangle \ = \ \mathrm{N}(3), \qquad \langle \mathbf{g}, \mathbf{C}(\mathcal{P},3)\,\mathbf{g}\rangle \ = \ \mathrm{N}(2).$$

Stanley's inequality (1.30) is equivalent to

$$\langle \mathbf{f}, \mathbf{C}(\mathcal{P},k)\,\mathbf{g}\rangle^2 \ \geq \ \langle \mathbf{f}, \mathbf{C}(\mathcal{P},k)\,\mathbf{f}\rangle \cdot \langle \mathbf{g}, \mathbf{C}(\mathcal{P},k)\,\mathbf{g}\rangle. \tag{14.5}$$

In this example, we have:

$$\mathrm{N}(4) \ = \ 3, \qquad \mathrm{N}(3) \ = \ 3, \qquad \mathrm{N}(2) \ = \ 2,$$

and the log-concavity in Stanley's inequality holds: $3^2 \geq 3 \times 2$.

14.8. **Posets with belts.** Let $\mathcal{P} = (X,\prec)$ be a poset with $|X| = n$ elements. We say that $\mathcal{P}$ has *belt* at $z \in X$ if $\mathrm{inc}(z)$ is either empty or a chain in $\mathcal{P}$. Note that $\mathrm{width}(\mathcal{P}) = 2$ if and only if $\mathcal{P}$ has a belt at every element $z \in X$. Below we show how to strengthen Theorem 1.35 for posets with belts.

Let $\omega : X \to \mathbb{R}_{>0}$ be an order-reversing weight function defined by (Rev), and fix element $z \in X$. Rather than use multiplicative formula (1.31) to extend ω to $\mathcal{E}$, we define $\omega : \mathcal{E} \to \mathbb{R}_{>0}$ by the *tropical formula*:

$$\mathrm{q}(L) \ := \ \max\bigl\{\omega(x) \, : \, L(x) < L(z)\bigr\}. \tag{14.6}$$

Theorem 14.10 (*Tropical Stanley inequality for posets with belts*)**.** *Let $\mathcal{P} = (X,\prec)$ be a poset with $|X| = n$ elements, and suppose $\mathcal{P}$ has a belt at $z \in X$. Let $\omega : X \to \mathbb{R}_{>0}$ be a positive order-reversing weight function. Define $\mathrm{q} : \mathcal{E} \to \mathbb{R}_{>0}$ by the tropical formula (14.6). Then, for every $1 < k < n$, we have:*

$$\mathrm{N_q}(k)^2 \ \geq \ \mathrm{N_q}(k-1) \cdot \mathrm{N_q}(k+1), \tag{14.7}$$

where $\mathrm{N_q}(k)$ is defined by (1.32).

More generally, let $\omega : \mathrm{Low}(\mathcal{P}) \to \mathbb{R}_{>0}$ be a weight function on the set of *lower ideals* of the poset $\mathcal{P}$. Suppose ω satisfies the following (*submodular property*):

$$\omega\bigl(S+x+y\bigr) \cdot \omega(S) \ \leq \ \omega\bigl(S+x\bigr)^2, \tag{Submod}$$

for all $x, y \in \mathrm{inc}(z)$, $x \prec y$, and for all $S \subset X$ such that $S, S+x, S+x+y \in \mathrm{Low}(\mathcal{P})$. We can then define

$$\mathrm{q}(L) \ := \ \omega(A), \quad \text{where} \quad A := \bigl\{x \in X \, : \, L(x) \prec L(z)\bigr\}. \tag{14.8}$$

Theorem 14.11 (*Submodular Stanley inequality for posets with belts*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements, and suppose $\mathcal{P}$ has a belt at $z \in X$. Let $\omega : \mathrm{Low}(\mathcal{P}) \to \mathbb{R}_{>0}$ be a positive weight function on the set of lower ideals of $\mathcal{P}$ which satisfies* (Submod)*. Define $\mathrm{q} : \mathcal{E} \to \mathbb{R}_{>0}$ by* (14.8)*. Then, for every $1 < k < n$, we have:*

$$\mathrm{N_q}(k)^2 \ \geq \ \mathrm{N_q}(k-1) \cdot \mathrm{N_q}(k+1), \quad \textit{where} \quad \mathrm{N_q}(m) := \sum_{L \in \mathcal{E}_m} \mathrm{q}(L), \quad \textit{for all} \ \ 1 \leq m \leq n. \tag{14.9}$$

Proof of Theorem 14.11. The result follows the same argument as Theorem 1.35 with two changes. First, in the proof of Lemma 14.8, the case (C1) does not need to be verified since $\mathcal{P}$ has a belt. Second, the case (C2) is instead verified through (Submod). We omit the details. □

Proof of Theorem 14.10. The result is a direct consequence of Theorem 14.11, as the tropical weight function in (14.6) clearly satisfies (Submod). The details are straightforward. □

15. Proof of equality conditions for linear extensions

In this section we extend and prove Theorem 1.40, see also §16.22.

15.1. **More equality.** Let $\mathcal{P} = (X_P, \prec)$ be a poset on $|X_P| = n$ elements, and let $\omega : X \to \mathbb{R}_{>0}$ be the order-reversing weight. Recall that in the notation of this section, $\mathrm{N}_\omega(\mathcal{P}, k) = \mathrm{N}_\omega(k)$, with the latter as defined in (1.32).

