<u>Thm</u> (Baire Category Theorem) If X is a complete metric space & $U_1, \dots$ is a sequence of dense open sets in X. Then $U = \bigcap_{i=1}^{\infty} U_i$ is also dense in X.

<u>Pf</u>: We have to show that if V is open and non-empty, then $V \cap U \neq \emptyset$. We will work iteratively. Since U_1 is dense, $U_1 \cap V \neq \emptyset$. Let $u_1 \in U_1 \cap V$. Since U_1 & V are open, we can find an $r_1 > 0$ s.t. $B_{r_1}(u_1) \subseteq U_1 \cap V$. By shrinking r_1, we can assume $r_1 < 1$ & $\overline{B}_{r_1}(u_1) \subseteq U_1 \cap V$. Now we repeat w/ $B_{r_1}(u_1)$ playing the role of V & U_2 replacing U_1. Continuing in this manner, we get a sequence u_i, with $u_i \in U_i$, and radii r_i, $0 < r_i < 1/i$, s.t. $\overline{B}_{r_i}(u_i) \subseteq U_i \cap B_{r_{i-1}}(u_{i-1})$ &

$$\overline{B}_{r_i}(u_i) \subseteq B_{r_{i-1}}(u_{i-1}) \subseteq \dots \subseteq B_{r_1}(u_1) \subseteq V.$$

In particular, for all $n \geq i$, $u_n \in B_{r_i}(u_i)$. Since $r_i \to 0$ as $i \to \infty$, this means the sequence is Cauchy. Since X is complete, $u = \lim u_i$ exists. For each n, u is the limit of the sub-sequence ask them to say why $u_{n+1}, \dots$ which is entirely in $B_{r_{n+1}}(u_{n+1})$. The limit is then in the closure, so u is also in the closed ball: $u \in \overline{B}_{r_{n+1}}(u_{n+1}) \subseteq B_{r_n}(u_n) \subseteq U_n$. $\Rightarrow u \in \bigcap_{i=1}^{\infty} U_i$. Since $B_{r_1}(u_1)$ is also in V, $u \in \left(\bigcap_{i=1}^{\infty} U_i\right) \cap V$. □

<u>Def</u> A set Y is <u>nowhere dense</u> if $\operatorname{int}(\overline{Y}) = \emptyset$.

<u>Cor</u> If $V_1, \dots$ is a collection of nowhere dense sets in a complete metric space X, then

$$V = \bigcup_{i=1}^{\infty} V_i$$ has empty interior.

<u>Pf</u>: If V_i is nowhere dense, then $X - \overline{V}_i$ is dense and open. Apply the Baire Category Theorem to the $X - \overline{V}_i$ to deduce that $\bigcap (X - \overline{V}_i)$ is dense and hence $X - (\bigcup \overline{V}_i)$ is dense. □

<u>Examples</u> A point in $\mathbb{R}$ (or $\mathbb{R}^n$) is nowhere dense. The corollary then implies that every countable set in $\mathbb{R}$ ($\mathbb{R}^n$) has empty interior. ask them why In particular, open sets are either empty or uncountable. Countable sets can be closed ($\mathbb{Z}$), dense ($\mathbb{Q}$), or neither ($\{1/n \mid n \in \mathbb{N}\}$).

We've said and used several times that $\mathbb{R}$ is complete. We should prove it. $\mathbb{R}$ has several nice properties:

<u>Properties I</u> $\mathbb{R}$ is an ordered field under $<$:

1) If $a, b \in \mathbb{R}$, then exactly one of $a<b$, $b<a$, $a=b$ holds.

2) If $a<b$, & $b<c$, then $a<c$.

3) If $a<b$ & $c \in \mathbb{R}$, then $a+c<b+c$ &

4) If $a<b$ & $c \in \mathbb{R}$ is ≥ 0, then $a \cdot c < b \cdot c$.

A bunch of things follow from this, like "if $a<b$, then $-b<-a$."

$\mathbb{R}$ has another property: least upper bounds (LUB).

<u>Def</u> $M \in \mathbb{R}$ is an <u>upper bound</u> for $S \subseteq \mathbb{R}$ if $\forall s \in S$, $s \leq M$. M' is a least upper bound if for all upper bounds M for S, $M' \leq M$.

<u>Least Upper Bound Axiom</u> Any $S \subseteq \mathbb{R}$ which has an upper bound has a least upper bound.

<u>Thm</u> $\mathbb{R}$ is complete.

<u>Pf</u>: Let $[x_n]$ be a Cauchy sequence in $\mathbb{R}$, and let $S = \{y \mid x_n < y \text{ for only finitely many } n\}$

If $y \in S$ & $z \leq y$, then $z \in S$, so if $y \in S$, then $(-\infty, y] \subseteq S$.

Since $[x_n]$ is Cauchy, for any $\epsilon > 0$, $\exists N$ s.t. $\forall n,m \geq N$, $d(x_n, x_m) = |x_n - x_m| < \epsilon$. In particular, $\forall n \geq N$, $x_n \in (x_N - \epsilon, x_N + \epsilon)$. Thus all but finitely many terms are $> x_N - \epsilon$, and $x_N - \epsilon \in S$.

Similarly, infinitely many x_n are $< x_N + \epsilon$, so $x_N + \epsilon$ is an upper bound for S, since if $y \in S$ were greater than $x_N + \epsilon$, then $x_N + \epsilon$ would be in S. Let x be the least upper bound of S. Then by assumption,

$x_N - \epsilon$ in S x the LUB

$x_N - \epsilon \leq x \leq x_N + \epsilon$, and $d(x_N, x) \leq \epsilon$. Thus for $n \geq N$,

$$d(x_n, x) \leq d(x_n, x_N) + d(x_N, x) < \epsilon + \epsilon = 2\epsilon.$$

Thus x is the limit. $\square$