My final talk comes from a different angle: spectral algebraic geometry. Descent ties this to equivariant homotopy.

<u>Lemma</u> If $\mathcal{O}$ is a sheaf of commutative ring spectra on $\mathcal{M}$ & $X \to Y$ is a G-Galois cover (over $\mathcal{M}$) then

1. $\mathcal{O}(X)$ has an action of G by com ring maps
2. $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)^{hG}$ is an equivalence.

Any spectrum with a G-action can be promoted to a genuine G-spectrum (in several way). In particular, $F(EG_+, \mathcal{O}(X))$ can be considered a genuine G-spectrum with fixed points $\mathcal{O}(Y)$.

<u>Thm</u> (Blumberg-Hill) If R is a naive E_∞-ring spectrum in G-spectra, then $F(EG_+, R)$ is a G-E_∞ ring spectrum.

So we took one problem and replaced it with a possibly trickier one. However, now we have more tools and directions to use for algebraic geometry. I'll focus on 2 aspects:

① Duality for $\mathcal{O}(Y)$ via $\mathcal{O}(X)$ & ② $\mathrm{Pic}(\mathcal{O}(Y))$ from $\mathrm{Pic}(\mathcal{O}(X))$.

I will also ground the discussion by focusing on an explicit example, w/ fun $RO(G)$.

<u>Thm</u> (Goerss-Hopkins-Miller, Lurie, H.-Lawson) There is a sheaf of E_∞-ring spectra on the (log-) étale site of the moduli of elliptic curves.

We can apply the Lemma above to geometrically meaningful examples.

<u>Ex</u> Let $\mathcal{M}_1(N)$ be the moduli stack of elliptic curves + a point of exact order N.

Let $\mathcal{M}_0(N)$ be the moduli stack of elliptic curves + a cyclic s.g. of order N. Then

$$\begin{array}{ccc} \mathcal{M}_1(N) & \longrightarrow & \mathcal{M}_0(N) \\ (E,e) & \longmapsto & (E,\langle e\rangle) \end{array}$$ is $(\mathbb{Z}/N)^\times$-Galois & over $\mathbb{Z}[1/N]$, $\mathcal{M}_0(N) \longrightarrow \mathcal{M}_{Ell}[1/N]$ is étale.

This is also true on compactifications.

<u>Def</u> $TMF_e(N) = \mathcal{O}^{der}(\mathcal{M}_e(N) \to \mathcal{M}_{Ell}[1/N])$, $Tmf_e(N) = \mathcal{O}^{der}(\overline{\mathcal{M}}_e(N) \to \overline{\mathcal{M}}_{Ell}[1/N])$.

In general, the moduli $\mathcal{M}_1(N)$ are schemes, and the corresponding homology theories are Landweber

Now let N be 3 or 5. By the above, $Tmf_1(3)$ is a C_2-spectrum & $Tmf_0(3)$ is the fixed pts.

fpqc locally, any elliptic curve, we can write our curve in Weierstrauss form and move e to (0,0).

$y^2 + a_1xy + a_3y = x^3$. This is smooth when $\Delta = a_3^3(a_1^3 - 27a_3)$ is a unit, and the only allowed symmetries are scaling. The compactification allows $\Delta = 0$, provided $a_3 \neq 0$. So we have

$\overline{\mathcal{M}}_1(3) = \mathbb{A}^2 - \{(0,0)\}/\mathbb{G}_m$, a weighted $\mathbb{P}^1$. They weighting is given by $|a_i| = 2i$

The descent SS for $\mathrm{Tmf}_1(3)$: $E_2^{s,t} = H^s(\overline{\mathcal{M}}_1(3); \omega^{\otimes t}) \Rightarrow \pi_{2t-s}()$ collapses and Serre duality gives

$$H^1(\overline{\mathcal{M}}_1(3); \omega^{\otimes t}) \cong \begin{cases} 0 & t \geq -3 \\ H^0(\overline{\mathcal{M}}_1(3); \omega^{\otimes(4-t)}) & t \leq -4. \end{cases}$$

The Serre dual class is $[1/a_1 \cdot a_3]$.