We add two more items to Theorem 1.40 and reformulate it in terms of words, to prove a stronger result:

Theorem 15.1 (cf. Theorem 1.40)**.** *Let $\mathcal{P} = (X_P, \prec)$ be a poset with $|X_P| = n$ elements, and let $\omega : X_P \to \mathbb{R}_{>0}$ be a positive order-reversing weight function. Fix element $z \in X_P$. Suppose that $\mathrm{N}_\omega(\mathcal{P}, k) > 0$. Then the following are equivalent:*

(a) $\mathrm{N}_\omega(\mathcal{P}, k)^2 \ = \ \mathrm{N}_\omega(\mathcal{P}, k-1) \cdot \mathrm{N}_\omega(\mathcal{P}, k+1)$,

(b) *there exists* $\mathrm{s} = \mathrm{s}(k, z) > 0$, *s.t.*
$$\mathrm{N}_\omega(\mathcal{P}, k+1) \ = \ \mathrm{s}\, \mathrm{N}_\omega(\mathcal{P}, k) \ = \ \mathrm{s}^2\, \mathrm{N}_\omega(\mathcal{P}, k-1),$$

(c) *there exists* $\mathrm{s} = \mathrm{s}(k, z) > 0$, *s.t.*
$$\mathrm{N}_\omega(\mathcal{P} - S - T, 3) \ = \ \mathrm{s}\, \mathrm{N}_\omega(\mathcal{P} - S - T, 2) \ = \ \mathrm{s}^2\, \mathrm{N}_\omega(\mathcal{P} - S - T, 1)$$
for every lower set S and upper set T of $\mathcal{P} - z$ satisfying $|S| = k-2$, $|T| = n-k-1$.

(d) *there exists $\mathrm{s} = \mathrm{s}(k, z) > 0$, s.t. $\omega(x_{k-1}) = \omega(x_{k+1}) = \mathrm{s}$, and for every $\gamma = x_1 \cdots x_n \in \mathcal{E}_k$, we have $z \,||\, x_{k-1}$, $z \,||\, x_{k+1}$.*

(e) *there exists $\mathrm{s} = \mathrm{s}(k, z) > 0$, s.t. $\omega(x_{k-1}) = \omega(x_{k+1}) = \mathrm{s}$, $f(x) > k$ for all $x \succ z$, and $g(x) > n - k + 1$ for all $x \prec z$,*

The direction (b) ⇒ (a) is trivial. For the direction (c) ⇒ (b), note that we have

$$\mathrm{N}_\omega(k) \ = \ \sum_{S,T} \omega(S)\, |\, \mathcal{E}(S)|\, |\, \mathcal{E}(T)|\, \mathrm{N}_\omega(\mathcal{P} - S - T, 2), \tag{15.1}$$

summed over all lower sets S and upper sets T of $\mathcal{P} - z$ satisfying $|S| = k-2$, $|T| = n-k-1$. Note that the analogous formulas also hold for $\mathrm{N}_\omega(k \pm 1)$. Together with (c), this implies that

$$\mathrm{N}_\omega(k) \ \leq \ \mathrm{s}\, \mathrm{N}_\omega(k-1), \qquad \mathrm{N}_\omega(k) \ \leq \ \frac{1}{\mathrm{s}}\, \mathrm{N}_\omega(k+1). \tag{15.2}$$

This in turn implies that $\mathrm{N}_\omega(k)^2 \leq \mathrm{N}_\omega(k-1)\, \mathrm{N}_\omega(k+1)$. On the other hand, by Theorem 1.35 we already know the inequality in the opposite direction: $\mathrm{N}_\omega(k)^2 \geq \mathrm{N}_\omega(k-1)\, \mathrm{N}_\omega(k+1)$. This implies the equality in (15.2), which in turn implies (b), as desired.

Below we prove (a) ⇒ (c), (c) ⇔ (d), and (d) ⇔ (e), thus completing the proof of Theorem 15.1.

15.2. **Proof of** (a) $\Rightarrow$ (c)**.** We start with the following preliminary result.

Lemma 15.2. *Let $\mathcal{P} = (X_P, \prec)$ be a finite poset, and let $\omega : X \to \mathbb{R}_{>0}$ be an order-reversing weight function, and let $k \in \{3, \ldots, |X_P| - 1\}$. Suppose that there exists $\mathrm{s} > 0$, such that*

$$\mathrm{N}_\omega(\mathcal{P}, k+1) \ = \ \mathrm{s}\, \mathrm{N}_\omega(\mathcal{P}, k) \ = \ \mathrm{s}^2\, \mathrm{N}_\omega(\mathcal{P}, k-1) \ > \ 0.$$

Then, for every $x \in \min(\mathcal{P})$,

$$\mathrm{N}_\omega(\mathcal{P} - x, k) \ = \ \mathrm{s}\, \mathrm{N}_\omega(\mathcal{P} - x, k-1) \ = \ \mathrm{s}^2\, \mathrm{N}_\omega(\mathcal{P} - x, k-2) \ > \ 0. \tag{15.3}$$

Proof. Let $\mathbb{A}(\mathcal{P}, k)$ be the combinatorial atlas defined in §14.2. Let $\mathbf{f}, \mathbf{g} \in \mathbb{R}^d$ be the characteristic vectors of Z_{down} and Z_{up}, respectively. Clearly, $\mathbf{f}, \mathbf{g}$ is a global pair for $\mathbb{A}$, i.e., they satisfy (Glob-Pos). This allows us to apply Theorem 7.1 in the reductions below.

