# Lecture 7 - More Isomorphism Theorems

Last time:

1st Iso Theorem: If $L: V \to U$, then $L$ induces an isomorphism $\tilde{L}: V/\ker(L) \to \operatorname{Im}(L)$.

One key feature is that we know the map in this isomorphism: it's just $L$ again. There's no special trick to it, so we can understand completely

The first isomorphism theorem gives us all the rest.

2nd Isomorphism Theorem If $U, W \leq V$, then

$$U/U\cap W \cong (U+W)/W .$$

Pf: Let $i: U \to (U+W)/W$ be $i(\bar{u}) = \bar{u}+W$. Then $i$ is clearly linear (it's the composite of $U \hookrightarrow V$ and $V \to V/W$).

$i$ is onto: any element of $(U+W)/W$ is of the form

$$(\bar{u}+\bar{w})+W = \bar{u}+W = i(\bar{u}).$$

$\ker(i) = \{\bar{u} \in U \mid \bar{u}+W = W\} = \{\bar{u} \in U \mid \bar{u} \in W\} = U \cap W$. Now the first isomorphism theorem gives the result. ☒

Cor If $U \cap W = \{\bar{0}\}$, then $U \cong (U+W)/W = (U \oplus W)/W$.

So here quotienting gives us a way to cancel out direct summands. The general case above just reminds us that elements of $U$ could also be in $V$.

Remark An application of homework problems shows we always have a surjective map $U \oplus W \to U+W \leq V$, $(\bar{u}, \bar{v}) \longmapsto \bar{u}+\bar{v}$.

The kernel is the "antidiagonal" $\{(\bar{v}, -\bar{v}),\ \bar{v} \in U \cap W\}$. The first iso theorem shows $(U \oplus W)/U\cap W \cong U+W$.

The second iso theorem helps us also understand complements to a subspace. Remember that if $W \subseteq V$, then we can find a $U \subseteq V$ s.t. $V = U \oplus W$. $U$ is in general very far from unique.

<u>Ex</u>: $V = \mathbb{R}^3$, $W = (x,y)$-plane. Then $\langle \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right] \rangle \oplus W = V = \langle \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right] \rangle + W$.

However, we learn the following:

<u>Cor</u> Every complement of $W$ in $V$ is naturally isomorphic to $V/W$.

<u>Pf</u>: $V = U \oplus W$, so $U = U/_{U \cap W} \cong V/W$.

<u>Remark</u> For infinite dimensional vector spaces, things can be a little weirder (essentially because $V$ is isomorphic to proper subspaces).

Thus, as was described last time, $V \cong W \oplus V/W$, but the isomorphism requires a choice of complement.

<u>3rd Isomorphism Theorem</u>: If $S \subseteq U \subseteq W \subseteq V$, then

$$\frac{W/S}{U/S} \cong W/U.$$

<u>Pf</u>: Consider

$$\begin{aligned} W/S &\longrightarrow W/U \\ \bar{w}+S &\longrightarrow \bar{w}+U \end{aligned}$$

(this is essentially $\pi_U$).

Then this is obviously surjective and linear. The kernel is $\{\bar{w}+S \mid \bar{w} \in U\} = U/S$. $\boxtimes$

So second iso theorem says $/$ isn't like division, while 3rd says it is. Confusing at first, but if you have the mantra "it works out as nicely as possible", it's easy

Another way to state the 1st Isomorphism theorem is "An epimorphism is a quotient map". Since $\mathrm{Im}(L) \cong V/\ker(L)$, $V \to \mathrm{Im}(L)$ does satisfy the universal property. We have a complementary notion:

"A monomorphism is an inclusion of a subspace"

Since $V \xrightarrow{\cong} \mathrm{Im}(L) \subseteq W$.

To drive home this point, we introduce a new notion.

__Def__ A sequence

$$U \xrightarrow{S} V \xrightarrow{T} W$$

is __exact__ at $V$ if $\ker(T) = \mathrm{Im}(S)$.

__Prop__ 1) $0 \to U \xrightarrow{S} V$ is exact at $U$ iff $S$ is a monomorphism

2) $V \xrightarrow{T} W \to 0$ is exact at $W$ if $T$ is epi.

__Pf__: 1) $\ker(S) = \mathrm{Im}(0) = \{\vec{0}\}$, so $S$ is injective, and vice versa

2) $\mathrm{Im}(T) = W \longleftrightarrow$ exactness at $W$ $(\ker(0) = W)$. ☒

__Def__ $0 \to U \xrightarrow{S} V \xrightarrow{T} W \to 0$ is short exact if it is exact at $U, V, W$.

__Prop__ If $0 \to U \to V \to W \to 0$ is a short exact sequence, then $U \cong \ker(V \to W)$ and $W \cong V/U$.

__Pf__ The first assertion is exactness, the second is the 1st isom theorem. ☒

There is an obvious symmetry here

$$\text{subspaces} \longleftrightarrow \text{quotients}$$

We can make this precise.

