POSITIVITY OF SCHUBERT COEFFICIENTS

IGOR PAK[★▲] AND COLLEEN ROBICHAUX[★♦]

ABSTRACT. *Schubert coefficients* $c_{u,v}^w$ are structure constants describing multiplication of Schubert polynomials. Deciding positivity of Schubert coefficients is a major open problem in Algebraic Combinatorics. We prove a positive rule for this problem based on two well known assumptions: the Generalized Riemann Hypothesis and the strong derandomization assumption by Miltersen–Vinodchandran.

דרך אמונה בחרתי

I chose the way of faith

Tehillim (Psalms) 119:30

1. INTRODUCTION

In the era of specialization we live in, it is common for experts in one area of mathematics to be unable to understand and evaluate results in another area. Occasionally, things turn for the worse, when one is unable to understand the *definitions*, or even the *notation*. The results then become expressions in a foreign language that take time, effort and dedication to translate.

Most recently, we found ourselves in an unenviable position of having made progress on a major open problem in Algebraic Combinatorics [PR24], yet our results are stated in the language of Computational Complexity that was deep and technical enough to be inaccessible to most people interested in the problem. Given a choice of being understood but unmotivated vs. being motivated and misunderstood, we chose the former.

The main goal of our study is to give a ***combinatorial interpretation for Schubert coefficients***, arguably the most important open problem in Schubert calculus. In [PR24], we analyze the ***Schubert positivity problem***, traditionally a stepping stone towards a combinatorial interpretation and a major open problem in own right. The main result in [PR24] is an upper bound for the complexity class of the Schubert positivity problem. We presented the results in a conventional CS theory style, while the proofs used some technical algebro-geometric arguments.

In this note, we address the Algebraic Combinatorics audience, giving implications of our main result in [PR24] using conventional combinatorics language. More precisely, we show that the Schubert positivity problem has a ***positive rule*** assuming two well known hypotheses. To streamline the presentation, we omit most of the Schubert calculus background that was included in [PR24], as well as the extensive complexity background which can be found, e.g., in [Gol08, Wig19].

2. THE RESULTS

2.1. **Background.** Below we give a brief review of the necessary background. We refer to [Mac91, Man01] for introductory surveys, to [Knu22] for an overview of recent results, to [AF24] for geometric aspects, and to [Pak24, §10] for computational complexity aspects.

Schubert polynomials $\mathfrak{S}_w \in \mathbb{N}[x_1, \ldots, x_n]$ indexed by permutations $w \in S_n$. They are celebrated generalizations of *Schur polynomials* but are not symmetric in general. Schubert polynomials were introduced by Lascoux and Schützenberger in 1982 to represent cohomology classes of Schubert

★ Department of Mathematics, UCLA, Los Angeles, CA 90095, USA. Email: {pak,robichaux}@math.ucla.edu.

▲ Supported by the NSF grant CCF-2007891.

♦ Supported by the NSF MSPRF grant DMS-2302279.

September 19, 2025.

varieties in the complete flag variety, and have been intensely studied from algebraic, combinatorial, geometric, and (more recently) complexity points of view.

Let $\mathfrak{S}_{w_\circ} := x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$, where $w_\circ := (n, n-1, \ldots, 1) \in S_n$ is the *long permutation*. *Schubert polynomials* are defined recursively: $\mathfrak{S}_{ws_i} := \partial_i \mathfrak{S}_w$ for all *descents* $w(i) > w(i+1)$. Here ∂_i is the *divided difference operator* defined as

$$\partial_i F := \frac{F - s_i F}{x_i - x_{i+1}}, \quad \text{where} \quad s_i := (i, i+1) \in S_n$$

is the transposition which acts on $F \in \mathbb{C}[x_1, \ldots, x_n]$ by transposing variables. Schubert polynomials coincide with Schur polynomials for permutations with at most one descent.

It is known that Schubert polynomials have integral coefficients: $[\boldsymbol{x}^\alpha]\mathfrak{S}_w \in \mathbb{N}$ are called *Schubert–Kostka numbers*, which generalize the usual *Kostka numbers*. Famously, Schubert–Kostka numbers have combinatorial interpretations in terms of *pipe dreams* (*RC-graphs*) and *bumpless pipe dreams*.