Let $v = t = 1 \in \Omega^1$. It then follows from the assumptions of the lemma and (Cfg) that the vertex v satisfies (s-Equ). Also note that v satisfies (Hyp) by Proposition 14.9. On the other hand, it is straightforward to verify that v is a functional vertex of Γ, i.e. it satisfies (Glob-Proj) and (h-Glob). By Theorem 7.1, every functional target of v also satisfies (s-Equ) with the same $\mathrm{s} > 0$. On the other hand, it is easy to see that the functional targets of v include vertices of the form $x \in \Omega^0$, where $x = x_{\text{down}} \in \min(\mathcal{P}, \text{down})$. Hence x satisfies (s-Equ), which implies that

$$\langle \mathbf{f}, \mathbf{C}(\mathcal{P} - x, k-1)\, \mathbf{f} \rangle \ = \ \mathrm{s}\, \langle \mathbf{f}, \mathbf{C}(\mathcal{P} - x, k-1)\, \mathbf{g} \rangle \ = \ \mathrm{s}^2\, \langle \mathbf{g}, \mathbf{C}((\mathcal{P} - x, k-1)\, \mathbf{g} \rangle.$$

It now follows from (Cfg) that

$$\mathrm{N}_\omega(\mathcal{P} - x, k) \ = \ s\, \mathrm{N}_\omega(\mathcal{P} - x, k-1) \ = \ s^2\, \mathrm{N}_\omega(\mathcal{P} - x, k-2).$$

Also note that $\mathrm{N}_\omega(\mathcal{P} - x, k-1) > 0$ by Lemma 14.1. The proof is now complete. $\square$

We can now prove (a) $\Rightarrow$ (c). Since $\mathrm{N}_\omega(\mathcal{P}, k) > 0$, it follows from (a) that

$$\mathrm{N}_\omega(\mathcal{P}, k+1) \ = \ \mathrm{s}\, \mathrm{N}_\omega(\mathcal{P}, k) \ = \ \mathrm{s}^2\, \mathrm{N}_\omega(\mathcal{P}, k-1) \ > \ 0 \qquad \text{for} \qquad \mathrm{s} := \frac{\mathrm{N}_\omega(k+1)}{\mathrm{N}_\omega(k)} > 0.$$

By applying Lemma 15.2 for $k-2$ many times, we have

$$\mathrm{N}_\omega(\mathcal{P} - S, 3) \ = \ \mathrm{s}\, \mathrm{N}_\omega(\mathcal{P} - S, 2) \ = \ \mathrm{s}^2\, \mathrm{N}_\omega(\mathcal{P} - S, 1) \ > \ 0, \tag{15.4}$$

for every lower set S of $\mathcal{P} - z$ satisfing $|S| = k-2$. Recall that $n := |X_P|$. Now note that, by applying (14.1), we have that (15.4) is equivalent to

$$\mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}} - S, n-k) \ = \ \mathrm{s}\, \mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}} - S, n-k+1) \ = \ \mathrm{s}^2\, \mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}} - S, n-k+2) \ > \ 0. \tag{15.5}$$

By applying Lemma 15.2 for another $n-k-1$ many times, (15.5) is equivalent to

$$\mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}} - S - T, 1) \ = \ \mathrm{s}\, \mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}} - S - T, 2) \ = \ \mathrm{s}^2\, \mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}} - S - T, 3) \ > \ 0, \tag{15.6}$$

for every upper set T of $\mathcal{P} - z$ satisfing $|T| = n-k-1$. Finally, by applying (14.1) again, it follows that (15.6) is equivalent to

$$\mathrm{N}_\omega(\mathcal{P} - S - T, 3) \ = \ \mathrm{s}\, \mathrm{N}_\omega(\mathcal{P} - S - T, 2) \ = \ \mathrm{s}^2\, \mathrm{N}_\omega(\mathcal{P} - S - T, 1) \ > \ 0,$$

and the proof is now complete. $\square$

15.3. **Proof of** (c) $\Rightarrow$ (d)**.** Let $S := \{x_1, \ldots, x_{k-2}\}$ and $T := \{x_{k+2}, \ldots, x_n\}$. Let $\mathbf{f}, \mathbf{g} \in \mathbb{R}^d$ be the characteristic vectors of Z_{down} and Z_{up}, respectively. It follows from (Cfg) and (c) that

$$\langle \mathbf{f}, \mathbf{C}(\mathcal{P} - S - T, 2)\, \mathbf{v} \rangle \ = \ \mathrm{s}\, \langle \mathbf{v}, \mathbf{C}(\mathcal{P} - S - T, 2)\, \mathbf{v} \rangle \ = \ \mathrm{s}^2\, \langle \mathbf{w}, \mathbf{C}(\mathcal{P} - S - T, 2)\, \mathbf{w} \rangle. \tag{15.7}$$

for some $\mathrm{s} > 0$.

Let $\mathbf{z} := \mathbf{f} - \mathrm{s}\,\mathbf{g}$. It follows from (15.7) that $\langle \mathbf{z}, \mathbf{C}(\mathcal{P} - S - T, 2)\, \mathbf{z} \rangle = 0$. By Lemma 7.2, this implies $\mathbf{C}(\mathcal{P} - S - T, 2)\ \mathbf{z} \ = \ \mathbf{0}$. On the other hand, the matrix $\mathbf{C}(\mathcal{P} - S - T, 2)$ is one of the seven matrices in (C1)–(C7) because $\mathcal{P} - S - T$ is a poset with three elements. From the seven matrices, only (C1) and (C2) can have $\mathbf{C}(\mathcal{P} - S - T, 2)\, \mathbf{z} = \mathbf{0}$. Now note that in both cases we have $x \,||\, z$ and $y \,||\, z$. By a direct calculation, in both cases we have:

$$\mathbf{C}(\mathcal{P} - S - T, 2)\, \mathbf{z} = \mathbf{0} \quad \Longleftrightarrow \quad \mathrm{s} = \omega(x) = \omega(y).$$

This proves (d), as desired. □

15.4. **Proof of (d) ⇒ (c).** It follows from (d), that, for every lower set S and upper set T of $\mathcal{P} - z$ satisfying $|S| = k-2$, $|T| = n-k-1.$, we have:

$$\mathrm{N}_\omega(\mathcal{P}-S-T,2) \ \leq\ \mathrm{s}\,\mathrm{N}_\omega(\mathcal{P}-S-T,1) \qquad \text{and} \qquad \mathrm{N}_\omega(\mathcal{P}-S-T,2) \ \leq\ \frac{1}{\mathrm{s}}\,\mathrm{N}_\omega(\mathcal{P}-S-T,3), \tag{15.8}$$

where $\mathrm{s} > 0$ is given in (d). Summing over all such S, T as in (15.1), we obtain:

$$\mathrm{N}_\omega(k) \ \leq\ \mathrm{s}\,\mathrm{N}_\omega(k-1), \qquad \mathrm{N}_\omega(k) \ \leq\ \frac{1}{\mathrm{s}}\,\mathrm{N}_\omega(k+1). \tag{15.9}$$

