CYCLIC COMODULES, THE HOMOLOGY OF j, AND j-HOMOLOGY

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1. Introduction

This short paper arose as a warm-up to the more difficult computations at the primes 2 and 3 of connective versions of Behrens' $Q(2)$ spectrum, a spectrum conjecturally describing half of the $K(2)$-local sphere [2]. The natural starting case is to understand the $K(1)$-local story, giving rise to the present discussion of the homology of the connective image of J spectrum. The homology results are by no means new: a great many authors, including Davis, Knapp, and Angeltveit-Rognes have given descriptions of the results [3, 5, 1]. The description herein contains an easier description using very general properties of comodules over a coalgebra or over a Hopf algebra, and so can be readily adapted to other situations. We also present a computation of the homotopy Hopf algebra of $H\mathbb{F}_p \wedge_j H\mathbb{F}_p$ for odd primes p, a Hopf algebra suitable for computing the j-homology of a finite j-module using a modified Adams spectral sequence similar to that employed by the author in [4].

2. Cyclic Comodules

In this section, we recast some easy and well-known statements about modules over an algebra A into statements about comodules over a graded coalgebra of finite type C. All algebras and coalgebras are taken over a ground field k, and everything considered will be graded and finite type. In particular, Hom is a functor to graded vector spaces over k. We begin with dual versions of generators and finite generation.

Definition 2.1. *Let N be a comodule with coaction ψ. A set of cogenerators for N is a collection $\{f_i | i \in I\}$ of elements of N^* such that the map $N \to \prod_I C$ defined by the composite*

$$N \xrightarrow{\psi} C \otimes_k N \xrightarrow{1 \otimes \prod f_i} \prod_I C$$

is injective.

A comodule is finitely cogenerated if the set I can be chosen to be finite.

A comodule is cyclic if the set I can be chosen to be the one point set.

These definitions are simply the linear dual of the corresponding statements for modules over an algebra. We will focus exclusively from this point on cyclic comodules.

Proposition 2.2. *If N is a C-comodule, then there is a natural isomorphism*

$$\mathrm{Hom}(N, C) \cong N^*.$$

Proof. This statement is the dual of the statement that if M is an A-algebra then $\mathrm{Hom}(A, M) = M$. Since everything is assumed to be finite type, the result follows from the isomorphism $\mathrm{Hom}(N, C) \cong \mathrm{Hom}(C^*, N^*)$.

The natural map $\mathrm{Hom}(N, C) \to N^*$ is defined by composing a comodule map $N \to C$ with the augmentation $C \to k$. □

Since cyclic comodules can be viewed as subcomodules of C, we can identify $\mathrm{Hom}(N, M)$ with a subspace of N^* whenever M is a cyclic comodule.

This set-up allows us to easily identify large pieces of the kernel of a map.

Lemma 2.3. *Let $F \in \mathrm{Hom}(N, C)$ correspond to a functional $f \in N^*$. If N' is a subcomodule of N that is in the kernel of f, then N' is in the kernel of F.*

Proof. Let ψ be the coaction on N. The map F is defined by the composite $(1 \otimes f) \circ \psi$. Since N' is a subcomodule, $Im(\psi|_{N'}) \subset C \otimes N'$, and this is annihilated by f. □

We conclude the section by strengthening several assumptions and getting some results about multiplicative structures. Let A be a connected Hopf algebra over k, and let N be a subcomodule of A which is also a subalgebra. All of the examples we will consider subsequently are of this form.

Lemma 2.4. *Let $F \in \mathrm{Hom}(N, A)$ correspond to an element $f \in N^*$. If f is primitive in the coalgebra structure induced by the algebra structure in N, then F is a derivation.*

Proof. We must show that $F(nm) = nF(m) + F(n)m$ for all $n, m \in N$. Let

$$\psi(n) = \sum_{i \geq 0} a_i \otimes n_i, \quad \psi(m) = \sum_{j \geq 0} b_j \otimes m_j,$$

where $n_0 = m_0 = 1$, $a_0 = n$, $b_0 = m$, and n_i and m_j are in the kernel of the augmentation map η for $i, j > 0$. Since N is a comodule algebra, we conclude that

$$F(nm) = \sum_{i,j \geq 0} (a_i b_j) f(n_i m_j).$$

Since f is primitive, we know that

$$f(n_i m_j) = f(n_i)\eta(m_j) + \eta(n_i) f(m_j),$$

and this implies that

$$F(nm) = \left(\sum_{i \geq 0} (a_i b_0) f(n_i) \right) + \left(\sum_{j \geq 0} (a_0 b_j) f(m_j) \right),$$

since n_i and m_j are in the kernel of η for $i, j > 0$. Pulling $b_0 = m$ out on the right and $a_0 = n$ out on the left gives the desired result. □

