Crash Course in equiv htpy Let G be a fixed, finite group. A G-space is a space X + a continuous action of G: $G \xrightarrow{\mu} \mathrm{Aut}(X)$. A continuous map $X \xrightarrow{f} Y$ is equivariant if $\forall g \in G, x \in X$, $f(g \cdot x) = g \cdot f(x)$. This gives us a category $\mathcal{T}op^G$, and can also do htpy.

If $H \subseteq G$, then we can "forget": $i_H^*: \mathcal{T}op^G \longrightarrow \mathcal{T}op^H$. This has both adjoints

(L) $\coprod_{G/H} - := G \times_H - : \mathcal{T}op^H \longrightarrow \mathcal{T}op^G$. $G \times_H X := G \times X / (gh, x) \sim (g, hx)$. $\Rightarrow [G \times_H X, Y]^G \cong [X, i_H^* Y]^H$.

(R) $\prod_{G/H} - := \mathrm{Map}^H(G, -) : \mathcal{T}op^H \to \mathcal{T}op^G \Rightarrow [X, \mathrm{Map}^H(G, Y)]^G \cong [i_H^* X, Y]^H$

The intuition is in representation theory: we have $i_H^*: \mathrm{Rep}^G \longrightarrow \mathrm{Rep}^H$ has

$$\mathrm{Ind}_H^G(-) = \mathbb{R}[G] \otimes_{\mathbb{R}[H]} (\ ) \equiv \mathrm{Hom}_{\mathbb{R}[H]}(\mathbb{R}[G], -) = \mathrm{CoInd}_H^G(-)$$

For spaces, $G \times_H X$ & $\mathrm{Map}^H(G, X)$ are very different.

Example: $X = *$. $G \times_H * \cong G/H$ while $\mathrm{Map}^H(G, *) \cong *$.

Genuine G-spectra fixes this, universally making $\coprod_T \simeq \prod_T$:

Thm (Blumberg) Genuine G-spectra are universally characterized by this w.e.

The standard treatment of $\mathcal{S}p^G$ are as $\mathrm{Rep}(G)$-graded sequences of G-spaces:

$$X_V \longrightarrow \Omega^W X_{V \oplus W} \qquad \forall\, V, W \in \mathrm{Rep}(G). \quad \text{\& this is the same data.}$$

Both induction & coinduction are the same construction: given a symmetric monoidal category $(\mathcal{C}, \otimes)$ we can always form $\bigotimes_{G/H} : \mathcal{F}un(BH, \mathcal{C}) \longrightarrow \mathcal{F}un(BG, \mathcal{C})$. For induction: $\amalg$, co: $\times$.

Spectra have another product: $- \wedge -$

descends to ho$\mathcal{S}p$ — commutes with smash

Thm (H.-Hopkins-Ravenel) There is a homotopically meaningful, symmetric monoidal functor

$N_H^G : \mathcal{S}p^H \longrightarrow \mathcal{S}p^G$ which commutes w/ sifted colimits & $N_H^G(S^V) \simeq S^{\mathrm{Ind}_H^G V}$.

essentially says how to compute

Ex/Prop: $N_H^G(\Sigma^\infty X_+) \simeq \Sigma^\infty \mathrm{Map}^H(G, X)_+$

If we apply this with $H = C_2$ to $MU_{\mathbb{R}}$, then we get

Def If $G = C_{2^n}$, let $MU^{((G))} := N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$. Everything saw for $MU_{\mathbb{R}}$ works

One last ingredient: the slice filtration. Saw last time that cells of the form $D(\mathbb{C}^n)$ are geometrically natural. These are attached by $S(\mathbb{C}^n) \simeq S^{n\rho - 1}$ ← regular rep

Since saw from N_H^G

For bigger groups, we want cells $\mathrm{Map}^{C_2}(G, D(\mathbb{C}^n)) \cong D(n\rho_G) \rightsquigarrow S^{n\rho_G - 1}$

Def A space/spectrum is slice $\geq n$ if it is in the localizing subcat generated by

$$G_+ \underset{H}{\wedge} S^{k\rho_H - \epsilon}, \qquad k \cdot |H| - \epsilon \geq n, \qquad \epsilon = 0, 1.$$

In other words, these are the attaching maps for cells of the form we want.