This implies that $\mathrm{N}_\omega(k)^2 \leq \mathrm{N}_\omega(k-1)\,\mathrm{N}_\omega(k+1)$. On the other hand, by Theorem 1.35 we already know the inequality in the opposite direction: $\mathrm{N}_\omega(k)^2 \geq \mathrm{N}_\omega(k-1)\,\mathrm{N}_\omega(k+1)$. This implies the equality in (15.9), which in turn implies the equality in (15.8), as desired. □

15.5. **Proof of (d) ⇔ (e).** Note that both items have the same weight function assumption, which reduces the claim to the following lemma of independent interest.

Lemma 15.3. *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements. Fix element $z \in X$ and suppose that $\mathrm{N}(k) > 0$. Then the following are equivalent:*

(i) *$f(y) > k$ for all $y \succ z$, and $g(y) > n-k+1$ for all $y \prec z$.*
(ii) *for every $\gamma = x_1 \cdots x_n \in \mathcal{E}_k$, we have $z \,||\, x_{k-1}$ and $z \,||\, x_{k+1}$.*

Proof. We first prove (i) ⇒ (ii). Suppose to the contrary that z is comparable to $y := x_{k+1}$. Then $z \prec y$, and it follows from (i) that $f(y) > k$. This implies that there are at least $(k+1)$ elements in γ that appear before y, contradicting the assumption that $y = x_{k+1}$. An analogous argument shows that z is incomparable to x_{k-1}.

We now prove (ii) ⇒ (i). Let $y \in X$ be such that $z \prec y$, and suppose to the contrary that $f(y) \leq k$. Let $Q \subseteq X$ be given by

$$Q \ :=\ \{x \in X \,:\, x \prec z,\, x \prec y\}.$$

Note that $|Q| \leq f(y) - 1 \leq k-1$, and that Q is a lower ideal of $\mathcal{P}$. Let $R \subseteq X$ be given by

$$R \ :=\ \big\{x \in X \,:\, x \prec z \ \text{or}\ x \,||\, z\big\}.$$

Note that R is a lower ideal of $\mathcal{P}$, that $z, y \notin R$, and that $Q \subseteq R$. Also note $|R| = n - g(z) - 1$. Since $g(z) \leq n-k$ by the assumption that $\mathrm{N}(k) > 0$, it follows that $|R| \geq k-1$.

We conclude that there exists a lower ideal U of $\mathcal{P}$ such that $Q \subseteq U \subseteq R$ and $|U| = k-1$. This in turn implies that there exists a linear extension $\gamma = x_1 \cdots x_n \in \mathcal{E}$, such that

$$U \ =\ \{x_1, \ldots, x_{k-1}\}, \quad x_k \ =\ z, \quad x_{k+1} \ =\ y.$$

It then follows from (ii) that z and y are incomparable, and we get a contradiction. The same argument shows that $g(y) > n-k+1$ for all $y \prec z$. This completes the proof of the lemma. □

16. Historical remarks

16.1. Unimodality is surprisingly difficult to establish even in some classical cases. For example, Sylvester in 1878 famously resolved Cayley's 1856 conjecture on unimodality of q-binomial coefficients $\binom{n}{k}_q$ using representations of $\mathrm{SL}(2,\mathbb{C})$, see [Sylv]. In 1982, a linear algebraic deconstruction was obtained by Proctor [Pro82]. The first purely combinatorial proof was obtained O'Hara's [O'H90] only in 1990, while the strict unimodality for $k, n-k \geq 8$ was proved in 2013, by the second author and Panova [PP13].

Log-concavity is an even harder property to establish. Over the years, a number of tools and techniques for log-concavity were found, across many areas of mathematics and applications, from elementary combinatorial to analytic, from Lie theoretic to topological. As Huh points out in [Huh18], sometimes there is only one known approach to the problem. We refer to surveys [Bre89, Bre94, Sta89] for an overview of classical unimodality and log-concavity results, to [Brä15] for a more recent overview emphasizing enumerative results and analytic methods, and to [SW14] for a survey on the role of log-concavity in analysis and probability.

16.2. Mason's matroid log-concavity conjectures were stated in [Mas72], motivated by the earlier work and conjectures in graph theory and combinatorial geometry. Many more similar and related conjectures were stated over the years. Some of them became famous quickly, and some were proved quickly, see e.g. a celebrated paper by Heilmann and Lieb [HL72] on log-concavity of the *matching polynomial* for a graph. On the other hand, *Rota's unimodality conjecture* was mentioned in passing in [Rota70], reiterated in [RH71, p. 209], generalized to log-concavity by Mason and Welsh, and proved only recently (Theorem 1.1). We refer to [Oxl92, §14.2] for a detailed overview of the early work on the subject.

16.3. In modern times, the algebraic approach was pioneered by Stanley, who used the *hard Lefschetz theorem* to establish the *Sperner property* of certain families of posets [Sta80b]. This easily implied the *Erdős–Moser conjecture* and laid ground for many recent developments. In fact, Stanley's approach was itself a rethinking of Sylvester's proof we mentioned above, see [Sta80a], and it was later deconstructed in [Pro82].

In the past decade, Huh and coauthors pushed the algebraic approach to resolve several conjectures which remained open for decades. They established the hard Lefschetz theorem and the Hodge–Riemann relations in a number of algebraic settings, which imply the log-concavity results. We will not attempt to review this work largely because it is thoroughly surveyed in Huh's ICM survey [Huh18]. Below is a quick recap of results used directly in this paper.

