HOMEWORK 3: STEENROD SQUARES

(1) One of the properties we didn't prove of the Squares is that $Sq^1 = \beta$, where β is the Bockstein associated to this. This problem will show the fundamental example for this.

(a) Consider the short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$. Let M_* be a (cohomological) DGM in which M_n is a free $\mathbb{Z}$-module for all n. Show that

$$0 \to M_* \otimes \mathbb{Z}/2 \to M_* \otimes \mathbb{Z}/4 \to M_* \otimes \mathbb{Z}/2 \to 0$$

is exact. The associated connecting homomorphism

$$H^*(M \otimes \mathbb{Z}/2) \to H^{*+1}(M \otimes \mathbb{Z}/2)$$

is the Bockstein β.

(b) Show that if M_* is the cellular cochain complex for $\mathbb{R}P^\infty$, then

$$\beta\colon H^1(\mathbb{R}P^\infty; \mathbb{F}_2) \to H^2(\mathbb{R}P^\infty; \mathbb{F}_2)$$

is an isomorphism (hint it suffices to consider the $\mathbb{R}P^2$ case).

(c) Conclude that on one dimensional cohomology classes, $Sq^1 = \beta$.

(2) There is another way to understand the cohomology of $BSU(n)$. If W_n is the Weyl group of $SU(n)$ (the normalizer of a maximal torus modulo the maximal torus), then W_n acts on T, the maximal torus, and it therefore acts on the cohomology. The maximal torus is a product of circles, so the classifying space BT is a product of copies of $\mathbb{C}P^\infty$ (all of this is true for any compact simple Lie group). By naturality, we conclude that there is an action of W_n on

$$H^*(BT) = \mathbb{F}_2[t_1, \dots, t_{n-1}].$$

For $SU(n)$, $W_n = \Sigma_n$, the symmetric group on n letters, acting on $t_1, \dots, t_n$ by permutation ($t_n = -t_1 - \cdots - t_{n-1}$, since we are looking at $SU(n)$, rather than $U(n)$). A theorem of Borel shows that the cohomology of $BSU(n)$ is the Weyl group invariants of the cohomology of BT (so it's the symmetric functions less σ_1).

Using this, compute the action of Sq^1, Sq^2, Sq^3, and Sq^4 on the generators of $H^*(BSU(2))$ and $H^*(BSU(3))$. This shows us that there is a non-trivial $Sq^2\colon H^3(SU(3)) \to H^5(SU(3))$, so the bundle $S^3 = SU(2) \to SU(3) \to S^5$ is non-trivial.

(3) From the previous problem set, we know that there are two fibrations $SU(2) \to G_2 \to S^6$ and $S^3 \to G_2 \to V_2(\mathbb{R}^7)$. It is a deep and subtle question about how the read out the Steenrod action, but use naturality (and the collapse) of these spectral sequences to deduce the action of Sq^1 and Sq^2 on the generators of the cohomology of G_2. From the Adem relation $Sq^3 = Sq^1 Sq^2$, conclude that the square of the 3-dimensional cohomology class is non-zero, and as an algebra, $H^*(G_2; \mathbb{F}_2) = \mathbb{F}_2[x_3, x_5]/x_3^4, x_5^2$.

(4) The Steenrod algebra has many distinguished subalgebras. Let $\mathcal{A}(n)$ denote the subalgebra generated by $Sq^1, \ldots, Sq^{2^n}$. For this problem, write out a basis for $\mathcal{A}(1)$. This algebra can be most easily understood by drawing a picture, in which curved lines represent left multiplication by Sq^2 and straight lines represent left multiplication by Sq^1. Draw the algebra.