MATH 227A – LECTURE NOTES

INCOMPLETE AND UPDATING!

1. Obstruction Theory

A fundamental question in topology is how to compute the homotopy classes of maps between two spaces. Many problems in geometry and algebra can be reduced to this problem, but it is monsterously hard. More generally, we can ask when we can extend a map defined on a subspace and then how many extensions exist.

Definition 1.1. *If $f\colon A \to X$ is continuous, then let*

$$M(f) = X \amalg A \times [0,1]/f(a) \sim (a,0).$$

*This is the **mapping cylinder** of f.*

The mapping cylinder has several nice properties which we will spend some time generalizing.

(1) The natural inclusion $i\colon X \hookrightarrow M(f)$ is a deformation retraction: there is a continous map $r\colon M(f) \to X$ such that $r \circ i = Id_X$ and $i \circ r \simeq_X Id_{M(f)}$. Consider the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A \times I \arrow[d] \arrow[ddr, bend left, "f\circ\pi_A"] & \\
X \arrow[r, "i"'] \arrow[drr, bend right, "Id"'] & M(f) \arrow[dr, dashed, "r"] & \\
 & & X.
\end{tikzcd}$$

Since $A \to A \times I$ is a homotopy equivalence relative to the copy of A, we deduce the same is true for i.

(2) The map $j\colon A \to M(f)$ given by $a \mapsto (a,1)$ is a closed embedding and there is an open set U such that $A \subset U \subset M(f)$ and U deformation retracts back to A: take $A \times (1/2, 1]$. We will often refer to a pair $A \subset X$ with these properties as "good".

(3) The composite $r \circ j = f$.

One way to package this is that we have factored any map into a composite of a homotopy equivalence r with a closed inclusion j.

We can also understand continuous maps out of $M(f)$. For this, let's assume that the image of A in X is closed (although viewing this as the pushout instead fixes this).

Proposition 1.2. *A continuous map $\tilde{G}\colon M(f) \to Y$ is*

(1) *a continuous map $g_0\colon X \to Y$ and*

(2) *a homotopy $G\colon A \times I \to Y$ that begins with $g_0 \circ f$.*

If we now insist that $G|_{A\times\{1\}}$ is constant, then we are asking for null-homotopies of $g_0 \circ f$. This is also represented.

Definition 1.3. *The mapping cone of f, $C(f)$, is the quotient space $M(f)/A$.*

Another way to describe this is as the pushout

$$\begin{array}{ccc} A & \longrightarrow & CA \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ X & \longrightarrow & C(f). \end{array}$$

Proposition 1.4. *A continuous map $g\colon CA \to Y$ is*

1. *a map $g_0\colon A \to Y$ and*
2. *a null-homotopy of g_0, ie a homotopy from g_0 to the constant map.*

As a corollary, we have the following by pushout out along this description.

Corollary 1.5. *A continuous map $C(f) \to Y$ is*

1. *a map $g_0\colon X \to Y$ and*
2. *a null-homotopy of $g_0 \circ f$.*

Remark 1.6. *There is also a pointed version for all of this; here we simply collapse $* \times I$ to a point everywhere.*

The case of $A = S^n$ is exact the basis for a cell complex. Here, we begin with

$$X^{[0]} = \{x_0, \dots\}$$

with the discrete topology, and then inductively define pushout squares

$$\begin{array}{ccc} \coprod_{I_n} S^{n-1} & \xrightarrow{\coprod e_i} & X^{[n-1]} \\ \big\downarrow & & \big\downarrow \\ \coprod_{I_n} D^n & \longrightarrow & X^{[n]}. \end{array}$$

Corollary 1.5 shows then that a map

$$f_n\colon X^{[n]} \to Y$$

is two pieces of data:

1. a map $f_{n-1}\colon X^{[n-1]} \to Y$ and
2. null-homotopies of $f_{n-1} \circ e_i$ for all $i \in I_n$.

Thus we have a series of “obstructions” to extending f_{n-1} over the n-skeleton of X: if for some $i \in I_n$ we have that $f_{n-1} \circ e_i$ is not homotopic to the constant map, then we cannot extend over that cell.

The maps $f_{n-1} \circ e_i$ are maps $S^{n-1} \to Y$, and when we keep track of basepoint, these are elements of a group $\pi_{n-1}(Y, y_0)$, which we will describe extensively below. We therefore get a function

$$I_n \to \pi_{n-1}(Y, y).$$

When $n \geq 3$, the target group is abelian, so this map extends linearly to a homomorphism

$$C_n^{cell}(X) \to \pi_{n-1}(Y, y_0),$$

and we have an extension if and only if this homomorphism is zero. Understanding maps out of the chains on X is the theory of cohomology, which we first study, and then we will pick up the thread and consider the homotopy groups. Almost all of algebraic topology is tied up in this story.

Part 1. Cohomology

2. Graded abelian groups and complexes

Definition 2.1. *A graded abelian group is a sequence of abelian groups A_i, $i \in \mathbb{Z}$.*

Remark 2.2. *We can talk about graded objects in any category. These are just sequences of objects in that category. Equivalently, it is functors from $\mathbb{Z}$, viewed as a category with only identity morphisms, to our chosen category.*

Graded abelian groups form a category, and in fact, a category enriched in itself.

Definition 2.3. *If $C_\bullet$ and $D_\bullet$ are chain complexes, then a homomorphism from $C_\bullet \to D_\bullet$ is a sequence of homomorphisms*

$$f_n \colon C_n \to D_n.$$

A map of degree k is a sequence of maps

$$f_n \colon C_n \to D_{n+k}.$$

Since the set of homomorphisms between two abelian groups form an abelian group, the set of map of graded abelian groups forms an abelian group, and the maps of various degrees gives an obvious grading.

Definition 2.4. *If $C_\bullet$ and $D_\bullet$ are chain complexes, then let*

$$\mathrm{Hom}(C_\bullet, D_\bullet)_k = \{f \colon C_\bullet \to D_\bullet \mid f \text{ has degree } k\},$$

Composition here is a bilinear map

$$\mathrm{Hom}(C_\bullet, D_\bullet)_k \otimes \mathrm{Hom}(B_\bullet, C_\bullet)_\ell \to \mathrm{Hom}(B_\bullet, D_\bullet)_{k+\ell}.$$

Definition 2.5. *A chain complex (resp. a cochain complex) is a graded abelian group $C_\bullet$ together with a map*

$$d \colon C_\bullet \to C_\bullet$$

of degree -1 (resp. 1) such that $d^2 = 0$.

Chain and cochain complexes form a very interesting category that is a kind of algebraization of the homotopy category of spaces.

Definition 2.6. *If $(C_\bullet, d^C)$ and $(D_\bullet, d^D)$ are chain complexes, then a map of chain complex is a map of graded abelian groups $f \colon C_\bullet \to D_\bullet$ such that $f \circ d^C = d^D \circ f$.*

Let $\mathcal{C}h_{\mathbb{Z}}$ be the category of chain complexes and $co\mathcal{C}h_{\mathbb{Z}}$ be the category of cochain complexes.

Since the only difference between chain and cochain complexes is the degree of d, the categories are actually isomorphic via the functor

$$C_\bullet \mapsto C_{-\bullet},$$

Every structure theorem holds then for both. We will focus for now on chain complexes.

Definition 2.7. *If $(C_\bullet, d)$ is a chain complex, then the cycles of $C_\bullet$ are the graded abelian group*

$$Z_k(C_\bullet, d) = \ker(d_k),$$

the boundaries are the graded abelian group

$$B_k(C_\bullet, d) = Im(d_{k+1}),$$

and the homology of $C_\bullet$ is the graded abelian group

$$H_k(C_\bullet, d) = Z_n(C_\bullet, d)/B_n(C_\bullet, d).$$

If $f \colon C_\bullet \to D_\bullet$ is a map of chain complexes, then let

$$H_k(f)\big([z]\big) = \big[f(z)\big].$$

In general, we will suppress explicitly naming the differential, as is common in algebra, using only the name of the graded abelian group.

Proposition 2.8. *Homology so defined is well-defined and gives a functor from chain complexes to graded abelian groups.*

Exercise 2.1. *Prove this.*

Definition 2.9. *Let A be an abelian group and let*

$$\mathrm{Hom}(-, A) \colon \mathcal{C}h_{\mathbb{Z}} \to co\mathcal{C}h_{\mathbb{Z}}$$

be the functor which takes a chain complex $(C_\bullet, d)$ to the cochain complex with graded abelian group

$$\mathrm{Hom}(C_\bullet, A)_k := \mathrm{Hom}(C_k, A),$$

and with differential

$$\delta_k := \mathrm{Hom}(d_{k+1}, A).$$

Proposition 2.10. *This is well-defined and a functor.*

The cohomology of this cochain complex is the composite of two functors. We can compare it to the composite in the other order.

Proposition 2.11. *If $C_\bullet$ is a chain complex, then we have a natural map*

$$H^k\big(\mathrm{Hom}(C_\bullet, A)\big) \longrightarrow \mathrm{Hom}\big(H_k(C_\bullet), A\big)$$

$$[\phi] \longrightarrow \phi|_{Z_k(C_\bullet)}$$

Proof. We first check that this is well-defined. Consider

$$\phi' = \phi + \delta\psi = \phi + \psi \circ d,$$

and let $z \in Z_k(C)$. Then

$$\phi'(z) = \phi(z) + \psi\big(d(z)\big) = \phi(z).$$

Similarly, if $z' = z + d(w)$, then

$$\phi(z') = \phi(z) + \phi\big(d(w)\big) = \phi(z) + (\delta\phi)(w) = \phi(z),$$

since ϕ is assumed to be a cycle. $\square$

Theorem 2.12. *Let $C\bullet$ be a chain complex where for each degree, C_k is a free abelian group. Then for every abelian group A, we have a natural short exact sequence*

$$0 \to \operatorname{Ext}\big(H_{n-1}(C), A\big) \to H^n\big(\operatorname{Hom}(C_\bullet, A)\big) \to \operatorname{Hom}\big(H_n(C), A\big) \to 0.$$

Moreover, the sequence always splits (but not naturally).

Proof. Since C_k is free abelian, so are $B_k(C) \subset Z_k(C) \subset C_k$. The defining sequence

$$B_k C \to Z_k C \to H_k C$$

is therefore a projective resolution of $H_k C$. Now we observe that we have an exact sequence of chain complexes

$$0 \to Z_\bullet(C) \to C_\bullet \xrightarrow{d} B_{\bullet-1}(C) \to 0,$$

where Z and B have trivial differential[1]. Additionally, since $B_k C$ is always free, this sequence splits, but there is no natural splitting. The splitting however guarantees that we have a corresponding short exact sequence of cochain complexes

$$0 \leftarrow Z_\bullet(C)^* \leftarrow C^*_\bullet \xleftarrow{\delta} B_{\bullet-1}(C)^* \leftarrow 0,$$

where here the $(-)^*$ notation refers to $\operatorname{Hom}(-, A)$. This induces a long exact sequence in cohomology. Since the differentials in the complexes $Z_\bullet(C)^*$ and $B_{\bullet-1}(C)^*$ are zero, these are canonically their cohomology. This gives us

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(C^*) \arrow[r] & Z_n(C)^* \arrow[r] & B_n(C)^* \arrow[dll, rounded corners] & \\
& H^{n+1}(C^*) \arrow[r] & Z_{n+1}(C)^* \arrow[r] & B_{n+1}(C)^* \arrow[r] & \cdots
\end{tikzcd}$$

By construction of the coboundary map in the long exact sequence, we know that the coboundary map

$$Z_n(C)^* \to B_n(C)^*$$

is the dual of the canonical inclusion

$$B_n(C) \to Z_n(C).$$

Since this inclusion is a projective resolution for $H_n(C)$, we deduce that the kernel of the coboundary is canonically $\operatorname{Hom}\big(H_n(C), A\big)$ and the cokernel is canonically $\operatorname{Ext}\big(H_n(C), A\big)$. Our long exact sequence then gives us short exact sequences

$$0 \to \operatorname{Ext}\big(H_{n-1}(C), A\big) \to H^n\big(\operatorname{Hom}(C_\bullet, A)\big) \to \operatorname{Hom}\big(H_n(C), A\big) \to 0.$$

$\square$

[1] the shift here is because the image of d_k is B_{k-1}

2.1. **Exercises!** Let R be a commutative, unital, associative ring, and let $\mathcal{C}h$ be the category of bounded below complexes of R-modules, homologically graded. Let $\mathrm{Hom}(C_*, D_*)$ denote the R-module of chain maps from C_* to D_*. Finally, all tensor products are assumed to be over R.

The first three problems have the same general form, and in fact, each is suborned by the next. You may simply prove the final one and use it to prove the earlier ones (though you may find a direct approach to the early ones gives intuition).

Definition 2.13. *Let $\Delta^1_\bullet$ be the chain complex defined by*

$$\Delta^1_k = \begin{cases} 0 & k \neq 0, 1 \\ R & k = 1 \\ R \oplus R & k = 0, \end{cases}$$

where the only interesting differential is $\partial_1(r) = (-r, r)$.

We should this of this as the cellular chain complex associated the standard cell structure on $[0, 1]$.

Definition 2.14. *If $C_\bullet$ and $D_\bullet$ are chain complexes, then let $(C \otimes D)_\bullet$ be the chain complex with*

$$(C \otimes D)_k = \bigoplus_{i+j=k} C_i \otimes D_j,$$

and

$$\partial(c \otimes d) = \partial_C(c) \otimes d + (-1)^{|c|} c \otimes \partial_D(d).$$

Exercise 2.2. *If $C_\bullet$ is any chain complex, show that*

$$\mathrm{Hom}(\Delta^1_\bullet \otimes C_\bullet, D_\bullet) \cong \{(f, g, F) | f, g \in \mathrm{Hom}(C_\bullet, D_\bullet), f \overset{F}{\simeq} g\}.$$

(In other words, a map from $\Delta^1_\bullet \otimes C_\bullet$ is the same data as a pair of maps from $C_\bullet$ together with a homotopy between them.)

Definition 2.15. *Let S^n be the unique chain complex with*

$$S^n_k = \begin{cases} R & k = n \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 2.16. *Let $d_i \colon S^0 \to \Delta^1_\bullet$ be the inclusion of the i^{th} summand. This induces a natural inclusion*

$$d_0 \colon C_\bullet \to \Delta^1_\bullet \otimes C_\bullet.$$

Let $f \colon C_\bullet \to D_\bullet$. Define the "mapping cylinder" of f, by

$$M(f) = (\Delta^1_\bullet \otimes C_\bullet) \oplus_{C_\bullet} D_\bullet,$$

where $\oplus_{C_\bullet}$ means we identify the copy of $C_\bullet$ in $\Delta^1_\bullet$ with $f(C_\bullet)$ in $D_\bullet$.

Exercise 2.3. *Show that for any f, we have a natural bijection*

$$\mathrm{Hom}(M(f)_\bullet, E_\bullet) \cong \{(g, h, F) | g \in \mathrm{Hom}(C_\bullet, E_\bullet), h \in \mathrm{Hom}(D_\bullet, E_\bullet), g \overset{F}{\simeq} h \circ f\}.$$

Definition 2.17. *Let $f \colon C_\bullet \to E_\bullet$ and $g \colon C_\bullet \to D_\bullet$ be maps. Define the* homotopy push out *(aka the double mapping cylinder) to be*

$$D_\bullet \oplus^h_{C_\bullet} E_\bullet = D_\bullet \oplus_{C_\bullet} (\Delta^1_\bullet \otimes C_\bullet) \oplus_{C_\bullet} E_\bullet,$$

where we have used the d_0 copy of $C_\bullet$ to attach $D_\bullet$ and the d_1 copy of $C_\bullet$ to attach $E_\bullet$.

The name here indicates that this is the push out "up to homotopy". The actual, categorical push out is just $D_\bullet \oplus_{C_\bullet} E_\bullet$, but this isn't well behaved.

Exercise 2.4. *Show that there is a natural bijection*

$$\mathrm{Hom}(D_\bullet \oplus^h_{C_\bullet} E_\bullet, F_\bullet) \cong \{(a,b,H) | a \in \mathrm{Hom}(D_\bullet, F_\bullet), b \in \mathrm{Hom}(E_\bullet, F_\bullet), a \circ g \overset{H}{\simeq} b \circ f\}.$$

Definition 2.18. *If $C_\bullet$ and $D_\bullet$ are chain complexes, then let $[C_\bullet, D_\bullet]$ denote the chain-homotopy classes of maps from $C_\bullet$ to $D_\bullet$.*

Exercise 2.5. *Consider the homotopy pushout square*

$$\begin{tikzcd} C_\bullet \arrow[r, "f"] \arrow[d, "g"'] & E_\bullet \arrow[d] \\ D_\bullet \arrow[r] & D_\bullet \oplus^h_{C_\bullet} E_\bullet \end{tikzcd}$$

Prove that the induced Mayer-Vietoris sequence:

$$[D_\bullet \oplus^h_{C_\bullet} E_\bullet, F_\bullet] \to [D_\bullet, F_\bullet] \oplus [E_\bullet, F_\bullet] \to [C_\bullet, F_\bullet]$$

is exact in the middle.

Definition 2.19. *If $f \colon C_\bullet \to D_\bullet$ is a chain map, then let $C(f)_\bullet$ be the homotopy pushout $0 \oplus^h_{C_\bullet} D_\bullet$.*

Exercise 2.6.

(1) *Describe the groups $C(f)_k$ and the differential in terms of the map f and the chain complexes $C_\bullet$ and $D_\bullet$.*

(2) *Deduce that the sequence*

$$C_\bullet \xrightarrow{f} D_\bullet \to C(f)_\bullet$$

is coexact in the homotopy category: for any complex $E_\bullet$, we have an exact sequence

$$[C(f)_\bullet, E_\bullet] \to [D_\bullet, E_\bullet] \to [C_\bullet, E_\bullet].$$

As a counter to this, we can ask "Why aren't we using the actual quotient?"

Exercise 2.7. *If $f \colon C_\bullet \to D_\bullet$, then let $D/C_\bullet$ be the quotient of $D_\bullet$ by the image of f. Show that for any $E_\bullet$, the sequence*

$$\mathrm{Hom}(D/C_\bullet, E_\bullet) \to \mathrm{Hom}(D_\bullet, E_\bullet) \to \mathrm{Hom}(C_\bullet, E_\bullet)$$

is exact in the middle. Given an example that explains why the corresponding statement about homotopy classes of maps need not be true.

3. Ext and Singular cohomology

3.1. **Computing** Ext. The Universal Coefficients Theorem shows that we can compute the cohomology groups of $\mathrm{Hom}(C_\bullet, A)$ for any A functorially out of the homology groups of $C_\bullet$ via Hom and its first derived functor Ext. Since the latter is perhaps less familiar, we start with this. So how do we compute $\mathrm{Ext}(M, N)$ for two module M and N? The general procedure does not use any features of $\mathbb{Z}$, so in particular, it works for any (commutative) ring R.

Definition 3.1. *An R-module M is projective if for every surjective map $N' \xrightarrow{f} N$ and for every map $g\colon M \to N$, there is a left $\tilde{g}\colon M \to N'$ such that $f \circ \tilde{g} = g$.*

The first step is to form a projective resolution of M. This is a long exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where each P_i is a projective R-module. If we throw away M here, then we have a chain complex $P_\bullet$, and the resolution identifies the homology of $P_\bullet$ with M via the map $P_0 \to M$. We can view this as being a map of complexes $P_\bullet \to M$, where here M is a complex consisting entirely of M in degree 0. To say that $P_\bullet$ is a projective resolution of M is then to say that each P_k is projective and that the map is an isomorphism in homology.

The second step is to build the associated cochain complex $\operatorname{Hom}_R(P_\bullet, N)$.

Definition 3.2. *If M and N are R-modules, then the groups $\operatorname{Ext}^k(M,N)$ are defined by*

$$\operatorname{Ext}^k_R(M,N) := H^k\big(\operatorname{Hom}_R(P_\bullet, N)\big)$$

for some choice of projective resolution $P_\bullet$ of M.

As written, this seems to depend very heavily on the choice of projective resolution of M. Moreover, while this is visibly functorial in N, we have no way to see that it is functorial also in M. However, in the Exercises, one shows that any two projective resolutions of M are chain homotopy equivalent, and hence the resulting cochain complexes are chain homotopy equivalent. Additionally, one shows that given any map $M \to M'$ and projective resolutions $P_\bullet$ and $P'_\bullet$ respectively, there is a map of chain complexes $P_\bullet \to P'_\bullet$ lifting the map. Together these gives independence of the choice of resolution and functoriality.

Proposition 3.3. *For any R-modules M and N, we have*

$$\operatorname{Ext}^0_R(M,N) \cong \operatorname{Hom}_R(M,N).$$

Proof. This follows from the left-exactness of Hom_R. □

Now some basic computations.

Proposition 3.4. *If M is projective, then $\operatorname{Ext}^k_R(M,N)$ vanishes for all $k > 0$.*

Proof. Since M is projective, $P_0 = M$ and $P_{>0} = 0$ gives a projective resolution of M. □

Proposition 3.5. *Let M_i, $i \in I$ be a collection of R-modules. Then*

$$\operatorname{Ext}^k_R\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \operatorname{Ext}^k_R(M_i, N).$$

Proof. For each $i \in I$, let $P^i_\bullet$ be a projective resolution for M_i. Since arbitrary direct sums of projects are projective, this gives that $\bigoplus P^i_\bullet$ is a projective resolution of $\bigoplus M_i$. The result follows from the fact that $\oplus$ is the coproduct in R-modules. □

3.2. **Unpacking** Ext **for** $\mathbb{Z}$. We restrict now to $R = \mathbb{Z}$, although everything we use works for an arbitary PID. Since subgroups of free abelian groups are free, if $\epsilon\colon P_0 \to M$ is any surjective map with P_0 free, then $P_1 = \ker(\epsilon)$ is also free. This gives a projective resolution[2]

$$\cdots \to 0 \to P_1 \to P_0.$$

Corollary 3.6. *For all $k > 1$ and for all M and N, $\operatorname{Ext}^k_{\mathbb{Z}}(M,N) = 0$.*

By the structure theory of finitely generated abelian groups, Propositions 3.5 and Proposition 3.4 will allow us to compute $\operatorname{Ext}^1(M,N)$ for any finitely generated abelian group M once we know how to compute $\operatorname{Ext}^1(\mathbb{Z}/m,N)$.

Proposition 3.7. *Let N be an abelian group. Then we have natural isomorphisms*

$$\operatorname{Hom}(\mathbb{Z}/m,N) \cong \{n \in N \mid m\cdot n = 0\} \text{ and } \operatorname{Ext}^1(\mathbb{Z}/m,N) \cong N/mN.$$

Proof. We have a simple projective resolution of $\mathbb{Z}/m$:

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z}.$$

Applying $\operatorname{Hom}(-,N)$ to this gives a complex

$$\ldots \leftarrow 0 \leftarrow N \xleftarrow{m} N,$$

where via the evaluation at 1, we have identified $\operatorname{Hom}(\mathbb{Z},N)$ with N. The result follows. $\square$

Remark 3.8. *If M is not finitely generated, then we can get extremely strange results. For example*

$$\operatorname{Ext}^1(\mathbb{Q},\mathbb{Z}) \cong \left(\prod_{p \text{ prime}} \mathbb{Z}_p^{\wedge}\right)/\mathbb{Z}$$

is an uncountable dimensional $\mathbb{Q}$-vector space.

3.3. **Singular Cohomology.** We now restrict the kinds of chain and cochain complexes which arise to those most natural in topology.

Definition 3.9. *If X is a space and M is an abelian group, then the singular cochains complex of X is*

$$C^\bullet(X;M) := \operatorname{Hom}\big(C_\bullet(X),M\big).$$

Similarly, if $A \subset X$, then the relative singular cochaines are given by

$$C^\bullet(X,A;M) := \operatorname{Hom}\big(C_\bullet(X,A),M\big).$$

The singular cohomology of X with coefficients in M is the cohomology of the cochain complex $C^*(X;M)$.

There are similar definitions with "cellular" replacing "singular"; nothing changes.