Schubert polynomials form a basis in the ring of polynomials, with integral structure constants:

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_\infty} c_{u,v}^w \, \mathfrak{S}_w .$$

It is known that $c_{u,v}^w \in \mathbb{N}$ for all $u, v, w \in S_\infty$, as they have both geometric and algebraic meanings, which generalize the number of intersection points of lines, see e.g. [AF24, BSY05]. These integers are called *Schubert coefficients* and generalize the *Littlewood–Richardson* (LR-) *coefficients*.

LR-coefficients have over 20 combinatorial interpretations, see [Pak24, §11.4]. By comparison, Schubert coefficients have two different *signed combinatorial interpretations*, see [Pak24, Prop. 10.2] and [PR25a]. Whether Schubert coefficients have a (unsigned) combinatorial interpretation is a major open problem, see [Sta00, Problem 11] and [Knu22, §1.4].

Schubert positivity is a problem whether Schubert coefficients are positive:

$$\{c_{u,v}^w >^? 0\}. \tag{2.1}$$

This problem was heavily studied and resolved in special cases. Notably, Knutson [Knu01], Purbhoo [Pur06], Billey and Vakil [BV08], and, most recently, St. Dizier and Yong [SY22], gave necessary conditions for Schubert positivity.

2.2. **Positive rule.** In Algebraic Combinatorics, a ***combinatorial interpretation*** is often informally defined as

"manifestly nonnegative formula", see e.g. [Knu22, §1.4]. (2.2)

This notion is best understood in context, especially when compared to standard combinatorial interpretations of Kostka numbers and LR-coefficients.

First, recall that a *closed formula* is best defined as a function computable by an efficient algorithm, see [Wilf82] (cf. [Sta12, §1.1]). Now, the word "formula" in (2.2) is meant to be a summation over combinatorial objects of closed formulas, rather than a stand alone closed formula. In fact, neither Kostka numbers nor LR-coefficients can be computed efficiently, see a discussion in [Pan24, §5.2].

Second, the word "manifestly" in this context is a reference to a sum of positive terms, since a signed summation is not self-evidently positive. Taken together, (2.2) can be rephrased as

"summation over combinatorial objects of positive closed formulas". (2.3)

In [Pak24, §11.4], the first author clarified this definition further as membership in #P, a standard computational complexity class. The meaning of "combinatorial interpretation" is then:

"the number of combinatorial objects, which are poly-time verifiable". (2.4)

Compared to (2.3), this specifies coefficients of the summations to be 1, and restricts the type of combinatorial objects to those whose validity can be verified in polynomial time. Still, definition (2.4) is broad enough to include the numerous combinatorial interpretations of LR-coefficients mentioned above. For example, one can verify in polynomial time if a given combinatorial object is indeed a *LR-tableau*, a *Gelfand–Tsetlin pattern*, a *Berenstein–Zelevinsky triangle*, a *Knutson–Tao puzzle*, etc.

The *positivity* of combinatorial numbers is often viewed in a similar spirit. Occasionally, the positivity can be decided in polynomial time; this is the case of Kostka and LR-coefficients (see e.g. [Pan24, §5.2]). Other times, this is known to be impossible, e.g. the positivity problem for Kronecker coefficients is NP-hard for Kronecker coefficients (ibid.) In many such cases, a ***positive rule*** is used to establish the positivity:

(2.5) *"positivity is given by a combinatorial object, which is poly-time verifiable".*

In other words, if the desired number is positive, then there is a combinatorial object as above, and vice versa. In particular, if the positivity is decidable in polynomial time, this algorithm by itself gives a positive rule (with input as the combinatorial object). In the language of Computational Complexity, this says that the problem is in NP. Clearly, *every* combinatorial object counted by a combinatorial interpretation (2.4) gives a positive rule (2.5). Thus a positive rule can in principle be easier to establish than a combinatorial interpretation.

For example, a single 3-coloring of a graph is a positive rule for of the number of 3-colorings. Similarly, a single LR-tableau is a combinatorial rule for positivity of a LR-coefficient. Let us emphasize that the existence of a such tableau is both a necessary and sufficient condition. On the other hand, following [Pur06, §3.2], a single *winning root game* suffices to show that a given Schubert coefficient is positive. This is only a sufficient condition for positivity, thus not a positive rule. Similarly, the existence of a *permutation array* is a necessary condition for positivity of Schubert coefficients [BV08], thus not a positive rule again.

