LECTURE 21: SYMMETRIC PRODUCTS AND ALGEBRAS

Symmetric Products

The last few lectures have focused on alternating multilinear functions. This one will focus on symmetric multilinear functions. Recall that a multilinear function $f\colon U^{\times m} \to V$ is symmetric if

$$f(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_j, \dots, \vec{v}_m) = f(\vec{v}_1, \dots, \vec{v}_j, \dots, \vec{v}_i, \dots, \vec{v}_m)$$

for any i and j, and for any vectors $\vec{v}_k$.

Just as with the exterior product, we can get the universal object associated to symmetric multilinear functions by forming various quotients of the tensor powers.

Definition 1. The m^{th} symmetric power of V, denoted $S^m(V)$, is the quotient of $V^{\otimes m}$ by the subspace generated by

$$\vec{v}_1 \otimes \dots \otimes \vec{v}_i \otimes \dots \otimes \vec{v}_j \otimes \dots \otimes \vec{v}_m - \vec{v}_1 \otimes \dots \otimes \vec{v}_j \otimes \dots \otimes \vec{v}_i \otimes \dots \otimes \vec{v}_m$$

where i and j and the vectors $\vec{v}_k$ are arbitrary.

Let $Sym(U^{\times m}, V)$ denote the set of symmetric multilinear functions $U^{\times m}$ to V. The following is immediate from our construction.

Lemma 1. *We have an natural bijection*

$$Sym(U^{\times m}, V) \cong \mathcal{L}(S^m(U), V).$$

We will denote the image of $\vec{v}_1 \otimes \dots \vec{v}_m$ in $S^n(V)$ by $\vec{v}_1 \cdot \dots \cdot \vec{v}_m$, the usual notation for multiplication to remind us that the terms here can be rearranged.

Unlike with the exterior product, it is easy to determine a basis for the symmetric powers.

Theorem 1. *Let* $\{\vec{v}_1, \dots, \vec{v}_m\}$ *be a basis for* V. *Then*

$$\{\vec{v}_{i_1} \cdot \dots \dot{\vec{v}}_{i_n} | i_1 \leq \dots \leq i_n\}$$

is a basis for $S^n(V)$.

Before we prove this, we note a key difference between this and the exterior product. For the exterior product, we have strict inequalities. For the symmetric product, we have non-strict inequalities. This makes a huge difference!

Proof. We first show the set spans. Since $\{\vec{v}_{i_1} \otimes \dots \otimes \vec{v}_{i_n}\}$ with no restrictions on the subscripts form a basis for $V^{\otimes n}$, the image of these elements spans $S^n(V)$ (this is equivalent to surjectivity). In the quotient $S^n(V)$, we can rearrange terms arbitrarily. This lets us rearrange any element of the form $\vec{v}_{i_1} \cdot \dots \cdot \vec{v}_{i_n}$ into one in which the subscripts are in the desired order.

For linear independence, we show that there are linear functionals which are non-zero on a chosen $\vec{v}_{i_1} \cdot \dots \cdot \vec{v}_{i_n}$ and zero on all others. This will show that these are linearly independent, by our usual arguments.

We can define an equivalence relation on the basis for $V^{\otimes n}$ by saying that two vectors are equivalent if their subscripts are just permutations of each other. In other words, two basis vectors are equivalent if they map to the same vector in $S^n(V)$. The equivalence classes form a partition of the basis. It's also clear that a linear functional on $V^{\otimes n}$ (so a multilinear functional) is symmetric if it takes the same value on equivalent vectors. Since linear functionals on $V^{\otimes n}$ are determined by their values on a basis, we can easily define a linear functional on $S^n(V)$ with the require property by considering the linear functional on $V^{\otimes n}$ which takes value 1 on the equivalence class corresponding to our vector and takes value 0 on all other equivalence classes. □

Why couldn't we do this for the wedge? This argument is a great deal simpler. The problem is one of the symmetric group and it's representations. We consider what happens under transpositions, and we use that to determine whether two vectors in the tensor product are to be equivalent. With the symmetric power, we have no worries about a sign (since we are saying a give tensor equals all of the ones related by swapping terms). For the exterior power, we do have to worry about a sign, and we therefore have to worry that our answer is independent of the ways we rearrange. Thus we have to build a non-trivial representation of the symmetric group and work with it.