Essentially all of the results we know from ordinary singular homology go through without change:

(1) $H^*(X,A;M)$ is a homotopy functor.
(2) If (X,A) is a pair, then we have an associated long exact sequence linking the cohomologies of A, X, and (X,A).
(3) Excisions / Mayer-Vietoris sequence.

[2]We can even make this more functorial by taking P_0 to be the free abelian group generated by the elements of M.

(4) A dimension axiom: $H^*(pt; M)$ is M if $* = 0$ and 0 otherwise.

We also have a "continuity" condition.

Proposition 3.10. *If X_i, $i \in I$ is a collection of spaces, then the inclusion maps $X_j \hookrightarrow \coprod X_i$ induce an isomorphism*

$$H^k\left(\coprod_{i\in I} X_i; M\right) \cong \prod_{i\in I} H^k(X_i; M).$$

Proof. Since simplices are connected, we have a splitting of cochain complexes

$$C_*\left(\coprod_{i\in I} X_i\right) \cong \bigoplus_{i\in I} C_*(X_i).$$

Applying $\mathrm{Hom}(-, M)$ and using the universal property of the direct sum gives the result. □

Here is an example of how one shows any of the other cases: the derivation of the long exact sequence for the pair. This has several interesting features in its own right.

Recall that for any pair (X, A), the relative chains are defined by an exact sequence

$$0 \to C_k(A) \to C_k(X) \to C_k(X, A) \to 0. \tag{1}$$

Since the inclusion map $C_k(A) \to C_k(X)$ is the inclusion of a direct summand, we have a splitting (of abelian groups!)

$$C_k(X) \cong C_k(X, A) \oplus C_k(A).$$

In particular, $\mathrm{Hom}(-, M)$ takes Equation 1 to an exact sequence

$$0 \leftarrow C^k(A; M) \leftarrow C^k(X; M) \leftarrow C^k(X, A; M) \leftarrow 0. \tag{2}$$

Taking cohomology then gives us our desired exact sequence.

Remark 3.11. *The relative cochains have a much more natural geometric description than the relative chains. The cochains on X are M-valued functionals on the singular simplicies in X. Equation 2 then shows that the relative cochains $C^k(X, A; M)$ are exactly those functionals on the singular simplicies in X which vanish on the singular simplicies in A.*

3.4. **Exercises!**

Exercise 3.1. *Let M_1 and M_2 be R-modules, and let P^i_* be a projective resolution of M_i.*

(1) *Show that any map $M_1 \to M_2$ induces a map of chain complexes $P^1_* \to P^2_*$.*
(2) *Show that any two lifts of a map $M_1 \to M_2$ are chain homotopic.*

Exercise 3.2. *If N is a finitely generated abelian group, determine*

$$\mathrm{Ext}(\mathbb{Z}/p^k, N)$$

for all p and all k.

Exercise 3.3. *Let A be a finitely generated abelian group and let $M(A,n)$ be a Moore space with*

$$\tilde{H}_k\big(M(A,n)\big) = \begin{cases} 0 & k \neq n \\ A & k = n. \end{cases}$$

(If you don't recall how to construct such a space, figure out how to make examples). Compute

$$H^*\big(M(A,n); N\big).$$

Just as Hom was a left-exact functor, $\otimes$ is a right exact functor: if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of R modules, then

$$A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is also for any R-module M.

Definition 3.12. *If N and M are R-modules, then let*

$$\mathrm{Tor}^i_R(N, M) = H_i(P_* \otimes_R M),$$

where P_* is a projective resolution of N.

Exercise 3.4. *Mirroring the proof of the universal coefficients theorem, prove the universal coefficients theorem for homology: if A is an abelian group, then for any chain complex C_* with C_k a free abelian group for all k, we have a natural short exact sequence*

$$0 \to H_n(C_*) \otimes A \to H_n(C_* \otimes A) \to \mathrm{Tor}^1(H_{n-1}(C_*), A) \to 0. \tag{3}$$

4. Cup Product

The Universal Coefficients Theorem shows us that the cohomology *groups* of a space X are functorially determined by the homology groups of X. Moreover, if X is a finite type complex, then we have a natural isomorphism of chain complexes

$$C_*^{cell}(X) \cong \mathrm{Hom}\left(C^*_{cell}(X;\mathbb{Z}), \mathbb{Z}\right) \cong \mathrm{Hom}\left(\mathrm{Hom}(C_*^{cell}(X), \mathbb{Z}), \mathbb{Z}\right),$$

since $C_k^{cell}(X)$ is a finitely generated free abelian group. Thus the cohomology of X determines the homology of X. So one might ask why we bother to study the cohomology at all! The answer is that we have *extra* natural structure on cohomology: a multiplication.

Definition 4.1 (Cup Product)**.** *Let R be an associative ring. The cup product on singular cochains with coefficients in R is the bilinear map*

$$- \smile -\colon C^k(X;R) \times C^\ell(X;R) \to C^{k+\ell}(X;R)$$

defined on a singular simplex

$$\sigma\colon \Delta^{k+\ell} = [v_0, \ldots, v_{k+\ell}] \to X$$

by

$$(\phi \smile \psi)(\sigma) := \big(\phi(\sigma|_{[v_0,\ldots,v_k]})\big) \cdot \big(\psi(\sigma|_{[v_k,\ldots,v_{k+\ell}]})\big).$$

We have constructed this to be bilinear. It is also obviously associative. One thing to note is that the vertex v_k occurs in both the initial k-simplex and the terminal ℓ-simplex. This can be helpful in picturing what is happening. Additionally, we will henceforth always use the shorthand

$$[v_i, \dots, v_j] := \sigma|_{[v_i,\dots,v_j]}.$$

Proposition 4.2. *The augmentation map $\epsilon\colon C_0(X) \to R$ which sends each singular simplex $\sigma\colon \Delta^0 = [v_0] \to X$ to 1 is the 2-sided multiplicative unit.*

Proof. We check this on singular simplices. Let $\phi \in C^k(X;R)$ and let $\sigma = [v_0, \dots, v_k]$ be a k-simplex in X.

$$(\epsilon \smile \phi)\big([v_0, \dots, v_k]\big) = \epsilon\big([v_0]\big) \cdot \phi\big([v_0, \dots, v_k]\big) = \phi\big([v_0, \dots, v_k]\big).$$

Thus $\epsilon \smile \phi$ and ϕ give the same functional on $C_k(X)$, and hence must agree. The other side is the same. $\square$

The cup product is also natural.

Proposition 4.3. *If $f\colon X \to Y$ is continuous, then*

$$f^*(\phi \smile \psi) = (f^*\phi) \smile (f^*\psi).$$

Proof. We check this on a singular $(k+\ell)$-simplex $[v_0, \dots, v_{k+\ell}]$ in X:

$$\begin{aligned} f^*(\phi \smile \psi)\big([v_0, \dots, v_{k+\ell}]\big) &= (\phi \smile \psi)\big(f_*[v_0, \dots, v_{k+\ell}]\big) \\ &= \phi\big(f_*[v_0, \dots, v_k]\big) \cdot \psi\big(f_*[v_k, \dots, v_{k+\ell}]\big) \\ &= (f^*\phi)\big([v_0, \dots, v_k]\big) \cdot (f^*\psi)\big([v_k, \dots, v_{k+\ell}]\big) \\ &= \big((f^*\phi) \smile (f^*\psi)\big)\big([v_0, \dots, v_{k+\ell}]\big). \end{aligned}$$

Since $f^*(\phi \smile \psi)$ and $(f^*\phi) \smile (f^*\psi)$ agree on all singular simplices of X, they are equal. $\square$

We now bring back in the boundary maps. There is essentially only one way to prove any statements like the one in the next Proposition: write out all of the terms and observe a linear dependence relation.

Proposition 4.4. *For $\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, we have*

$$\delta(\phi \smile \psi) = (\delta\phi) \smile \psi + (-1)^k \phi \smile (\delta\psi).$$

Proof. Let $[v_0, \dots, v_{k+\ell+1}]$ be a singular $(k+\ell+1)$-simplex. Then we have

$$\begin{aligned} \delta(\phi \smile \psi)\big([v_0, \dots, v_{k+\ell+1}\big) &= (\phi \smile \psi)\big(d[v_0, \dots, v_{k+\ell+1}]\big) \\ &= (\phi \smile \psi)\left(\sum_{j=0}^{k+\ell+1} (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_{k+\ell+1}]\right) \\ &= \sum_{j=0}^{k+\ell+1} (-1)^j (\phi \smile \psi)\big([v_0, \dots, \hat{v}_j, \dots, v_{k+\ell+1}]\big). \end{aligned}$$

Similarly, we have

$$\begin{aligned}\big((\delta\phi)\smile\psi\big)\big([v_0,\dots,v_{k+\ell}]\big) &= (\delta\phi)\big([v_0,\dots,v_{k+1}]\big)\cdot\psi\big([v_{k+1},\dots,v_{k+\ell+1}]\big)\\ &= \sum_{j=0}^{k+1}(-1)^j\phi\big([v_0,\dots,\hat{v}_j,\dots,v_{k+1}]\big)\cdot\psi\big([v_{k+1},\dots,v_{k+\ell+1}]\big)\\ &= \left(\sum_{j=0}^{k}(-1)^j(\phi\smile\psi)\big([v_0,\dots,\hat{v}_j,\dots,v_{k+1},\dots,v_{k+\ell+1}]\big)\right)\\ &\qquad + (-1)^{k+1}\phi\big([v_0,\dots,v_k]\big)\psi\big([v_{k+1},\dots,v_{k+\ell+1}]\big).\end{aligned}$$

Finally, we have

$$\begin{aligned}(-1)^k\big(\phi\smile(\delta\psi)\big)\big([v_0,\dots,v_{k+\ell+1}]\big) &= (-1)^k\phi\big([v_0,\dots,v_k]\big)\cdot(\delta\psi)\big([v_k,\dots,v_{k+\ell+1}]\big)\\ &= \sum_{j=k}^{k+\ell+1}(-1)^j\phi\big([v_0,\dots,v_k]\big)\cdot\psi\big([v_k,\dots,\hat{v}_j,\dots,v_{k+\ell+1}]\big)\\ &= \left(\sum_{j=k+1}^{k+\ell+1}(-1)^j(\phi\smile\psi)\big([v_0,\dots,v_k,\dots,\hat{v}_j,\dots,v_{k+\ell+1}]\big)\right)\\ &\qquad + (-1)^k\phi\big([v_0,\dots,v_k]\big)\cdot\psi\big([v_{k+1},\dots,v_{k+\ell+1}]\big).\end{aligned}$$

The result follows. □

Corollary 4.5. *The cup product of cocycles is a cocycle. The collection of coboundaries form an ideal in the ring of cycles.*

Proof. The first part is immediate. For the second, if $\phi=\delta\phi'$ and ψ is a cocycle, then $\phi\smile\psi=\delta(\phi'\smile\psi)$ and similarly for $\psi\smile\phi$. □

Corollary 4.6. *The cup product gives a natural, unital, associative product on $H^*(X;R)$ for any unital, associative ring R:*

$$[\phi]\smile[\psi] := [\phi\smile\psi].$$

Theorem 4.7. *If R is a commutative ring, then the cup product is graded commutative: if $[\phi]\in H^k(X;R)$ and $[\psi]\in H^\ell(X;R)$, then*

$$[\phi]\smile[\psi] = (-1)^{k\ell}[\psi]\smile[\phi].$$

Proof. We modify the "prism" operator used to show homotopy invariance to build a chain homotopy between $\phi\smile\psi$ and $(-1)^{k\ell}\psi\smile\phi$. Let $n=k+\ell$.

For each natural number j, let

$$\epsilon_j = (-1)^{\frac{j(j+1)}{2}}.$$

For each simplex $[v_0,\dots,v_n]$, let

$$\tau\big(v_0,\dots,v_n\big) = \epsilon_n[v_n,\dots,v_0].$$

We will first build a chain homotopy between the identity and τ. Consider the prism $\Delta^n\times I$, and let $\{v_0,\dots,v_n\}$ be the vertices of the bottom face and $\{w_0,\dots,w_n\}$ be the vertices of the top face. In the proof that homology is a homotopy invariant, we gave this one cell structure. Now we give a different one, taking for each $0 \le j \le n$ the $(n+1)$-simplex with vertices

$$[v_0, \dots, v_j, w_n, \dots, w_j].$$

Note that this takes place in our abstract simplicial complex $\Delta^n \times I$. The projection map $q\colon \Delta^n \times I \to \Delta^n$ then gives us a map back to Δ^n, which we can compose with σ. In particular, we have produced a singular $(n+1)$-simplex of X. Let

$$P(\sigma) = \sum_{j=0}^{n} (-1)^j \epsilon_{n-j} \sigma \circ q\big([v_0, \dots, v_j, w_n, \dots, w_j]\big).$$

By the standard induction argument used for homotopy invariance, this gives a chain homotopy between the identity map and τ, so P^* gives a chain homotopy between the identy and τ^* on $C^*(X;R)$. Now we compute

$$(\tau^*\phi)\big([v_0, \dots, v_k]\big) = \epsilon_k \phi\big([v_k, \dots, v_0]\big)$$
$$(\tau^*\psi)\big([v_k, \dots, v_n]\big) = \epsilon_\ell \psi\big([v_n, \dots, v_k]\big).$$

This implies that

$$\epsilon_k \epsilon_\ell (\tau^*\phi \smile \tau^*\psi) = \epsilon_{k+\ell} \tau^*(\psi \smile \phi).$$

Since τ^* is chain homotopic to the identity, it induces the identity on cohomology, giving the result. $\square$

Remark 4.8. *There is a slightly more general version where we allow R to be a graded commutative ring. The result still holds there.*

5. Künneth Theorem

We have very few computations we can do directly with the definition of the cup product.

Example 5.1. *For any n, $H^*(S^n;R) \cong E_R(x_n) = R[x_n]/x_n^2$, where $|x_n| = n$, since there is no room for any non-trivial products in the cohomology.*

Example 5.2. *We have a splitting of rings*

$$H^*(X \amalg Y;R) \cong H^*(X;R) \times H^*(Y;R).$$

The corresponding idempotents are the augmentations which send singular 0-simplices in X to 1 and those in Y to 0 (and vice versa).

Example 5.3. *We can write $\mathbb{R}P^2$ as a quotient of the 2-simplex $[v_0, v_1, v_2]$, where we identify $[v_0, v_1]$ with $[v_1, v_2]$ and send $[v_0, v_2]$ to a point. This gives a CW-structure with 1 0-, 1-, and 2-cell. Let x_1 be the class dual to the 1-cell and x_2 the class dual to the 2-cell. Then we have*

$$(x_1 \smile x_1)\big([v_0, v_1, v_2]\big) = x_1\big([v_0, v_1]\big) \cdot x_1\big([v_1, v_2]\big) = 1.$$

Thus $x_1 \smile x_1$ must also be the dual basis vector to the 2-cell: $x_1 \smile x_1 = x_2$.

To build more spaces, we use an external version of the cup product, the cross product.

Definition 5.4. *Let X and Y be space. If $\phi \in H^k(X;R)$ and $\psi \in H^\ell(Y;R)$, then let*

$$\phi \times \psi = (q_X^*\phi) \smile (q_Y^*\psi) \in H^{k+\ell}(X \times Y;R),$$

where $q_?$ is the projection onto $?$. This is the "cross product" of ϕ and ψ.

Remark 5.5. *The cross product also determines the cup product. Take $Y = X$, and consider the diagonal map $\Delta\colon X \to X \times X$. Then since $q_i \circ \Delta = Id$, where q_i now is the projection onto the ith factor,*

$$\Delta^*(\phi \times \psi) = \phi \smile \psi.$$

We can describe this operation in more categorical terms, using the tensor product in graded R-modules. For this, recall the tensor product of chain complexes from Definition 2.14. Forgetting the differential, this also gives us the tensor product of graded abelian groups. We prolong this to an operation on graded commutative R-algebras.

Definition 5.6. *If $A_\bullet$ and $B_\bullet$ are graded commutative R-algebras, then $(A \otimes_R B)_\bullet$ becomes a graded commutative algebra via*

$$(a \otimes b) \cdot (a' \otimes b') := (-1)^{|a'||b|}(aa') \otimes (bb').$$

Remark 5.7. *The degree here arises from graded commutativity: we are swapping a' and b.*

The usual argument from algebra shows the following.

Proposition 5.8. *The tensor product is the coproduct in the category of graded commutative R-algebras.*

This gives us a way to interpret the cross product. The projection maps q_X and q_Y, being maps of spaces, induce maps of graded commutative R-algebras

$$H^*(X;R) \xrightarrow{q_X^*} H^*(X \times Y;R) \xleftarrow{q_Y^*} H^*(Y;R).$$

By the universal property of the coproduct, this gives us a unique map from the tensor product to $H^*(X \times Y;R)$. Unpacking the definition, we see that this is exactly the cross product.

The cross product is again a natural operation. Given maps $f\colon X \to X'$ and $g\colon Y \to Y'$, we have a commutative diagram

$$\begin{tikzcd}
H^k(X';R) \otimes_R H^\ell(Y';R) \arrow[r, "\times"] \arrow[d, "f^* \otimes g^*"'] & H^{k+\ell}(X' \times Y';R) \arrow[d, "(f \times g)^*"] \\
H^k(X;R) \otimes_R H^\ell(Y;R) \arrow[r, "\times"'] & H^{k+\ell}(X \times Y;R)
\end{tikzcd}$$

since all of our constructions are functorial. In particular, if we now hold Y constant, we have two functors of X:

$$X \mapsto \begin{cases} H^k(X;R) \otimes_R H^\ell(Y;R) \\ H^{k+\ell}(X \times Y;R), \end{cases}$$

and the above shows that the cup product is a natural transformation from the first to the second. Summing these together, we see that the cross product is a natural transormation between two functors from spaces to graded abelian groups:

$$X \mapsto \begin{cases} H^*(X;R) \otimes_R H^*(Y;R) \\ H^*(X \times Y;R). \end{cases}$$

The Künneth Theorem gives us conditions in which this is a natural isomorphism.

Definition 5.9. *A cohomology theory on CW-complexes is a functor on pairs*

$$h^*: \mathcal{T}op^2 \to gr\mathcal{A}b$$

together with a sequence of natural trasnformations

$$h^m(A, \emptyset) \xrightarrow{\delta} h^{m+1}(X, A)$$

for any pair (X, A). such that

1. *h^m is a homotopy functor for all m*
2. *If $X = A \cup B$, then $h^m(X, A) \cong h^m(B, A \cap B)$ induced by the inclusions.*
3. *The map δ and the maps $(A, \emptyset) \to (X, \emptyset) \to (X, A)$ induce a long exact sequence upon applying h^*.*
4. *h^* takes disjoint unions to products.*

Definition 5.10. *A natural transformation of cohomology theories $h^* \Rightarrow k^*$ is a sequence of natural transformations $h^m \Rightarrow k^m$ that commute with the corresponding boundary maps δ_h and δ_k.*

Proposition 5.11. *If $H^*(Y; R)$ is degreewise a finitely generated projective R-module, then*

$$(X, A) \mapsto H^*(X, A; R) \otimes_R H^*(Y; R)$$

is a cohomology theory.

Proof. Most of the properties are inherited from $H^*(-, -; R)$. The only ones we need to check are the long exact sequence and the disjoint union axioms. For the former, finitely generated projective R-modules are flat, and hence tensoring over R with $H^k(Y; R)$ is tensoring with something flat, and hence preserves exact sequences. Direct sums are also exact, and this gives the result. For the disjoint union, we need finite generation. If $H^k(Y; R)$ is finitely generated free, generated by some set I_k, then

$$H^*(X; R) \otimes_R H^k(Y; R) \cong \bigoplus_{I_k} H^*(X; R) \cong \prod_{I_k} H^*(X; R),$$

since I_k is finite. Products always commute, so this is again the product. In any given degree, there are only finitely many k for which $H^k(Y; R)$ can contribute, so again, we get the result. A retract of an isomorphism is an isomorphism, so the same is true for any summand of a finite sum of frees, hence any projective. □

It is obvious that

$$(X, A) \mapsto H^*(X \times Y, A \times Y; R)$$

is a cohomology theory, since ordinary cohomology is.

Lemma 5.12. *If h^* and k^* are cohomology theories and $F: h^* \Rightarrow k^*$ is a natural transformation, then F is an isomorphism on finite dimensional CW complexes if it is an isomorphism on a point.*

Proof. Since points are finite dimensional CW complexes, one direction is clear. For the reverse, the Mayer-Vietoris sequence, together with the 5-Lemma, shows that if F is an isomorphism on a point, then it is so on a sphere. Let X be a finite dimensional CW complex. We now argue by induction on the dimension. The case that the dimension is zero is either the assumption or the Milnor axiom, so we assume that F is an isomorphism on the $(n-1)$-skeleton of X. If we consider the long exact sequence for the pair $(X^{[n]}, X^{[n-1]})$ (which by excision is a disjoint union of a bunch of disks relative to their boundaries), then the 5-Lemma again shows that F is an isomorphism. □

Theorem 5.13 (Künneth Theorem). *Let Y be a space such that for all k, $H^k(Y;R)$ is a finitely generated projective R-module. Then for any pair (X,A) of finite dimensional CW-complexes the cross product induces an isomorphism*

$$H^*(X,A;R) \otimes_R H^*(Y;R) \cong H^*(X \times Y, A \times Y; R).$$

Theorem 5.14. *Both sides of the desired isomorphism are cohomology theories, and the cross product is a natural transformation of the associated functors. By assumption, the cross product is an isomorphism for $(X,A) = (*, \emptyset)$. We will be able to use Lemma 5.12 if we know that the cross product commutes with the relevant coboundary maps. This is an exercise.*

Exercise 5.1. *Show that the cross product commutes with the coboundary map for the pair (X,A): we have a commutative diagram*

$$\begin{tikzcd} H^k(A;R) \otimes_R H^\ell(Y;R) \arrow[r, "\delta"] \arrow[d, "- \times -"'] & H^{k+1}(X,A;R) \otimes_R H^\ell(Y;R) \arrow[d, "- \times -"] \\ H^{k+\ell}(A \times Y;R) \arrow[r, "\delta"'] & H^{k+\ell+1}(X \times Y, A \times Y; R). \end{tikzcd}$$

Corollary 5.15. *Let $X = S^{n_1} \times \cdots \times S^{n_k}$. Then*

$$H^*(X;R) \cong \bigotimes_{i=1}^{k} \Lambda_R(x_{n_i}),$$

where $|x_{n_i}| = n_i$

6. The Milnor Sequence

An obvious downside to Lemma 5.12 is that it allowed only finite dimensional CW complexes. This meant that our version of the Künneth Theorem was similar hamstrung. The Milnor sequence fixes this. We first need some algebraic constructions.

Definition 6.1. *An inverse system of abelian groups (R-modules, etc) is a collection $A_1, \ldots$ together with maps*

$$A_1 \xleftarrow{f_1^2} A_2 \xleftarrow{f_2^3} \leftarrow \ldots.$$

Remark 6.2. *There is an obvious extension of this to any category, and the full-strength concept is dual to the concept of a direct system introduced in Definition 9.2 below. We will not need anything more than what we use here.*

Example 6.3. *If*

$$X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \ldots,$$

then applying $H^k(-;R)$ to this sequence of spaces gives an inverse system of R-modules.

The product over all $n \in \mathbb{N}$ of the A_n has two interesting maps on it: the identity and the map which applies f_n^{n+1} in the appropriate component.

Definition 6.4. *If $A_\bullet$ is an inverse system, then let*

$$\theta\colon \prod_{n=1}^{\infty} A_n \to \prod_{n=1}^{\infty} A_n$$

be defined by

$$\theta\big((a_1, a_2, \dots)\big) = \big(f_1^2(a_2), f_2^3(a_3), \dots\big).$$

Since all of the maps f_i^{i+1} are homomorphisms, this is again a homomorphism.

Definition 6.5. *If $A_\bullet$ is an inverse system, then let*

$$\varprojlim A_\bullet = \ker(1 - \theta)$$

be the ***inverse limit*** *and let*

$$\varprojlim{}^1 A_\bullet = \operatorname{coker}(1 - \theta)$$

be the ***first derived functor of inverse limit****.*

The definition of the kernel gives us a universal property for the inverse limit.