3. The Odd Primary Case

The odd primary connective j spectrum is defined by the fiber sequence

$$j \to \ell \xrightarrow{\psi_k - 1} \Sigma^{2p-2} \ell,$$

where k is chosen to be a topological generator of $\mathbb{Z}_p^\times$. Using the cyclic comodule idea, we can easily identify the effect in homology and compute $H_*(j)$ and a comodule algebra.

We first recall that

$$H_*(\ell) = \mathbb{F}_p[\bar{\xi}_1, \dots] \otimes E(\bar{\tau}_2, \dots) \subset \mathcal{A}_*,$$

where the inclusion is induced by the zeroth Postnikov section composed with reduction modulo p. In particular, this shows that $H_*(\ell)$ is a subcomodule algebra of $\mathcal{A}_*$. Proposition 2.2 then immediately implies that

$$\mathrm{Hom}(H_*(\ell), \Sigma^{2p-2}\mathcal{A}_*) = \mathbb{F}_p,$$

generated by the functional "cap with $\mathcal{P}^1$". This functional is the dual basis vector to ξ_1 in the monomial basis. We will soon see that every one of these comodule maps has image lying in $\Sigma^{2p-2}H_*(\ell)$.

We start by describing some properties of this map.

Lemma 3.1. *Let $F \in \mathrm{Hom}(H_*(\ell), \Sigma^{2p-2}\mathcal{A}_*)$, and let f denote the corresponding functional. Then the sub-comodule algebra*

$$(\mathcal{A}//\mathcal{A}(1))_* = \mathbb{F}_p[\xi_1^p, \bar{\xi}_2, \dots] \otimes E(\bar{\tau}_2, \dots)$$

is in the kernel of F.

Proof. We apply Lemma 2.3. The comodule map F corresponds to a scalar multiple of the element dual to ξ_1 in the basis dual to the monomial basis. This functional annihilates all of $(\mathcal{A}//\mathcal{A}(1))_*$, giving the result. □

Since $\mathcal{P}^1$ is a primitive in $\mathcal{A}$, it is one in the dual of $H_*(\ell)$. This means we can use Lemma 2.4 to conclude the following corollary.

Corollary 3.2. *If $p \in (\mathcal{A}//\mathcal{A}(1))_*$, then $F(p\xi_1^i) = pF(\xi_1^i)$.*

Corollary 3.3. *The inclusion $\Sigma^{2p-2}H_*(\ell) \to \mathcal{A}_*$ induces an isomorphism*

$$\mathrm{Hom}(H_*(\ell), \Sigma^{2p-2}H_*(\ell)) \cong \mathrm{Hom}(H_*(\ell, \Sigma^{2p-2}\mathcal{A}_*).$$

To finish our analysis of $H_*(j)$, we need to understand F slightly better. Since over $\mathbb{F}_p$ the actual non-zero scalars all give isomorphic results, we need only determine if F is 0.

Lemma 3.4. *The map $(\psi_k - 1)_*$ is non-zero if k generates $\mathbb{Z}_p^\times$.*

Proof. We assume to the contrary and show that if $(\psi_k - 1)_*$ is zero, then homotopy of j is wrong. If $(\psi_k - 1)_* = 0$, then $H_*(j)$ admits a filtration such that the associated graded as a comodule algebra is $H_*(\ell) \otimes E(x_{2p-3})$, where x_{2p-3} is in degree $2p-3$ and is primitive. This filtration gives rise to a spectral sequence computing the Adams E_2 term of the form

$$E_1 = \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_p, H_*(\ell) \otimes E(x_{2p-1})) \Longrightarrow \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_p, H_*(j))$$

and in which the differentials look like Adams d_1 differentials.

Since $H_*(\ell) = \mathcal{A}_* \square_{E(\tau_0,\tau_1)} \mathbb{F}_p$, we can use a change of rings argument to conclude that the E_1 term is actually

$$\mathrm{Ext}_{E(\tau_0,\tau_1)}(\mathbb{F}_p, E(x_{2p-3})) = \mathbb{F}_p[v_0, v_1] \otimes E(x_{2p-3}),$$

where the bidegrees, written in the Adams indexing of $(t-s, s)$, are $|v_0| = (0,1)$, $|v_1| = (2p-2, 1)$, and $|x_{2p-3}| = (2p-3, 0)$. A picture of this for $p = 3$ is given in Figure 1.

