MIDTERM (MATH 184, WINTER 2018)

Your Name: ____ **(must be in ink)**

UCLA id: ____ **(must be in ink)**

Date: _______________________

The rules:
You MUST simplify completely and BOX all answers with an **INK PEN**.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Points:

1	
2	
3	
4	
5	

Total: **(out of 100)**

Problem 1. (20 points)

Let $m = 5$, $n = 6$. Compute the number of shortest grid walks $(0, 0) \to (m, n)$, such that:
$a)$ they do NOT go through $(2, 3)$ and $(3, 4)$,
$b)$ they make at most two turns.

Problem 2. (20 points)

Let $n = 15$, $k = 5$. Let A be a random subset of $[n] = \{1, 2, \ldots, n\}$ of size k. Let X be the number of odd integers in A. Compute the following:

$a)$ $P(X = 0)$
$b)$ $P(X = 1)$
$c)$ $E[X]$
$d)$ $Var(X)$.

Problem 3. (20 points)

Let $n = 10$. Let σ be a random permutation of $[n] = \{1, 2, \dots, n\}$. Compute the following probabilities:

a) probability that σ has no cycles of length 9.

b) probability that σ has no cycles of length 5.

Problem 4. (20 points)

Recall the coupon collector's story with $n = 50$ coupons.

$a)$ Compute the probability exactly 3 coupons are collected after 5 trials.

$b)$ Supposed after 100 trials all coupons are collected. Compute the expected number of coupons collected at least twice. Same question with exactly twice.

Problem 5. (20 points, 2 points each) **TRUE or FALSE?**

You MUST circle correct answer with INK. No explanation required.

T **F** (1) The number of permutations in S_n with k cycles = that with $(n-k+1)$ cycles

T **F** (2) Recall the "Princess problem" (or "secretary problem" per WP), where one needs to choose one best candidates of the n applicants, $n = 100$. Then there is no way to make the probability of success $\geq 1/3$.

T **F** (3) Same question for the 100 prisoners problem.

T **F** (4) $n! = o(n^n)$

T **F** (5) $2^n = \Theta(3^n)$

T **F** (6) In the 100 rabbits problem, the exact probabilities of k rabbits surviving can be expressed in Stirling numbers of first kind

T **F** (7) In the coupon collector's problem with $n = 50$ coupons the expected time to collect them all is ≥ 200.

T **F** (8) The number of permutations of [15] with one cycle of length 15 is smaller than those with one cycle of length 14.

T **F** (9) $E[X - Y] = E[X] - E[Y]$ for all X, Y random variables

T **F** (10) $E[X \cdot Y] = E[X] \cdot E[Y]$ for all X, Y random variables