Proposition 6.6. *Let $A_\bullet$ be an inverse system, and let B be any other abelian group. Then*

$$\operatorname{Hom}(B, \varprojlim A_\bullet) = \{(g_1, \dots) \mid f_n^{n+1} \circ g_{n+1} = g_n\} \subset \prod_{n=1}^{\infty} \operatorname{Hom}(B, A_n).$$

Proof. A map

$$G\colon B \to \prod_{n=1}^{\infty} A_n$$

lands in the inverse limit if and only if $G = \theta \circ G$. By the universal property of the product, G is a sequence of maps $g_n\colon B \to A_n$. The condition $G = \theta \circ G$ is exactly the condition

$$f_n^{n+1} \circ g_{n+1} = g_n.$$

□

We can think of this is a collection of maps from B to the terms A_n in the sequence, compatible with the structure maps. A more universal description is to observe that B gives an inverse system where all the structure maps are the identity, and the map in question is exactly the obvious notion of a map of inverse systems.

Theorem 6.7 (Milnor Sequence)**.** *Assume we have given a sequence of CW inclusions*

$$X_1 \subset X_2 \subset \dots \subset X = \bigcup_{k=1}^{\infty} X_k.$$

Then we have a natural short exact sequence

$$0 \to \varprojlim{}^1 H^{n-1}(X_k; R) \to H^n(X; R) \to \varprojlim H^n(X_k; R) \to 0.$$

Here the maps are as in Example 6.3.

Proof. For each k, let ι_k denote the inclusion of $X_k \to X_{k+1}$. Observe that without loss of generality, we can replace X with the infinite mapping telescope of the various inclusions:

$$T = \left(\coprod_{k=1}^{\infty} X_k \times [k, k+1]\right) /(x, k+1) \sim \big(\iota_k(x), k+1\big).$$

The inclusion of T into $X \times \mathbb{R}$ is a homotopy equivalence, as is the projection $X \times \mathbb{R} \to X$, so it suffices to prove the result for T. Here we have a decomposition into two subcomplexes:

$$T_e = \coprod_{k=1}^{\infty} X_{2k} \times [2k, 2k+1]$$

$$T_o = \coprod_{k=1}^{\infty} X_{2k+1} \times [2k+1, 2k+2]$$

Note that at the attachment points for the pieces of the telescope, we only include the part from the "thinner" part of the telescope.

The intersection is

$$T_e \cap T_o = \coprod_{k=1}^{\infty} X_k \times \{k+1\}.$$

We can now use the Milnor axiom and the Mayer-Vietoris sequence to compute the homology of $T = T_o \cup T_e$ now. We have

$$H^n(T_e; R) \times H^n(T_o; R) \cong \prod_{k=1}^{\infty} H^n(X_k; R),$$

which of course is the same for the intersection. Now the key map in the Mayer-Vietoris sequence is the map induced by the inclusion of the intersection into the two pieces. For both of these, they are of two types (illustrated for T_e):

(1) $X_{2k} \hookrightarrow T_e$ as $X_{2k} \times \{2k+1\}$ (which induces the obvious projection map) and
(2) $X_{2k-1} \hookrightarrow T_e$ as $X_{2k-1} \times \{2k\} \hookrightarrow X_{2k} \times \{2k\}$, which induces ι_{2k-1}^* on that factor.

For T_o, the roles of odd and even are switched, and we have a global minus sign. In our Mayer-Vietoris sequence, we have

$$\prod_{k=1}^{\infty} H^n(X_k; R) \to \prod_{k=1}^{\infty} H^n(X_k; R),$$

where on coordinates this looks like

$$(\dots, \phi_k, \dots) \mapsto (\dots, (-1)^k(\phi_k - \iota_{k+1}^* \phi_{k+1}), \dots).$$

If we apply the global automorphism that switched all of the odd terms with their additive inverse, then we see that this is exactly the map $1 - \theta$ above. $\square$

Our proof of the Milnor sequence never used the dimension axiom, so in particular, it applies to a general cohomology theory.

Corollary 6.8. *If h^* and k^* are cohomology theories and $F\colon h^* \Rightarrow k^*$ is a natural transformation, then F is an isomorphism of CW complexes if and only if it is an isomorphism on a point.*

Proof. Since F is an isomorphism on finite complexes (Lemma 5.12), we know that F induces an isomorphism on the corresponding limits and derived limits. The result follows from the 5-Lemma. $\square$

Corollary 6.9. *If Y is a space such that for all n, $H^n(Y;R)$ is finitely generated projective, then for all pairs (X,A), we have a natural isomorphism*
$$H^*(X,A;R)\otimes_R H^*(Y;R)\cong H^*(X\times Y,A\times Y;R).$$

7. the James construction

We briefly digress to compute the cohomology of an interesting class of spaces: the James construction applied to spheres.

Definition 7.1. *If X is a pointed space, pointed by x, then let*
$$J_n(X)=\left(\coprod_{i=1}^{n}X^{\times i}\right)/(x_1,\dots,x_{i-1},x,x_{i+1},\dots,x_k)\sim(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k).$$
Let
$$J(X)=\bigcup_{n=1}^{\infty}J_n(X)$$
with the colimit topology.

This is very analogous to the tensor algebra of modules. We can use this intuition to make this more precise.

Definition 7.2. *A unital, associative monoid in spaces is a space M together with continous maps*
$$\mu\colon M\times M\to M \text{ and } \iota\colon *\to M,$$
such that
$$\mu\circ(\mu\times Id_M)=\mu\circ(Id_M\times\mu)$$
and if $m=\iota(*)$, then
$$\mu(m,x)=x=\mu(x,m)$$
for all $x\in M$.

A homomorphism of unital, associative monoids in spaces is a continous map that commutes with all the structure.

Definition 7.3. *Let $\mathcal{A}ssoc$ denote the category of associative, unital monoids in spaces. Let $U\colon\mathcal{A}ssoc\to\mathcal{T}op$ denote the forgetful functor.*

Proposition 7.4. *The James construction lifts to a functor*
$$J\colon\mathcal{T}op\to\mathcal{A}ssoc$$
which is left-adjoint to the forgetful functor:
$$\mathcal{A}ssoc\big(J(X),M\big)\cong\mathcal{T}op\big(X,U(M)\big).$$

Exercise 7.1. *Prove Proposition 7.4. As a hint: the multiplication is the obvious concatination.*

We can compute the homology of the James construction. It is helpful here to have a slightly different reworking of the equivalence relation.

Definition 7.5. *Let X be a pointed space, pointed by x. Then the nth **fat wedge** of X is*

$$F_n(X) = \{(x_1, \ldots, x_n) \mid \exists i, x_i = x\} \subset X^n.$$

By construction, $F_n(X)$ is exactly the space we kill off if we form the iterated smash powers of X, so the following is immediate.

Proposition 7.6. *For any space X, we have a natural homeomorphism*

$$X^n / F_n(X) \cong X^{\wedge n}.$$

Additionally, if X is a CW complex with x as a zero-cell, then $F_n(X)$ is a subcomplex of X^n. In particular, the inclusions $F_n(X) \to X^n$ are nice embeddings.

Now the spaces $F_n(X)$ are also exactly the subspace on which we have an interesting equivalence relation for the James construction. If $\vec{x} \notin F_n(X)$, then the equivalence class of $\vec{x} \in J_n(X)$ is just $\vec{x}$ itself. The points of $F_n(X)$ are the points which get folded into the lower summands.

Proposition 7.7. *Let X be a space. Then we have a pushout diagram of spaces*

$$\begin{tikzcd} F_n(X) \arrow[r] \arrow[d, "\nabla"'] & X^{\times n} \arrow[d] \\ J_{n-1}(X) \arrow[r] & J_n(X), \end{tikzcd}$$

where $\nabla \colon F_n(X) \to J_{n-1}(X)$ is the map

$$\nabla\big(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n\big) = \big[(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)\big].$$

As written, it is not clear that ∇ is even well-defined. If there is only one copy of x, then this is unambiguous, but if there is more than one, it appears that we had to make a choice. This also complicated continuity. This is actually the only piece that is possibly confusing, however, since as we observed above, the subcomplex $F_n(X)$ is the only one one which we see the equivalence relation non-trivially.

We can make well-definedness and continuity precise by unpacking $F_n(X)$ slightly.

Definition 7.8. *If $I \subset \{1, \ldots, n\}$, then let*

$$F_n^I(X) = \{(x_1, \ldots, x_n) \mid \forall i \in I, x_i = x\}.$$

The following features of these subspaces are immediate.

Proposition 7.9.

(1) *For each $I \subset \{1, \ldots, n\}$, the space $F_n^I(X)$ is a subcomplex of $X^{\times n}$.*
(2) *If $I, J \subset \{1, \ldots, n\}$, then*

$$F_n^I(X) \cap F_n^J(X) = F_n^{I \cup J}(X).$$

(3) *The space $F_n(X)$ is the union of the $F_n^I(X)$ where I ranges over the non-empty subsets.*

Exercise 7.2. *Prove Proposition 7.9.*

One way to restate the last part is that

$$F_n(X) = \bigcup_{i=1}^{m} F_n^{\{i\}}(X),$$

but by including the other spaces, we see that these fit into a natural stratification which mirrors the one of the James construction. The key feature of the proof of Proposition 7.7 is then the following.

Proposition 7.10. *The maps*

$$\nabla_i \colon F_n^{\{i\}}(X) \to J_{n-1}(X)$$

defined by

$$\nabla_i\big(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n\big) = \big[(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)\big]$$

are each continous and agree on the intersections $F_n^{\{i,j\}}(X)$.

Remark 7.11. *The spaces* $F_n^I(X)$ *as* I *ranges over the subsets of* $\{1, \dots, n\}$ *form a kind of stratification of* $X^{\times n}$. *In this, the points in* $F_n^I(X)$ *map to* $J_{n-|I|}(X)$ *under the canonical map* $X^n \to J_n(X)$.

We can use the pushout square of Proposition 7.7 to compute the cohomology for particular X as a ring.

Theorem 7.12. *The cohomology groups of* $J(S^{2n})$ *are*

$$H^*\big(J(S^{2n}); R\big) \cong \begin{cases} R & * \equiv 0 \mod 2n \\ 0 & \textit{otherwise.} \end{cases}$$

Let x_{2kn} *generate* $H^{2kn}\big(J(S^{2n}); R\big)$. *Then we can choose these so that*

$$x_{2kn} \smile x_{2jn} = \binom{k+j}{k} x_{2(k+j)n}.$$

Proof. For both parts, we use the pushout square of Proposition 7.7. Note that since for any CW complex X, Proposition 7.6 shows that we have a cofiber sequence

$$F_n(X) \to X^{\times n} \to X^{\wedge n}.$$

This implies that we have a cofiber sequence

$$J_{m-1}(X) \to J_m(X) \to X^{\wedge m}.$$

Now specialize to $X = S^{2n}$. By induction on n, this implies that we have a cell-structure for $J_m(S^{2n})$ where we have cells in dimensions

$$0, 2n, 4n, \dots, 2mn.$$

In particular, cellular cohomology immediate gives us the first part. Note also that the map $J_m(S^{2n}) \hookrightarrow J_{m+1}(S^{2n})$ induces an isomorphism in cohomology through dimension $2n(m+1) - 1$.

This result did not depend on R, so it is in particular true for $R = \mathbb{Z}$. The general result then again follows from this by tensoring with R, so it suffices to compute the ring structure for $R = \mathbb{Z}$. Exercise 7.3 below shows that it suffices to show that we can choose the x_{2kn} such that

$$x_{2n}^k = k! x_{2kn},$$

and this is what we will show.

The pushout square of Proposition 7.7 actually shows a little more. As constructed, we see that this is a cellular map, where we are simply identifying all of the $2kn$-cells in $(S^{2k})^{\times m}$. In particular, the map on cohomology is injective, and hence as a ring,

$$H^*\big(J_m(S^{2n})\big) \hookrightarrow H^*\big((S^{2n})^{\times m}\big) \cong \Lambda_{\mathbb{Z}}(e_{(2n,1)}, \dots, e_{(2n,m)}),$$

where for $1 \leq i \leq m$, $|e_{(2n,i)}| = 2n$. Since all m of the $2n$-cells of $(S^{2n})^{\times m}$ are identified with the same cell in $J_1(S^{2n})$, we deduce that in cohomology, we have

$$x_{2n} \mapsto \sum_{i=1}^{m} e_{(2n,i)}.$$

An induction argument shows that more generally, we can choose the x_{2nk} such that

$$x_{2nk} \mapsto \sum_{I_k} e_{2n,j_1} \smile \cdots \smile e_{2n,j_k},$$

where I_k is the set of k-element subsets of $\{1, \dots, m\}$. Here, the only choice we make is that of the sign; this amount to a choice of sign for the $2n$-cell, which then gives one for all others by the Künneth theorem. The result follows. $\square$

Exercise 7.3. *Let $\Gamma(x)$ be the $\mathbb{Z}$-algebra whose underlying abelian group is free on the set $\{\gamma_i(x) \mid 0 \leq i \leq \infty\}$ (here, $\gamma_0(x) = 1$). If for all k, $\gamma_1(x)^k = k!\gamma_k(x)$, then show that*

$$\gamma_i(x) \cdot \gamma_j(x) = \binom{i+j}{i} \gamma_{i+j}(x).$$

Remark 7.13. *In fact, more is true. For a general X, we have an isomorphism of R-modules*

$$\tilde{H}^*\big(J(X); R\big) \cong \bigoplus_{k=1}^{\infty} \tilde{H}^*\big(X^{\wedge k}; R\big).$$

This follows from a stable splitting of the James construction (in fact, it splits after a single suspension). If $H^*(X; R)$ is a finitely generated projective R-module in every degree, then the Künneth theorem allows us to refine this, giving an isomorphism of graded R-modules:

$$H^*\big(J(X); R\big) \cong \bigoplus_{k=0}^{\infty} \big(\tilde{H}^*(X; R)\big)^{\otimes k}.$$

8. Lecture 8 - the cohomology of projective space

The cohomology of projective spaces gives an important example of spaces whose cohomology we can compute fairly directly. This also underscores the beautiful, geometric content for the cup product: here it exactly measures the number of intersections of particularly generic submanifolds of complementary dimension.

Theorem 8.1. *For all n, we have*

$$H^*(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2[x_1]/x_1^{n+1}, |x_1| = 1$$
$$H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[x_2]/x_2^{n+1}, |x_2| = 2$$
$$H^*(\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[x_4]/x_4^{n+1}|x_4| = 4.$$

We also have

$$H^*(\mathbb{O}P^2; \mathbb{Z}) \cong \mathbb{Z}[x_8]/x_8^3,$$

but this is the last octonionic projective space.

Proof. We will show the $\mathbb{R}P^n$ case; all others are the same. We also will shorten notation by writing P^i for $\mathbb{R}P^i$ and to stress the field independence.

Note that the $(n-1)$-skeleton of P^n is P^{n-1}, and since all cellular boundary and coboundary maps are zero (being multiplication by 0 or 2 which is zero mod 2, or being actually 0 for the complex and quaternionic cases), we know that the inclusions map induces an isomorphism

$$H^k(P^n;\mathbb{F}_2) \to H^k(P^{n-1};\mathbb{F}_2)$$

for $0 \leq k \leq n-1$. In particular, given any two elements whose degrees add up to less than n, their product is detected in P^{n-1}. By induction on n, this is know to be truncated polynomial. It therefore suffices to show that if $i+j=n$, then

$$H^i(P^n;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^j(P^n;\mathbb{F}_2) \xrightarrow{\smile} H^n(P^n;\mathbb{F}_2)$$

is an isomorphism of one dimensional $\mathbb{F}_2$-vector spaces. We argue this by judiciously unpacking the pieces.

Recall the homogeneous coordinates on P^n:

$$[x_0 : \cdots : x_n] \in (\mathbb{R}^{n+1} - \vec{0})/\mathbb{R}^\times.$$

Inside P^n, let

$$P^i = \big\{[x_0 : \cdots : x_i : 0 : \cdots : 0]\big\}$$

be the standard embedding of P^i in P^n, and let

$$P^j = \big\{[0 : \cdots : 0 : x_i : \cdots : x_{n+1}]\big\}.$$

These intersect at

$$\big\{[0 : \cdots : 0 : x_i : 0 : \cdots : 0]\big\} = P^0,$$

a point. We have a relative cup product:

$$H^j(P^n, P^n - P^i) \otimes_{\mathbb{F}_2} H^i(P^n, P^n - P^j) \xrightarrow{\smile} H^n(P^n, P^n - P^0).$$

By definition,

$$P^n - P^i = \big\{[x_0 : \cdots : x_{n+1} \mid (x_{i+1}, \ldots, x_{n+1}) \neq \vec{0}\big\}.$$

The straight-line homotopy taking the first $(i+1)$-coordinates to zero then gives us a deformation retraction of this onto

$$P^{j-1} \subset P^j,$$

and similarly for $P^n - P^j$. We therefore have that

$$H^j(P^n, P^n - P^i) \cong H^j(P^n, P^{j-1}).$$

Since P^{j-1} can be used as the $(j-1)$-skeleton of P^n (simply place the cells from right to left instead of the usual way), we have a commutative square

$$\begin{tikzcd}
H^j(P^n) \otimes H^i(P^n) \arrow[r, "\smile"] & H^n(P^n) \\
H^j(P^n, P^n - P^i) \otimes H^i(P^n, P^n - P^j) \arrow[u, "\cong"] \arrow[r, "\smile"'] & H^n(P^n, P^n - P^0). \arrow[u, "\cong"']
\end{tikzcd}$$

We wish to show that the top line is an isomorphism, so it suffices to show the bottom is. Now we begin excising. The points with a fixed coordinate non-zero give us a subspace homeomorphic to $\mathbb{R}^n$ in P^n, and the complement is P^{n-1}. Excision therefore shows us that

$$H^n(P^n, P^n - P^0) \cong H^n(\mathbb{R}^n, \mathbb{R}^n - \vec{0}).$$

Similarly, the inclusions

$$(P^i, P^i - P^0) \hookrightarrow (P^n, P^n - P^j)$$

induce an isomorphism of H^i, and these can then be further excised away to $(\mathbb{R}^i, \mathbb{R}^i - \vec{0})$. By restriction, we therefore get a commutative diagram

$$\begin{array}{ccc} H^i(P^n, P^n - P^j) \otimes H^j(P^n, P^n - P^i) & \xrightarrow{\smile} & H^n(P^n, P^n - P^0) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ H^i(\mathbb{R}^i, \mathbb{R}^i - \vec{0}) \otimes H^j(\mathbb{R}^j, \mathbb{R}^j - \vec{0}) & \xrightarrow[\times]{} & H^n(\mathbb{R}^n, \mathbb{R}^n - \vec{0}). \end{array}$$

The Künneth theorem then gives the result, since the map is the cross product for the standard identification

$$(\mathbb{R}^i, \mathbb{R}^i - \vec{0}) \times (\mathbb{R}^j, \mathbb{R}^j - \vec{0}) = (\mathbb{R}^n, \mathbb{R}^n - \vec{0}).$$

□

Combining this with the Milnor sequence, we have

Corollary 8.2. *We have*

$$\begin{aligned} H^*(\mathbb{R}P^\infty; \mathbb{F}_2) &\cong \mathbb{F}_2[x_1] \\ H^*(\mathbb{C}P^\infty; \mathbb{Z}) &\cong \mathbb{Z}[x_2] \\ H^*(\mathbb{H}P^\infty; \mathbb{Z}) &\cong \mathbb{Z}[x_4]. \end{aligned}$$

And since these are free over their corresponding ground rings, the Künneth theorem then allows us to build any truncated polynomial algebra in classes in degrees 1, 2, and 4 by crossing these spaces together.

Part 2. Poincaré Duality

Theorem 8.3. *Let M be a closed, R-oriented, compact, connected n-manifold. Then*

1. *$H_n(M;R) \cong R$, generated by a "fundamental class" $[M]$.*
2. *The cap product induces an isomorphism*

$$H^k(M;R) \otimes_R H_n(M;R) \xrightarrow{-\frown-} H_{n-k}(M;R).$$

If M is not compact, then we get an isomorphism considering instead compactly supported cohomology.

Corollary 8.4. *Let R be a field and let M be a closed, R-oriented, compact, connected n-manifold. Then the cup product induces a perfect pairing*

$$H^k(M;R) \otimes_R H^{n-k}(M;R) \xrightarrow{-\smile-} H^n(M;R) \cong H_n(M;R) \cong R.$$

Proof. The map in question is

$$(\phi \otimes \psi) \mapsto [M] \frown (\phi \smile \psi).$$

Since the cap product is a right module action, this is

$$([M] \frown \phi) \frown \psi.$$

Theorem 11.4 identifies $H^k(M;R)$ with $H_{n-k}(M;R)$ via capping with $[M]$. The cap product

$$H_{n-k}(M;R) \otimes_R H^{n-k}(M;R) \to H_0(M;R) \cong R$$

is the canonical action of cohomology on homology, and since R is a field, this is a perfect pairing. □

We have a lot to define to even prove this theorem. In particular, we must say what "orientable" means, what the "fundamental class" is, and what "compactly supported cohomology" is. Our proof of Poincaré duality is via an open cover, so we necessarily will consider non-compact manifolds. We will start there.

9. Compactly Supported Cohomology

For de Rham cohomology, we have an obvious notion of compact support for a form. The support of $\omega \in \Omega^k(M)$ is the set of points $m \in M$ where $\omega(m) \neq 0$. Compact support here then means the obvious thing: the support is contained in a compact set. In other words, outside of a compact set, the form vanishes.

Definition 9.1. *A cochain ϕ has compact support if there is a compact $K \subset M$ such that $\phi \in C^k(M, M - K)$.*

The compactly supported k-cochains are

$$C^k_c(M;R) := \bigcup_{K \subset M} C^k(M, M-K;R).$$

The compactly supported cochains are visibly closed under the coboundary map. What is not clear is that the sum of these for different K are also compactly supported. For this, we need a general, categorical notion.

Definition 9.2. *A directed set is a poset $(I, \leq)$ such that if $a, b \in I$, then there is a $c \in I$ such that $a, b \leq c$.*

A direct system in a category $\mathcal{C}$ is a functor $I \to C$.

In this definition, we are viewing a poset as a category with object set I and with $I(a,b)$ a point if $a \leq b$ and empty otherwise.

Definition 9.3. *Let $A_\bullet$ be a direct system of abelian groups (or R-modules for some R). Then the direct limit of $A_\bullet$ is*

$$\lim_{\to} A_\bullet = \Big(\bigoplus_{i\in I} A_i\Big) / \big(f_i^j(a_i) - a_i\big),$$

where for each $i \leq j$, $f_i^j\colon A_i \to A_j$ is the structure map in the direct system.

The following is a straightforward application of the universal properties of the quotient and of the direct sum.

Proposition 9.4. *If B is any abelian group, and $A_\bullet$ is a direct system of abelian groups, then*

$$\operatorname{Hom}(\lim_{\to} A_\bullet, B) = \{f_i\colon A_i \to B \mid f_j \circ f_i^j = f_i\}$$

If all of the maps in the direct system are inclusions, then the direct limit is also the union. In particular, if we can recast the definition of compactly supported cochains, then we will have that this is a subcochain complex. For this, we use the following.

Proposition 9.5. *The collection of all compact subsets of a manifold form a directed set under inclusion.*

Proof. The union of finitely many compacts is compact. $\square$

Proposition 9.6. *The assignment*

$$K \mapsto C^k(M, M-K; R)$$

gives a direct system of R-modules.

Proof. A cochain ϕ on M is in $C^k(M, M-K; R)$ if and only if it vanishes on $C_k(M-K; R)$. If $K \subset L$, then $M-L \subset M-K$, so in particular, $C^k(M, M-K; R) \subset C^k(M, M-L; R)$. $\square$

Exercise 9.1. *We have a natural isomorphism*

$$H_c^k(M; R) \cong \lim_{\to} H_c^k(M, M-K; R).$$

We need also to be able to compute these more efficiently.

Definition 9.7. *A subset C of a directed set I is cofinal if for all $i \in I$, there is a $c \in C$ such that $i \leq c$.*

In particular, cofinal subsets are necessarily directed sets (which is not true for a general sub-poset). A useful exercise with the definition is the show the following.