16.4. Matroids are often associated with several important sequences, including the f-vector whose components are the numbers $\mathrm{I}(k)$, and the h-vector, which can be computed by a certain linear transformation of the f-vector. Both are coefficients of specializations of the *Tutte polynomial* associated with the matroid. We refer to [Bry82, BO92] for the introduction and further references.

16.5. In their celebrated paper [AHK18], Adiprasito, Huh and Katz proved the log-concavity of the *characteristic polynomial* of a matroid, which is a generalization of the graph chromatic polynomial, and a specialization of the Tutte polynomial. They deduce the Welsh–Mason Conjecture (1.1) indirectly, via an observation by Brylawski [Bry77] (see also [Lenz13]). This culminated a series of previous papers [Huh12, Huh15, HK12] on the subject (see also [AS16]).

The inequality (1.3) is the strongest of the *Mason's conjectures* [Mas72]. This inequality was recently proved independently by Brändén and Huh [BH18, BH20], and by Anari et. al [ALOV18] in the third paper of the series. These papers use interrelated ideas, and avoid much of the algebraic technology in [AHK18]. Let us mention a notable application in [ALOV19] which proved that the base exchange random walk mixes in polynomial time. This was yet another long standing open problem in the area [FM92].

16.6. Brylawski [Bry82, §6] and Dawson [Daw84, Conj. 2.5] conjectured that matroid h-vectors are log-concave. This was resolved in [ADH20] and [BST20]. The latter paper proves a stronger version of log-concavity, while the former proves further results for the *no broken circuit* (NBC) complex, another popular matroid construction, see [Bry77].

For how log-concavity of h-vectors implies log-concavity of f-vectors, see e.g. [Bre94, Cor. 8.4], [Bry82, Prop. 6.13], and [Daw84, Prop. 2.7][17]. As we mentioned in the Introduction (see §1.4), Lenz [Lenz11] showed that log-concavity of the h-vector implies *strict log-concavity* of the f-vector. See also [DKK12] for many low-dimensional examples.

16.7. The matroid in Example 1.12 is a special case of a matroid *realizable* over $\mathbb{F}_q$, see e.g. [Oxl92, §6.5]. In Example 1.14, we consider a subclass of *paving matroids* defined as matroids $\mathcal{M}$ with $\mathrm{girth}(\mathcal{M}) = \mathrm{rk}(\mathcal{M})$, see [Wel76]. Our construction of Steiner matroids follows [Jer06, Kahn80]. Notably, Jerrum considers matroid corresponding to $\mathrm{Stn}(5, 8, 24)$. We refer to [Dem68] for more on finite geometries arising in this example.

16.8. Theorem 1.15 for morphism of matroids is proved by Eur and Huh in [EH20], by extending the approach in [BH20]. The notion of the morphisms is quite elegant, and follows a long series of combinatorial papers of Las Vergnas on the subject, which includes a definition of the Tutte polynomial in this case. We refer to [EH20] for an overview and many references, and to [Chm21] for the extensive survey of generalizations of the Tutte polynomial to general topological embeddings.

16.9. Discrete polymatroids are also called *integral polymatroids* in [Edm70], and appear in the context of discrete convex sets [Mur03] and integral generalized permutohedra [Pos09]. We refer to [HH02] for their history and algebraic motivation. Note that discrete polymatroids are explicitly treated in [Mur03, §4.1] and [BH20] under the equivalent formulation of *M-convex sets*. They are a part the definition of Lorentzian polynomials, so in fact weighted polymatroids and Lorentzian polynomials are closely related notions.[18] Although Theorem 1.20 is not stated in this form, it follows easily from the results in [BH20]. Indeed, we need Theorem 3.10 combined with

[17] There is an unfortunate typo in the statement of the Dawson's proposition.

[18] The *completely log-concave polynomials* considered in [ALOV18] are not necessarily homogeneous and thus more general; they coincide with Lorentzian polynomials in the homogeneous case, see [BH20, p. 826].

taking derivatives and limits in proof of Theorem 4.14, and where Theorem 2.10 is substituted with Theorem 2.30 (all in [BH20]). The details are straightforward.

16.10. We refer to [BKP20, §14] and [Pos09, §12], for the background on hypergraphical polymatroids in Example 1.22, and further references. Note that there are many notions of "hypertree" and "hyperforest" available in the literature. We refer to [GP14, §10.2] for a quick overview, and to [Ber89] for background on hypergraphs and more traditional definitions.

16.11. The notion of *weight function* originates in statistical physics and is now standard in probability and graph theory. In the context of graph polynomials it comes up in connection to the *Potts model* which is equivalent to the *random cluster model*. We refer to [Sok05] for an extensive introduction, and to [Gri06] for a thorough treatment.

16.12. The equality conditions have long emerged an important counterpart to the inequalities, see e.g. [BB65, HLP52]. They serve as a key check on the inequality: if the equality occurs rarely or never, perhaps there is a way to sharpen the inequality either directly or by introducing additional parameters. *Strict log-concavity* inequalities are especially suggestive of possible quantitative results.

For example, in his pioneer paper [Huh12], Huh proved the log-concavity of the *chromatic polynomial* of a graph, establishing several conjectures going back to [Read68]. In a followup paper [Huh15], Huh proved a strict log-concavity conjecture of Hoggar [Hog74]. There are no explicit stronger bounds implying strict log-concavity in the style of Theorem 1.16 and [BST20].

In the opposite direction, when there are many special cases when the inequality becomes an equality, the equality conditions are unlikely to be very precise. It seems, this is the case of our equality conditions for matroid log-concavity given in §1.9 (see also §1.13). In the context of this paper, the only nontrivial equality condition known prior to this work for matroid inequalities is Theorem 1.8 proved by Murai, Nagaoka and Yazawa in [MNY21] using an algebraic argument built on [BH20].