We conclude by noting that there are nonnegative combinatorial functions with no positive rules (unless polynomial hierarchy PH collapses), see [CP24a, IP22]. Notably, in [IPP24] we showed that non-vanishing problem of the S_n character $\big\{|\chi^\lambda(\mu)| >^? 0\big\}$ does not have a positive rule, implying that the squared character does not have a combinatorial interpretation.

2.3. **Main theorem.** Our main result is a positive rule for Schubert coefficients under two assumptions. This is the first general claim towards the long sought combinatorial interpretation.

Theorem 2.1. *Assuming* GRH *and* MVA*, the Schubert positivity problem* (2.1) *has a positive rule.*

Here the GRH is the *Generalized Riemann Hypothesis* which states that all nontrivial zeros of L-functions $L(s, \chi_k)$ have real part $\frac{1}{2}$. In fact, the *Extended Riemann Hypothesis* (ERH, see e.g. [B+08, §6]), or an even weaker assumption in [Roj03, Thm 2(2)], also suffice.

The MVA is the *Miltersen–Vinodchandran Assumption*, a strong derandomization assumption which implies a collapse of two complexity classes: NP = AM. Formally, MVA states that some language in $\mathsf{NE} \cap \mathsf{coNE}$ requires nondeterministic circuits of size $2^{\Omega(n)}$. This assumption was introduced in [MV05, Thm 1.5] as an interactive proof analogue of the *Impagliazzo–Wigderson Assumption* (IWA), that some problem in E (say, SAT), requires circuits of size $2^{\Omega(n)}$. The IWA is a classic derandomization assumption which implies P = BPP, i.e., that all probabilistic polynomial time algorithms can be made deterministic.

The assumption MVA is best viewed as a substantial strengthening of the $\mathsf{P} \neq \mathsf{NP}$ conjecture, far beyond the *Exponential Time Hypothesis* (ETH). We refer to [Gol08, §8.3, §9.1] and [Wig19, §7.2] for more of these results in the context of computational complexity, and to [MV05, §7] for prior work, stronger assumptions, and an overview of followup results.

Proof. We deduce the result from [PR24] and [MV05]. We showed in [PR24, Lemma 1.10], that a modification of the *lifted formulation* given in [HS17] can be used to show that the Schubert positivity problem reduces to *Parametric Hilbert's Nullstellensatz* (HNP). Here HNP asks if the polynomial system $f_1 = \ldots = f_m = 0$ has a solution over $\overline{\mathbb{C}(y_1, \ldots, y_k)}$, where $f_i \in \mathbb{Z}(y_1, \ldots, y_k)[x_1, \ldots, x_s]$ for all $1 \leq i \leq m$. In [A+25, Thm 1], the authors extend Koiran's celebrated result [Koi96], to show that HNP is in AM assuming GRH. In [MV05, Thm 1.5], the authors show that AM = NP assuming MVA. Thus, the Schubert positivity problem (2.1) is in NP given both assumptions, as desired. □

Remark 2.2. Since this paper was first written, the authors made a substantial progress both extending the result in [PR24, v1] to other root systems (see v2, ibid.), to larger products of Schubert polynomials, and to Gromov–Witten invariants [PRX25]. Most notably, we prove that the Schubert positivity problem is in $\mathsf{AM} \cap \mathsf{coAM}$, i.e., that the *Schubert vanishing problem* also has a positive rule under the same two assumptions [PR25b]. In fact, the MVA assumption can be somewhat weakened, see [PR25b, §1.7] for details.

3. Discussion

3.1. **The meaning of Theorem 2.1.** This is the first positive rule for Schubert positivity in full generality. It is also the first general result in favor of Schubert coefficients having a combinatorial interpretation, which would contradict Conjecture 10.1 in [Pak24]. The theorem by itself gives no indication as to whether Schubert positivity is NP-hard asked in [ARY19, Question 4.3], but if the answer is positive (as we expect), this suggests that (2.1) is NP-complete.[1]

We deduce Theorem 2.1 from a known complexity theoretic result [MV05], a recent breakthrough [A+25], and one of our results in [PR24]. As we explain in [PR24], our result extends verbatim to root systems B and C, but not to D, since the corresponding lifted formulation in the latter case does not satisfy HNP's requirements.[2]

Whether the positive rule we obtain in Theorem 2.1 is especially *combinatorial* is in the eye of the beholder. Roughly, the positive rule one gets from our proof combined with the algorithm in [A+25, Koi96], consists of solutions of the lifted formulation system over multiple primes whose existence is guaranteed by GRH. The variables happen to be matrix entries of matrices which can be interpreted as flags over the finite fields. Now, since one needs random bits to test the resulting polynomial identities, these are provided by a *pseudorandom number generator* constructed in [MV05] using combinatorial tools from hard problems such as SAT or HamiltonCycle, which are believed to have exponential circuit complexity. All in all, it's an involved rule that is deeply combinatorial in its construction.