Proposition 9.8. *If $C \subset I$ is cofinal, then $\lim_C = \lim_I$.*

Corollary 9.9. *If M is compact, then $H_c^k(M; R) = H^k(M; R)$.*

Proof. The set $\{M\}$ is a cofinal subset of the set of compact subsets of M. $\square$

Proposition 9.10. *For any n,*

$$H_c^k(\mathbb{R}^n; R) \cong \begin{cases} R & k = n \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The set of closed balls of radius $m \in \mathbb{N}$ forms a cofinal subset of the compact subsets of $\mathbb{R}^n$. The relevant homologies are

$$H^k(\mathbb{R}^n, \mathbb{R}^n - B; R),$$

and the inclusion $B_m \subset B_{m+1}$ induces an isomorphism. The result follows. $\square$

9.1. **Exercises.**

Exercise 9.2. *Show that homology commutes with direct limits. In other words, if I is a directed set and C^i is a direct system of chain complexes, then show that the direct limits of $C^(-)$ is naturally a chain complex and*

$$\lim_{\rightarrow} H_k(C^i) \cong H^k(\lim_{\rightarrow} C^i),$$

Definition 9.11. *A map $f\colon X \to Y$ is* proper *if for every compact subspace $K \subset Y$, $f^{-1}(K)$ is compact.*

Exercise 9.3. *Show that a proper map induces a homomorphism on cohomology with compact support.*

Exercise 9.4 (Hatcher, 3.3.21)**.** *If X is a space, let X^+ denote the 1-point compactification. If X^+ has the property that there is a neighborhood of ∞ that is a cone with cone point ∞, then show that the natural map*

$$H^k_C(X) \to H^k(X^+, \infty)$$

is an isomorphism.

Exercise 9.5. *Compute the cup product structure on $(S^6 \times S^{25}) \# (S^{10} \times S^{21})$, where here $\#$ denotes the connect sum of manifolds.*

10. Orientations

If M is a manifold, then excision provides us with several interesting isomorphisms. For all $m \in M$ and for all open balls $B \subset M$, we have isomorphisms

$$H_n(M, M - \{m\}; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \vec{0}; R) \cong R \cong H_n(M, M - B; R).$$

In particular, a choice of element $r \in H_n(M, M - B; R)$ determines elements in $H_n(M, M - \{m\}; R)$ for all $m \in B$ via

$$i_{m*}\colon H_n(M, M - B; R) \to H_n(M, M - \{m\}; R).$$

Loosely speaking, an orientation of M is a continuous choice of basis for this free, rank 1 R-module. In particular, it is a continuous choice of unit in R. To make sense of what "continuous" means here, since the target is technically changing with each point of m, we must assemble these.

Definition 10.1. *If R is a commutative ring, then let*

$$M_R = \Big\{(m, \mu) \mid m \in M, \mu \in H_n\big(M, M - \{m\}; R\big)\Big\}.$$

Topologize this by taking as a basis

$$U_{B,\mu_B} = \Big\{(m, \mu_m) \mid m \in B \subset M, \mu_m = i_{m*}(\mu_B)\Big\}.$$

We have a natural map $M_R \to M$ which forgets the second coordinate.

Proposition 10.2. *The map $M_R \to M$ is a covering map of degree $|R|$.*

We in fact built the topology so this would be so. We used the local affineness of the manifold to glue together copies of $U \times R$, where R was given the discrete topology.

Definition 10.3. *An R-orientation of M is a section s of $M_R \to M$ such that for all $m \in M$, $s(m)$ is a generator of $H_n(M, M - \{m\}; R)$.*

An n-manifold is R-orientable if there exists an orientation.

The property of being a generator of a free, rank 1 R-module is invariant under any R-module automorphisms. In particular, we have a sub-covering space

$$M_{R^\times} = \{(m, \mu_m) \mid \mu_m \text{ generates}\} \subset M_R$$

which is a cover of degree $|R^\times|$. We can think of orientations as sections of this covering space.

Proposition 10.4. *If $R = \mathbb{Z}/2$, then every manifold is R-orientable.*

Proof. Here, $R^\times = \{1\}$, and $M_{R^\times} = M$. □

We next address the universal case of $R = \mathbb{Z}$, the initial ring. Here, $R^\times = \{\pm 1\}$ so $M_{R^\times} \to M$ is a double cover.

Proposition 10.5. *For any n-manifold M, $M_{\mathbb{Z}^\times}$ is $\mathbb{Z}$-orientable.*

Proof. The assignment $(m, \mu_m) \mapsto \mu_m$ gives a continuous choice of basis. □

Proposition 10.6. *A connected n-manifold is $\mathbb{Z}$-orientable if and only if $M_{\mathbb{Z}^\times}$ has 2 components.*

Proof. Since this is a double cover, if $M_{\mathbb{Z}^\times}$ has 2 components, then it splits as $M \amalg M$. There are thus 2 sections. If $M_{\mathbb{Z}^\times}$ has one component, then $M_{\mathbb{Z}^\times}$ corresponds to an index 2 subgroup of $\pi_1(M, m)$. In particular, by the lifting lemma, there are no sections. □

Proposition 10.7. *If M is connected, then a section of $M_R \to M$ or of $M_{R^\times} \to M$ is determined by its value on a point.*

Proof. This is the lifting lemma. □

For more general R, these *still* describe the structure!

Proposition 10.8. *For any commutative ring R, we have a splitting of M_R as*

$$M_R \cong \left(\coprod_{2a=0} M \right) \amalg \left(\coprod_{a \neq -a} M_{\mathbb{Z}^\times} \right),$$

and hence of $M_{R^\times}$ as

$$M_{R^\times} \cong \begin{cases} \left(\coprod_{a \in R^\times} M\right) & 2 = 0 \in R \\ \left(\coprod_{a \in R^\times/\{\pm 1\}} M_{\mathbb{Z}^\times}\right) & 2 \neq 0 \in R. \end{cases}$$

Proof. This is an interesting exercise in naturality. A priori, we would expect that we would have to group points according to the action of $R^\times$ on $H_n(M, M-\{m\}; R)$, since this corresponds to the possible choices of basis. In fact, we just see the action of ± 1. First observe that since $H_n(M, M-B; \mathbb{Z})$ is free for any ball B in an $\mathbb{R}^n \subset M$, we have a natural (in B and in R) isomorphism

$$H_n(M, M-B; \mathbb{Z}) \otimes_{\mathbb{Z}} R \xrightarrow{\cong} H_n(M, M-B; R).$$

In particular, the dependence of the right-hand side on B is determined by the dependence of $H_n(M, M-B; \mathbb{Z})$ on B. Put another way, if $B' \subset B$, then we have a commutative square

$$\begin{tikzcd}
H_n(M, M-B; \mathbb{Z}) \otimes_{\mathbb{Z}} R \arrow[r, "\cong"] \arrow[d, "\cong \otimes Id_R"'] & H_n(M, M-B; R) \arrow[d, "\cong"] \\
H_n(M, M-B'; \mathbb{Z}) \otimes_{\mathbb{Z}} R \arrow[r, "\cong"] & H_n(M, M-B'; R)
\end{tikzcd}$$

Thus the only isomorphisms which can show up in the homology are those which arose from the isomorphisms as a $\mathbb{Z}$-module.

Now let's unpack slightly the covering space itself. To build the covering space, we choose an open cover of M by open balls $B \subset \mathbb{R}^n \subset M$. For each of these, choose an isomorphism

$$\tau_B \colon H_n(M, M-B; \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z},$$

and hence by the above an isomorphism

$$\tau_B \otimes R \colon H_n(M, M-B; R) \xrightarrow{\cong} R.$$

The isomorphism τ_B determines for all $m \in B$ an isomorphism $H_n(M, M-\{m\}; \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ via the commutative diagram

$$\begin{tikzcd}
H_n(M, M-B) \arrow[r, "\tau_B"] \arrow[d, "\iota_*^{m,B}"'] & \mathbb{Z}, \\
H_n(M, M-\{m\}) \arrow[ur, "\tau_{B,m}"'] &
\end{tikzcd}$$

where $\iota^{m,B}$ is the map of pairs $(M, M-B) \subset (M, M-\{m\})$. Again, for any coefficients R, this gives us an isomorphism $H_n(M, M-\{m\}; R) \cong R$ that depends only on the underlying case of $R = \mathbb{Z}$.

Now we build our covering space by starting with the space

$$\tilde{M} := \coprod_{B \subset \mathbb{R}^n \subset M} B \times R,$$

and we glue together along the intersections of open balls in the following way. Given two open balls $B_1, B_2 \subset \mathbb{R}^n \subset M$ and a point $m \in B_1 \cap B_2$, we have two ways to identify $H_n(M, M-\{m\}; R)$ with R: $\tau_{B_1,m}$ and $\tau_{B_2,m}$. In particular, we have an automorphism

$$\gamma_{B_2}^{B_1} = \tau_{B_1,m} \circ \tau_{B_2,m}^{-1} \colon R \xrightarrow{\cong} R,$$

which, since it comes from the underlying case of $R = \mathbb{Z}$, is either ± 1. Now for each $m \in B_1 \cap B_2$, we identify

$$B_1 \times R \ni (m, r) \sim (m, \gamma_{B_2}^{B_1} r) \in B_2 \times R.$$

In particular, the identifications are always taking a point (m, r) to either (m, r) or to $(m, -r)$. We never see any other elements of $R^\times = GL_1(R)$ showing up here.

If $r = -r$, then we never did anything, and hence we got a copy of M. If $r \neq -r$, then the entire construction we have done is the same as for $\mathbb{Z}^\times$, we are just naming the fibers $\{\pm r\}$ rather than $\{\pm 1\}$. The result follows. □

One might be concerned that we made a lot of choices here (namely the possible identifications. The homeomorphism type of the space we write down doesn't care about these. First note that a different choice of isomorphism $H_n(M, M - B) \cong \mathbb{Z}$ will just replace certain maps by -1 times themselves, and all of the maps we consider will adjust accordingly (in other words, we are just globally switching a bunch of directions). One might also be concerned that we might identify pieces in $B \times R$ with themselves as we pass through our identifications. This could destroy our covering space property! The following exercise is key to showing that this does not happen.

Exercise 10.1. *Show that the identification maps $\gamma_{B_j}^{B_i}$ satisfy a "cocycle condition": if $m \in B_1 \cap B_2 \cap B_3$, then*

$$\gamma_{B_1}^{B_3} = \gamma_{B_2}^{B_3} \circ \gamma_{B_1}^{B_2},$$

and $\gamma_B^B = Id$ for all B.

Corollary 10.9. *If M is $\mathbb{Z}$-orientable, then M is R-orientable for all R.*

Proof. The decomposition of $M_{R^\times}$ into copies of M and of $M_{\mathbb{Z}^\times}$ together with Proposition 10.6 shows that if M is $\mathbb{Z}$-oriented, then $M_{R^\times}$ is a disjoint union of copies of M. In particular, we have sections. □

Now we add in some extra structure. This is where we use the more general construction above (having an arbitrary A, rather than just $R^\times$).

Proposition 10.10. *The sections $\Gamma(M_R)$ of $M_R \to M$ form an R-module under pointwise addition and multiplication.*

Proposition 10.11. *If M is R-orientable and connected, then an R-orientation μ gives an identification of covering spaces*

$$R \times M \xrightarrow{-\cdot-} M_R,$$

where $\cdot$ is the map $(r, m) \mapsto r \cdot \mu(m)$.

Proof. Composing μ with the "multiplication by r" map gives $|R|$ distinct sections of p_R which mutually cover M_R. In particular, M_R splits as a disjoint union of copies of M via these sections. This is exactly the map we defined. □

Corollary 10.12. *If R is a ring of characteristic not 2, then R-orientability implies $\mathbb{Z}$-orientability.*

Proof. Since the characteristic of R is not 2, no unit u satisfies $2u = 0$. In particular, $M_{R^\times}$ is a collection of copies of $M_{\mathbb{Z}^\times}$. Since we have a section, this must split by Proposition 10.11. Hence by Proposition 10.6, we deduce $\mathbb{Z}$-orientability. □

The sections of $R \times M$ are easy to determine: they are just the continuous maps $M \to R$ (where R has the discrete topology). In particular, they are locally constant functions on M.

Corollary 10.13. *If M is a connected, R-orientable manifold, then $\Gamma(M_R) \cong R$. If M is not R-orientable, then $\Gamma(M_R) \cong \{r \in R \mid 2r = 0\}$.*

Theorem 10.14. *If M is a closed, connected compact n-manifold, then*

(1) *M is R-orientable if and only if $H_n(M;R) \cong R$.*
(2) *if M is not R-orientable, then $H_n(M;R) \cong \{r \mid 2r = 0\} \subset R$*
(3) *for $m > n$, $H_m(M;R) = 0$.*

To prove this, we will use Corollary 10.13 and a "characteristic function" $H_n(M;R) \to \Gamma(M_R)$ defined by
$$[\sigma] \mapsto [\sigma_m] \in H_n(M, M - \{m\}; R).$$
This is locally constant, and hence a continuous section of $M_R \to M$. We will show that this map is an isomorphism. We will prove this building up along compact subspaces.

Lemma 10.15. *Let $K \subset M$ be a compact subset.*

(1) *If $a \in \Gamma(M_R)$ is a section, then there is a unique $[\alpha] \in H_n(M, M-K; R)$ such that for all $x \in K$, $[\alpha]$ maps to $a(x)$ under the natural map*
$$H_n(M, M-K; R) \to H_n(M, M - \{x\}; R).$$
(2) *For all $m > n$, $H_m(M, M-K; R) = 0$.*

If M is compact, then taking $K = M$ in Lemma 10.15 gives Theorem 10.14.

Sketch of Proof of Lemma 10.15. The Mayer-Vietoris sequence shows that if the lemma is true for K_1, for K_2, and for $K_1 \cap K_2$, then it is true for $K_1 \cup K_2$. The Mayer-Vietoris sequence here looks like

$$0 = H_{n+1}\big(M, M - (K_1 \cap K_2)\big) \longrightarrow H_n\big(M, M - (K_1 \cup K_2)\big)$$
$$\hookrightarrow H_n(M, M - K_1) \oplus H_n(M, M - K_2) \longrightarrow H_n\big(M, M - (K_1 \cap K_2)\big) \longrightarrow \dots$$

Given a, by assumption we can find α_{K_1} and α_{K_2} that satisfy the conditions of the lemma. The restrictions of each to $H_n\big(M, M - (K_1 \cap K_2)\big)$ works for a on $K_1 \cap K_2$, so by uniqueness, they must agree. Exactness then produces $\alpha_{K_1 \cup K_2}$. Continuing the Mayer-Vietoris sequence to the left also shows that the groups $H_m(M, M - (K_1 \cup K_2))$ vanish for $m > n$.

We now reduce to simpler situations. Since $K \subset M$ and M is covered by open affines, we can interatively use the Mayer-Vietoris sequence and induction on the number of affines reduce to the case that K is a compact subset of $\mathbb{R}^n$. Now the distance from any representative of $[z] \in H_n(M, M-K)$ to K is non-zero, so we can without loss of generality replace K by a union of convex sets (slightly fattening up K). Now by induction, we are reduced to the case that K is a closed ball of finite radius, and here the statement from the definition of the topology on M_R. □

10.1. Exercises.

Definition 10.16. *If M and N are oriented n-manifolds and $f\colon M \to N$ is a map, then the degree of f is the integer such that*
$$f_*([M]) = deg(f)[N].$$

Exercise 10.2 (Hatcher 3.3.7)**.** *If M is a closed, connected, orientable n-manifold, then show there is a degree 1 map*
$$M \to S^n.$$

11. Proof of Poincare Duality

Theorem 11.1. *If M is a closed, connected orientable n-manifold, then the orientation gives a homomorphism*

$$D_M \colon H_c^k(M;R) \to H_{n-k}(M;R).$$

Proof. Compactly supported cohomology is a direct limit, so to define a map out of it is the same things as defining a collection of maps

$$H^k(M, M-K;R) \xrightarrow{D_M^K} H_{n-k}(M;R)$$

such that if $K \subset L$, then we have a commutative triangle

$$\begin{tikzcd} H^k(M, M-K;R) \arrow[r, "D_M^K"] \arrow[d] & H_{n-k}(M;R) \\ H^k(M, M-L;R) \arrow[ur, "D_M^L"'] & \end{tikzcd}$$

An orientation is a choice of nice section s of $M_R \to M$, and Lemma 10.15 shows that for each $K \subset M$ compact, there is a unique $[\mu_K]$ that restricts to the value of s at each point of K. If $K \subset L$, then let

$$i_K^L \colon (M, M-L) \to (M, M-K)$$

be the inclusion. In $H_n(M, M-K;R)$ we then have two classes: $[\mu_K]$ and $(i_K^L)_*[\mu_L]$, and for all $m \in K \subset L$, the class $[\mu_L]$ restricts to $s(m)$. By uniqueness of the class $[\mu_K]$, we deduce that

$$(i_K^L)_*[\mu_L] = [\mu_K].$$

Now let

$$D_M^K(\phi) = [\mu_K] \frown \phi.$$

The push-pull formula (expressing naturality of the cap product) then shows that

$$D_M^L\big((i_K^L)^*\phi\big) = [\mu_L] \frown (i_K^L)^*\phi = (i_K^L)_*[\mu_L] \frown \phi = [\mu_K] \cap \phi = D_M^K(\phi).$$

In particular, these assemble into a map out of the direct system and hence give a map out of the direct limit. □

Proposition 11.2. *When $M = \mathbb{R}^n$, then the map D_M from Theorem 11.1 is an isomorphism.*

Proof. First note that we only have compactly supported cohomology for $k = n$. This reduces the cases we have to consider.

Excision shows that we have isomorphisms

$$H_n(\mathbb{R}^n, \mathbb{R}^n - B;R) \cong H_n(\Delta^n, \partial\Delta^n;R) \text{ and } H^n(\mathbb{R}^n, \mathbb{R}^n - B;R) \cong H^n(\Delta^n, \partial\Delta^n;R).$$

The identity map on Δ^n gives a generator $[\Delta]$ of $H_n(\Delta^n, \partial\Delta^n;R)$, and a choice of orientation amounts to a unit u in R, giving the class $u[\Delta^n]$. The Universal Coefficients Theorem (Theorem 2.12) shows that

$$H^n(\Delta^n, \partial\Delta^n;R) \cong \operatorname{Hom}_R\big(H_n(\Delta^n, \partial\Delta^n;R), R\big).$$

In particular, capping with a unit multiple of $[\Delta]$ gives an isomorphism. However, this is just the maps giving D_M. □

Since manifolds are unions of affines homeomorphic to $\mathbb{R}^n$, a Mayer-Vietoris type argument will allow us to conclude this. Here we need a technical lemma.

Lemma 11.3. *Let $M = U \cup V$, where both U and V are open. Then we have a commutative diagram*
(4)

$$\begin{tikzcd}
\dots \arrow[r] & H_c^k(U \cap V; R) \arrow[r] \arrow[d, "D_{U\cap V}"] & H_c^k(U; R) \oplus H_c^k(V; R) \arrow[r] \arrow[d, "D_U \oplus D_V"] & H_c^k(U \cup V; R) \arrow[r, "\delta"] \arrow[d, "D_{U\cup V}"] & H_c^{k+1}(U \cap V; R) \arrow[d, "D_{U\cap V}"] \\
\dots \arrow[r] & H_{n-k}(U \cap V; R) \arrow[r] & H_{n-k}(U; R) \oplus H_{n-k}(V; R) \arrow[r] & H_{n-k}(U \cup V; R) \arrow[r, "\partial"'] & H_{n-k-1}(U \cap V; R)
\end{tikzcd}$$

with exact rows.

Theorem 11.4 (Poincaré Duality)**.** *If M is a closed, R-orientable n-manifold, then*

$$D_M \colon H_c^k(M; R) \xrightarrow{\cong} H_{n-k}(M; R).$$

Proof. We build up our manifold in pieces.

If M is an open set in $\mathbb{R}^n$, then

$$M = \bigcup_{i=1}^{\infty} U_i,$$

where each U_i is an open ball in $\mathbb{R}^n$. If we let

$$V_j = \bigcup_{i=1}^{j} U_i,$$

then

$$M = \bigcup_{j=1}^{\infty} V_j,$$

but here, $V_{j-1} \cap U_j$ is a union of $(n-1)$ open sets which are either empty or homeomorphic to $\mathbb{R}^n$. By induction on Lemma 11.3, the maps D_{V_j} are all isomorphisms. This means that the direct limit of the maps D_{V_j} is also an isomorphism. By Lemma **??**, D_M is an isomorphism.

If M is a union of countably many open sets homeomorphic to $\mathbb{R}^n$, then we can apply the same argument (where here $V_{j-1} \cap U_j$ is an open in $\mathbb{R}^n$, and hence we use the previous part) to deduce that D_M is an isomorphism.

For an arbitrary M (so not second countable), we choose a maximal open U for which D_U is an isomorphism. If $m \in M - U$, then we can find an affine chart of M that contains m. Lemma 11.3 and the first part then show that $D_{U \cup \mathbb{R}^n}$ is also an isomorphism. Zorn's Lemma gives the result. □

Proof of Lemma 11.3. There are two things we have to prove: exactness and the commutativity of the resulting diagrams.

Exactness and the rows. The bottom row is just the Mayer-Vietoris sequence for $U \cup V$. For the top row, Exercise 11.1 below shows that the collection of compacts of the form $K \cup L$, where $K \subset U, L \subset V$ are compact, give a cofinal system of compacts in $U \cup V$. Similarly, all compacts of $U \cap V$ are of the form $K \cap L$. So in all cases, the compactly supported cohomology is the direct limit over the compacts

$K \subset U, L \subset V$. We have a Mayer-Vietoris sequence in cohomology associated to $(M, M-K) \cup (M, M-L)$:

$$\begin{tikzcd} \dots \arrow[r] & H^k\big(M, M-(K\cap L)\big) \arrow[r] & H^k(M, M-K) \oplus H^k(M, M-L) \arrow[dll, rounded corners] \\ H^k\big(M, M-(K\cup L)\big) \arrow[r, "\delta"] & H^{k+1}\big(M, M-(K\cap L)\big) \arrow[r] & \dots \end{tikzcd}$$

Excision gives isomorphisms

$$H^k(U\cap V, U\cap V - K\cap L) \cong H^k\big(M, M-(K\cap L)\big),$$
$$H^k(U, U-K) \oplus H^k(V, V-L) \cong H^k(M, M-K) \oplus H^k(M, M-L).$$

Taking the direct limit over $K \subset U, L \subset V$ then gives us the desired long exact sequence.

Commutativity of the squares. Since the top row is the direct limit along the compacts $K \subset U, L \subset V$, it suffices to show that the squares commute for any particular pair of compacts. We therefore have 4 different squares to show that they commute. The squares involving maps that do not shift degree are all the same: we show the first one here.

Proposition 11.5. *Let $K \subset U, L \subset V$ be compacts. Then we have a commutative square*

$$\begin{tikzcd} H^k\big(U\cap V, U\cap V-(K\cap L); R\big) \arrow[r] \arrow[d, "-\frown[\mu_{K\cap L}]"'] & H^k(U, U-K; R) \arrow[d, "(-\frown[\mu_K])"] \\ H_{n-k}(U\cap V; R) \arrow[r] & H_{n-k}(U; R). \end{tikzcd}$$

Proof. Consider the following inclusions of pairs:

$$(U\cap V, U\cap V - K\cap L) \xrightarrow{i^U_{U\cap V}} (U, U-K\cap L) \xleftarrow{i^{U-K\cap L}_{U-K}} (U, U-K).$$

Excision shows that the map $(i^U_{U\cap V})^*$ is an isomorphism. The top map is therefore the composite

$$(i^{U-K\cap L}_{U-K})^* \circ \big((i^U_{U\cap V})^*\big)^{-1},$$

and we can insert a phantom $H^k(U, U-K\cap L; R)$ between the two groups at the top. It is convenient to start with this group and trace around.

The bottom map in the square is just $(i^U_{U\cap V})_*$.

