Compactness is an important measure of "smallness" in topology.

Def A collection of open sets $\{U_i\}_{i\in I}$ in X is an open cover of $Y\subset X$ if $Y\subseteq \bigcup_{i\in I} U_i$.

So an open cover of Y is just a collection of opens U_i with $y\in Y \Rightarrow \exists j$ s.t. $y\in U_j$.

Def $Y\subseteq X$ is <u>compact</u> if for every open cover $\{U_i\}_{i\in I}$ of Y, there is a finite set $\{i_1,\dots,i_n\}$ s.t. $Y\subseteq U_{i_1}\cup\dots\cup U_{i_n}$.

The slogan is that "every cover has a finite subcover."

<u>Ex</u>: 1) Any finite set is compact. (there are only finitely many things to check)

2) $\mathbb{R}$ is not compact: $\{U_n=(n-2/3, n+2/3)\}_{n\in\mathbb{Z}}$ is an open cover w/ no finite subcover, since n is only in U_n.

3) If $[x_n]$ is a sequence in X & $x_n\longrightarrow x$, then $\{x,x_1,\dots\}$ is compact: Any open set U s.t. $x\in U$ also contains all but finitely many of the x_n. We need only finitely many more opens to cover these (by 1)).

Prop Let X be compact & let $f: X\longrightarrow Y$ be continuous. Then $f(X)\subseteq Y$ is compact.

Pf: Let $\{U_i\}_{i\in I}$ be an open cover of $f(X)$ in Y. Then since f is continuous, $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of $f^{-1}(f(X))$ and hence of X. Since X is compact, for some $\{i_1,\dots,i_n\}\subseteq I$, $X\subseteq f^{-1}(U_{i_1})\cup\dots\cup f^{-1}(U_{i_n}) \Rightarrow f(X)\subseteq f(f^{-1}(U_{i_1}))\cup\dots\cup f(f^{-1}(U_{i_n}))\subseteq U_{i_1}\cup\dots\cup U_{i_n}$. Hence we have a finite subcov.

This makes compact a very special kind of property: it is preserved by any continuous map.

Prop If X is compact & $V\subseteq X$ is closed, then V is compact.

Pf: Let $\{U_i\}_{i\in I}$ be an open cover of V. Since V is closed, $X-V$ is open, and hence $\{X-V\}\cup\{U_i\}_{i\in I}$ is an open cover of X. Since X is compact, this has a finite subcover $\{X-V\}\cup\{U_{i_1},\dots,U_{i_n}\}$ (without loss of generality, we can assume $X-V$ is in the subcover, since adding it in doesn't change "finite"). Then $\{U_{i_1},\dots,U_{i_n}\}$ is a finite subcover of $\{U_i\}_{i\in I}$, and V is compact. $\square$

We have a converse to this.

Prop If $V\subseteq X$ is compact, then V is closed.

Pf: We have to show that for all $x \in X-V$, $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq X-V$. For each $v \in V$, let

$\epsilon_v = d(x,v)/2$. Then $B_{\epsilon_v}(v) \cap B_{\epsilon_v}(x) = \emptyset$: If $y \in B_{\epsilon_v}(v) \cap B_{\epsilon_v}(x)$, then

$2\epsilon_v = d(x,v) \leq d(x,y) + d(y,v) < \epsilon_v + \epsilon_v = 2\epsilon_v$, a contradiction. Now $\{B_{\epsilon_v}(v)\}_{v \in V}$ is an open cover of V, so $\exists v_1, \ldots, v_n$ s.t. $\{B_{\epsilon_{v_1}}(v_1), \ldots, B_{\epsilon_{v_n}}(v_n)\}$ is also an open cover. Then let $\epsilon = \min\{\epsilon_{v_1}, \ldots \epsilon_{v_n}\}$

Then $B_\epsilon(x) = \bigcap_{i=1}^{n} B_{\epsilon_{v_i}}(x)$, and since $B_{\epsilon_{v_i}}(x) \cap B_{\epsilon_{v_i}}(v_i) = \emptyset$, $\emptyset = B_\epsilon(x) \cap \bigcup_{i=1}^{n} B_{\epsilon_{v_i}}(v_i) \supseteq B_\epsilon(x) \cap V$. $\square$

Cor If X is compact, then V is compact $\leftrightarrow$ V is closed & if $f: X \to Y$ is continuous, then f is closed.

Cor If X is compact, then if $f: X \to Y$ is a continuous bijection, then f is a homeomorphism.

This is an incredibly powerful tool for showing that two spaces are homeomorphic.