The use of the GRH assumption is somewhat unfortunate, and possibly avoidable by a more involved number theoretic argument. The use of the MVA assumption may seem surprising but should not, since these complexity assumptions are often left unstated. Indeed, given a *signed* combinatorial interpretation for Schubert coefficients, *every* combinatorial interpretation as in (2.2) becomes trivial from the complexity point of view, if $\mathsf{P} = \mathsf{NP}$, for example. For the same reason one can never prove a "nonexistence of a combinatorial interpretation" result without some kind of complexity assumptions.

One can ask if our rule for Schubert positivity (2.5) should be viewed as an indication towards a combinatorial interpretation (2.4). While we favor a positive answer, there are two negative arguments to keep in mind. First, note that Koiran's algorithm [Koi96] at the heart of our construction is based on Hilbert's Nullstellensatz and does not extend to counting the number of solutions, so a completely different approach is needed.

Second, we now know of a natural problem, the *defect of the strong Mason's inequality* for binary matroids, see [CP24c, §14], whose positivity is NP-complete (see [CP24c, Cor. 15.3]), while the counting is (conjecturally) not in $\#\mathsf{P}$. Since Schubert positivity is (conjecturally) NP-hard (see [PR24, Conj. 1.5]), this argument suggests that the problem of combinatorial interpretations is more intricate than we initially thought.

3.2. **The meaning of derandomization.** Note that GRH is probably the most famous conjecture in all of mathematics, with countless references extolling its powers (see e.g. [B+08]), and there is a universal belief that it holds (cf. [Far22]). By contrast, MVA and the whole area of *Derandomization* may seem unfamiliar and even counterintuitive. While we cannot give it justice in a few paragraphs, let us try nonetheless to give some motivation for this direction. For a proper introduction we refer to lecture notes [Vad11], textbook [Gol08], and a book-length survey [Wig19].

[1] Here and elsewhere we are assuming that permutations are given in their natural presentation.

[2] David Speyer recently announced how our construction can modified to work in type D (personal communication).

On a basic level, derandomization aims to "simulate random bits" by an algorithm which produces a binary sequence that can be used by a given probabilistic algorithm as if it was truly random. While the idea is rather old and practical, theoretical results are both difficult and technical. In a major breakthrough, Nisan–Wigderson (1994) and later Impagliazzo–Wigderson (1997), gave reasonable hardness assumptions to imply that $\mathsf{BPP} = \mathsf{P}$. Later work extended this *hardness-randomness tradeoff* to other points on the hardness spectrum, and to other complexity classes.

Most relevant to this work, Miltersen–Vinodchandran [MV05] generalized this phenomenon to interactive proof systems (of which AM is the first nontrivial class). They extended prior work by Arvind–Köbler (2001) and Klivans–van Melkebeek (2002), which proved the $\mathsf{AM} = \mathsf{NP}$ collapse under stronger assumptions. In the opposite direction, the authors weakened MVA at the expense of making a smaller collapse from AM to a quasipolynomial version of NP, see [MV05, Thm 1.9].

Part of the effort to derandomize AM comes from applications to GraphIsomorphism, after it was shown by Goldreich–Micali–Wigderson (1986), that GraphNonIsomorphism is in AM. Since GraphIsomorphism is trivially in NP, the MVA implies that this problem is in $\mathsf{NP} \cap \mathsf{coNP}$, the class which contains IntegerFactoring. Babai's unconditional quasipolynomial time algorithm (2016) was a major development that opened a possibility of a poly-time algorithm.

A long chain of successes in the area of derandomization led to a large 98% majority belief in the $\mathsf{BPP} = \mathsf{P}$ conjecture, according to the latest poll of experts [Gas19, p. 21]. This is shy of 99% majority that $\mathsf{P} \neq \mathsf{NP}$ but very close to a universal belief. By contrast, only 70% of experts believe that GraphIsomorphism is in P, and only 31% of experts believe that IntegerFactoring is in P. This suggests serious doubts in the $\mathsf{NP} \cap \mathsf{coNP} = \mathsf{P}$ conjecture, and a substantial belief that quantum algorithms are more powerful that deterministic algorithms: $\mathsf{BQP} \neq \mathsf{P}$ (ibid.)