We now compute. Let $\bar\varphi \in H^k(U, U-K\cap L; R)$. Going left, we have

$$(i^U_{U\cap V})_*\big(\mu^{U\cap V}_{K\cap L} \frown (i^U_{U\cap V})^*\bar\varphi\big) = (i^U_{U\cap V})_*\mu^{U\cap V}_{K\cap L} \frown \bar\varphi = (*),$$

by the push-pull formula. By the usual "uniqueness" argument, we have

$$(i^U_{U\cap V})_*\mu^{U\cap V}_{K\cap L} = \mu^U_{K\cap L},$$

since they have the same restrictions to points in $K\cap L$ (as always, by excision). We then have

$$(*) = \mu^U_{K\cap L} \frown \bar\varphi.$$

Going instead around the right, we have

$$\mu^U_K \frown (i^{U-L\cap K}_{U-K})^*\bar\varphi = (i^{U-L\cap K}_{U-K})_*(\mu^U_{K\cap L} \frown (i^{U-K\cap L}_{U-K})^*\bar\varphi = \mu^U_{K\cap L} \frown \bar\varphi,$$

giving the result. $\square$

Commutativity involving δ and ∂ is trickier. Recall that to compute δ, we write a relative cohomology class

$$[\varphi] \in H^k\big(M, (M-K) \cap (M-L); R\big)$$

as $\varphi_{M-L} + \varphi_{M-K}$, where each $\varphi_{M-?} \in C^k(M, M-?; R)$. Then

$$\delta\big([\varphi]\big) = \big[\delta(\varphi_{M-L})\big] = -\big[\delta(\varphi_{M-K})\big].$$

An almost identical formula is true for ∂. We now have a lengthy aside.

Since $U-L$, $U \cap V$, and $V-K$ mutually cover $M = U \cup V$ and are all open, the inclusion map

$$C_*(U-L;R) + C_*(U \cap V;R) + C_*(V-K;R) \hookrightarrow C_*(U \cup V;R)$$

is a quasi-isomorphism. We can therefore replace any class $\sigma \in C_*(M;R)$ with

$$\sigma_{U-L} + \sigma_{U\cap V} + \sigma_{V-K},$$

where the subscripts indicate the chains on which space we have. Note that here we are working with the absolute chains, where the Mayer-Vietoris sequence is easiest. Since we want to compute $\varphi \frown -$, where φ is a relative cycle, this is fine: $\varphi \frown -$ will not see any of the classes we would kill for the relative case. Now since

$$\sigma_{U-L} \in C_*(U-L;R) \subset C_*(M-L;R),$$

we have

$$[\sigma_{U\cap V} + \sigma_{V-K}] = [\sigma] \in C_*(M, M-L).$$

We have obviously similar formulae for the other relative groups.

Let

$$\sigma = \sigma_{U-L} + \sigma_{U\cap V} + \sigma_{V-K} \in C_n(M;R)$$

be a class such that

$$[\sigma] \in C_n(M, M-K \cup L; R)$$

represents $\mu_{K\cup L}$. By the above observations, and by the uniqueness of the classes $\mu_?$, we have that

$$[\sigma_{U\cap V} + \sigma_{V-K}] = \mu_L, [\sigma_{U\cap V}] = \mu_{K\cap L}, \text{ and } [\mu_{U-L} + \sigma_{U\cap V}] = \mu_K.$$

Let $[\varphi] \in H^k(M, M-K\cup L; R)$ be represented by $\varphi_{M-L} + \varphi_{M-K}$. Then $[\mu_{K\cap L}] \frown \delta[\varphi]$ is represented by

$$\sigma_{U\cap V} \frown \delta(\varphi_{M-L}) = -\sigma_{U\cap V} \frown \delta(\varphi_{M-K}).$$

By the boundary formula

$$\partial(\sigma_{U\cap V} \frown \phi_{M-K}) = (-1)^k\big((\partial\sigma_{U\cap V} \frown \varphi_{M-K} - \sigma_{U\cap V} \frown (\delta\phi_{M-K})\big),$$

the class $[\mu_{K\cap L}] \frown \delta[\varphi]$ is also represented by $(\partial\sigma_{U\cap V}) \frown \varphi_{M-K}$.

Now we check the other direction.

$$[\mu_{K\cup L}] \frown [\varphi] = \big[(\sigma_{U-L}+\sigma_{U\cap V}+\sigma_{V-K}) \frown \varphi\big] = \big[(\sigma_{U-L} \frown \varphi)+\big((\sigma_{U\cap V}+\sigma_{V-K}) \frown \varphi\big)\big].$$

The first grouping is in $C_{n-k}(U;R)$, while the second is in $C_{n-k}(V;R)$. So the boundary here is

$$\partial\big([\mu_{K\cup L}] \frown [\varphi]\big) = \big[\partial(\sigma_{U-L} \frown \varphi)\big].$$

Since φ is a cocycle, we have

$$\partial(\sigma_{U-L} \frown \varphi) = (-1)^k(\partial\sigma_{U-L}) \frown \varphi.$$

Now, $\partial\sigma_{U-L} \in C_{n-1}(M-L;R)$, so φ_{M-L} vanishes identically on it. We therefore deduce that

$$(-1)^k(\partial\sigma_{U-L}) \frown \varphi = (-1)^k\big(\partial\sigma_{U-K} \frown \varphi_{M-K}\big).$$

Now, $[\sigma_{U-L} + \sigma_{U\cap V}] \in C_n(M, M-K;R)$ is a cycle (since it represents μ_K), so

$$\partial(\sigma_{U-L} + \sigma_{U\cap V}) \in C_{n-1}(M-K;R).$$

Since φ_{M-K} vanishes on these,

$$(-1)^k(\partial\sigma_{U-L} \frown \varphi_{M-K}) = (-1)^{k+1}(\partial\sigma_{U\cap V} \frown \varphi_{M-K}),$$

as desired.

11.1. Exercises.

Exercise 11.1. *If U and V are locally compact, then show that the compact subsets of the form $K \cup L$, where $K \subset U$ and $L \subset V$ are compact form a cofinal subsystem of all compact subsets of $U \cup V$. (Hint: first think of the case that U and V are themselves σ-compact).*

Exercise 11.2 (Hatcher, 3.3.34)**.** *For a compact manifold M, verify the following diagram relating Poincaré duality for $(M, \partial M)$ and ∂M is commutative:*

$$\begin{tikzcd}
H^{k-1}(\partial M;R) \arrow[r, "\delta"] \arrow[d, "{[\partial M]\frown -}"'] & H^k(M,\partial M;R) \arrow[r] \arrow[d, "{[M]\frown -}"'] & H^k(M;R) \arrow[r] \arrow[d, "{[M]\frown -}"'] & H^k(\partial M;R) \arrow[d, "{[\partial M]\frown -}"'] \\
H_{n-k}(\partial M;R) \arrow[r] & H_{n-k}(M;R) \arrow[r] & H_{n-k}(M,\partial M;R) \arrow[r] & H_{n-k-1}(\partial M;R).
\end{tikzcd}$$

12. Applications of Orientability & Poincaré duality

Our first application is really one of orientability.

Theorem 12.1. *Let M be a closed, compact, n-manifold.*

1. *If M is $\mathbb{Z}$-orientable, then $H_{n-1}(M;\mathbb{Z})$ is torsion free.*
2. *If M is not $\mathbb{Z}$-orientable, then*

$$H_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \text{ (torsion free).}$$

Proof. We use the Universal Coefficients for Homology: there is a natural short exact sequence (unnaturally split)

$$0 \to H_n(M;\mathbb{Z}) \otimes \mathbb{Z}/p \to H_n(M;\mathbb{Z}/p) \to \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}), Z/p) \to 0. \tag{5}$$

If M is $\mathbb{Z}$-orientable, then M is $\mathbb{Z}/p$-orientable for all primes p. Theorem 10.14 then says that

$$H_n(M;\mathbb{Z}) \cong \mathbb{Z} \text{ and } H_n(M;\mathbb{Z}/p) \cong \mathbb{Z}/p.$$

Plugging these into Equation 5 shows then that

$$\mathrm{Tor}\,\big(H_{n-1}(M;\mathbb{Z}), \mathbb{Z}/p\big) = 0.$$

The standard projective resolution

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p$$

shows that for any abelian group A, we have a natural isomorphism

$$\mathrm{Tor}(A, \mathbb{Z}/p) \cong \{a \in A \mid p \cdot a = 0\}.$$

We therefore conclude that we have no p-torsion in $H_{n-1}(M;\mathbb{Z})$ for any p, and hence it is torsion free.

If M is not $\mathbb{Z}$-orientable, then it is not orientable for any $\mathbb{Z}/p$ with p odd. Since there are no 2-torsion points in $\mathbb{Z}$ or $\mathbb{Z}/p$ for p odd, Theorem 10.14 says that

$$H_n(M;\mathbb{Z}) \cong H_n(M;\mathbb{Z}/p) \cong 0.$$

Plugging this into Equation 5 again shows that

$$\mathrm{Tor}\left(H_{n-1}(M;\mathbb{Z}), \mathbb{Z}/p\right) = 0,$$

so we again have no p-torsion for odd p.

Every manifold is $\mathbb{Z}/2$-orientable, and moreover, the simple 2-torsion points of $\mathbb{Z}/2^k$ is also a copy of $\mathbb{Z}/2$, so Theorem 10.14 gives

$$H_n(M;\mathbb{Z}/2^k) \cong \mathbb{Z}/2.$$

Equation 5 then shows that for all k,

$$\mathrm{Tor}\left(H_{n-1}(M;\mathbb{Z}), \mathbb{Z}/2^k\right) \cong \mathbb{Z}/2.$$

The same argument as above says that this Tor group is the points of order dividing 2^k, so we conclude that the only 2-torsion is a $\mathbb{Z}/2$. Since there is no odd torsion, this completes the proof. $\square$

Example 12.2. *Let p be an odd prime, and let $\mu_p \subset S^1$ be the subgroup of pth roots of unity. This acts freely (and properly discontinuously) on $S(\mathbb{C}^n) = S^{2n-1}$, so the orbits are a manifold L_p^{n-1}. This description identifies μ_p with the fundamental group of L_p^{n-1} for all $n > 1$, so we deduce that*

$$H_1(L_p^{n-1};\mathbb{Z}) \cong \mathbb{Z}/p.$$

Cellular homology shows the the homology is very similar to projective space:

$$H_k(L_p^{n-1};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, 2n-1 \\ 0 & k \text{ even or } > 2n-1 \\ \mathbb{Z}/p & k = 2j-1, 1 \leq j \leq n. \end{cases}$$

Of course, since π_1 has no index 2 subgroups, Proposition 10.6 implies this is orientable[3] *and Theorem 10.14 then gives that the top homology group is $\mathbb{Z}$.*

As an immediate consequence, we see that the $2k$-skeleton of L_p^{n-1} can never be a manifold: the codimension 1 homology has p-torsion. If $p = 2$, this is of course different: there L_2^{2k} is just $\mathbb{R}P^{2k}$ is a manifold, but it is not orientable.

Working modulo p, Universal Coefficients shows that

$$H^k(L_p^{n-1};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p & 0 \leq k \leq 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Let x_1 generate H^1 and y_2 generate H^2. By graded commutativity,

$$x_1^2 = -x_1^2 = 0.$$

By induction on n, together with the Cup Product form of Poincaré Duality then gives the ring structure:

$$H^*(L_p^{n-1};\mathbb{Z}/p) \cong E(x_1) \otimes \mathbb{Z}/p[y_2]/y_2^n.$$

[3] which also follows here from being a quotient of the unit sphere in $\mathbb{C}^n$

Exercise 12.1. *Compute the cellular homology groups of L_p^{n-1}. (Hint: describe a μ_p-equivariant cell structure of S^{2n-1}).*

We can also use the Cup Product form to get an amusing structure result about manifolds.

Proposition 12.3. *If M is a closed, connected, compact n-manifold and $M \simeq \Sigma Y$, then $H_*(M;\mathbb{Z}) \cong H_*(S^n;\mathbb{Z})$.*

For this, we need a small lemma about the cup product.

Lemma 12.4. *If $X = A \cup B$ with A and B open and contractible, then for $i, j > 0$, the cup product map*

$$H^i(X;R) \otimes_R H^j(X;R) \to H^{i+j}(X;R)$$

is automatically zero.

Proof. Consider the diagram given by naturality of the cup product and the various inclusions

$$\begin{tikzcd}
H^i(X,x;R) \otimes_R H^j(X,x;R) \arrow[r, "\cup"] & H^{i+j}(X,x;R) \\
H^i(X,A;R) \otimes_R H^j(X,B;R) \arrow[r, "\cup"'] \arrow[u, "\cong"] & H^{i+j}(X,A\cup B;R) = 0. \arrow[u]
\end{tikzcd}$$

Since A and B are contractible, we have by the long exact sequence for the triple $(X, -, *)$ that the left vertical map is an isomorphism for all i and j. Naturality of the cup product then shows that the cup product of any two relative classes in (X, x) factors through zero. The assumption on i and j implies that the relative and absolute cases agree here. □

Remark 12.5. *The "cup length" of a space is the length of the longest sequence of elements in non-zero degree for which the cup product is non-zero. Lemma 12.4 shows that if X is covered by 2 contractible open sets, then the cup length is 1. More generally, if X is covered by n contractible open sets, then the cup length is at most $(n-1)$.*

Proof of Proposition 12.3. Lemma 12.4 shows that all cup products in M vanish for any coefficients. The Cup Product form of Poincaré duality shows that if we have any cohomology between degrees 0 and n exclusive, then there must be a corresponding dual class that cups with it to get a generator of $H^n(M;F)$ for all fields F for which M is orientable. Applying this to $F = \mathbb{Z}/2$, we deduce that M has no cohomology with coefficients in $\mathbb{Z}/2$ between degrees 0 and n exclusive, and hence no homology there. In particular, we have no $H_{n-1}(M;\mathbb{Z}/2)$, which means that M is orientable for $\mathbb{Z}$. The same argument then shows that we have no homology with any finite coefficients or $\mathbb{Q}$ between 0 and n exclusive. □

Part 3. Homotopy Theory

13. Homotopy Co-exactness

We now turn to the study of homotopy groups. Recall the construction of the mapping cone from Lecture **??**. There is an obvious notion for pairs of spaces.

Definition 13.1. *Let $f = (f_0, f_1)\colon (X, A) \to (Y, B)$ be a map of pairs. The mapping cone of f is the pair (Cf_0, C_{f_1}).*

By construction, and by the arguments from Lecture **??**, we have an identification of the maps out of the mapping cone.

Proposition 13.2. *Let $(f_0, f_1)\colon (X, A) \to (Y, B)$ be a map of pairs. Then a map of pairs $(Cf_0, Cf_1) \to (Z, D)$ is a pair:*

1. *$g\colon (Y, B) \to (Z, D)$.*
2. *A nullhomotopy G of $g \circ f_0$ restricting to a nullhomotopy of $g \circ f_1$.*

Definition 13.3. *A sequence of pairs of spaces*

$$(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$$

is ***coexact*** *if for all pairs (W, D), we have an exact sequence of points sets:*

$$\big[(X, A), (W, D)\big] \xleftarrow{f^*} \big[(Y, B), (W, D)\big] \xleftarrow{g^*} \big[(Z, C), (W, D)\big].$$

"Exact" here means $Im(g^*)(f^*)^{-1}(0)$.

The following is a restatement of Proposition 13.2. This is an extremely important example that will motivate much of what we do.

Theorem 13.4. *Let $f = (f_0, f_1)\colon (X, A) \to (Y, B)$ be a map of pairs. Then the sequence*

$$(X, A) \xrightarrow{f} (Y, B) \xrightarrow{i} (Cf_0, Cf_1)$$

is coexact.

Corollary 13.5. *If $f = (f_0, f_1)\colon (X, A) \to (Y, B)$ is a map of pairs, then we have a long coexact sequence*

$$(X, A) \xrightarrow{f} (Y, B) \xrightarrow{i} (Cf_0, Cf_1) \xrightarrow{j} (Ci_0, Ci_1) \xrightarrow{f^1} (Cj_0, Cj_1) \xrightarrow{i^1} (Cf_0^1, Cf_1^1) \to \dots$$

This is just a repeated application of Theorem 13.4. Every sequence three term sequence is a direct application of Theorem 13.4. We want to rewrite this in a more useful form. In particular, we would like to be able to identify the spaces $(C(-)_0^k, C(-)_1^k)$ without having to describe all of the previous ones. Moreover, we would like extra structure here. Recall that the space Y is "nicely embedded" in the space Cf_0 (). Here we again mean that we have a neighborhood that deformation retracts back onto Y. This is a more homotopically robust condition, and we expand on that slightly.

Definition 13.6. *A pair (X, A) has the* ***homotopy extension property*** *if for all spaces Y and for all pairs of maps*

1. *$f\colon X \to Y$ and*
2. *$F\colon A \times I \to Y$ such that $F(a, 0) = f(a)$ for all $a \in A$,*

then there is a map $\tilde{F}\colon X \times I \to Y$ such that $\tilde{F}(x, 0) = f(x)$ for all $x \in X$ and $\tilde{F}(a, t) = F(a, t)$ for all $(a, t) \in A \times I$.

If A is closed in X, then we can combine the continuous map $A \times I \to Y$ and $X \to Y$ to get a map

$$\hat{F}\colon X \cup_A A \times I := X \times \{0\} \cup_{A\times\{0\}} A \times I \to Y.$$

Proposition 13.7. *A pair (X, A) with A closed in X has the homotopy extension property if and only if $X \cup_A A \times I$ is a retract of $X \times I$.*

Proof. Assume (X, A) has the homotopy extension property. If we take $Y = X \cup_A A \times I$ and $\hat{F}$ to be the identity, then the homotopy extension property gives us an extension

$$X \times I \xrightarrow{r} X \cup_A A \times I,$$

extending the identity. This is the retraction.

Now assume that $r\colon X \times I \to X \cup_A A \times I$ is a retraction. Since A is closed in X, a map

$$\hat{F}\colon X \cup_A A \times I \to Y,$$

is the same data as that in Definition 13.6. Our of this, we can form $\hat{F} \circ r\colon X \times I \to Y$. This is the desired extension. $\square$

Exercise 13.1. *If A is any space, then show that (CA, A) has the homotopy extension property. (Hint: what does a map out of CA mean).*

Since I is compact (and hence locally compact) and Hausdorff, we have for any spaces X and Y a natural homeomorphism

$$\mathrm{Map}(X \times I, Y) \cong \mathrm{Map}\big(X, \mathrm{Map}(I, Y)\big),$$

where here we endow all of the mapping spaces with compact open topology. We can use this to rewrite the conditions of the homotopy extension property.

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d] & Y^I \arrow[d, "ev_0"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{F}"] & Y, \end{tikzcd} \tag{6}$$

where here ev_0 is the map which sends a path $\gamma\colon Y^I \to Y$ to $\gamma(0)$.

This has an added advantage that we can restate this in terms of just the map $A \to X$.

Definition 13.8. *A map $A \to X$ is a* ***cofibration*** *if for every space Y and for every solid diagram in Equation 6, the dashed map exists.*

This reformulation allows us to use more categorical constructions.

Lemma 13.9. *Cofibrations are closed under pushouts. If $A \to X$ is a cofibration and $g\colon A \to B$ is any map, then $B \to X \cup_A B$ is a cofibration.*

Proof. Let Y be a space, and assume given map $F\colon B \to Y^I$ and $f\colon X \cup_A \to Y$ making a commutative square. We can extend this to a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d] \arrow[rr, bend left, "G"] & B \arrow[r, "F"] \arrow[d] & Y^I \arrow[d, "ev_0"] \\
X \arrow[r] \arrow[urr, dashed, "\tilde{G}"] \arrow[rr, bend right, "\tilde{f}"'] & X \cup_A B \arrow[r, "f"] \arrow[ur, dotted, "\tilde{F}"] & Y
\end{tikzcd}$$

The map $\tilde{G}$ exists making the solid diagram commute by applying the definition to the curved arrows. The map $\tilde{F}$ then exists making the solid diagram commute by the universal property of the pushout. $\square$

Corollary 13.10. *If $f\colon X \to Y$ is continuous, then $i\colon Y \to Cf$ is a cofibration.*

Proof. The map i is the pushout of the map $X \hookrightarrow CA$ along $X \to Y$. By Exercise 13.1 and Lemma 13.9, we have that this is a cofibration. $\square$

Corollary 13.11. *If $f\colon X \to Y$ is continuous and $i\colon Y \to Cf$ is the inclusion of Y into the mapping cone, then $CY \to Ci$ is a cofibration.*

Proof. The map is the pushout of i along the inclusion $Y \hookrightarrow CY$. Corollary 13.10 and Lemma 13.9 then give the result. $\square$

Theorem 13.12. *If (X, A) has the homotopy extension property and $A \simeq *$, then the canonical quotient map $q\colon X \to X/A$ is a homotopy equivalence.*

Proof. Let $F\colon A \times I \to A$ be a homotopy between the identity map of A and the constant map at the basepoint. Then viewing this as a map $A \times I \to X$ via the inclusion $A \hookrightarrow X$, we get a map

$$\hat{F}\colon X \cup_A A \times I \to X.$$

Since (X, A) has the homotopy extension property, we can find an extension $\tilde{F}\colon X \times I \to X$. Let $p(x) = \tilde{F}(x, 1)$. By construction, for all $a \in A$, $p(a) = *$. In particular, p descends to a map

$$\bar{p}\colon X/A \to X.$$

Note that $\bar{p} \circ q = p$, so by construction, $\bar{p} \circ q \simeq Id_X$. For the other direction, note that since F took all of $A \times I$ back to A, $q \circ \tilde{F}$ descends to a map $X/A \times I \to X/A$. By construction, it is a homotopy $q \circ \bar{p} \simeq Id_{X/A}$, and these are homotopy inverses. $\square$

Since CY is contractible, we can apply Theorem 13.12 to Corollary 13.11.

Corollary 13.13. *If $f\colon X \to Y$ is continuous and $i\colon Y \to Cf$ is the inclusion of Y into the mapping cone, then we have a homotopy equivalence*

$$Ci \simeq \Sigma X = Ci/CY = Cf/Y.$$

Everything done so far obviously applies to pairs. This gives us a reformulation of the Puppe sequence.

Theorem 13.14. *Let $f\colon (X, A) \to (Y, B)$ be continous. Then we have a long co-exact sequence*

$$(X, A) \xrightarrow{f} (Y, B) \xrightarrow{i} (Cf_0, Cf_1) \xrightarrow{j} (\Sigma X, \Sigma A) \xrightarrow{\Sigma f} (\Sigma Y, \Sigma B) \xrightarrow{\Sigma i} (\Sigma Cf_0, \Sigma Cf_1) \to \dots$$

14. co-H-spaces

At the end of the last lecture, we saw that all of the spaces after the first 3 in the Puppe sequence are suspensions. This gives us extra structure on the natural exact sequences of maps out of these.

Definition 14.1. *A co-H-space*[4] *is a pointed space X together with maps*

$$\mu\colon X \to X \vee X \text{ and } \iota\colon X \to X$$

such that the following diagrams commute in the homotopy category:

(7)
$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "\mu"'] & X \vee X, \arrow[d, "Id \vee \mu"] \\ X \vee X \arrow[r, "\mu \vee Id"'] & X \vee X \vee X \end{tikzcd}$$

(8)
$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[d, "\mu"] \arrow[dr, equal] & \\ X & X \vee X \arrow[l, "\epsilon \vee Id"] \arrow[r, "Id \vee \epsilon"'] & X \end{tikzcd}$$

(9)
$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "\epsilon"'] & X \vee X \arrow[r, "Id \vee \iota"] & X \vee X \arrow[d, "\nabla"] & X \arrow[r, "\mu"] \arrow[d, "\epsilon"'] & X \vee X \arrow[r, "\iota \vee Id"] & X \vee X \arrow[d, "\nabla"] \\ * \arrow[rr] & & X & * \arrow[rr] & & X \end{tikzcd}$$

Here the map ϵ is the canonical map to a point, and the unlabeled map is the inclusion of the basepoint. The map ∇ is the fold map.

*The map μ is the **comultiplication**, the map ι is the **coïnversion**.*

*A co-H-space X is **cocommutative** if moreover*

(10)
$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "\mu_X"'] & X \vee X \arrow[d, "\tau"] \\ & X \vee X, \end{tikzcd}$$

where τ is the twist map interchanging the two summands.

