In words, $\mathring{Y}$ is the largest open set contained in Y.

Open sets are recording "closeness".

<u>Def</u> A set $Y \subseteq X$ is <u>closed</u> if the complement $X-Y$ is open.

A random set will be neither closed nor open. Also note that the notions of closed and open are not opposed: sets can be both!

<u>Prop</u> The intersection of closed sets is closed. Finite unions of closed sets are closed.

<u>Pf</u>: In general, $X-\left(\bigcap_I Y_i\right) = \bigcup_I (X-Y_i)$ & $X-\left(\bigcup_I Y_i\right) = \bigcap_I (X-Y_i)$. Since open sets are closed under arbitrary unions & finite intersections, we conclude closed sets are the reverse. □

<u>Def</u> The <u>closure</u> of $Y \subseteq X$ is the complement of the interior of $X-Y$:

$$\overline{Y} = X - (\mathring{X-Y}).$$

In particular, $\overline{Y}$ is closed. What does it mean for $y \in \overline{Y}$?

<u>Prop</u> $Y \subseteq \overline{Y}$.

<u>Pf</u>: $Y \cap (X-Y) = \emptyset$ & $(\mathring{X-Y}) \subseteq X-Y \Rightarrow (\mathring{X-Y}) \cap Y = \emptyset$. □

<u>Prop</u> If $y \in \overline{Y}$, then $\forall r>0$, $B_r(y) \cap Y \neq \emptyset$.

<u>Pf</u>: $y \in \overline{Y}$ is equivalent to $y \notin (\mathring{X-Y})$. Now $y \in \mathring{X-Y}$ iff $\exists\, \epsilon>0$ s.t. $B_\epsilon(y) \subseteq X-Y$, or equivalently, $B_\epsilon(y) \cap Y = \emptyset$. So $y \notin (\mathring{X-Y})$ iff $\forall \epsilon>0$, $B_\epsilon(y) \cap Y \neq \emptyset$. □

<u>Def</u> A point $x \in X$ is an <u>adherent point</u> for $Y \subseteq X$ if $\forall r>0$, $B_r(x) \cap Y \neq \emptyset$.

In other words, x is an adherent point for Y if there are points of Y arbitrarily close to x. This ties into the notion of the limit of a sequence $[x_n]$ in X.

<u>Def</u> Let $x_1, \dots$ be a sequence of elements in X. Then $x \in X$ is a <u>limit</u> of the sequence if for all $r>0$, $\exists\, N_r \geq 1$ s.t. $\forall n \geq N_r$, $x_n \in B_r(x)$.

This <u>is</u> the ordinary definition from calculus, but we never needed that we were in $\mathbb{R}^n$!

<u>Prop</u> If $x_1, \dots$ is a sequence in X, then if x & y are limits of the sequence, then $x=y$.

<u>Pf</u>: Let $\epsilon>0$ be arbitrary. Then we can find N_1 s.t. $\forall n \geq N_1$ $d(x,x_n) < \epsilon/2$. Similarly, we can find N_2 s.t. $\forall n \geq N_2$, $d(x_n,y) < \epsilon/2$. So for all $n \geq \max\{N_1,N_2\}$, the triangle

inequality implies $d(x,y) \leq d(x,x_n) + d(x_n,y) < \epsilon/2 + \epsilon/2 = \epsilon$. This implies $d(x,y) = 0$, so $x = y$. □

<u>Prop</u> $x \in X$ is an adherent point for Y iff $\exists$ a sequence $y_1, \ldots$ in Y s.t. $\lim y_n = x$.

<u>Pf</u>: This follows from the fact that for any $\epsilon > 0$, there is an N s.t. $\forall n > N$, $\frac{1}{n} < \epsilon$. This means that the open balls $B_{1/n}(x)$ evetually sit in any open set. □

<u>Cor</u> The closure of Y is the set of limits of sequences of elements in Y.