Having talked about spaces, we start talking about functions that play nicely with the topology.

<u>Def</u> Let X & Y be metric spaces, and let $f: X \to Y$ be a function. Then f is <u>continuous</u> if for every open set $U \subseteq Y$, $f^{-1}(U)$ is open in X.

Let's unpack this. First, inverse image plays nicely with union & intersection:

$$f^{-1}\Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I} f^{-1}(U_i) \quad \& \quad f^{-1}\Big(\bigcap_{i\in I} U_i\Big) = \bigcap_{i\in I} f^{-1}(U_i).$$

$$\Big(x \in f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \longleftrightarrow f(x) \in \bigcup_{i\in I} U_i \longleftrightarrow \exists\, i \in I \text{ s.t. } f(x) \in U_i \leftrightarrow \exists\, i \in I \text{ s.t. } x \in f^{-1}(U_i) \leftrightarrow x \in \bigcup_{i\in I} f^{-1}(U_i)\Big)$$

swap w/ $\forall$ for $\cap$

Since every open set is a union of open balls, this lets us rewrite our condition.

<u>Prop</u> $f: X \to Y$ is continuous iff for all $x \in X$ and $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.

<u>Pf</u>: Since open sets are unions of open balls, $f: X \to Y$ is continuous iff for all $y \in Y$ & $\epsilon > 0$, $f^{-1}(B_\epsilon(y))$ is open in X. $f^{-1}(B_\epsilon(y))$ is open iff $\forall\, x \in f^{-1}(B_\epsilon(y))$, $\exists\, \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$. Since $x \in f^{-1}(\{f(x)\})$, this condition covers all the points of X. $\square$

Continuous functions also play nicely with limits, giving another version of continuity.

<u>Prop</u> $f: X \to Y$ is continuous iff for every convergent sequence $[x_n]$ in X, $f(\lim x_n) = \lim f(x_n)$.

<u>Pf</u>: Let $x = \lim x_n$, $y_n = f(x_n)$, and $y = f(x)$. Assume f is continuous. Then for every $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\epsilon(y)$. Since $x_n \to x$, all but finitely many terms of $[x_n]$ are in $B_\delta(x)$, so all but finitely many y_n are in $B_\epsilon(y)$. If f is not continuous, then for some $x \in X$, & $\exists\, \epsilon > 0$ s.t. $\forall\, \delta > 0$, $\exists\, z_\delta \in X$ w/ $d(x, z_\delta) < \delta$ but $d(f(x), f(z_\delta)) \geq \epsilon$. For each $k \geq 1$, let z_k be such a $z_{1/k}$. Then $z_k \to x$, but $d(f(z_k), f(x))$ is always bound below by ϵ. $\square$

<u>Cor</u> If B is a dense subspace of X and $f: X \to Y$ is continuous, then f is determined by $f|_B$.

<u>Pf</u> Any $x \in X$ is a limit of a sequence of elements of B. $\square$

<u>Def</u> A function $f: X \to Y$ is an <u>isometry</u> if $\forall\, x, y \in X$, $d(x,y) = d(f(x), f(y))$.

<u>Prop</u> Isometries are continuous.

<u>Pf</u>: Take $\delta = \epsilon$, and hence $\delta > d(x,y) = d(f(x), f(y))$. $\square$

<u>Prop</u> For any X, Id_X is continuous.

Prop If d_1, d_2 are equivalent metrics, then $Id: (X, d_1) \longrightarrow (X, d_2)$ and $Id: (X, d_2) \longrightarrow (X, d_1)$ are cont.

Pf: Equivalent metrics, by definition, have the same open sets. $\square$

Prop If $g: X \longrightarrow Y$ & $f: Y \longrightarrow Z$ are continuous, then $f \circ g: X \longrightarrow Z$ is continuous.

Pf: Let $U \subseteq Z$ be open. Since f is continuous, $f^{-1}(U)$ is open in Y. Since g is continuous, $g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U)$ is also open. Hence $(f \circ g)$ is continuous. $\square$

Def A function $f: X \longrightarrow Y$ is <u>open</u> (resp <u>closed</u>) if for all open (closed) $U \subseteq X$, $f(U)$ is open (closed).

A function can be open, closed, both or neither.

Prop If f is a bijection with inverse g, then f is continuous iff g is open.

Pf: If U is open in Y, then $f^{-1}(U) = g(U)$ is open if f is cont or g is open. $\square$

Def A map $f: X \longrightarrow Y$ is a <u>homeomorphism</u> if f is a bijection, continuous, and open.

From the point of view of topology, a homeomorphism indicates the two spaces are indistinguishable.