LECTURE 22: INNER PRODUCTS

For the remaining lectures, we will work exclusively over $\mathbb{R}$ or $\mathbb{C}$. We'll also look at a very special kind of bilinear form.

Definition 1. An inner product is a function $\langle -, - \rangle \colon V \times V \to \mathbb{F}$ such that

(1) $\langle \vec{v}, \vec{v} \rangle \geq 0$ with equality iff $\vec{v} = \vec{0}$.

(2) $\langle \vec{v}, \vec{u} \rangle = \begin{cases} \langle \vec{u}, \vec{v} \rangle & \mathbb{F} = \mathbb{R} \\ \overline{\langle \vec{u}, \vec{v} \rangle} & \mathbb{F} = \mathbb{C}. \end{cases}$

(3) $\langle a\vec{v} + b\vec{u}, \vec{w} \rangle = a\langle \vec{v}, \vec{w} \rangle + b\langle \vec{u}, \vec{w} \rangle$.

The first condition is that the inner product is "positive definite", the second is that it is (conjugate) symmetric, and the third is that it is linear in the first factor.

Proposition 1. *If $\langle -, - \rangle$ is an inner product, then*

$$\langle \vec{v}, a\vec{u} + b\vec{w} \rangle = \bar{a}\langle \vec{v}, \vec{u} \rangle + \bar{b}\langle \vec{v}, \vec{w} \rangle,$$

and so if $\mathbb{F} = \mathbb{R}$, then an inner product is bilinear.

Proof. Use conjugate symmetry to swap the two coordinates, then use linearity in the first and conjugate symmetry again. □

Over $\mathbb{C}$, this is not bilinearity. The inner product is additive in the second factor, but it has a conjugation that appears in the scalar multiplication. In this case, we call it sesquilinear (the prefix "sesqu-" means "one and a half": it's linear in the first factor and half linear in the second). Of course, we understand complex conjugation completely, so sesquilinearity is as good as bilinearity.

Now some examples.

(1) $\mathbb{R}^n$ with the dot product. This is the prototype of all inner product spaces.

(2) $\mathbb{C}^n$ with the standard inner product:
$$\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle = a_1\bar{b}_1 + \dots + a_n\bar{b}_n.$$

(3) The space $\ell_2 = \{(a_0, \dots) | \sum |a_i|^2 < \infty\}$ with the inner product
$$\langle (a_1, \dots), (b_1, \dots) \rangle = \sum a_i\bar{b}_i.$$
This is the [countably] infinite dimensional analogue of the previous ones.

(4) $C[a, b]$, the set of continuous functions on $[a, b]$ with inner product
$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}w(x)dx,$$
where $w(x) > 0$ for all $x \in [a, b]$. These spaces are essentially all the same (and all the same as ℓ_2, though this is all too analytical to be fun).

(5) If W is a subspace of an inner product space V, then the inner product of V restricted to W gives an inner product on W.

There are several very important consequences of the existence of an inner product. The first is a notion of distance.

Definition 2. The norm of $\vec{v}$ is

$$||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle}.$$

Since the inner product is positive definite, this is always greater than or equal to zero, and moreover, we know that it equals zero iff $\vec{v} = \vec{0}$. We know from sesquilinearity that

$$||a\vec{v}|| = |a| \cdot ||\vec{v}||.$$

This is one of the nice features of sesquilinearity.

We have two more nice relations: the Cauchy-Schwarz inequality (which will link the inner product to a notion of angles) and the triangle inequality (which will ensure that our norm gives a good notion of distance).

Proposition 2 (Cauchy-Schwarz)**.** *For all $\vec{v}$ and $\vec{u}$ in an inner product space V, we have*

$$|\langle \vec{v}, \vec{u} \rangle| \leq ||\vec{v}|| \cdot ||\vec{u}||.$$

Proof. We first note that if $\vec{v}$ or $\vec{u}$ is zero, then the result is obvious. We therefore assume that neither is zero.

We will consider the square of the norm of $\vec{v} - r\vec{u}$ for various r and then deduce the desired result from this.

$$||\vec{v} - r\vec{u}||^2 = \langle \vec{v} - r\vec{u}, \vec{v} - r\vec{u} \rangle = ||\vec{v}||^2 - r\langle \vec{u}, \vec{v} \rangle - \bar{r}(\langle \vec{v}, \vec{u} \rangle - r||\vec{u}||^2) \geq 0.$$

This holds for all r, so in particular, it holds for the value of r that simplifies the expression: $r = \langle \vec{v}, \vec{u} \rangle / ||\vec{u}||^2$. With this, our previous expression becomes

$$0 \leq ||\vec{v}||^2 - \frac{\langle \vec{v}, \vec{u} \rangle}{||\vec{u}||^2} \langle \vec{u}, \vec{v} \rangle = ||\vec{v}||^2 - \frac{|\langle \vec{v}, \vec{u} \rangle|^2}{||\vec{u}||^2},$$

by sesquilinearity. Rearranging gives the result. $\square$

What does this buy us? Let's say that $\vec{v}$ is a unit vector if $||\vec{v}|| = 1$. If $||\vec{v}|| \neq 0$, then we can always divide $\vec{v}$ by its length to get a unit vector. Thus we can restate the previous result as

Proposition 3. *If $\vec{v}$ and $\vec{u}$ are unit vectors, then*

$$|\langle \vec{v}, \vec{u} \rangle| \leq 1.$$

Thus we can find a unique $\theta \in [0, \pi]$ such that

$$\cos(\theta) = \langle \vec{v}, \vec{u} \rangle.$$

Definition 3. The angle between $\vec{v}$ and $\vec{u}$ is that θ such that

$$\langle \vec{v}, \vec{u} \rangle = ||\vec{v}|| ||\vec{u}|| \cos(\theta).$$

With this, we can get a refined triangle inequality.

Proposition 4 (Triangle Inequality)**.** *For all $\vec{v}$ and $\vec{u}$, we have*

$$||\vec{v} + \vec{u}|| \leq ||\vec{v}|| + ||\vec{u}||.$$

Proof. We show the square of the inequality.

$$||\vec{v} + \vec{u}||^2 = \langle \vec{v} + \vec{u}, \vec{v} + \vec{u} \rangle = ||\vec{v}||^2 + ||\vec{u}||^2 + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle.$$

Since $\langle \vec{v}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle$ is twice the real part of $\langle \vec{v}, \vec{u} \rangle$, we know

$$\langle \vec{v}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle \leq 2|\langle \vec{v}, \vec{u} \rangle| \leq 2||\vec{v}|| \cdot ||\vec{u}||.$$

This gives

$$||\vec{v}+\vec{u}||^2 \leq ||\vec{v}||^2 + 2||\vec{v}|| \cdot ||\vec{u}|| + ||\vec{u}||^2 = (||\vec{v}|| + ||\vec{u}||)^2. \qquad \square$$

We can do better, if we are working over $\mathbb{R}$: we can rederive the law of cosines. Recalling that

$$\langle \vec{v}, \vec{u} \rangle = \langle \vec{u}, \vec{v} \rangle,$$

we can simplify the earlier formula and derive the following.

Proposition 5. *For any $\vec{u}$ and $\vec{v}$, we have*

$$||\vec{u}-\vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}|| \cdot ||\vec{v}|| \cos(\theta),$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.