**Def** A linear functional is a linear transformation
$$V \longrightarrow \mathbb{F}.$$

**Def** The dual of $V$ is the vector space
$$V^* = L(V, \mathbb{F}).$$

**Prop** If $L: V \to W$, then we get a linear transformation $L^*: W^* \to V^*$ defined by
$$(L^*(f))(\bar{v}) = f(L(\bar{v})).$$

This is another "categorical" construction: we know what to do to vector spaces and what to do to maps. With duality, we can link sub and quotient spaces.

**Thm** If $0 \to U \xrightarrow{S} V \xrightarrow{T} W \to 0$ is exact, then
$$0 \leftarrow U^* \xleftarrow{S^*} V^* \xleftarrow{T^*} W^* \leftarrow 0 \quad \text{is exact.}$$

Pf: Exact at $W^*$:

If $T^*(f) = 0 \longleftrightarrow T^*(f)(\bar{v}) = \bar{0} \quad \forall \bar{v} \in V$

$\longleftrightarrow f(T(\bar{v})) = \bar{0} \quad \forall \bar{v} \in V$

$\longleftrightarrow f(\bar{w}) = 0 \quad \forall \bar{w} \in W = T(V).$

Exact at $V^*$:

If $S^*(f) = 0$, then $S^*(f)(\bar{u}) = 0 \ \forall \bar{u} \in U$

$\longleftrightarrow f(S(\bar{u})) = 0 \quad \forall \bar{u} \in U$

So $f$ annihilates $\mathrm{Im}(S) = \ker(T)$

$\longleftrightarrow f(\bar{v}) = \tilde{f}(\bar{v} + \ker(T)) = \tilde{f}(T(\bar{v}))$

$\longleftrightarrow f = T^*(\tilde{f}).$

Exact at $U^*$:

If $g \in U^*$, then $g$ is a map $U \to \mathbb{F}$

Choose a complement of $S(U)$ in $V$:

$V \cong S(U) \oplus W$, and define $\hat{g}$ by

$\hat{g}(S(\bar{u}) + \bar{w}) = g(\bar{u})$. Then $S^*(\hat{g}) = g$. ☒

So duality swaps subspaces and quotients! This shows they are inexorably linked. To give one is to give the other.

What about matrices? Assume $V, W$ finite dimensional, with bases $\mathcal{B} = \{\bar{v}_1, \dots, \bar{v}_n\}$ and $\mathcal{C} = \{\bar{w}_1, \dots, \bar{w}_m\}$, and write $A = {}_{\mathcal{C}}[L]_{\mathcal{B}}$.

<u>Def</u> The <u>dual</u> <u>basis</u> to $\mathcal{B}$, $\mathcal{B}^*$ is

$$\mathcal{B}^* = \{\delta_{\bar{v}_i}\}, \qquad \delta_{\bar{v}_i}(\bar{v}_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

<u>Prop</u> This is a basis.

First an observation:

$$\left(\sum_{k=1}^{n} a_k \delta_{\bar{v}_k}\right)(\bar{v}_j) = a_j \qquad \text{and}$$

$$\delta_{\bar{v}_j}\left(\sum_{k=1}^{n} a_k \bar{v}_k\right) = a_j .$$

This is just linearity (so we are pulling out coefs).

<u>Pf</u> Assume $\sum a_k \delta_{\bar{v}_k} = 0$. Then evaluating on $\bar{v}_j$ gives 0 for all $j$, but

$$\left(\sum a_k \delta_{\bar{v}_k}\right)(\bar{v}_j) = a_j.$$

So lin. ind.

For spanning, any $f: V \to \mathbb{F}$ is determined by $f(\bar{v}_i)$, $1 \leq i \leq n$

So $f = \sum_{i=1}^{n} f(\bar{v}_i) \cdot \delta_{\bar{v}_i}$. $\boxtimes$

Prop ${}_{\mathcal{B}^*}[L^*]_{\mathcal{C}^*} = A^t$.

Pf: The $i^{th}$ column of $[L^*]$ is the vector of coords of $L^* \delta_{\bar{w}_i}$.

So the $(j,i)^{th}$ element is the coefficient of $\delta_{\bar{v}_j}$ in $L^* \delta_{\bar{w}_i}$

$= (L^* \delta_{\bar{w}_i})(\bar{v}_j)$

$= \delta_{\bar{w}_i}(L(\bar{v}_j)) =$ coefficient of $\bar{w}_i$ in $L(\bar{v}_j)$

$= a_{i,j}$. $\boxtimes$.

Thus with an ordered basis, $V \longleftrightarrow V^*$ is essentially transposition.