At this point, we reach our contradiction. For degree reasons x_{2p-3} is a permanent cycle. However, for degree reasons, the smallest v_0 multiple of it which can

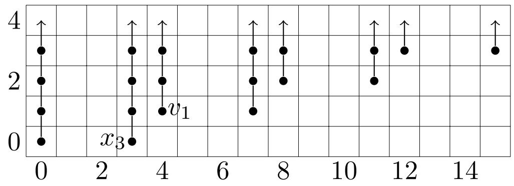

FIGURE 1. The E_1 term of the Spectral Sequence for $\pi_*(j)$ at $p = 3$

be killed is v_0^2. This would force $\pi_{2p-3}(j) = \mathbb{Z}/p^n$ where $n > 1$, contradicting the known result that $\pi_{2p-3}(j) = \mathbb{Z}/p$. $\square$

The end result is that we have computed the homology of j.

Theorem 3.5. *As a comodule algebra,*

$$H_*(j) = (\mathcal{A}//\mathcal{A}(1))_* \otimes E(x_{2p(p-1)-1}),$$

where $x_{2p(p-1)-1}$ is primitive and where we have a hidden comodule extension:

$$\psi(\xi_1^p) = \xi_1^p \otimes 1 + 1 \otimes \xi_1^p + \tau_0 \otimes x_{2p(p-1)-1}.$$

Proof. For degree reasons, the algebra structure and the coproduct on $x_{2p(p-1)-1}$ must be as stated. We resolve the comodule extension by computing Ext of the associated graded given to us by the fiber sequence. Since as an $\mathcal{A}_*$ comodule algebra,

$$Gr(H_*j) = \mathcal{A}_* \square_{\mathcal{A}(1)_*} E(x_{2p(p-1)-1}),$$

a change of rings argument establishes that

$$\operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_p, Gr(H_*j)) = \operatorname{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_p, \mathbb{F}_p) \otimes E(x_{2p(p-1)-1}).$$

As an algebra, $\operatorname{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_p, \mathbb{F}_p)$ is generated by classes v_0, v_1^p, β_1, and α_i for $1 \leq i \leq p-1$, in bidegrees $(0,1)$, $(2p(p-1), p)$, $(2p(p-1)-2, 2)$, and $(2i(p-1)-1, i)$ respectively, subject to the relations

$$v_0\alpha_i = 0, \quad \alpha_i\alpha_j = \begin{cases} 0 & i+j \neq p \\ v_0^{p-2}\beta_1 & i+j = p. \end{cases}$$

A picture of the E_1 term for the algebraic spectral sequence is presented in Figure 2 for $p = 3$. The salient features we will need are identical for large primes.

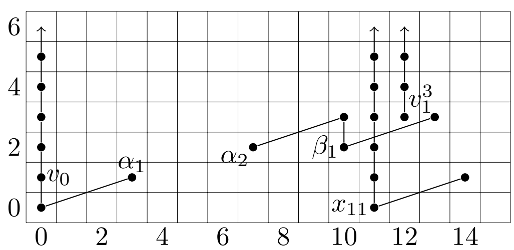

FIGURE 2. The Algebraic E_1 term from $Gr(H_*j)$

We see for degree reasons that in the first possible algebraic differential is on v_1^p, hitting $v_0^{p+1}x_{2p(p-1)-1}$. In particular, the algebraic E_1 page is the Adams E_2 page in the range $t - s \leq 2p(p-1) - 1$ and $s < p$.

Since the class β_1 is not present in the homotopy of j, we know that it cannot survive the Adams spectral sequence. For degree reasons, it can support neither algebraic nor Adams differentials, and it cannot be the target of any algebraic differentials. We must therefore kill this class with an Adams differential, and for degree reasons, we must conclude that

$$d_2(x_{2p(p-1)-1}) = \beta_1.$$

We complete the argument using the fact that j is a ring spectrum and therefore admits a natural map from the sphere. For $t-s < 2p(p-1)-1$, the map on Adams E_2 terms induced by the unit map from the sphere is surjective. In the Adams spectral sequence for the sphere, the class $v_0\beta$ is killed by an Adams d_2 originating on the class $h_{1,1}$, the class whose bar representative is $[\xi_1^p]$. By naturality of the Adams spectral sequence, this allows us to conclude that the element $h_{1,1}$ in the Adams spectral sequence for the sphere maps to the element $v_0 x_{2p(p-1)-1}$. This in turn implies the desired comodule extension, since v_0 multiplication detects τ_0 comultiplication.