If there is one conclusion to be made from these unscientific polls, it's that there is a widespread belief among complexity theorists that probabilistic algorithms are in fact no more powerful than deterministic algorithms, even if this might take centuries to prove (since MVA, IWA and other derandomization assumptions are stronger than $\mathsf{P} \neq \mathsf{NP}$). This may feel contrary to the obvious and numerous successes of Monte Carlo algorithms in many applications, but suggests we should take these derandomization assumptions very seriously.

4. Two critiques

4.1. **Not interesting.** First, one can argue that the positive rule for Schubert positivity given by the proof above is substantially different from any combinatorial objects that had been studied before, that such notion is not useful in applications and thus not interesting. While this argument can be neither formalized nor refuted, it is still worth addressing for the argument's sake.

The first author argued in [Pak24] that the complexity class $\#\mathsf{P}$ is the right notion to define a "combinatorial interpretation", as encompassing and generalizing all standard examples in the area. We emphasized that without a proper definition one can never argue *against* existence of combinatorial interpretation. We also noted that this "not in $\#\mathsf{P}$" approach had already been fruitful in several interesting cases [CP24a, CP24b, IP22, IPP24]. As we then explain in [PR25a, §2.2(2)], our main result can also be viewed as the opposite direction, as an obstacle to having a proof of this kind.

Curiously, the literature rarely addresses the possibility of a positive but extremely involved solution. One exception is the problem whether a given knot is an *unknot*, see e.g. [Pak24, §4.6]. In this case, a positive rule (NP witness) is a sequence of Reidemeister moves which can be easily verified, but the crux of the argument is that such sequence always exists of polynomial length. Even more impressively, testing whether a knot is *not an unknot* is also in NP. This is proved by a highly sophisticated argument using Gabai's construction of taut foliations as a way of certifying the genus of a knot, see a discussion in [Lac17, §3]. Viewed through the prism of combinatorial interpretations, this construction is highly unintuitive compared to Reidemeister sequences, and at least as technical as our construction.

As we mentioned above, in the area of Enumerative Combinatorics, there is an old tradition of using involved efficient algorithms for a "closed formula" [Wilf82]. See also [Pak18] for an extensive overview

of this approach. Once you cross the bridge from "nice closed formulas" to poly-time algorithms, it is not a big leap to go from "nice combinatorial interpretations" (2.2) to poly-time certificates (2.4). Arguably, this is how one turns art into science.

Finally, the relative lack of progress in the positive direction and the exceedingly cumbersome nature of recent combinatorial interpretations for certain families of Schubert coefficients [KZ17, KZ23], suggest that it is unlikely there is a nice and fully satisfactory combinatorial interpretation, or even a positive rule. If history is the guide, one might want to let go of this dream and consider our more general notions:

> *"Bien des fois déjà on a cru avoir résolu tous les problèmes, ou, tout au moins, avoir fait l'inventaire de ceux qui comportent une solution. Et puis le sens du mot solution s'est élargi, les problèmes insolubles sont devenus les plus intéressants de tous et d'autres problèmes se sont posés auxquels on n'avait pas songé."* (Henri Poincaré, 1913)
>
> "Many times already men have thought that they had solved all the problems, or at least that they had made an inventory of all that admit of solution. And then the meaning of the word solution has been extended; the insoluble problems have become the most interesting of all, and other problems hitherto undreamed of have presented themselves." [Poi14, p. 22]

4.2. **Lack of certainty.** One can also argue that our results are quite weak since they rely on two major unproven hypotheses, each of them stronger than a `Millennium Problem`. This is true, of course, but somewhat misleading since in other areas assuming conjectures is part of the culture.

Notably, there are hundreds of papers in Number Theory and its applications which assume variants of the GRH as well as other standard conjectures: the *ABC conjecture*, the *Bateman–Horn conjecture*, the *Birch and Swinnerton-Dyer* (BSD) *conjecture*, the *Cohen–Lenstra heuristics*, *Schanuel's conjecture*, the *Shafarevich–Tate conjecture*, etc. Sometimes, later arguments manage to remove or weaken these conjectures as an assumption,[3] but all such results are still considered very valuable.