Spectra have a kind of duality: Brown-Comenetz & Andersen: Since $\mathbb{Q}/\mathbb{Z}$ is injective, $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is exact & $X \mapsto \mathrm{Hom}(\pi_{-k}^s X, \mathbb{Q}/\mathbb{Z})$ gives a cohomology theory. This is representable, so have a spectrum $I_{\mathbb{Q}/\mathbb{Z}}$ & "duality" $X \mapsto F(X, I_{\mathbb{Q}/\mathbb{Z}}) \leftarrow$ BC dual. Can play the same game w/ $\mathbb{Q}$ & get maps $D_{\mathbb{Q}}X \to D_{\mathbb{Q}/\mathbb{Z}}X$. The fiber is $D_{\mathbb{Z}}X$ Andersen Dual

Thm (Stojanowska) $D_{\mathbb{Z}}\mathrm{Tmf}_1(3) \simeq \Sigma^9 \mathrm{Tmf}_1(3)$, $D_{\mathbb{Z}}\mathrm{Tmf}_0(3)$ is an invertible $\mathrm{Tmf}_0(3)$ module.

The first part is basically Serre duality. Phrased correctly, the second part is too!

Thm (H.-Meier) There are classes $\bar{a}_1$ & $\bar{a}_3$ s.t. $\bar{a}_i \in \pi_{i\rho}\mathrm{Tmf}_1(3)$ s.t. equivariantly $\mathrm{Tmf}_1(3)$ looks like $\mathbb{P}^1$ weighted accordingly. $D_{\mathbb{Z}}^{C_2}\mathrm{Tmf}_1(3) \simeq_{C_2} \Sigma^{5+2\rho}\mathrm{Tmf}_1(3)$.

$\Rightarrow D_{\mathbb{Z}}\mathrm{Tmf}_0(3) \simeq \left(\Sigma^{5+2\rho}\mathrm{Tmf}_1(3)\right)^{C_2}$.

Thus passing to the $RO(C_2)$-graded context gives us a closer Serre duality statement.

① $\mathrm{Tmf}_1(3)$ is Real oriented: there is a map $MU_{\mathbb{R}} \to \mathrm{Tmf}_1(3)$ & $\mathrm{Tmf}_1(3)[\bar{\Delta}^{-1}]$ is Real Landweber exact. (They all look like $E_{\mathbb{R}}(1)$ or $E_{\mathbb{R}}(2)$).

② The Slice associated graded of $\mathrm{Tmf}_1(3)$ is

$$\bigvee_p S^{-1-4\rho-|p|} \wedge H\underline{\mathbb{Z}} \vee \bigvee_p S^{|p|} \wedge H\underline{\mathbb{Z}}$$

Thm (Ullman) The dual of the slice tower is the slice tower of the dual.

③ $D_{\mathbb{Z}}^{C_2}\mathrm{Tmf}_1(3)$: $\bigvee_p S^{1+4\rho+|p|} \wedge H\underline{\mathbb{Z}}^* \vee \bigvee_p S^{-|p|} \wedge H\underline{\mathbb{Z}}^*$. Now $H\underline{\mathbb{Z}}^* \simeq S^{4-2\rho} \wedge H\underline{\mathbb{Z}}$. $\Rightarrow$

Duality class is in degree $-(1+4\rho+4-2\rho) = -5-2\rho$, as desired.

A similar argument applies to $\mathrm{Tmf}_1(5)$ for C_4. Even more exciting!

<u>Thm</u> (Meier) 2-locally, $\mathrm{Tmf}_1(N) \underset{C_2}{\simeq} \mathrm{Tmf}_1(3) \vee \bigvee S^{\cancel{2}\rho i} \wedge \mathrm{Tmf}_1(3)$.

$\Rightarrow$ Can understand duality for any $\mathrm{Tmf}_0(N)$ in the same way.