These form a category, with the obvious notion of a homomorphism.

Definition 14.2. *Let X and Y be co-H-spaces. A continuous map $f\colon X \to Y$ is a homomorphism if*

$$(f \vee f) \circ \mu_X \simeq \mu_Y \circ f \text{ and } f \circ \iota_X \circ \iota_Y \circ f.$$

The axioms are most easily understood in their dual form. The wedge is the coproduct in pointed spaces and in the homotopy category of points spaces:

$$\mathrm{Map}_*(X \vee Y, Z) \cong \mathrm{Map}_*(X, Z) \times \mathrm{Map}_*(Y, Z) \text{ and } [X \vee Y, Z] \cong [X, Z] \times [Y, Z].$$

This gives us another way to interpret the axioms.

[4]the "co" here is in the category theory sense, while H stands for "Hopf"

Proposition 14.3. *If X is a co-H-space, then for any pointed space Y, the set*

$$X(Y) := [X, Y]$$

is a group, with multiplication given by

$$[X, Y] \times [X, Y] \cong [X \vee X, Y] \xrightarrow{\mu^*} [X, Y],$$

and with inversion given by ι^*.

If $f\colon Y \to Z$ is any continuous map, then

$$f_*\colon [X, Y] \to [X, Z]$$

is a group homomorphism

Proof. Since pre- and post-composition commute, the second part follows immediately from the first. For the first, note that associativity is recorded by Equation 7, that the unit condition is exactly recorded by Equation 8, and Equations 9 shows that ι^* is a two sided inverse. □

Exercise 14.1. *Verify the claims in the proof of Proposition 14.3. You may find it helpful to dualize the axioms for a coH-space and write the axioms for a group object in a category.*

Remark 14.4. *The assignment $Y \mapsto [X, Y]$ gives a covariant functor from spaces to pointed sets. Since this is given by the* Hom*s in the category out of a fixed object X, this is a "representable" functor. The Yoneda Lemma can be used to show that if this functor has a lift to a functor from spaces to groups, then X must be a co-H-space.*

Example 14.5. *The space S^1 is a co-H-space. Here, the comultiplication is just*

$$S^1 \mapsto S^1/\{\pm 1\} \cong S^1 \vee S^1,$$

and the coïnversion is $z \mapsto \bar{z}$ (in both, we have viewed $S^1 \subset \mathbb{C}$).

Remark 14.6. *Proposition 14.3 can be used to deduce that $[S^1, X]$ has a natural group structure for any space X: this is the fundamental group.*

Proposition 14.7. *Let X be a co-H-space and let Y be any pointed space. Then the maps*

$$\mu_{X \wedge Y} := \mu_X \wedge Id_Y \text{ and } \iota_{X \wedge Y} := \iota_X \wedge Id_Y$$

make $X \wedge Y$ into a co-H-space.

If $f\colon Y \to Z$ is any continuous map, then $Id_X \wedge f$ is a homomorphism.

In this, we of course also use the canonical distributivity isomorphism

$$(X \vee X) \wedge Y \cong (X \wedge Y) \vee (X \wedge Y).$$

Exercise 14.2. *Prove Proposition 14.7.*

Remark 14.8. *If Y is locally compact and Hausdorff, then we can use the Yoneda form of a co-H-space. We have a natural isomorphism of functors of Z:*

$$[X \wedge Y, Z] \cong [X, \mathrm{Map}(Y, Z)].$$

The co-H-space structure on X endows the right-hand side with a canonical group structure for all spaces Z, so the left-hand side inherits one via this natural isomorphism.

Combining Example 14.5 and Proposition 14.7, we immediately deduce a huge amount of extra structure on the Puppe sequence. All of the spaces beyond the first three terms are suspensions, and all of the maps are suspensions, so all of these spaces are canonical co-H-spaces and the maps are homomorphisms. There is yet *more* structure.

Definition 14.9. *If X and Y are co-H-spaces, then $X \vee Y$ becomes a co-H-space with comultiplication*

$$X \vee Y \xrightarrow{\mu_X \vee \mu_Y} (X \vee X) \vee (Y \vee Y) \xrightarrow{Id_X \vee \tau \vee Id_Y} (X \vee Y) \vee (X \vee Y).$$

Lemma 14.10. *Let X be a space and assume that X has 2 co-H-space structures and that each is a homomorphism for the other. Then the two comultiplications agree and are co-commutative.*

Proof. Let μ and μ' denote the two comultiplications. Our assumption is then that we have a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "\mu"] \arrow[d, "\mu'"'] & & X \vee X \arrow[d, "\mu' \vee \mu'"] \\
X \vee X \arrow[r, "\mu \vee \mu"'] \arrow[rr, bend right, "\mu_{X \vee X}"'] & X \vee X \vee X \vee X \arrow[r, "Id \vee \tau \vee Id"'] & X \vee X \vee X \vee X
\end{tikzcd}$$

Equation 8 shows that we have

$$(\epsilon \vee Id_X) \circ \mu \simeq (Id_X \vee \epsilon) \circ \mu \simeq Id_X$$

and identically for μ'.

We now compute

$$\begin{aligned}
(11) \quad \mu = (Id_X \vee Id_X) \circ \mu &\simeq \big((Id_X \vee \epsilon \vee \epsilon \vee Id_X) \circ (\mu' \vee \mu')\big) \circ \mu \\
\simeq (Id_X \vee \epsilon \vee \epsilon \vee Id_X) \circ (Id_X \vee \tau \vee Id_X) \circ (\mu \vee \mu) \circ \mu' &\simeq (Id_X \vee \epsilon \vee \epsilon \vee Id_X) \circ (\mu \vee \mu) \circ \mu' \\
&\simeq (Id_X \vee Id_X) \circ \mu' = \mu'.
\end{aligned}$$

Using $(\epsilon \vee Id_X \vee Id_X \vee \epsilon)$ instead gives us that $\mu \simeq \tau\mu'$, which then gives cocommutativity. □

Corollary 14.11. *For any spaces X, Y and for any $k \geq 2$, $[\Sigma^k X, Y]$ is naturally an abelian group. If $f \colon X' \to X$ and $g \colon Y \to Y'$, then f^* and g_* are maps of abelian groups.*

Putting this all together, we get a shocking amount of structure on the Puppe sequence.

Theorem 14.12. *If $f \colon (X, A) \to (Y, B)$, and if (Z, D) is any other pair, then we have a long exact sequence*

$$\dots \longleftarrow \big[\Sigma^k(X, A), (Z, D)\big] \xleftarrow{\Sigma^k f^*} \big[\Sigma^k(Y, B), (Z, D)\big] \xleftarrow{\Sigma^k i^*} \big[\Sigma^k(Cf_0, Cf_1), (Z, D)\big] \longleftarrow \dots$$

when $k \geq 1$, these are all groups and group homomorphisms; when $k \geq 2$, these are all abelian.

We have one last piece of structure here.

Definition 14.13. *Let X be a co-H-space, and let Y be a space. A* ***coäction*** *of X on Y is a map*

$$\nu\colon Y \to Y \vee X$$

such that the following diagram commutes up to homotopy

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[d, "\nu"'] & Y \vee X \arrow[d, "\nu \vee Id_X"] \\ Y \vee X \arrow[r, "Id_Y \vee \mu"'] & Y \vee X \vee X. \end{tikzcd} \tag{12}$$

A map of spaces coacted on by X is the obvious thing.

Remark 14.14. *Equation 12 is the dual of the usual formula expressing the action of a group on a set or space: this is encoding the [co]associativity. Hence we naturally get an action when mapping out of these.*

Proposition 14.15. *Let X be a co-H-space that coacts on Y. Let Z be any other space. Then the pointed set $[Y, Z]$ has a natural action of the group $[X, Z]$.*

Proof. Apply $[-, Z]$ to the diagram in Equation 12. ☐

Proposition 14.16. *Let $f\colon A \to X$ be any map, and let $C = Cf$ be the mapping cone. Then the map*

$$Cf \to Cf/\big(A \times \{1/2\}\big) \cong Cf \vee \Sigma A$$

gives a coaction of ΣA on Cf.

Proof. The space X here really plays no role; we might as well actually show that the map described gives a ΣA coaction on CA. We refer to the square in Equation 12. Tracing along the top and the right, we get the map

$$CA \to CA/\big(A \wedge \{\tfrac{1}{4}, \tfrac{1}{2}\}_+\big).$$

Tracing along the left and bottom, we get the map

$$CA \to CA/\big(A \wedge \{\tfrac{1}{2}, \tfrac{3}{4}\}_+\big).$$

Rescaling the interval gives the desired homotopy between these. ☐

The fact that the proof of Proposition 14.16 took place far from the copy of X also immediately gives us the following proposition.

Proposition 14.17. *The map*

$$Cf \to \Sigma A$$

is a map of spaces coacted on by ΣA, where the coaction on ΣA is via μ itself.

Corollary 14.18. *If $f\colon (X, A) \to (Y, B)$ is continous, and (Z, D) is arbitrary then*

(1) *The map*

$$\big[\Sigma(X, A), (Z, D)\big] \xrightarrow{j^*} \big[(Cf_0, Cf_1), (Z, D)\big]$$

is a map of pointed $\big[\Sigma(X, A), (Z, D)\big]$-sets.

(2) *The map*

$$\big[(Cf_0, Cf_1), (Z, D)\big] \to \big[(Y, B), (Z, D)\big]$$

factors through the $\big[\Sigma(X, A), (Z, D)\big]$ orbits.

Proof. The first part is just Proposition 14.17 combined with Proposition 14.15. For the second part, observe that the coaction takes place near the cone point of Cf, and hence never impacts the restriction to the base. ☐

15. Homotopy Groups

Definition 15.1. *If (X, A) is a pair then let*

- $\pi_k(X, x) = [\Sigma^k S^0, X]$ *for* $k \geq 0$ *and*
- $\pi_k(X, A, x) = \left[\Sigma^{k-1}(D^1, S^0), (X, A)\right]$ *for* $k \geq 1$.

These are the ***homotopy groups*** *of X and the* ***relative homotopy groups*** *of (X, A) respectively.*

The name is slightly misleading, but is almost always true.

Proposition 15.2.
(1) *For any pointed space X, the pointed sets $\pi_k(X, x)$ are groups for $k \geq 1$ and abelian groups for $k \geq 2$.*
(2) *For any pair (X, A), the pointed sets $\pi_k(X, A, x)$ are groups for $k \geq 2$ and abelian for $k \geq 3$.*

Proof. Abelianness is immediate from Corollary 14.11. The group structure is immediate from Propositions 14.7 and Example 14.5. ☐

Theorem 15.3. *If (X, A) is any pair, then we have a natural long exact sequence*

$$\pi_0(X, x) \xleftarrow{i_*} \pi_0(A, x) \xleftarrow{\partial} \pi_1(X, A, x) \xleftarrow{j_*} \pi_1(X, x) \longleftarrow \dots$$

The maps i_* and j_* are induced by the obvious inclusions. The map ∂ is just the restriction of a map of pairs $(D^k, S^{k-1}) \to (X, A)$ to the map $S^{k-1} \to A$.

Proof. Apply the Puppe long exact sequence to the map of pairs

$$(S^0, *) \hookrightarrow (S^0, S^0).$$

The mapping cone of this is just (D^1, S^0). This gives the long exact sequence. Now note that a map of pairs $(S^k, *) \to (X, A)$ is just a map $S^k \to X$, while a map of pairs $(S^k, S^k) \to (X, A)$ is just a map $S^k \to A$. ☐

In general, homotopy groups are extremely difficult to compute. We shall see below that they are much more powerful than homology groups, however. We have some basic results.

Theorem 15.4 (Jordan Curve Theorem)**.** *If $k < \ell$ then every map*

$$S^k \to S^\ell$$

is null-homotopic.

Corollary 15.5. *If $k < \ell$ then*

$$\pi_k(S^\ell, s) = 0.$$

Definition 15.6. *A pair (X, A) is n-connected if*
(1) *every path component of X intersects A and*
(2) *for every choice of basepoint $a \in A$ and for every $k \leq n$, $\pi_k(X, A, a) = 0$.*

One way we can restate Corollary 15.5 is that the pair $(S^n, *)$ is $(n-1)$-connected. This plus the long exact sequence gives us a result for disks.

Proposition 15.7. *For any n, the pair (D^n, S^{n-1}) is $(n-1)$-connected.*

Proof. Since $D^n \simeq *$ and since homotopy groups are obviously a homotopy invariant, we know that for all k,

$$\pi_k(D^n, *) = 0.$$

Corollary 15.5 shows that for $k-1 < n-1$

$$\pi_{k-1}(S^{n-1}, *) = 0.$$

The long exact sequence from Theorem 15.3 squeeze $\pi_k(D^n, S^{n-1}, *)$ between these two groups, giving the result. □

We can most directly tie these to obstruction theory.

Proposition 15.8. *A class $[F] \in \pi_k(X, A, x)$ is zero if and only if $F \simeq_{S^1} F'$, where*

$$F' \colon (D^k, S^{k-1}, *) \to (A, A, x).$$

Proof. If F represents 0 then $F \simeq *$ as a map of pairs. Choose such a null-homotopy. This gives a map

$$H \colon D^k \times I \to X$$

such that $H|_{S^{k-1}\times I}$ lands in A. In all cases, $H(\vec{v}, 1) = x \in A$. If we consider the disks

$$D^k \times \{t\} \cup S^{k-1} \times [0, t],$$

then we see that our homotopy H can also be viewed as a homotopy relative to S^{k-1} of F to a map

$$H|_{D^k\times\{1\}\cup S^{k-1}\times I} \colon D^k \to A.$$

For the converse, if F is homotopic relative to S^{k-1} to a map $(D^k, S^{k-1}, *) \to (A, A, x)$, then composing with a nullhomotopy of the identity of D^k shows F is null. □

Corollary 15.9. *The relative group $\pi_k(X, A, x) = 0$ if and only if every map*

$$(D^k, S^{k-1}) \to (X, A)$$

is homotopic relative to its boundary to a map with image in A.

Proposition 15.10. *Let (X, A) be a relative CW-complex, and let (Y, B) be a pair. If whenever (X, A) has an n-cell, we have $\pi_n(Y, B, y) = 0$, then any map $(X, A) \to (Y, B)$ is homotopic relative to A to a map with image in B.*

Proof. We proceed by induction on the skeleta. Assume we have already compressed the $(n-1)$-skeleton to B:

$$X^{[n-1]} \xrightarrow{f_{n-1}} B.$$

For each n cell e_α of X, we have a characteristic map

$$(D^n, S^{n-1}) \xrightarrow{e_\alpha} (X^{[n]}, X^{[n-1]}),$$

and we can compose this with f to get an element in $\pi_n(Y, B, y)$. Since by assumption this group is zero, we know that the composite $f \circ e_\alpha$ is homotopic relative to the boundary S^{n-1} to a map to B. Gluing these together for all of the n-cells, we get a homotopy

$$f \simeq_{X^{[n-1]}} f_n,$$

where now f_n takes the n-skeleton of X to B. For an infinite complex, note that our homotopies move only finitely many cells, so they transfinitely glue. □

This gives us an absurdly strong result: the Whitehead Theorem.

Theorem 15.11 (Whitehead Theorem)**.** *If X and Y are connected CW-complexes, then $f\colon X \to Y$ is a homotopy equivalence if and only if $f_*\colon \pi_n(X,x) \to \pi_n(Y, f(x))$ is an isomorphism for all n.*

If $X \subset Y$ induces an isomorphism in homotopy groups, then X is a deformation retraction of Y.

Proof. Since homotopy groups are a homotopy functor, the forward implication is immediate: a homotopy inverse for f gives the inverse homomorphism.

For the other implication, CW-approximation allows us replace f with a cellular map which is homotopic to it. In this case, the mapping cylinder of f is again a CW complex, and the map $X \to Mf$ is a CW-inclusion. Since the map $Mf \to Y$ is always a homotopy equivalence, it will suffice to show that $X \to Mf$ is. This is essentially the statement of the second part, so we are reduced to proving that.

Now the long exact sequence of the pair (Theorem 15.3) shows that for all n, the relative homotopy groups

$$\pi_n(Y, X, x) = 0.$$

Applying Proposition 15.10 to the identity map $(Y, X) \to (Y, X)$ shows that the identity map is homotopy equivalent, relative to X, to a map $(Y, X) \to (X, X)$. This is exactly the statement that X is a deformation retraction of Y. $\square$

16. Homotopy groups and fibrations

In general, computing π_k is extremely difficult. There is a second case where we have a long exact sequence, and this is extremely useful.

Definition 16.1. *A map $p\colon E \to B$ is a **[Serre] fibration** if for all n and for any solid diagram*

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & B \end{tikzcd}$$

the dashed arrow exists making the entire diagram commutative.

We say that a Serre fibration "has the homotopy lifting property" for the intervals.

Remark 16.2. *A map $p\colon E \to B$ is a Hurewicz fibration if we have the analogous diagram but with I^n replaced with an arbitrary space.*

Example 16.3. *If $\tilde{X} \xrightarrow{p} X$ is a covering map, then it is a fibration.*

Proposition 16.4. *If $A \to X$ is a cofibration and both A and X are "nice" (locally compact, Hausdorff), then for any space B, the induced map*

$$B^X \xrightarrow{i^*} B^A$$

is a fibration.

Proof. Consider the solid diagram

$$\begin{tikzcd} I^n \arrow[r, "f"] \arrow[d] & B^X \arrow[d, "i^*"] \\ I^n \times I \arrow[r, "H"'] \arrow[ur, dashed, "G"] & B^A. \end{tikzcd}$$

Via the canonical adjunctions, the map $f\colon I^n \to B^X$ is equivalent to $I^n \times X \xrightarrow{\hat{f}} B$ is equivalent to $X \xrightarrow{\bar{f}} B^{I^n}$. Similarly, the map H is equivalent to a map $\bar{H}\colon A \to (B^{I^n})^I$. In particular, our diagram is equivalent to the diagram

$$\begin{tikzcd} A \arrow[r, "\bar{H}"] \arrow[d, "i"'] & (B^{I^n})^I \arrow[d] \\ X \arrow[r, "\bar{f}"'] \arrow[ur, dashed, "\bar{G}"] & B^{I^n}. \end{tikzcd}$$

Since $A \to X$ is a cofibration, the map $\bar{G}$ exists, and unpacking, this gives the map G making the first diagram commute. $\square$

Just as with cofibrations, the diagramatic description gives us a way to deduce certain closure properties.

Proposition 16.5. *If $p\colon E \to B$ is a fibration and $f\colon X \to B$ is any map, then*

$$p' = f^*(p)\colon X \times_B E \to X$$

is a fibration.

Proof. Exercise 16.1. $\square$

Exercise 16.1. *Prove Proposition 16.5.*

Definition 16.6. *Let $b \in B$, and let*

$$P(B,b) = \{\gamma\colon I \to B \mid \gamma(0) = b\} \subset \operatorname{Map}(I,B)$$

be the space of paths in B based at b.

Corollary 16.7. *For any $b \in B$, the evaluation map $ev_1\colon P(B,b) \to B$ is a fibration.*

Proof. The map $\{0,1\} \to I$ is a cofibration, and so the dual map

$$B^I \to B^{\{0,1\}} \cong B \times B$$

is a fibration by Proposition 16.4. The map ev_1 is the pullback of this along the map $B \to B \times B$ given by $x \mapsto (b,x)$, and hence Proposition 16.5 give the result. $\square$

Exercise 16.2. *Give a direct proof that the based path space gives a fibration by describing the lift of a homotopy.*

Serre fibrations actually have lifts along a much broader collection of spaces. We can lift over any CW-complex.

Proposition 16.8. *If K is a finite CW complex and $p\colon E \to B$ is a Serre fibration, then we can complete any solid diagram*

$$\begin{tikzcd} K \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ K \times I \arrow[r] \arrow[ur, dashed] & B. \end{tikzcd}$$

More generally, if (X, A) is a relative CW-complex, then we can complete any solid diagram

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B. \end{tikzcd}$$

Proof. Taking $A = \emptyset$ in the second part gives the first, so we prove the second. For this, observe that when $(X, A) = (D^n, S^{n-1})$, then there is a lift, since we have a homeomorphism of pairs

$$(D^n \times I, D^n \times \{0\} \cup S^{n-1} \times I) \cong (I^n \times I, I^n \times \{0\})$$

given by unwrapping the disk along the cylinder. For the general case, we induct over the cells. The base case is the "(-1)-skeleton", which is A itself. Here, the statement is immediate, since we have given ourselves the extension over $A \times I$.

Assume now that we have built our extension over the $(n-1)$-skeleton. This gives us a map

$$f_{n-1}\colon X \times \{0\} \cup X^{[n-1]} \times I \to E$$

lifting H. We therefore have a solid diagram

$$\begin{tikzcd} X \times \{0\} \cup X^{[n-1]} \times I \arrow[r, "f_{n-1}"] \arrow[d] & E \arrow[d, "p"] \\ X \times \{0\} \cup X^{[n]} \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B. \end{tikzcd}$$

Restricting to each n-cell in $X^{[n]}$ then gives us a diagram

$$\begin{tikzcd} D^n \times \{0\} \cup S^{n-1} \times I \arrow[r, "f_{n-1} \circ e_\alpha"] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r, "H \circ e_\alpha"'] \arrow[ur, dashed, "\tilde{H}"] & B, \end{tikzcd}$$

and here we have a lift by the above homeomorphism. Gluing these together gives the lift

$$f_n\colon X \times \{0\} \cup X^{[n]} \times I \to E.$$

□

We now come to by far the most useful property of fibrations: the long exact sequence in homotopy.

Theorem 16.9. *Let $p\colon E \to B$ be a fibration, and let E be pointed by e and B be pointed by $b = p(e)$. If $A \subset B$, then p_* induces an isomorphism*

$$p_*\colon \pi_k(E, p^{-1}A, e) \to \pi_k(B, A, b).$$

Proof. Both parts follow from homotopy lifting.

Let $F\colon (D^n, S^{n-1}) \to (B, A)$ be a pointed map. Ignoring A, the map $F\colon D^n \to B$ is homotopic to c_b, the constant map at b. Let $H\colon D^n \times I \to B$ be a homotopy from c_b to F. This gives us a solid diagram

$$\begin{tikzcd} D^n \times \{0\} \arrow[r, "c_e"] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B. \end{tikzcd}$$

Since p is a fibration, the map $\tilde{H}\colon D^n \times I \to E$ exists, lifting H. Let $\tilde{F} = \tilde{H}(-, 1)$. By construction,

$$p \circ \tilde{F} = F,$$

so in particular, we deduce first that

$$\tilde{F}(S^{n-1}) \subset p^{-1}(A)$$

and then that $[\tilde{F}] \in \pi_n(E, p^{-1}A, e)$ maps to $[F]$ under p_*.

For injectivity, let $\bar{F}\colon (D^n, S^{n-1}) \to (E, p^{-1}A)$ be such that $p\circ\bar{F}$ represents zero in $\pi_n(B, A, b)$. Proposition 15.8 then says that the map $p\circ\bar{F}$ is homotopic (relative to S^{n-1}) to a map $F\colon (D^n, S^{n-1}) \to (A, A)$. Let H be such a homotopy. We then have a solid diagram

$$\begin{tikzcd} D^n \times \{0\} \arrow[r, "\bar{F}"] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r, "H"'] \arrow[ur, dashed, "\bar{H}"] & B, \end{tikzcd}$$

and since p is a fibration, the map $\bar{H}$ exists. If we let $\tilde{F} = \bar{H}(-, 1)$, then we know that $p \circ \tilde{F} = F$. In particular, for all $z \in D^n$, $\tilde{F}(z) \in p^{-1}(A)$, and hence $\bar{F}$ is homotopic to a map which takes all of D^n into $p^{-1}A$. Proposition 15.8 again shows that this is zero. $\square$

Corollary 16.10. *If $F = p^{-1}(b)$ is the fiber at $b \in B$, then we have a long exact sequence*

$$\cdots \to \pi_k(F, e) \xrightarrow{i_*} \pi_k(E, e) \xrightarrow{p_*} \pi_k(B, b) \xrightarrow{\partial} \pi_{k-1}(F, e) \to \dots.$$

Proof. This is the long exact sequence for the pair (E, F), where using Theorem 16.9, we replace the relative homotopy group with the absolute homotopy group of the base. $\square$

17. Base-points and π_1 actions

In class, the question was asked "Why do we not just consider higher homotopy groupoids rather than the higher homotopy groups?" This is a great question, and it allows us to talk about the action of the fundamental group on the higher homotopy groups and also on the fibers in a fibration.

