Localizations of Equivariant Commutative Rings

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(joint work with Michael J. Hopkins)

In this talk, I discussed joint work with Hopkins which addresses the question: "When is the localization of a commutative G-equivariant ring a commutative G-equivariant ring?". In all that follows, let G be a finite group. The talk sketches a proof of the following theorem.

Theorem 1. *If for all acyclics Z for a localization L and for all subgroups H, $N_H^G Z$ is acylic, then for all commutative G-ring spectra R, $L(R)$ is a commutative G-ring spectrum.*

The proof is modeled on the standard non-equivariant proof in EKMM [1]. The essential twist is understanding the interplay between the G-action on the E_∞ operad and the norm.

Already in the statement of the theorem we have used the norm (as described in Hill-Hopkins-Ravenel [2]). This is a symmetric monoidal functor $N_H^G\colon \mathcal{S}_H \to \mathcal{S}_G$ from the category of H-spectra (with its smash product) to the category of G-spectra. This has the distinguished feature of also refining to the left adjoint to the forgetful functor from commutative G-ring spectra to commutative H-ring spectra. Thus there is for any commutative G-ring spectrum R a canonical map of commutative G-ring spectra

$$N_H^G Res_H^G(R) \to R.$$

These satisfy axioms analogous to the norm maps in Tambara functors, making commutative G-rings into spectral Tambara functors (analogous to Guillou-May's description of equivariant spectra).

1. Localizations Need Not Be Commutative

We first sketch a counterexample to the obvious conjecture. Let $\mathcal{P}$ denote the family of proper subgroups of G, and let $\tilde{E}\mathcal{P}$ denote the cofiber of the natural map from the classifying space $E\mathcal{P}_+$ to S^0. The spectrum $\tilde{E}\mathcal{P}$ is a localization of S^0: we kill all maps from induced cells. Since G is finite, we can also realize $\tilde{E}\mathcal{P}$ as $S^0[a_{\bar{\rho}}^{-1}]$, where $a_{\bar{\rho}}$ is the inclusion of $\{0,\infty\}$ into the representation sphere associated to $\bar{\rho}$, the quotient of the real regular representation by its trivial summand.

This spectrum does not admit maps from the norms of its restrictions. For any proper subgroup H, the commutative H-ring spectrum $Res_H^G(\tilde{E}\mathcal{P})$ is contractible. This is the terminal commutative H-ring, and since $\tilde{E}\mathcal{P}$ is not contractible, we cannot have a commutative ring map

$$* \simeq N_H^G Res_H^G(\tilde{E}\mathcal{P}) \to \tilde{E}\mathcal{P}.$$

The example already underscores the role the norm will play. Here there is an obstruction to being a commutative ring spectrum. One way to interpret our theorem is that this is the obstruction; if localization "plays nicely with the norm", then it takes commutative ring objects to commutative ring objects. On the other hand, the spectrum $\tilde{E}\mathcal{P}$ is a the result of inverting a map from an invertible element ($S^{-\bar{\rho}}$) to the symmetric monoidal unit S^0. For formal reasons, this is guaranteed to be "infinitely coherently commutative". Thus we need to understand the different ways a G-commutative ring spectrum can be commutative.

2. Flavors of E_∞

In the non-equivariant context, the model of the E_∞ operad used for commutative rings is the linear isometries operad on an separable Hilbert space U. In the equivariant context, we have an additional choice: how does the group act on U? We consider only universes (so if an irreducible representation occurs, it does so infinitely often) that contain a trivial summand. This gives a hierarchy of operads, all of which are underlain by the ordinary linear isometries operad. In all cases, the key determination is which subgroups H are such that G/H embeds in U (just as with transfers in the additive context). The extremal cases are where U is a trivial universe (so only G/G embeds within), giving the "naive E_∞ operad" and U a complete universe (so all G/H embed within), giving the commutative operad.

There is a huge difference between the algebras over these operads. Operads over the naive E_∞ operad are "coherently homotopy commutative", but for a free algebra over this operad on Z, the geometric fixed points is the free algebra on the geometric fixed points of Z. In particular, for $Z = G_+$, the geometric fixed points are S^0. In stark contrast, for the commutative operad, the geometric fixed points can be much more complicated.