We remark that naturality of the Adams spectral sequence also forces the aforementioned differential on v_1^p, since otherwise a v_0 torsion element would map to a v_0 torsion free element. The presence of any algebraic differentials implies a non-trivial comodule extension, and for degree reasons, this is the only possible extension. $\square$

The Adams operations are E_∞ ring maps, and this fiber sequence realizes j as the connective cover of an E_∞ ring spectrum, making it an E_∞ ring spectrum. We can apply and our computation of the homology this to compute

$$H_*^j H = \pi_*(H\mathbb{F}_p \wedge_j H\mathbb{F}_p).$$

Since $H_*^j H$ is a flat $\mathbb{F}_p$ module, it is a Hopf algebra, Ext over which computes the E_2 term of a variant of the Adams spectral sequence in j-modules.

Theorem 3.6. *If M is a j-module, then there is an Adams spectral sequence for the homotopy of the p-completion of M of the form*

$$E_2 = \operatorname{Ext}_{H_*^j H}(\mathbb{F}_p, \pi_*(H\mathbb{F}_p \wedge_j M)) \Longrightarrow \pi_*(M_p^\wedge).$$

Theorem 3.7. *As a Hopf algebra,*

$$H_*^j H = \pi_*(H\mathbb{F}_p \wedge_j H\mathbb{F}_p) = \Gamma(\xi_1) \otimes E(\tau_0, \tau_1),$$

where $\Gamma(-)$ denotes the "divided powers" Hopf algebra.

Proof. There is a topological change of ring equivalence of the form

$$H\mathbb{F}_p \wedge_j H\mathbb{F}_p = H\mathbb{F}_p \wedge_{(H\mathbb{F}_p \wedge j)} (H\mathbb{F}_p \wedge H\mathbb{F}_p).$$

If we take homotopy, then there is a Künneth spectral sequence computing the homotopy of the right hand side with E_2 term given by

$$\operatorname{Tor}^{H_*(j)}(\mathbb{F}_p, \mathcal{A}_*).$$

Since this Tor is computed just using the algebra structure, we conclude that as an algebra, the E_2 term is

$$\Gamma(\sigma x_{2p(p-1)-1}) \otimes \mathcal{A}(1)_* = \mathbb{F}_p[\xi_1]/\xi_1^p \otimes E(\tau_0, \tau_1) \otimes \Gamma(\sigma x_{2p(p-1)-1}).$$

For degree reasons, this spectral sequence collapses with no hidden multiplicative extensions, and thus this is the associated graded Hopf algebra corresponding to a filtration of $H_*^j H$.

We resolve the issue of hidden comultiplications by computing the Adams E_2 term computing the homotopy of j as a j-module. The Künneth spectra sequence produced an associated graded Hopf algebra of $H_*^j H$, and Ext over this Hopf algebra is the E_1 term of a spectral sequence computing Ext over $H_*^j H$. The corresponding Ext group is presented in Figure 3 for $p = 3$.

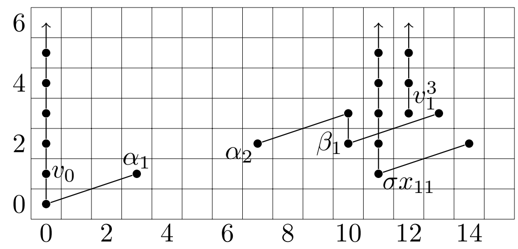

FIGURE 3. The E_1 term computing the Adams E_2 term for $\pi_* j$ at $p = 3$

In this, we see that the class β_1 is present, and since this class is not in the homotopy of j, it cannot survive both the algebraic and the Adams spectral sequences. For degree reasons, it cannot be the source of any algebraic or Adams differentials, and it also cannot be the target of any Adams differentials. This means it must be the target of an algebraic differential, and the only candidate for the source of such a differential is the class represented in the bar complex by $[\sigma x_{2p(p-1)-1}]$. The target of an algebraic differential records the actual coproduct on an element, so we conclude that

$$\psi(\sigma x_{2p(p-1)-1}) = (\sigma x_{2p(p-1)-1}) \otimes 1 + 1 \otimes (\sigma x_{2p(p-1)-1}) + \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} \xi_1^i \otimes \xi_1^{p-i}.$$

However, this is exactly the formula that realizes $\sigma x_{2p(p-1)-1}$ as the p^{th} divided power of ξ_1. $\square$

4. The story at 2

At $p = 2$ the story is slightly trickier. The only novel point of the computation is the use of homology, rather than cohomology. The main results about the structure of the induced map in homology and the resolution of extensions is identical to the treatment of Davis [3]. We will therefore omit these arguments. Here our model of the connective image of j spectrum is given by the fiber sequence

$$j \to ko \xrightarrow{\psi_3 - 1} bspin,$$

where $bspin = \Sigma^4 kspin$. We recall briefly the homologies involved.