Thm (HHR) The slice associated graded of $MU^{(G)}$ is $\left(\bigvee C_{2^n +} \underset{C_{2^k}}{\wedge} S^{m\rho_{2^k}}\right) \wedge H\underline{\mathbb{Z}}$.

Doug spoke about this here, so I'll focus on other aspects & consequences. In particular "even"

Thm (H.-Hopkins) For all n, $\Omega^\infty(S^{n\rho} \wedge MU^{(G)})$ has cells of the form $G_+ \underset{H}{\wedge} D(m\rho_H)$.

Thm (HHR) For all n & H, $[G_+ \underset{H}{\wedge} S^{n\rho_H}, MU^{(G)}]^G$ is f.g. free & $[G_+ \underset{H}{\wedge} S^{n\rho_H - 1}, MU^{(G)}]^G = 0$. part of Gap Thm

This is our prototype for even.

Def X is even if it is simply connected, built out of cells of the form $G_+ \underset{H}{\wedge} D(m\rho_H)$, &

$$[S^{m\rho_H - 1}, X]^H = 0 \qquad \forall m, H.$$

Ex. If X is even for H, then $\mathrm{Map}^H(G, X)$ is even for G.

Prop If X is even, then X is an H-space (with norms!)

So can do everything from before, including building primitive elements (on-going).

I haven't said anything about cohomology yet. The first issue is coefficients. This is back w/ induction, coinduction, etc. For any X, have $X \longrightarrow \mathrm{Map}^H(G, i_H^* X)$ for all $H \subseteq G$.

$\Rightarrow$ $X^G \longrightarrow (\mathrm{Map}^H(G,X))^G \cong X^H$. If induction is coinduction, then have a map back.

This gives me a Mackey functor.

Def A Mackey functor is a pair of functors $\underline{M}_*, \underline{M}^*$ on finite G-sets s.t.

⓪ $\underline{M}_*(T) = \underline{M}^*(T)$ & $\underline{M}_*(T_1 \amalg T_2) \cong \underline{M}_*(T_1) \oplus \underline{M}_*(T_2)$

① $\begin{array}{ccc} S' & \xrightarrow{h'} & S \\ {\scriptstyle f'}\downarrow & \text{p.b.} & \downarrow{\scriptstyle f} \\ T' & \xrightarrow[h]{} & T \end{array} \Rightarrow (h')_* \circ (f')^* = f^* \circ h_*$ $\qquad f_* =$ "transfer", $f^* =$ "restriction"

Thm (Bredon) There is a good homology/cohomology theory w/ coefs in any Mackey functor.

This is harder to work with. In particular, the natural transformations are not known.

Thm (Oruç) If $\underline{M}$ is a Mackey field, then the $\underline{M}$-Steenrod alg is "essentially classical".

Thm (Hu-Kriz) The $RO(C_2)$-graded $\underline{\mathbb{F}}_2$- Steenrod algebra is generated by certain lifts of the Steenrod squares to $RO(C_2)$-graded classes.

We get something similar for $G = C_{2^n}$! Here slices come in.

Thm (HHR) There is an associative ring map $A = S^0[G\cdot x_1, \ldots] \longrightarrow MU^{(G)}$ with

$$MU^{(G)} / G\cdot x_1, \ldots = MU^{(G)} \underset{A}{\wedge} S^0 = H\underline{\mathbb{Z}}.$$

Now $H\underline{\mathbb{F}}_2$ is orientable: $\quad H\underline{\mathbb{F}}_2 \wedge MU^{(G)} = H\underline{\mathbb{F}}_2 \wedge B$, where $B \simeq A$ (but not as A-modules)

$$\Rightarrow \quad H\underline{\mathbb{F}}_2 \wedge H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{F}}_2 \wedge \left(MU^{(G)} \underset{A}{\wedge} S^0\right) \simeq \left(H\underline{\mathbb{F}}_2 \wedge B\right) \underset{A}{\wedge} S^0.$$

So it suffices to understand $A \longrightarrow \left(H\underline{\mathbb{F}}_2 \wedge MU^{(G)}\right)$, which is purely algebraic.

Caveat Audientis: $\quad C_{2+} \longrightarrow S^0$ has cofiber $S^{\rho - 1}$.