The spaces in these linear isometries operads are universal spaces for families of subgroups of $G \times \Sigma_n$. For the naive operad, the family is those subgroups contained in G. For the commutative operad, the family is those subgroups $H \subset G \times \Sigma_n$ such that $H \cap \Sigma_n = \{e\}$. As universal families, they have nice simplicial decompositions, and the only cells which appear are those with stabilizer in the families associated to the operad.

In all cases, Σ_n acts freely. Since $G \times \Sigma_n$ need not, there can be fixed points produced. The easiest way to describe how these interact, and to see the result, is to utilize the natural enrichment of the symmetric monoidal structure on G-spectra.

2.1. **Tensoring over G-spaces.**

It is well known that the category of commutative G-ring spectra is tensored over G-spaces (see for instance Mandell-May or the appendix to Hill-Hopkins-Ravenel [3]). The universal property defining the tensor structure establishes canonical equivalences

$$G/H \otimes R \simeq N_H^G Res_H^G(R).$$

This tells us how to tensor any commutative G-ring spectrum with any finite G-set (and therefore by the usual tricks with any G-space, though this will not directly be needed). What is perhaps more exciting is that while the left-hand side is not defined for R an arbitrary element of $\mathcal{S}_G$, the right-hand side is. This means that via the norm, we can define $X \otimes Z$ for a finite G-set X and a G-spectrum Z, and the underlying spectrum is just the $|X|$-fold smash power of Z.

The key step in proving the theorem is to identify

$$(G \times \Sigma_n/H)_+ \wedge_{\Sigma_n} Z^{\wedge n} = (G \times \Sigma_n/H)_+ \wedge_{\Sigma_n} (\underline{n} \otimes Z),$$

where $\underline{n}$ denotes the set with n-elements and a trivial G-action. Depending on H, smashing over Σ_n converts $\underline{n}$ into a different G-set (possibly with a non-trivial G-action). This means that smashing $G \times \Sigma_n/H$ over Σ_n with $\underline{n} \otimes Z$ yields a wedge of spectra of the form $X \otimes Z$ for various G-sets (determined by H).

The key example is as follows. Let $n = 2$, and let $G = C_2$. There is a subgroup, $\mathbb{Z}/2$ of $G \times \Sigma_2$ given by the diagonal (it's ludicrous, since all the groups are the same). Now consider

$$G \times \Sigma_2/H_+ \wedge_{\Sigma_2} (\underline{2} \otimes Z).$$

The quotient by H shows that we identify the canonical Σ_2-action on $\underline{2}$ with a C_2-action. This converts $\underline{2}$ into C_2 as a C_2-space, and hence

$$G \times \Sigma_2/H_+ \wedge_{\Sigma_2} (\underline{2} \otimes Z) = C_2 \otimes Z = N_e^{C_2} Res_e^{C_2}(Z).$$

A similar analysis holds in the general case.

3. Sketch of the proof

We need to show that if Z is acyclic, then $Z^{\wedge n}/\Sigma_n$ is acyclic. Knowing this will allow us to simply copy EKMM. We can replace $Z^{\wedge n}/\Sigma_n$ with $\mathcal{L}_n(U)_+ \wedge_{\Sigma_n} Z^{\wedge n}$, where $\mathcal{L}_n(U)$. Using the cell decomposition and the described analysis of the possible stabilizer subgroups, we see that this has a filtration with associated graded suspensions of wedges of smash products of norms. EKMM arguments handle the wedges and smash products, and by assumption, the norms are also acyclic.

We can do something better. We described above a hierarchy of commutative operads. The stronger statement, showed in essentially the same way, is as follows.

Theorem 2. *If for all L-acyclic spectra Z and for all G/H embedding in U the spectrum $N_H^G Res_H^G(Z)$ is L-acyclic, then for all commutative G-ring spectra R, $L(R)$ is an algebra over $\mathcal{L}(U)$.*

References

[1] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[2] Michael A Hill, Michael J Hopkins, and Douglas C Ravenel. On the non-existence of elements of Kervaire invariant one. arXiv.org: 0908.3724.

[3] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. *Mem. Amer. Math. Soc.*, 159(755):x+108, 2002.