16.13. Greedoids were defined and heavily studied by Korte and Lovász as set systems on which the *greedy algorithm* provably works, thus the name. They generalize matroids, which in turn generalize graphs, where the greedy algorithm is classically defined to compute the *minimal spanning tree* (MST). For general greedoids, the reader should think of the (greedy) *Prim's algorithm* for the MST in undirected graphs, rather than *Kruskal's algorithm*, as a starting point of the generalization. The approach to greedoids in terms of languages goes back to original papers. We refer to [KLS91] for a foundational monograph on the subject, and to [BZ92] for a relatively short and digestible survey.

16.14. Antimatroids is a subclass of greedoids named after the *anti-exchange property*, which is a key axiom in their definition via set systems [KLS91, §3.1]. There are many examples of antimatroids coming from graph theory (e.g. *branching process*) and discrete geometry (e.g. *shelling process*), although poset antimatroids have a combinatorial nature they also have some geometric aspects (see e.g. [KL13]). Much of the terminology in the area is rather unfortunate and can be somewhat confusing, so we refer the reader to the top of page 335 in [BZ92], which defines classes of greedoids in terms of properties of the corresponding lattices of feasible sets. See Figure 17.1 below for the diagram of relationships between main greedoid classes (see also [KLS91, p. 301] for a larger diagram).

16.15. Standard Young tableaux (see Example 1.27) are fundamental in algebraic combinatorics. They play a key role in representation theory of S_n and $\mathrm{GL}(N,\mathbb{C})$, and the geometry of the Grassmannian, see e.g. [Ful97] and [Sta99, §7]. Numbers $f^{\lambda/\mu} = |\operatorname{SYT}(\lambda/\mu)|$ have an elegant *Aitken–Feit determinant formula* [Ait43, Feit53], see also [Sta99, Cor. 7.16.3]. For the sequence $\{b_k\}$ in Example 1.27, see e.g. [FS09, Ex. VIII.5].

16.16. Enumeration of increasing arborescences (also called *branchings* and *search trees*) in Example 1.32 without graphical constraints is common in enumerative combinatorics, see e.g. [BBL98, FS09]. Maximal arborescences (also called *directed spanning trees*) also appear in connection to the *reachability problem* in network theory, see [BP83, GJ19], and can be sampled by the *loop-erased random walk* and its relatives, see [GP14, Wil96].

16.17. Linear extensions of a finite poset $\mathcal{P}$ are in obvious bijection with maximal chains in the lattice $\mathbb{L}(\mathcal{P})$ of lower order ideals of $\mathcal{P}$. Lattice $\mathbb{L}(\mathcal{P})$ is always distributive, and by *Birkhoff's representation theorem* (see e.g. [Sta99, Thm 3.4.1]), every finite distributive lattice can be obtained that way. We refer to [BrW00, Tro95] for definitions and standard results on posets and linear extensions.

16.18. Stanley's inequality (1.30) was originally conjectured by Chung, Fishburn and Graham in [CFG80], extending an earlier unimodality conjecture by R. Rivest (unpublished). The proof in [Sta81] is a simple application of the *Alexandrov–Fenchel inequality*. Until now, no direct combinatorial proof of Stanley's inequality was known in full generality, although [CFG80] gives a simple proof for posets of width two (see also [CPP21]). Most recently, the authors and Panova obtained a q- and multivariate analogues of Stanley's inequality for posets of width two [CPP21]. These notions are specific to the width two case and are incompatible with the weighted analogue (Theorem 1.35) nor the case of posets with belts (Theorem 14.11).

16.19. The connection between linear extensions of two dimensional posets and lower order ideals of Bruhat order used in Example 1.37 has been discovered a number of times in varying degree of generality, see [BW91, FW97] (see also [DP18]). Statistics $\beta : S_n \to \mathbb{N}$ in that example seems different from other permutation statistics which appear in the context of log-concavity, see e.g. [Brä15, Bre89].

Statistic γ on the alternating permutations in Example 1.38 is more classical. Note, however, a major difference: while much of the literature studies permutation statistics as polynomials in $\mathbb{N}[q]$ whose coefficients can sometimes form a log-concave sequence, we study values of these polynomials at fixed $q \in \mathbb{R}$. For more on the Euler and Bernoulli numbers and the connection between them, see e.g. [FS09, §IV.6.1]. For log-concavity of Entringer numbers and their generalizations, see [B+19, GHMY21].

16.20. The Alexandrov–Fenchel inequality is a classical result in convex geometry which remains mysterious despite a number of different proofs, see e.g. [BuZ88, §20] and [Sch14, §7.3]. It generalizes the *Brunn–Minkowski inequality* to *mixed volumes*, and has remarkable applications to the *van der Waerden conjecture*, see e.g. [vL82]. Let us single out one of the original proofs by Alexandrov using polytopes [Ale38], the inspirational (independent) proofs by Khovanskii and Teissier using Hodge theory, see [BuZ88, §27] and [Tei82], a recent concise analytic proof by Cordero-Erausquin et al. [CKMS19], and the proof by Shenfeld and van Handel [SvH19], which partly inspired this paper.

16.21. For geometric inequalities such as the *isoperimetric inequalities*, the equality conditions are classical problems going back to antiquity (see e.g. [BuZ88, Sch14]). In many cases, the equality conditions are equally important and are substantially harder to prove than the original inequalities. For example, in the *Brunn–Minkowski inequality*, the equality conditions are crucially used in the proof of the *Minkowski theorem* on existence of a polytope with given normals and facet volumes (see e.g. §7.7 and §36.1 in [Pak19]). For poset inequalities, the equality conditions are surveyed in [Win86].

16.22. The equality conditions for Stanley's inequality for the case when ω is uniform (Theorem 1.39), were recently obtained by Shenfeld and van Handel in [SvH20, Thm 15.3]. They used a sophisticated geometric analysis to prove equality conditions of the Alexandrov–Fenchel inequality for convex polytopes. We should mention that part (d) of Theorem 15.1 is inspired by our results in [CPP21, §8] for the Kahn–Saks inequality, which in some sense are more general. Finally, the q-analogue of the equality condition was obtained in the same paper [CPP21, Thm 1.5] for posets of width two.