Similarly, in Computational Complexity, Cryptography and related areas of Theoretical Computer Science, a large majority of results use *some* complexity assumptions. In addition to $\mathsf{P} \neq \mathsf{NP}$, $\mathsf{BPP} = \mathsf{P}$, the ETH, and non-collapse of the polynomial hierarchy PH, standard assumptions include the *decisional Diffie–Hellman assumption*, the *existence of one-way functions assumption*, the *learning with errors assumption*, the *small set expansion hypothesis*, the *unique games conjecture*, etc. Despite this apparent lack of certainty, the area learned to survive and prosper decades ago.

Returning to Schubert positivity, it is certainly possible that there is an unconditional positive rule of explicit combinatorial nature. Unfortunately, the known special cases are most definitely not general enough to remain hopeful. Still, we are completely persuaded by Theorem 2.1 that there is *some* (unconditional) positive rule for this problem, despite the uncertainty stemming from conjectures in the assumptions of the theorem.

We believe the time has come for the area to embrace the uncertainty as well. Having to rely on unproven assumptions can be rather uncomfortable, of course, but it is preferable over the alternatives, such as having a blind conviction, or believing in nothing at all. We are reminded of Voltaire's famous dictum made in a theological context:

> *"Le doute n'est pas un état bien agréable, mais l'assurance est un état ridicule."*
> "Doubt is not a pleasant condition, but certainty is an absurd one," Voltaire, 1770.

[3]For example, this is what happened to the "not an unknot" result discussed above, as the original Kuperberg's argument used the GRH in a manner similar to [Koi96]. Later, Agol removed this assumption, see [Lac17, §3].

References

[ARY19] Anshul Adve, Colleen Robichaux and Alexander Yong, Vanishing of Littlewood–Richardson polynomials is in P, *Comput. Complexity* **28** (2019), 241–257.

[A+25] Rida Ait El Manssour, Nikhil Balaji, Klara Nosan, Mahsa Shirmohammadi and James Worrell, A parametric version of the Hilbert Nullstellensatz, in *Proc. 8th SOSA* (2025), 444–451.

[AF24] David Anderson and William Fulton, *Equivariant cohomology in algebraic geometry*, Cambridge Univ. Press, Cambridge, UK, 2024, 446 pp.

[BV08] Sara Billey and Ravi Vakil, Intersections of Schubert varieties and other permutation array schemes, in *Algorithms in algebraic geometry*, Springer, New York, 2008, 21–54.

[B+08] Peter Borwein et al. (Eds.), *The Riemann hypothesis*, Springer, New York, 2008, 533 pp.

[BSY05] Anders S. Buch, Frank Sottile and Alexander Yong, Quiver coefficients are Schubert structure constants, *Math. Res. Lett.* **12** (2005), 567–574.

[CP24a] Swee Hong Chan and Igor Pak, Equality cases of the Alexandrov–Fenchel inequality are not in the polynomial hierarchy, *Forum Math. Pi* **12** (2024), Paper No. e21, 38 pp.

[CP24b] Swee Hong Chan and Igor Pak, Computational complexity of counting coincidences, *Theoret. Comput. Sci.* **1015** (2024), Paper No. 114776, 19 pp.

[CP24c] Swee Hong Chan and Igor Pak, Equality cases of the Stanley–Yan log-concave matroid inequality, preprint (2024), 36 pp.; `arXiv:2407.19608`.

[Far22] David W. Farmer, Currently there are no reasons to doubt the Riemann Hypothesis, preprint (2022), 76 pp.; `arXiv:2211.11671`.

[Gas19] William I. Gasarch, The third $\mathsf{P} =? \mathsf{NP}$ poll, *ACM SIGACT News* **50** (2019), no. 1, 38–59.

[Gol08] Oded Goldreich, *Computational complexity. A conceptual perspective*, Cambridge Univ. Press, Cambridge, UK, 2008, 606 pp.

[HS17] Nickolas Hein and Frank Sottile, A lifted square formulation for certifiable Schubert calculus, *J. Symbolic Comput.* **79** (2017), 594–608.

[IP22] Christian Ikenmeyer and Igor Pak, What is in #P and what is not?, preprint (2022), 82 pp.; extended abstract in *Proc. 63rd FOCS* (2022), 860–871; `arXiv:2204.13149`.

[IPP24] Christian Ikenmeyer, Igor Pak and Greta Panova, Positivity of the symmetric group characters is as hard as the polynomial time hierarchy, *Int. Math. Res. Not.* **2024** (2024), no. 10, 8442–8458.