To describe the groupoid action, we fix some notation. Let D^n_2 denote the disk of radius 2 in R^n, and let $D^n \to D^n_2$ denote the obvious inclusion.

Definition 17.1. *Let $\gamma\colon I \to X$ be a path from x_1 to x_2. Let $[f] \in \pi_k(X, x_2)$ be represented a map $f\colon S^k \to X$, which we view as a map $D^k \to X$ which takes S^{k-1} to x_2. Now let*

$$\gamma_*\big([f]\big)\colon (D_2^k, S^{k-1}) \to (X, x_1)$$

be the map which is f on D^k, and on closure of the complement

$$\overline{D_2^k - D_k} \cong S^{k-1} \times I \xrightarrow{q_I} I \xrightarrow{\gamma} X.$$

On the intersection $S^{k-1} \subset D_k$, these maps are both constant at x_2, and hence glue.

This gives a functorial homomorphism.

Theorem 17.2. *The construction $[f] \mapsto \gamma_*\big([f]\big)$ gives a homomorphism*

$$\gamma_*\colon \pi_k(X, x_2) \to \pi_k(X, x_1).$$

If $\gamma = \gamma_1 \cdot \gamma_2$, then

$$\gamma_* = \gamma_{1*} \circ \gamma_{2*}.$$

Proof. We first note that the map γ_* is well-defined. All of our homotopies defining $\pi_k(X, x)$ are defined relative to the basepoint. In particular, if we have two representatives, f, f', of the same homotopy class, then there is a homotopy

$$D^k \times I \to X$$

between them which takes all of $S^{k-1} \times I$ to the basepoint. We can then just glue this homotopy to the constant homotopy on the complementary annulus, showing that this did not depend on f but rather only on its homotopy class.

To show that this is a homomorphism, we slightly recast the addition yet again. Radial projection gives a homeomorphism of pairs $(I^n, \partial I^n) \cong (D^n, S^{n-1})$. The addition is then given by observing that we have a homeomorphism

$$I^n \cup_{I^{n-1}} I^n \cong I^n,$$

where the I^{n-1} embeds in as $I^{n-1} \times \{1\}$ in the first summand and as $I^{n-1} \times \{0\}$ in the second. Since all of ∂I^n is sent to the basepoint under all of the maps (by assumption), these maps glue. On every face, we can glue a copy of I, along which we run γ. This gives us $\gamma_*(f_1 + f_2)$. A choice of null-homotopy of $\gamma \cdot \gamma^{-1}$ gives a homotopy between $\gamma_*(f_1 + f_2)$ and the map $\gamma_*(f_1) + \gamma_*(f_2)$.

Functoriality is immediate from considering instead a disk of radius 3 for the construction of γ_*. □

Proposition 17.3. *When $k = 1$, this gives us the conjugation action of π_1 on itself.*

There is a similar construction and similar results for the relative homotopy groups.

Definition 17.4. *Let $\gamma\colon I \to A$ be a path from a_1 to a_2. If $f\colon (D^n, S^{n-1}) \to (X, A)$, then concatinating with γ gives a map*

$$\gamma_*(f)\colon (D_2^n, S^{n-1}) \to (X, A).$$

Proposition 17.5. *The assignment $\gamma \mapsto \gamma_*$ gives an action of $\pi_1(A, a)$ on $\pi_k(X, A, a)$ for any basepoint $a \in A$.*

In fact, the entire long exact sequence for the pair is natural for the action of $\pi_1(A, a)$.

17.1. **Fiber Bundles.** We can actually do all of this in families, which gives us information about the fibers in a fibration.

Definition 17.6. *Let $p\colon E \to B$ be a fibration. For each $b \in B$, let $F_b = p^{-1}(B)$ be the fiber over b.*

Theorem 17.7. *Let $p\colon E \to B$ be a fibration such that each of the fibers F_b are CW complexes. Then the assignment*

$$b \mapsto F_b$$

extends to a contravariant functor from the fundamental groupoid of B to the homotopy category of CW complexes.

Proof. Let $\gamma\colon I \to B$ be a path, starting at b_0 and ending at b_1. Composing γ with the projection $F_{b_0} \times I \to I$ gives a map $\hat{\gamma}$ which fits into a commutative diagram

$$\begin{tikzcd} F_{b_0} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ F_{b_0} \times I \arrow[r, "\hat{\gamma}"'] \arrow[ur, dashed, "\bar{\gamma}"] & B, \end{tikzcd}$$

where $F_{b_0} \to E$ is the natural inclusion. Since F_{b_0} is a CW complex and $E \to B$ is a fibration, we know that the map $\bar{\gamma}$ exists. Since this is a lift of γ, we have

$$\tilde{\gamma} := \bar{\gamma}(-, 1)\colon F_{b_0} \to F_{b_1}.$$

These visibly glue.

Now if $\gamma \simeq \gamma'$, then choose a homotopy H realizing this. Note also that we have a homeomorphism of pairs

$$(I^2, I \times \{0\}) \cong (I^2, \{0,1\} \times I \cup I \times \{0\}).$$

The natural inclusion of F_{b_0} into E together with the maps $\tilde{\gamma}$ and $\tilde{\gamma}'$ give us a map

$$F_{b_0} \times (\{0,1\} \times I \cup I \times \{0\}) \to E,$$

which fits into a commutative square with the homotopy H:

$$\begin{tikzcd} F_{b_0} \times (\{0,1\} \times I \cup I \times \{0\}) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ F_{b_0} \times I \times I \arrow[r, "H"'] \arrow[ur, dashed, "\bar{H}"] & B. \end{tikzcd}$$

We know that the map $\bar{H}$ exists, as before, so this gives a homotopy between $\tilde{\gamma}$ and $\tilde{\gamma}'$. ☐

Corollary 17.8. *If $p\colon E \to B$ is a fibration such that all fibers have the homotopy types of CW complexes, then all of the fibers in a particular path component are homotopy equivalent.*

Proof. Given two points in the path component, functoriality is exactly the statement that the induced map $\tilde{\gamma}$ is an isomorphism in the homotopy category. ☐

Corollary 17.9. *Let $b \in B$ and let $p\colon E \to B$ be a fibration such that all fibers are CW complexes. Then $\pi_1(B, b)$ acts on F in the homotopy category.*

Definition 17.10. *We say that $p\colon E \to B$ is simple if the action of $\pi_1(B, b)$ on the fiber is trivial in the homotopy category.*

Part 4. Spectral Sequences

18. Filtrations and Spectral Sequences

The concept of a filtration is fundamental in algebraic topology. By filtering away any confusing or difficult aspects of a computation, we can reduce it to a series of less complicated steps. This is recorded in a spectral sequence.

Definition 18.1. *An **increasing filtration** on M is a collection of submodules F^iM, $i \in \mathbb{Z}$, such that if $i < j$, then*

$$F^iM \subset F^jM.$$

*A **decreasting filtration** on M is a collection of submodules F_iM, $i \in \mathbb{Z}$ such that if $i < j$, then*

$$F_iM \supset F_jM.$$

Definition 18.2. *If M and M' are filtered modules, then a homomorphism of filtered modules is a homomorphism $f\colon M \to M'$ such that for all $i \in \mathbb{Z}$, we have*

$$f\big(F^iM\big) \subset F^iM'.$$

Remark 18.3. *We can view a filtration as a functor from the poset $\mathbb{Z}$ to the category of R-modules and injective maps. The decreasing filtrations are contravariant functors, so one can make all of the analogous definitions there. Because of this, it is obvious what we mean by filtered objects in other categories.*

We have two extremal cases:

$$F^{-\infty}M = \bigcap_{i\in\mathbb{Z}} F^iM \text{ and } F^{\infty}M = \bigcup_{i\in\mathbb{Z}} F^iM.$$

Definition 18.4. *An increasing filtration is **Hausdorff** if $F^{-\infty}M = 0$.*
*An increasing filtration is **exhaustive** if $F^{\infty}M = M$.*

Example 18.5. *Let M be a module, and let $r \in R$ be an element. Define the r-Bockstein filtration by*

$$F^iM = \begin{cases} M & i \geq 0 \\ Im\big(r^{-i}\cdot(-)\big) & i \leq 0. \end{cases}$$

The most important part of a filtration is the associated graded, where we approximate any M by simpler modules.

Definition 18.6. *If M is a filtered module, then for each $i \in \mathbb{Z}$, let*

$$Gr_i(M) := F^iM/F^{i-1}M.$$

*This gives a graded modules, the **associated graded**.*

Example 18.7. *If M is a finitely generated free abelian group, and if $p \in \mathbb{Z}$ is prime, then the associated graded for the p-Bockstein filtration is*

$$Gr_iM = \begin{cases} M \otimes \mathbb{Z}/p & i \leq 0 \\ 0 & i > 0. \end{cases}$$

In particular, these are all $\mathbb{Z}/p$-vector spaces.

Proposition 18.8. *If $f\colon M \to M'$ is a map of filtered modules, then f induces a homomorphism*

$$Gr_*(f)\colon Gr_*(M) \to Gr_*(M').$$

Proof. By assumption, f induces homomorphisms:

$$\begin{tikzcd} F^iM \arrow[r, "f"] & F^iM' \\ F^{i-1}M \arrow[u, "\subset"] \arrow[r, "f"'] & F^{i-1}M'. \arrow[u, "\subset"'] \end{tikzcd}$$

The universal property of the quotient then gives the desired map $Gr_i(f)$. □

Definition 18.9. *A filtration on a chain complex $C_\bullet$ is a sequence of subchain complexes*

$$F^iC_\bullet \subset F^{i+1}C_\bullet \subset \cdots \subset C_\bullet.$$

Proposition 18.10. *A filtered chain complex is a filtered graded abelian group together with a map d of filtered graded abelian groups of degree -1 which satisfies $d^2 = 0$.*

Proposition 18.11. *If $C_\bullet$ is a filtered chain complex, then $Gr_*C_\bullet$ is a (bigraded) chain complex with differential*

$$d_0 = Gr_*(d).$$

Proof. Since d is a map of filtered modules, it induces a map on associated gradeds. The fact that it squares to zero is visibly unchanged, as is the fact that it is of degree -1. □

It is important here to note that we have two degrees in the associated graded:

(1) a filtration degree (the i in Gr_iC_k), and
(2) the internal degree (the k in Gr_iC_k).

On the associated graded, the differential *always* drops the internal degree by 1 and preserves the filtration.

Note that since we have inclusions of complexes $F^iC_\bullet \to C_\bullet$, we have a natural filtration of the homology by:

$$F^i\big(H(C_\bullet)\big) = Im\big(H(F^iC_\bullet)\big).$$

Warning 18.12. *We have no reason to believe that $H(Gr_*C_\bullet) \cong Gr_*H(C_\bullet)$. A spectral sequence allows us to deduce the latter from the former in good circumstances.*

Here is an important example.

Example 18.13. *Consider the complex*

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \dots,$$

and endow this with the 2-Bockstein filtration. The associated graded is the bigraded complex

$$G_{i,k} = \begin{cases} \mathbb{Z}/2 & (i \leq 0)\&(k = 0,1) \\ 0 & \textit{otherwise}. \end{cases}$$

The map d was multiplication by 2, so it gives zero on the associated graded. In particular, the homology is the associated graded:

$$H(G_{*,\bullet}) = G_{*,\bullet}.$$

So we traded making the differential much simpler for having a much larger homology.

Unpacking Example 18.13 a little more also shows us how we should fix this. Let's say that a class

$$[z] \in Gr_i C_k$$

is a d_0-cycle. This means that for some lift z' in $F^i C_k$, we have

$$d(z') \in F^{i-1} C_{k-1} \subset F^i C_{k-1}.$$

We want to compose this with the canonical quotient to $Gr_{i-1}C_{k-1}$ to get a new map, but here we run into well-definedness. First note that since we defined this relative to d itself, we know that the image of this is a d-cycle, and hence the image in $Gr_{i-1}C_{k-1}$ is a d_0-cycle for any choice of lift.

If z_0 is some other class that represents $[z]$, then

$$z' - z_0 \in F^{i-1} C_k,$$

and hence the difference between $d(z')$ and $d(z_0)$ in $Gr_{i-1}C_{k-1}$ is a boundary. In particular, although our function was not well-defined as a map to cycles, it is well-defined as a map to homology.

Definition 18.14. *Let $C_\bullet$ be a filtered chain complex. Let*

$$E^1_{i,k} = H\big(Gr_i C_k\big),$$

and let

$$d_1 \colon E^1_{i,k} \to E^1_{i-1,k-1}$$

be the map

$$[z] \mapsto [d(z)].$$

Since d is a differential, this gives us a new differential.

Proposition 18.15. *If $C_\bullet$ is a filtered chain complex, then (E^1, d_1) is a chain complex.*

We can continue the analysis from above, unpacking the cycles for first d_1 and then later ones. This gives our definition.

Definition 18.16. *A **spectral sequence** is a collection of chain complexes (E^r, d_r) (the **pages**) such that for all r, we have*

$$E^{r+1} \cong H(E^r).$$

A map of spectral sequences is a collection of maps of chain complexes:

$$f^r \colon E^r_1 \to E^r_2$$

such that

$$H(f^r) = f^{r+1}.$$

The spectral sequences which have historically been best studied are those which arise as above. These are double graded in that each page is a bigraded complex. Other natural examples are singly graded or have even more gradings! We reserve flexibility here. In general, the procedure above takes a filtered differential graded object (with some number of gradings), and returns a spectral sequence with one more grading.

Warning 18.17. *Many texts describe only* ***homological*** *or* ***cohomological*** *spectral sequences, defining these as those for which the differentials d_r shift the bigrading in the same way the Serre spectral sequence will. This often results in very unnatural manipulations to force a spectral sequence in nature into this form! We should instead remember that the filtration and internal degrees are often give to us, and then we can* choose *how to grade everything in a way that makes the most sense for our computations.*

19. The Serre Spectral Sequence

The Serre spectral sequence is the primary tool for computing the cohomology of the total space of a fibration. We should think of this as generalizing the Künneth filtration, allowing us to handle "twisted" products as well.

Theorem 19.1 (Homological Serre Spectral Sequence)**.** *Let $p\colon E \to B$ be a fibration, as let F be the fiber over $b \in B$. Assume also that for all $k \in \mathbb{N}$, the action of $\pi_1(B,b)$ on $H_k(F;R)$ is trivial. Then we have a spectral sequence with*

$$E^2_{p,q} = H_p\big(B; H_q(F;R)\big)$$

and converging to $H_{p+q}(E;R)$.

The d_r differential changes bidegree as:

$$d_r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

There is also a cohomological version, which is more highly structured (reflecting the additional structure on cohomology). Here we need a slight bit of terminology.

Definition 19.2. *A spectral sequence of algebras is a spectral sequence (E_r, d_r) such that*

1. *for each r, E_r is an algebra and d_r is a derivation and*
2. *for each r, the isomorphism $E_{r+1} \cong H(E_r)$ is an isomorphism of algebras.*

Theorem 19.3 (Cohomological Serre Spectral Sequence)**.** *Let $p\colon E \to B$ be a fibration, as let F be the fiber over $b \in B$. Assume also that for all $k \in \mathbb{N}$, the action of $\pi_1(B,b)$ on $H^k(F;R)$ is trivial. Then we have a spectral sequence of algebras with*

$$E_2^{p,q} = H^p\big(B; H^q(F;R)\big)$$

and converging to $H^{p+q}(E;R)$.

The d_r differential changes bidegree as:

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

The multiplicative structure puts tremendous constraints on the spectral sequence.

Exercise 19.1. *Let $C_\bullet$ be a differential graded algebra. Show that the differential is completely determined by its value on a collection of algebra generators.*

Note that in both cases, if we plot where the spectral sequence possibly has non-trivial values, then we find that we only see groups in the 1st quadrant. Because of these, these are sometimes called first quadrant spectral sequences. We can make other observations as well.

Proposition 19.4. *For each r, $E^r_{0,q}$ is a quotient of $H_0\big(B; H_q(F;R)\big)$.*
For each r, $E_r^{p,0}$ is a quotient of $H^p\big(B; H^0(F;R)\big)$.

Proof. The possible targets of any differentials originating from these groups are all zero by assumption (they would land in those groups for which p or q is negative). In particular, every element is a cycle for all differentials, and hence the homology is just an iterated quotient. □

Proposition 19.5. *For all (p,q), there is an r_0 such that for all $r > r_0$,*

$$E^r_{p,q} \cong E^{r+1}_{p,q} \cong \dots.$$

Proof. If $r > p$, then

$$d_r \colon E^r_{p,q} \to E^r_{p-r,q+r-1} = 0,$$

and hence every element of $E^r_{p,q}$ is a d_r-cycle. Similarly, if $r > q+1$, then

$$d_r \colon 0 = E^r_{p+r,q-r+1} \to E^r_{p,q},$$

and hence the only boundary in $E^r_{p,q}$ is zero. Taking r_0 to be anything greater than p and $q+1$ then shows that every element of $E^r_{p,q}$ is a cycle and that the only boundary is zero, as desired. □

Definition 19.6. *Let $E^\infty_{p,q}$ be the group $E^r_{p,q}$ for $r >> 0$.*

Nothing we used here depended on our use of homological rather than cohomological Serre Spectral Sequences, so they also work there with the obvious modifications.

Theorem 19.7. *Let $p \colon E \to B$ be a fibration.*

(1) *There is a natural filtration of $H_k(E;R)$ such that the associated graded is*
$$Gr_p\big(H_k(E;R)\big) = E^\infty_{p,k-p}.$$
(2) *There is a natural filtration of $H^k(E;R)$ such that the associated graded is*
$$Gr_p\big(H^k(E;R)\big) = E_\infty^{k-p,p}.$$

Remark 19.8. *Proving this is a bit tricky, but describing the filtration is not. Consider a CW decomposition of B: $B = \bigcup B^{[k]}$, and let*

$$F^k C_*(E;R) = C_*\big(p^{-1}B^{[k]};R\big) \subset C_*(E;R).$$

The Serre Spectral Sequence is the spectral sequence associated to this filtration, and the filtration on the homology of E is just the filtration by the images of the homologies of the filtered pieces.

Let's shift focus to consider how this works in practice. We have two basic kinds of fibrations we most often consider:

(1) If X is a space with basepoint x, then $P(X,x) \xrightarrow{p=ev_1} X$ is a fibration with fiber the based loop space of X: $\Omega^1 X$.
(2) If $H \subset G$ is a closed subgroup of a compact Lie group G, then $G \to G/H$ is a fibration.

Let X be a simply connected space. Then in the fibration

$$\Omega^1 X \to P(X,x) \to X,$$

the total space $P(X,x)$ is contractible. We therefore know a huge amount of information about its cohomology: it's all zero except in degree 0.

Proposition 19.9. *Let X be a simply connected space. Then in the Serre Spectral Sequence for $\Omega^1 X \to P(X,x) \to X$, we have*

$$E^{\infty}_{p,q} = E^{p,q}_{\infty} = 0$$

for all $(p,q) \neq (0,0)$, and is R if $p = q = 0$.

Proof. The only subgroups of the zero group are zero, so this gives the answer for $p + q \neq 0$. The final case is just the observation that for degree 0, the only groups which can contribute are $E^r_{0,0}$. □

Since we have

$$E_2^{p,q} = H^p\big(X; H^q(\Omega^1 X; R)\big),$$

we know that everything has to cancel out via differentials. We can use this to produce some very nice inductive results.

An example: loops on a sphere. Consider $X = S^3$. Since $\pi_1 S^3 = 0$ (Corollary 15.5), we know that we have a Serre Spectral Sequence of the form

$$E_2^{p,q} = H^p\big(S^3; H^q(\Omega S^3; R)\big).$$

We also know

$$H^*(S^3; R) \cong E_R(x_3),$$

for any ring R, where $|x_3| = 3$. Since this is always free, the Künneth theorem gives an isomorphism

$$H^*\big(S^3; H^\bullet(\Omega S^3; R)\big) \cong E_R(x_3) \otimes_R H^\bullet(\Omega S^3; R),$$

where viewed bigradedly, x_3 has bidegree $(3,0)$ and where anything in $H^k(\Omega S^3; R)$ has bidegree $(0,k)$.

Now consider the cohomology groups of the fiber. Since the E_∞ page is zero in positive degrees, we must have

$$d_2 \colon H^1(\Omega S^3; R) \xrightarrow{\cong} H^2(S^3; R) \cong 0.$$

Since x_3 is a permanent cycle, we must then have that it is the target of a differential. A d_2 differential would originate in

$$E_2^{1,1} \cong H^1\big(S^3; H^1(\Omega S^3; R)\big) = 0,$$

so it must instead be the target of a d_3 differential:

$$d_3 \colon H^2(\Omega S^3; R) \twoheadrightarrow H^3(S^3; R).$$

Moreover,

$$d_2 \colon H^2(\Omega S^3; R) \to H^2\big(S^3; H^1(\Omega S^3; R)\big) = 0,$$

so we deduce that d_3 is an isomorphism: there is a class

$$y_2 \in H^2(\Omega S^3; R) \cong R$$

such that $d_3(y_2) = x_3$.

Continuing in this manner, we see the following proposition.

Proposition 19.10.

$$H^k(\Omega S^3;R)\cong\begin{cases}0 & k\equiv 1 \mod 2\\ R\cdot y_k & k\equiv 0 \mod 2.\end{cases}$$

We can also deduce the ring structure.

Proposition 19.11. *The cohomology of ΩS^3 is the divided power algebra on y_2.*

Proof. By Exercise 7.3, it suffices to show that

$$y_2^k = k!y_{2k}$$

for all k. We argue by induction, the case of $k=1$ be a tautology. Assume that

$$y_2^{k-1}=(k-1)!y_{2(k-1)}.$$

Since d_3 is a derivation, we have

$$d_3(y_2^k)=ky_2^{k-1}d_3(y_2)=k!y_{2(k-1)}x_3.$$

Now y_{2k} was defined by the property that the d_3 differential on it hit $y_{2(k-1)}x_3$, and since d_3 is an injection, this gives the result. □

Note that nothing here depended in an essential way on 3: everything we did works just as well for S^{2k+1}.

Exercise 19.2. *Show that*

$$H^*(\Omega S^{2k};\mathbb{Z})\cong E(x_{2k-1})\otimes\Gamma(y_{4k-2}),$$

where the degrees of the elements are the subscripts.

An example: Unitary Groups. We recall a basic fact about homogeneous spaces for a compact group G.

Proposition 19.12. *Let X be a Hausdorff G space on which G acts transitively, and let $x\in X$ be a point. Let $H=Stab(x)$ be the stabilizer subgroup of H. Then the map*

$$f_x\colon G/H\to X$$

which sends gH to gx is a homeomorphism of G-spaces.

Proof. First note that if $x\in X$, then $Stab(x)$ is a closed subgroup: if $h_1,\dots$ is any convergent sequence of points in $Stab(x)$, then

$$(\lim_{k\to\infty}h_k)\cdot x=\lim_{k\to\infty}(h_k\cdot x)=x,$$

so the limit is again in $Stab(x)$. The map $G/H\to X$ is visibly continuous and surjective. Now

$$f_x(gH)=f_x(g'H),$$

if and only if

$$g\cdot x=g'\cdot x,$$

and hence $g^{-1}g'\in Stab(x)$. Thus these are the same coset. Hence the map is injective. Since the source is compact and the target is Hausdorff, this is a homeomorphism. □

Corollary 19.13. *We have a homeomorphism*

$$U(n)/U(n-1) \cong S^{2n-1},$$

where $U(n-1) \hookrightarrow U(n)$ is the map

$$A \mapsto \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}.$$

Proof. The unitary group $U(n)$ acts on the unit sphere in $\mathbb{C}^n$. Grahm-Schmidt shows that this action is transitive, and the stabilizer of the first standard basis vector is exactly the described copy of $U(n-1)$. □

Corollary 19.14. *We have a fiber sequence*

$$U(n-1) \to U(n) \to S^{2n-1}.$$

Since for $n > 1$ the bases are all simply connected, we again can apply Theorem 19.3.