17. Final remarks and open problems

17.1. Unimodality is so natural, sooner or later combinatorialists start seeing it everywhere, generating a flood of conjectures. In the spirit of the "strong law of small numbers" [Guy88], many such conjectures do in fact hold in small examples but fail in larger cases. Sometimes, it takes years of real or CPU time until large counterexamples are found (see e.g. [RR91]), in which case they are published. Notable unimodality disproofs can be found in [Bjö81, Stan90, Ste07], all related to poset inequalities in some way.

Log-concavity is a stronger property than unimodality, but is also more natural. Indeed, in the absence of symmetry there is no natural location of the mode (maximum) of the sequence. While the mode location is critical in establishing unimodality, it is irrelevant for log-concavity. Moreover, as was pointed out in [RT88, p. 38], log-concavity of polynomial coefficients is preserved under multiplication of polynomials, an important property of poset polynomials. Similarly, it was shown in [Lig97] (see also [Gur09]), that ultra-log-concavity is preserved under *convolution*, yet another property of some poset polynomials.

17.2. In his discussion of influence of Rota on matroid theory, Kung writes that Rota was motivated in his unimodality conjecture (see §16.2) in part by the mixed volumes which are "somewhat analogous" to the Whitney numbers, see [Kung95, §3.1]. This seems extremely prescient from the point of view of this paper, as we prove matroid log-concavity with a technology that originates in the "right" proof of the Alexandrov–Fenchel inequality. One could argue that we inadvertently fulfilled Rota's unstated prediction (cf. [AS16]).

17.3. As we mentioned in the introduction, traditionally matroids are viewed as a subclass of lattices, see e.g. [Oxl92, Wel76]. Similarly, greedoids are usually defined by their feasible sets a more general subclass of posets (cf. §16.13). Thus, the title of the paper.

17.4. Our proofs in Section 5 borrows heavily from [SvH19], although they are written in a very different language (see also Remark 5.4). According to the authors, the idea of this proof can be traced back to the work of Lichnerowicz [Lic58], see [SvH19, §6.3] for a further discussion.

The proof of Theorem 7.1 is a modification on the argument in [Ale38], which in turn is based on [Weyl]. In the draft of the paper, we were not aware of the connection and used a similar but longer argument. This simplification was kindly proposed to us by Ramon van Handel (personal communication).

17.5. In the proof of Theorem 1.31 given in §8.4, at a critical step (in the base of induction), we employed *Cauchy's interlacing theorem*. In fact, interlacing of eigenvalues is surprisingly powerful, see e.g. [Hua19, MSS15] for notable recent applications.

17.6. As we mentioned earlier, our proof of Theorem 1.35 is inspired by the approach of Shenfeld and van Handel [SvH19]. Indeed, the mixed volumes in Alexandrov–Fenchel inequality can be converted into inner products in (Hyp), where the vectors are given by the support functions of the polytopes. We present this proof in [CP22a]. Technically, one can object that we assumed the diagonal entries of $\mathbf{M}$ are assumed to be nonnegative. In fact, this assumption is made for convenience as nonnegativity holds in our examples, but allowing M_{ii} to be negative does not change the proof.

Now, it is shown in [SvH19, §5], that the corresponding matrices and vectors for simple, strongly-isomorphic polytopes satisfy all conditions of Theorem 5.2. Note that in that setting (Pull) is always an equality, see [SvH19, Eq. (1.2)] and [SvH20, Eq. (5.23)] for the proof. On the other hand, the inequality (Pull) can be strict in our setting. The comparison between our proof Theorem 15.1 and the proof of Theorem 1.39 in [SvH20] is also curious and we don't fully understand it. We should mention the crucial use of the opposite poset $\mathcal{P}^{\mathrm{op}}$, which does not seem to show up in this context. It would be interesting to find further applications in this "duality" approach (cf. §17.15).

17.7. Although Theorem 1.8 says that there are no interesting examples of equality of log-concavity for matroids, the examples in §1.7 suggest that the family of matroids with equality in Theorem 1.6 is rather rich. While our Theorem 1.9 gives some natural necessary and sufficient conditions, it would be interesting to see if this description can be used to obtain a full classification of such matroids in terms of known classes of set systems.

17.8. Our work is completely independent of the algebraic approach in [AHK18], yet some glimpses of similarity to more recent developments are noticeable if one squints hard enough. For instance, we need the element null in the proof of Theorem 1.31, for roughly the same technical reason that papers [B+20a, B+20b] need to use the *augmented Bergman fan* in place of the (usual) *Bergman fan* employed in [AHK18].

17.9. The connection between our proof and Lorentzian polynomial approach is somewhat indirect to make any formal conclusions. On the one hand, we can use combinatorial atlases to emulate everything Lorentzian polynomial do [CP22a]. On the other hand, the atlas we construct for matroids and polymatroids is sufficiently flexible to allow our refined inequalities. On a technical level, in notation of Section 6, the matrix $\mathbf{K} = (\mathrm{K}_{ij})$ which arises when we emulate Lorentzian polynomials, is always zero (cf. Remark 6.3). Thus, it would be interesting to see if the tools in [ALOV18] and [BH20] can be modified to yield our Theorems 1.6 and 1.21.

17.10. Most recently, Brändén and Leake showed in [BL21] how to obtain the log-concavity of the characteristic polynomial of a matroid using a purely Lorentzian polynomial approach, avoiding the use of algebra altogether. While it is too early to say, we intend to see if the combinatorial atlas technology can be combined with that approach.

17.11. It would be interesting to see if one can derive Theorems 1.35 from the Alexandrov–Fenchel inequality. If this is possible, do the tools in [SvH20] extend to prove Theorem 1.40?