[Knu01] Allen Knutson, Descent-cycling in Schubert calculus, *Experiment. Math.* **10** (2001), 345–353.

[Knu22] Allen Knutson, Schubert calculus and quiver varieties, in *Proc. ICM* (2022, virtual), Vol. VI, EMS Press, 4582–4605.

[KZ17] Allen Knutson and Paul Zinn-Justin, Schubert puzzles and integrability I: invariant trilinear forms, preprint (2017), 51 pp.; `arXiv:1706.10019`.

[KZ23] Allen Knutson and Paul Zinn-Justin, Schubert puzzles and integrability III: separated descents, preprint (2023), 42 pp.; `arXiv:2306.13855`.

[Koi96] Pascal Koiran, Hilbert's Nullstellensatz is in the polynomial hierarchy, *J. Complexity* **12** (1996), 273–286.

[Lac17] Marc Lackenby, Elementary knot theory, in *Lectures on geometry*, Oxford Univ. Press, Oxford, UK, 2017, 29–64.

[Mac91] Ian G. Macdonald, *Notes on Schubert polynomials*, Publ. LaCIM, UQAM, Montreal, 1991, 116 pp.

[Man01] Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS, Providence, RI, 2001, 167 pp.

[MV05] Peter B. Miltersen and N. Variyam Vinodchandran, Derandomizing Arthur–Merlin games using hitting sets, *Comput. Complexity* **14** (2005), 256–279.

[Pak18] Igor Pak, Complexity problems in enumerative combinatorics, in *Proc. ICM Rio de Janeiro*, Vol. IV, World Sci., Hackensack, NJ, 2018, 3153–3180.

[Pak24] Igor Pak, What is a combinatorial interpretation?, in *Open Problems in Algebraic Combinatorics*, AMS, Providence, RI, 2024, 191–260.

[PR24] Igor Pak and Colleen Robichaux, Vanishing of Schubert coefficients, preprint (2024), `arXiv:2412.02064`; `v1` is 24 pp., `v2` includes Appendix C joint with David E Speyer, 30 pp.; extended abstract in *Proc. 57th STOC* (2025), 1118–1129.

[PR25a] Igor Pak and Colleen Robichaux, Signed combinatorial interpretations in algebraic combinatorics, *Algebraic Combinatorics* **8** (2025), 495–519.

[PR25b] Igor Pak and Colleen Robichaux, Vanishing of Schubert coefficients is in $\mathsf{AM} \cap \mathsf{coAM}$ assuming the GRH, preprint (2025), 18 pp.; to appear in *Forum Math. Sigma*; `arXiv:2504.03004`.

[PRX25] Igor Pak, Colleen Robichaux and Weihong Xu, On vanishing of Gromov–Witten invariants, preprint (2025), 11 pp.; `arXiv:2508.15715`.

[Pan24] Greta Panova, Computational complexity in algebraic combinatorics, in *Current Developments in Mathematics*, Int. Press, Boston, MA, 2024, 241–280.

[Poi14] Henri Poincaré, *Science and method* (translated from French), Nelson, London, 1914, 288 pp.

[Pur06] Kevin Purbhoo, Vanishing and nonvanishing criteria in Schubert calculus, *Int. Math. Res. Notices* **2006** (2006), Art. 24590, 38 pp.

[Roj03] J. Maurice Rojas, Dedekind zeta functions and the complexity of Hilbert's Nullstellensatz, preprint (2003), 16 pp.; `arXiv:math/0301111`.

[SY22] Avery St. Dizier and Alexander Yong, Generalized permutahedra and Schubert calculus, *Arnold Math. J.* **8** (2022), 517–533.

[Sta00] Richard P. Stanley, Positivity problems and conjectures in algebraic combinatorics, in *Mathematics: frontiers and perspectives*, AMS, Providence, RI, 2000, 295–319.

[Sta12] Richard P. Stanley, *Enumerative Combinatorics*, vol. 1 (Second ed.), Cambridge Univ. Press, 2012, 626 pp.

[Vad11] Salil P. Vadhan, Pseudorandomness, *Found. Trends Theor. Comput. Sci.* **7** (2011), 1–336.

[Wig19] Avi Wigderson, *Mathematics and computation*, Princeton Univ. Press, Princeton, NJ, 2019, 418 pp.

[Wilf82] Herbert S. Wilf, What is an answer?, *Amer. Math. Monthly* **89** (1982), no. 5, 289–292.