Corollary 1. *Let V have dimension n. Then the dimension of $S^i(V)$ is $\binom{n+i-1}{i}$.*

All of the dimensions we have seen so far have combinatorial meaning. We consider the problem of choosing i objects out of a box of n items. We have two parameters we can vary:

(1) The order in which we choose the objects does or doesn't matter.
(2) The objects are drawn with or without replacing them.

We summarize the classical combinatorics results in a table. Let V be an n-dimensional vector space.

	With Replacement	Without
Ordered	$\dim V^{\otimes i} = n^i$	$n!$
Unordered	$\dim S^i(V) = \binom{n+i-1}{n-1}$	$\dim \Lambda^i(V) = \binom{n}{i}$.

The one that doesn't have a vector space associated to it would be the distinction between "alternating" and "skew-symmetric". Since for vector spaces, these are the same, we don't have a corresponding vector space for that position.

How should we interpret the symmetric powers? We should think of these as polynomials in a basis. The i^{th} symmetric power is the collection of degree i polynomials, where we think of the elements of V itself as the degree 1 polynomials. Thus a basis gives us a name for our variables, and the symmetric powers are spanned by the monomials in the basis.

Algebras

The relationship between symmetric powers and polynomials can be made more precise, and in doing so, we can draw out more connections with the other products we've discussed.

Definition 2. An <u>algebra</u> over $\mathbb{F}$ is a vector space V together with a bilinear function

$$*\colon V \times V \to V,$$

the multiplication.

The "bilinear" condition expresses that the multiplication distributes over the addition (and that it commutes with the scalar multiplication). By our earlier work, we know that the multiplication is also expressible as a linear map $V \otimes V \to V$. Just as with the addition, we denote $*(\vec{v} \otimes \vec{u})$ by $\vec{v} * \vec{u}$.

We have made no assumptions on the multiplication beyond it distributing over addition. In particular, we don't require it to be associative, commutative, unital, etc.

Our three flavors of products assemble to give algebras. For all of them, we define the zeroth power to be the ground field $\mathbb{F}$.

Definition 3. Let V be a vector space.

(1) The tensor algebra on V is $T(V) = \bigoplus_{i \geq 0} V^{\otimes i}$.
(2) The exterior algebra on V is $\Lambda(V) = \bigoplus_{i \geq 0} \Lambda^i(V)$.
(3) The symmetric algebra on V is $S(V) = \bigoplus_{i \geq 0} S^i(V)$.

The products on the second two are induced from the first. There the product is induced by concatenation.

We spell out a little bit more the product in the first case. We first begin with a straightforward result.

Proposition 1. *Let V_i, $i \in I$, and W_j, $j \in J$ be two collections of vector spaces. Then*

$$\left(\bigoplus_{i \in I} V_i\right) \otimes \left(\bigoplus_{j \in J} W_j\right) \cong \bigoplus_{(i,j) \in I \times J} V_i \otimes W_j.$$

In words, "tensor product distributes over direct sum". Thus we can understand easily linear maps out of $T(V) \otimes T(V)$: they are collections of linear maps $V^{\otimes i} \otimes V^{\otimes j} \to V^{\otimes k}$. As we talked about before, we have a very natural map $V^{\otimes i} \otimes V^{\otimes j} \to V^{\otimes i+j}$. We therefore define our multiplication to be this map when $k = i + j$ and to be zero when $k \neq i + j$. Tracing through, this means

$$(\vec{v}_1 \otimes \cdots \otimes \vec{v}_i) \otimes (\vec{u}_1 \otimes \cdots \otimes \vec{u}_j) \mapsto \vec{v}_1 \otimes \cdots \otimes \vec{v}_i \otimes \vec{u}_1 \otimes \cdots \otimes \vec{u}_j.$$

Now a universal property.

Let A and B be algebras. A linear map $L \colon A \to B$ is an algebra map if

$$L(\vec{v} * \vec{u}) = L(\vec{v}) * L(\vec{u}).$$

We can talk about the set of algebra maps from A to B (or even the set of maps that are associative or commutative algebra maps), and let's call it $Alg(A, B)$. Then the following is an easy exercise.

Proposition 2. *We have a natural bijection*

$$Alg(T(V), B) \cong \mathcal{L}(V, B).$$

Thus $T(V)$ is the "free associative algebra" on V.

Similarly, we have

$$Comm(S(V), B) \cong \mathcal{L}(V, B),$$

meaning $S(V)$ is the "free commutative algebra on V.