Proposition 4.1. *As a subcomodule algebra of $\mathcal{A}_*$,*

$$H_*(ko) = \mathbb{F}_2[\xi_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \ldots].$$

As a subcomodule of $\mathcal{A}_*$, $H_*(kspin)$ is the $H_*(ko)$ submodule generated by 1, ξ_1^2, and $\bar{\xi}_2$.

In particular, all of the lemmata from Section 2 will apply in this context. In particular, Proposition 2.2 tells us that

$$\mathrm{Hom}(H_*(ko), \Sigma^4\mathcal{A}_*) = \mathbb{F}_2,$$

generated by the class $Sq^4 \in H^*(ko)$. Since Sq^4 is primitive, Lemma 2.4 tells us that the non-zero comodule homomorphism is a derivation of $H_*(ko)$-modules. We can again also identify immediately a large piece (in fact all) of the kernel of the non-zero map.

Proposition 4.2. *The subcomodule algebra*

$$(\mathcal{A}//\mathcal{A}(2))_* = \mathbb{F}_2[\xi_1^8, \bar{\xi}_2^4, \bar{\xi}_3^2, \bar{\xi}_4, \dots]$$

is in the kernel.

Proof. The non-zero map is determined by the element dual to ξ_1^4 in the basis dual to the monomial basis. Since this is not an element of $(\mathcal{A}//\mathcal{A}(2))_*$, Lemma 2.3 gives the result. □

The two maps are therefore determined by their values on their coimages.

Proposition 4.3. *The non-zero map is determined by the derivation from $(\mathcal{A}(2)//\mathcal{A}(1))_*$ to $(\mathcal{A}(2)//\mathcal{A}(1))_*\{1, \xi_1^2, \bar{\xi}_2\}$ given on generators by*

$$\xi_1^4 \mapsto 1, \quad \bar{\xi}_2^2 \mapsto \xi_1^2, \quad \bar{\xi}_3 \mapsto \bar{\xi}_2.$$

Corollary 4.4. *The kernel of the non-zero map is precisely $(\mathcal{A}//\mathcal{A}(2))_*$, and the cokernel is the quotient of $(\mathcal{A}//\mathcal{A}(1))_*\{1, \xi_1^2, \bar{\xi}_2\}$ by the subcomodule*

$$(\mathcal{A}//\mathcal{A}(2))_*\{1,\, \xi_1^2,\, \bar{\xi}_2,\, \bar{\xi}_2^2 + \xi_1^6,\, \bar{\xi}_3 + \xi_1^4\bar{\xi}_2,\, \xi_1^2\bar{\xi}_3 + \bar{\xi}_2^3,\, \bar{\xi}_2^2\bar{\xi}_3 + \xi_1^6\bar{\xi}_3 + \xi_1^4\bar{\xi}_2^3\}.$$

Just as with the odd primary case, it is easy to show that the map in homology is the non-zero map.

Theorem 4.5 ([3])**.** *The map $(\psi_3 - 1)_*$ is the non-zero map. As a corollary,*

$$0 \to \Sigma^3 M \to H_*j \to (\mathcal{A}//\mathcal{A}(2))_* \to 0,$$

where M is the quotient described above, is a short exactly sequence of $\mathcal{A}_*$ comodules.

Just as with $p > 2$, there is a non-trivial extension of comodules to give H_*j.

Lemma 4.6 ([3])**.** *The short exact sequence*

$$0 \to \Sigma^3 M \to H_*j \to (\mathcal{A}//\mathcal{A}(2))_* \to 0$$

is not split in comodules. There is a hidden coproduct generated by

$$\psi(\xi_1^8) = \xi_1^8 \otimes 1 + 1 \otimes \xi_1^8 + \xi_1 \otimes \xi_1^4 e_3,$$

where e_3 is the lowest dimensional class in $\Sigma^3 M$.

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