Convergence of a sequence is also a topological property.

Def A sequence $[x_n]$ in X is Cauchy if $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n,m > N$, $d(x_n, x_m) < \epsilon$.

Prop If $[x_n]$ converges, then $[x_n]$ is Cauchy.

Pf: Let $\epsilon > 0$, and let $x = \lim x_n$. This means that $\exists N > 0$ s.t. $\forall n > N$, $d(x_n, x) < \epsilon/2$.
In particular, $\forall n,m > N$ (same N!), $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \epsilon/2 + \epsilon/2 = \epsilon$. □

Prop If $[x_n]$ is a Cauchy sequence & $[x_{n_k}]$ is a subsequence converging to x, then $[x_n]$ itself converges to x.

Pf: Let $\epsilon > 0$. Since $x_{n_k} \to x$, $\exists N_1$ s.t. $\forall k \geq N_1$, $d(x_{n_k}, x) < \epsilon/2$. Since $[x_n]$ is Cauchy, $\exists N_2$ s.t. $\forall n,m \geq N_2$, $d(x_n, x_m) < \epsilon/2$. Then if $N = \max(N_2, n_{N_1})$, then $\forall$ $n, n_k > N$, $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon/2 + \epsilon/2$. □

Def A metric space X is complete if every Cauchy sequence in X converges.

Prop If $Y \subseteq X$ is a subset, then $d|_{Y \times Y}$ defines a metric on Y.

Pf: The axioms hold for all points in X, so in particular for all points in Y.

Thm A closed subspace of a complete metric space is complete.

Pf: Let $[y_n]$ be a Cauchy seq. in $Y \Rightarrow [y_n]$ is a Cauchy seq in $X \Rightarrow x = \lim y_n$ exists.
Y closed $\Rightarrow$ limits of seq. in Y are in Y, so $x \in Y$ & $[y_n]$ converged in Y. □

Thm A complete subspace of a metric space X is closed.

Pf Let $[y_n]$ be a sequence in Y converging to $x \in X$. Then $[y_n]$ is Cauchy $\Rightarrow$ limit exists in Y $\Rightarrow x \in Y$ & Y is closed. □

Def A set Y is dense in X if $\overline{Y} = X$.

Prop