17.12. In the Example 1.5, the asymptotic constant 3/2 is probably far from tight for *dense graphs*, say with $\Omega(\mathrm{N}^2)$ edges. What's the right constant then?

17.13. When it comes to interval greedoids, there are more questions than answers. For example, since there is a Tutte polynomial for greedoids defined in [GM97], does it make sense to define an NBC complex? Are there any log-concavity results for characteristic polynomials in some special cases? Can one define morphism of antimatroids or interval greedoids? Are there any other interesting classes of interval greedoids whose log-concavity is worth studying?

17.14. *Weak local greedoids* introduced in §3.2 by the weak local property (WeakLoc), is a new class of greedoids. It contains poset antimatroids, matroids, discrete polymatroids, and *local poset greedoids*, see Figure 17.1. We do not consider the latter in this paper, but they play an important role in greedoid theory, see [KLS91, Ch. VII]. To understand the relationship between weak local greedoids and local poset greedoids, note the excluded minor characterization of local poset greedoids in [KLS91, Cor. VII.3.2]. By contrast, weak local greedoids exclude the same minor under contraction, but not necessarily deletion.

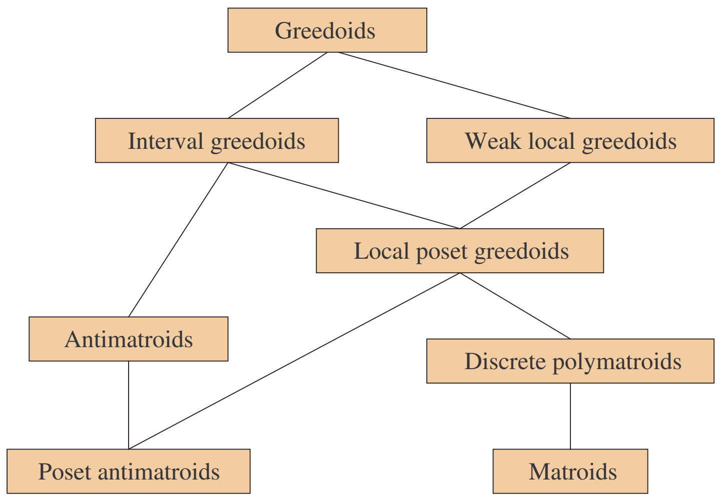


FIGURE 17.1. Diagram of inclusions of greedoid classes.

17.15. Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid, and let $\mathcal{B} \subseteq \mathcal{L}$ be the set of feasible words $\alpha = x_1 \cdots x_\ell$ of maximum length $\ell = \mathrm{rk}(\mathcal{G})$. Denote by $\mathcal{B}^{\mathrm{op}}$ the set of words $\alpha^{\mathrm{op}} := x_\ell \cdots x_1$. An interval greedoid is called *reversible* if $\mathcal{B}^{\mathrm{op}}$ is the set of basis feasible words of an interval greedoid. Note that matroids, polymatroids and poset antimatroids are examples of reversible greedoids.

Let us note that our proof of Stanley's inequality (1.30) can be generalized to reversible interval greedoids. Unfortunately, in the examples above the corresponding generalization of Stanley's inequality is trivial. It would be interesting to characterize reversible greedoids or at least find new interesting examples.

17.16. From the *computational complexity* point of view, one can distinguish "easy inequalities" from "hard inequalities", depending whether the components (or their differences) are computationally easy or hard. For example, *Hoffman's bound* (see e.g. [Big74, Thm 8.8]), relates the *independence number of a graph* which is NP-hard, to the ratio of graph eigenvalues which can be computed in polynomial time. Assuming P $\neq$ NP, one would expect such bound not to be sharp in many natural cases.

By contrast, Alon's lower bound on the number of spanning trees in regular graphs (see [Alon90]) has both sides computable in polynomial time. This suggests that complexity approach may not necessarily capture the mathematical difficulty of the result.

In this context, the inequalities in this paper are the "hardest" of all. For the Mason's conjectures, even in the simplest case of graphical matroids (Example 1.5), the number of k-forests is known to be #P-complete, see e.g. [Wel93]. Similarly, in Stanley's inequality (Theorem 1.34), the number of linear extensions of a poset is #P-complete even for posets of height two or dimension two, see [BrW91, DP18].

17.17. Another computational complexity approach to combinatorial inequalities is to understand whether their difference of two sides is nonnegative for combinatorial reason, i.e. whether it has a combinatorial interpretation. This is a natural question we previously discussed in [Pak19].

For example, observe that both sides in Stanley's inequality (1.30) are #P-functions, i.e., they have a natural combinatorial interpretation. The difference of LHS and RHS is then a function in GapP = #P − #P. Now the problem whether it lies in #P. Although our proof is elementary, this question remains unresolved.

Similarly, in the case of graphical matroid, the equation (1.1) also corresponds to a nonnegative function in GapP. Again, no combinatorial interpretation is known in this case. This is in sharp contrast, e.g., with the Heilmann–Lieb theorem (see §16.2) on log-concavity of the matching polynomial, where a combinatorial interpretation of the difference follows from Krattenthaler's combinatorial proof [Kra96], see also [Pak19]. We intend to return to this problem in the future.[19]

17.18. Going back to the discussion in Foreword §1.1 and Final Remark §17.1 above, it seems, the importance of poset log-concavity conjectures is yet to be settled. Back in 1989, Francesco Brenti wrote in this context:

> *"In this author's opinion, conjectures and open problems in mathematics are not so much interesting and important 'per se' but because they are symptoms that our knowledge is not complete in some area. Their greatest value is not whether they are true or false but that they stimulate and lead us into deeper knowledge."* [Bre89, p. 6]

One can disagree with these sentiments, but speaking for ourselves we certainly owe these conjectures a debt of gratitude, as we find ourselves in the midst of unexplored territory we neither sought nor expected to discover.

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