Proposition 19.15. *We have a spectral sequence with*

$$E_2^{p,q} = H^p\big(S^{2n-1}; H^q(U(n-1); R)\big)$$

converging to the cohomology of $U(n)$.

Theorem 19.16. *For all n, we have an isomorphism of rings*

$$H^*(U(n); R) \cong E_R(x_1, x_3, \ldots, x_{2n-1}),$$

where the degree of x_i is i.

Proof. We proceed by induction on n. The base case of $n = 1$ is the computation of the cohomology of $U(1) \cong S^1$, so assume we have proved this for $U(n-1)$. Proposition 19.15 gives us a spectral sequence of algebras with

$$E_2^{p,q} = H^p\big(S^{2n-1}; H^q(U(n-1); R)\big) \cong E(x_{2n-1}) \otimes_R E(x_1, x_3, \ldots, x_{2n-3}),$$

where here we have used the inductive hypothesis and the Künneth theorem. The bidegrees of the elements are:

$$|x_{2n-1}| = (2n-1, 0), |x_1| = (0, 1), \ldots, |x_{2n-3}| = (0, 2n-3).$$

Since the differentials are derivations, they are completely determined by their value on the generators. The class x_{2n-1} is a permanent cycle since it comes from the cohomology of the base; the classes x_{2i-1} for $1 \leq i \leq n-1$ are permanent cycles since the possible targets are all zero. Hence

$$E_2^{p,q} = E_\infty^{p,q}.$$

Everything in sight is a free R-module, and hence all of the exact sequences:

$$F_{p+1}H^{p+q}(U(n); R) \hookrightarrow F_pH^{p+q}(U(n); R) \twoheadrightarrow E_\infty^{p,q}$$

split. This shows the result additively. For the multiplicative statement, note that any element of odd degree must square to something simple 2-torsion:

$$2x_{2i-1}^2 = 0.$$

Considering the case of $R = \mathbb{Z}$, we see that in fact, any choice of lifts x_{2i-1} must be exterior, and hence we have it as algebras. □

20. The Gysin and Wang Sequences

The two big examples from last time were greatly simplified by the fact that the base in both cases was a sphere. In general, if the base or the fiber is a sphere, then the Serre spectral sequence is much simpler. In general, we can only have one differential, and it often has significant geometric content.

20.1. **Gysin Sequence.** Let $S^n \to E \xrightarrow{p} B$ be a fibration, and assume that the action of $\pi_1(B,b)$ on S^n is trivial. Then our Serre Spectral Sequence is especially nice. We have

$$E_2^{p,q} = H^p\big(B; H^q(S^n;R)\big),$$

and since the cohomology of a sphere is a free R-module, the Künneth theorem gives us an algebra isomorphism

$$E_2^{*,\bullet} \cong H^*(B;R) \otimes_R H^\bullet(S^n;R) \cong H^*(B;R) \otimes_R E_R(x_n).$$

The bigrading of anything in $H^*(B;R)$ is $(*,0)$, and hence all are permanent cycles. The bigrading of x_n is $(0,n)$. The spectral sequence is non-zero only in 2 rows: 0 and n, and so the only differential which we can have is

$$d_{n+1}\colon E_{n+1}^{p,n} = E_2^{p,n} \to E_2^{p+n+1,0} = E_n^{p+n+1,0}.$$

Exercise 19.1 shows that the differential is completely determined by its value on x_n, since all other algebra generators are permanent cycles.

Definition 20.1. *If $S^n \to E \to B$ is a fibration with a trivial action of $\pi_1(B,b)$ on the fiber, then the class*

$$e = d_{n+1}(x_n) \in H^{n+1}(B;R)$$

*is the **Euler class** of the spherical fibration E.*

Proposition 20.2. *If $S^n \to E \to B$ is a fibration with a trivial action of $\pi_1(B,b)$ on the fiber, then we have*

$$E_\infty^{p,q} = \begin{cases} \operatorname{coker}\big(H^{p-n-1}(B;R) \xrightarrow{-\smile e} H^p(B;R)\big) & q=0 \\ \ker\big(H^p(B;R) \xrightarrow{-\smile e} H^{p+n+1}(B;R)\big) & q=n \\ 0 & \textit{otherwise.} \end{cases}$$

We can do better here, since we have extremely simple sequences. For this, we also look at a more universal situation.

Proposition 20.3. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a simple fibration. Then*

$$E_\infty^{p,0} = Im\big(p^*\colon H^p(B;R) \to H^p(E;R)\big),$$

and

$$E_\infty^{0,q} = Im\big(i^*\colon H^q(E;R) \to H^q(F;R)\big).$$

Proof. Consider the diagram

$$\begin{tikzcd} F \arrow[r, "="] \arrow[d, "="'] & F \arrow[r] \arrow[d, "i"] & * \arrow[d] \\ F \arrow[r, "i"] \arrow[d] & E \arrow[r, "p"] \arrow[d, "p"] & B \arrow[d, "="] \\ * \arrow[r] & B \arrow[r, "="'] & B. \end{tikzcd}$$

Each column is a fibration. Naturality of the Serre Spectral Sequence then says that we have a map of spectral sequences for each of the maps between columns, and that this converges to the map on the total spaces. The Serre Spectral Sequences for the left and right columns is very easy, since we have a point as one of the two outer slots. In particular, for the left-hand side, we have

$$E_2^{0,q} = E_\infty^{0,q} = H^q(F;R),$$

and all other groups are zero. The right-hand side has

$$E_2^{p,0} = E_\infty^{p,0} = H^p(B;R),$$

and all other groups are zero. The maps on E_2 terms are just given by the obvious projections. The result follows from naturality. $\square$

Now let's put all of these back together with the Gysin sequence. Proposition 20.2 gives us a short exact sequence

$$0 \to E_\infty^{k,n} \to H^k(B;R) \xrightarrow{-\smile e} H^{k+n+1}(B;R) \to E_\infty^{k+n+1,0} \to 0.$$

The filtration on $H^k(E)$ takes the form of a short exact sequence

$$0 \to E_\infty^{k,0} \to H^k(E;R) \to E_\infty^{k-n,n} \to 0.$$

Splicing these together give a long exact sequence.

Theorem 20.4 (Gysin Sequence)**.** *If $S^n \to E \xrightarrow{p} B$ is a simple fibration, then we have a natural long exact sequence*

$$\cdots \to H^{k+n}(E;R) \to H^k(B;R) \xrightarrow{-\smile e} H^{k+n+1}(B;R) \xrightarrow{p^*} H^{k+n+1}(E;R) \to \dots,$$

where e is the Euler class of the fibration.

20.2. **Gysin Examples.**

Remark 20.5. *Euler classes arise from the theory of vector bundles (where they are an obstruction to have a nowhere vanishing section). If B is paracompact and if $E \to B$ is a vector bundle, then we can choose a metric on E (so a continuous inner product on the fibers) and form the associated sphere bundle. The Euler class for the Gysin sequence is then the same as the Euler class here. If E is the tangent bundle to an oriented manifold M, then*

$$e = \chi(M) \cdot [1]^*,$$

where $[1]^*$ is the Poincaré dual to 1.

Definition 20.6. *Let $V_2(\mathbb{R}^{n+1})$ be the space of orthogonal pairs of unit vectors in $\mathbb{R}^{n+1}$.*

There are many ways we can topologize this. First, this is a homogeneous space for $O(n+1)$, where we send an ordered pair$(\vec{v},\vec{w}) \to (A\vec{v}, A\vec{w})$. Alternatively, we can connect this to tangent bundles.

Proposition 20.7. *The space $V_2(\mathbb{R}^n)$ is the unit sphere bundle in the tangent bundle to S^n.*

Proof. A model for the tangent bundle to S^n is

$$TS^n = \{(\vec{x}, \vec{v}) \mid \vec{x} \cdot \vec{x} = 1, \vec{x} \cdot \vec{v} = 0\},$$

and the map to S^n is the obvious projection onto the first coordinate. The unit sphere bundle simply imposes the condition that $\vec{v} \cdot \vec{v} = 1$, which gives us $V_2(\mathbb{R}^{n+1})$. □

Thus we have a sphere bundle

$$S^{n-1} \to V_2(\mathbb{R}^{n+1}) \to S^n.$$

We also know the Euler class of the n-sphere:

$$e = \chi(S^n)x_n = \big(1 + (-1)^n\big)x_n.$$

If n is odd, then this is zero. If n is even, then this is 2.

Theorem 20.8. *If $n = 2k + 1$, then*

$$H^*\big(V_2(\mathbb{R}^{n+1}); \mathbb{Z}\big) \cong E(x_{2k}, x_{2k+1}).$$

If $n = 2k$, then

$$H^*\big(V_2(\mathbb{R}^{n+1}); \mathbb{Z}\big) \cong \begin{cases} \mathbb{Z} & * = 0, 2n-1 \\ \mathbb{Z}/2 & * = n \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. These computations are immediate from the Gysin sequence (Theorem 20.4). The multiplicative structure is the only possibility. □

Remark 20.9. *The spaces $V_2(\mathbb{R}^{n+1})$ are smooth, $\mathbb{Z}$-orientable manifolds. When n is odd, the cohomology is as is expected for Poincaré duality. When n is even, we see that considering only the torsion free pieces is necessary for deducing that the cup product is a perfect pairing.*

Exercise 20.1. *Let $S^1 \to E \to \mathbb{C}P^\infty$ be the spherical fibration[5] with Euler class $2x \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$. Compute the cohomology of E using the Gysin sequence.*

20.3. **The Wang Sequence.** We have a completely analogous story when the base is instead a sphere.

Exercise 20.2. *Let $F \to E \to S^n$ be a fibration. Unpacking the Serre Spectral Sequence for homology, show that we have a long exact sequence connecting the homologies of F and E.*

21. The Hurewicz Theorem

Using the Serre spectral sequence, we can finally start determining homotopy groups (oddly enough). Recall this fundamental theorem of Poincaré:

Theorem 21.1. *If X is a path-connected space, then the map*

$$h\colon \pi_1(X, x) \to H_1(X; \mathbb{Z})$$

which sends a loop γ to the same loop viewed as a singular 1-cycle is abelianization:

$$H_1(X; \mathbb{Z}) \cong \pi_1(X, x)^{ab}.$$

[5]This is the sphere bundle associated to the tensor square of the tautological line bundle on $\mathbb{C}P^\infty$, and the total space $E \simeq \mathbb{R}P^\infty$

Definition 21.2. *Let*

$$h\colon \pi_k(X,x) \to H_k(X,x;\mathbb{Z}) \cong \tilde{H}_k(X;\mathbb{Z})$$

be the map which takes $f \in \pi_k(X,x)$ to

$$f_*([\Delta^k]).$$

*This is the **Hurewicz homomorphism**.*

Remark 21.3. *If f and f' differ by the action of an element of $\pi_1(X,x)$, then*

$$h(f) = h(f').$$

In particular, we factor through the action of the fundamental group (which when $k=1$ realizes abelianization).

Theorem 21.4 (Hurewicz Theorem)**.** *If X is a $(k-1)$-connected space space, then*

$$H_i(X;\mathbb{Z}) \cong \begin{cases} 0 & 0 < i < k, \\ \pi_k(X,x)^{ab} & i = k. \end{cases}$$

The isomorphism in dimension k is given by h.

We will prove this via the Serre spectral sequence. We first produce a nice, long exact sequence in good contexts.

Theorem 21.5 (Homology Serre Exact Sequence)**.** *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a simple fibration. Assume also that for all $1 \leq p \leq k-1$ and $1 \leq q \leq \ell - 1$, we have*

$$H_p(B;\mathbb{Z}) \cong H_q(F;\mathbb{Z}) \cong 0.$$

Then we have a natural long exact sequence

$$H_{k+\ell-1}(E;R) \xrightarrow{p_*} H_{k+\ell-1}(B;R) \xrightarrow{\tau} H_{k+\ell-2}(F;R) \xrightarrow{i_*} H_{k+\ell-2}(E;R) \xrightarrow{p_*} \dots.$$

Proof. The conditions on the Serre spectral sequence ensure that the E^2 page is quite simple:

$$E^2_{p,q} = 0, 1 \leq p \leq k, 1 \leq q \leq \ell.$$

In particular, the first place where we have an element in $E^2_{p,q}$ with both p and q non-zero is possibly $E^2_{k,\ell}$. This is in total degree $k+\ell$, so we see that the only things which can contribute to $H_r(E;R)$ for $r < k+\ell-1$ are the homologies of the base and the fiber. This makes the Serre spectral sequence incredibly sparse in a long range. In particular, we notice that the only differential possible on

$$H_p(B;R) = E^2_{p,0} = E^p_{p,0} \xrightarrow{d_p} E^p_{0,p-1} = E^2_{0,p-1} = H_{p-1}(F;R),$$

for $1 \leq p \leq k+\ell-1$. We deduce that the $E^\infty_{p,0}$ is the kernel of d_p, while $E^\infty_{p-1,0}$ is the cokernel. Unpacking the filtration gives the exact sequence. □

Remark 21.6. *The surprising introduction of a -1 in bounds like $r < k+\ell-1$ is because we could have a d_ℓ differential on $E^2_{k,\ell}$ which would hit $H_{\ell-1}(F;R)$. This is giving us the slightly strange behavior.*

Using Theorem 21.5, we can prove the Hurewicz theorem.

Lemma 21.7. *Let B be a simply connected space.*

(1) *If* $H_p(B;\mathbb{Z}) = 0$ *for all* $1 \leq p \leq k-1$*, then* $H_p(\Omega B;\mathbb{Z}) = 0$ *for all* $1 \leq p \leq k-2$ *and*
$$d_k = \tau\colon H_k(B;\mathbb{Z}) \xrightarrow{\cong} H_{k-1}(\Omega B;\mathbb{Z}).$$
(2) *If* $H_p(\Omega B;\mathbb{Z}) = 0$ *for all* $1 \leq p \leq k-2$*, then* $H_p(B;\mathbb{Z}) = 0$ *for all* $1 \leq p \leq k-1$ *and*
$$d_k = \tau\colon H_k(B;\mathbb{Z}) \xrightarrow{\cong} H_{k-1}(\Omega B;\mathbb{Z}).$$

Proof. We know that $PB \simeq *$, so all homology groups beyond the zeroth vanish. In particular, all classes in $H_k(B;\mathbb{Z})$ and $H_k(\Omega B;\mathbb{Z})$ must support or be the target of differentials. If for all $1 \leq p \leq k-1$, the homology groups
$$H_p(B;\mathbb{Z}) = 0,$$
then for all $1 \leq p \leq k-1$, we have $E^2_{p,q} = 0$ for all q. In particular, there are no sources for differentials targeting $H_{p-1}(\Omega B;\mathbb{Z})$. Thus these groups must be zero. Switching the roles of p and q and running the same kind of argument gives the second part.

In both cases, Theorem 21.5 then gives the desired isomorphisms. □

Theorem 21.8 (Hurewicz Theorem)**.** *If* X *is* $(k-1)$*-connected, then*
$$H_k(X;\mathbb{Z}) \cong \pi_k(X,x)^{ab}.$$

Proof. We show this by induction on k. If X is connected, then this is Theorem 21.1. So assume this is true for a $(k-2)$-connected spaces. Now if X is $(k-1)$-connected, then ΩX is $(k-2)$-connected. So we have a natural isomorphism by the induction hypothesis
$$\pi_k(X,x) \cong \pi_{k-1}(\Omega X, x) \cong H_{k-1}(\Omega X;\mathbb{Z}).$$
Lemma 21.7 then shows that we have a natural isomorphism
$$H_{k-1}(\Omega X;\mathbb{Z}) \cong H_k(X;\mathbb{Z}).$$
□

We have not yet technically shown that our map is the Hurewicz map described above. We can show this in a somewhat roundabout way.

Corollary 21.9. *For all* k*, we have*
$$\pi_k(S^k, s) \cong \mathbb{Z}$$
generated by the identity map on S^k*.*

Proof. Theorem 21.8 shows that
$$\pi_k(S^k, s) \cong H_k(S^k;\mathbb{Z}) \cong \mathbb{Z}.$$
Naturality gives a map
$$\pi_k(S^k, s) \to \mathrm{End}\left(H_k(S^k;\mathbb{Z})\right)$$
which is surjective, sending the identity map to the identity. Any surjective map $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism. □

Now we have an opportunity to focus on a fantastically useful lemma: the Yoneda Lemma.

Definition 21.10. *If $\mathcal{C}$ and $\mathcal{D}$ are categories and if $F, G\colon \mathcal{C} \to \mathcal{D}$ are functors, then let*

$$\mathrm{Nat}(F, G) = \{F \Rightarrow G\}$$

be the set of natural transformations from F to G.

Lemma 21.11 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category, and let $X \in \mathcal{C}$. Let*

$$h_X\colon \mathcal{C} \to \mathcal{S}et$$

be the functor

$$Y \mapsto \mathcal{C}(X, Y).$$

If $F\colon \mathcal{C} \to \mathcal{S}et$ is any other functor, then we have an isomorphism

$$\mathrm{Nat}(h_X, F) \cong F(X).$$

Proof. The entire argument hangs on the fact that we have a special element

$$Id_X \in h_X(X) = \mathcal{C}(X, X).$$

This has the property that for any $f \in h_X(Y)$, we have

$$f = h_X(f)(Id_X).$$

Thus any natural transformation $\eta\colon h_X \Rightarrow F$ is completely determined by the value of η_X on Id_X. Conversely, any element in $u \in F(X)$ gives a natural transformation by saying that the value on

$$f \in h_X(Y)$$

is $F(f)(u)$. $\square$

Corollary 21.12. *If X is $(k-1)$-connected, then the Hurewicz map*

$$h\colon \pi_k(X, x) \to H_k(X; \mathbb{Z})$$

is an isomorphism.

Proof. The Yoneda Lemma shows that we have an isomorphism

$$\mathrm{Nat}\left(\pi_k(-), H_k(-;\mathbb{Z})\right) \cong H_k(S^k;\mathbb{Z}) \cong \mathbb{Z}.$$

Implicit in the Hurewicz map is a choice of orientation, which corresponds to plus or minus 1. The result follows. $\square$

We obviously have an analogous statement for cohomology. We leave the proof as an exercise.

Theorem 21.13 (Cohomology Serre Exact Sequence)**.** *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a simple fibration. Assume also that for all $1 \leq p \leq k-1$ and $1 \leq q \leq \ell - 1$, we have*

$$H^p(B;\mathbb{Z}) \cong H^q(F;\mathbb{Z}) \cong 0.$$

Then we have a natural long exact sequence

$$H^{k+\ell-1}(E;R) \xleftarrow{p^*} H^{k+\ell-1}(B;R) \xleftarrow{\tau} H^{k+\ell-2}(F;R) \xleftarrow{i^*} H^{k+\ell-2}(E;R) \xleftarrow{p^*} \dots.$$

Exercise 21.1. *Prove Theorem 21.13.*

22. Eilenberg-MacLane spaces

We can produce a large number of spaces with interesting properties from the Hurewicz Theorem.

Proposition 22.1. *If I is a set and if X_i is a collection of $(k-1)$-connected spaces for $k \geq 2$, then $X = \bigvee_{i \in I} X_i$ is also $(k-1)$-connected.*

Proof. The Van Kampen Theorem shows that X is simply connected, so we can apply the Hurewicz theorem which says that the first non-zero homology and homotopy groups occur in the same dimension and agree. Homology takes wedges to direct sums, so we see that the lowest dimension in which $H_k(X; \mathbb{Z})$ can be non-zero is k. The same is therefore true for the homotopy groups. □

Proposition 22.2. *Let I be a set and let $k > 1$. Then*

$$\pi_k\left(\bigvee_{i \in I} S^k, s\right) \cong \bigoplus_{i \in I} \mathbb{Z}.$$

Proof. This is an immediate application of the Hurewicz Theorem. □

We can say more here, in that we can also understand what happens when we cone off elements of π_k for a $(k-1)$-connected space.

Proposition 22.3. *Let X be a $(k-1)$-connected space with $k \geq 2$, and let $f \in \pi_k(X, x)$. Then*

$$\pi_k(Cf, x) \cong \pi_k(X, x)/(f).$$

Proof. Since $k \geq 2$, we know that attaching the cone on f does not produce new elements of π_1 (by the Van Kampen Theorem), so Cf is again simply connected. The Hurewicz theorem then says that the first non-vanishing homology and homotopy groups agree, so we must find the bottom homology group of Cf. Consider the long exact sequence for the cofiber sequence

$$S^k \xrightarrow{f} X \to Cf.$$

The map f_* on homology in degree k takes 1 to the element f, viewed as an element of

$$\pi_k(X, x) \cong H_k(X; \mathbb{Z}).$$

The result follows by exactness. □

Working more generally, we can cone off families of maps by instead considering a wedge of spheres mapping in to X. The analogous result holds.

Definition 22.4. *Let A be an abelian group, and for each $k \geq 1$, let*

$$M(A, k)$$

be a space with

$$\tilde{H}_\ell\big(M(A, k); \mathbb{Z}\big) \cong \begin{cases} A & \ell = k \\ 0 & \text{otherwise.} \end{cases}$$

These are ***Moore spaces*** *for A.*

Proposition 22.5. *For any A and k, a Moore space $M(A, k)$ exists.*

Proof. Choose a presentation of A:

$$\bigoplus_{i\in I_1}\mathbb{Z} \xrightarrow{f_0} \bigoplus_{i\in I_0}\mathbb{Z} \to A.$$

Proposition 22.2 shows that

$$\pi_k\left(\bigvee_{i\in I_0} S^k, s\right) \cong \bigoplus_{i\in I_0}\mathbb{Z},$$

and moreover, the map f_0 can be realized as a map

$$\bigvee_{i\in I_1} S^k \xrightarrow{f} \bigvee_{i\in I_0} S^k.$$

Let $M(A,k) = Cf$. The long exact sequence in homology for Cf is exactly the resolution for A, which completes the proof. □

Corollary 22.6. *For any abelian group A and $k \geq 2$, there is a space X with*

$$\pi_j(X,x) = \begin{cases} 0 & j < k \\ A & j = k. \end{cases}$$

We can do better still: we can continue to kill off homotopy groups.

Theorem 22.7. *Let X be a path connected space, and let $f \in \pi_k(X,x)$. Then the map*

$$\pi_j(X,x) \to \pi_j(Cf,x)$$

is an isomorphism for $j < k$ and in dimension k, it is the quotient by the subgroup generated by f.

Corollary 22.8. *For any abelian group A and for any k, there is a space*

$$K(A,k)$$

with

$$\pi_j\big(K(A,k),x\big) \cong \begin{cases} A & j = k \\ 0 & j \neq k. \end{cases}$$

Proof. We build this inductively, beginning with the Moore space $M(A,k) = X_0$. Assume that we have built a space X_n with the property that

$$\pi_j(X_n) \cong \begin{cases} 0 & j < k \\ A & j = k \\ 0 & k < j \leq k+n. \end{cases}$$

Choose a surjection

$$\bigoplus_{i\in I_n}\mathbb{Z} \xrightarrow{f_n} \pi_{k+n+1}X_n,$$

and let

$$X_{n+1} = Cf_n.$$

Theorem 22.7 shows that the homotopy groups of X_n and X_{n+1} agree through dimension $(k+n)$ and that $\pi_{k+n+1}X_{n+1} = 0$ as well. Passing to the colimit gives us the desired space $K(A,n)$. □

23. Steenrod Operations

Definition 23.1. *An Eilenberg-MacLane space is one which has a single non-vanishing homotopy groups.*

One of the most interesting features of the Eilenberg-MacLane spaces is the functor which they represent. We first note that the Hurewicz and Universal Coefficients theorems give.

Proposition 23.2. *Let A be an abelian group and let $k \geq 1$. Then for any M, a choice of isomorphism*

$$\phi\colon \pi_k\big(K(A,k),x\big) \cong A$$

gives an isomorphism

$$H^k\big(K(A,k);M\big) \cong \mathrm{Hom}(A,M).$$

Proof. The Hurewicz Theorem, together with the map ϕ gives an isomorphism

$$A \xrightarrow{\phi} \pi_k\big(K(A,k),x\big) \cong H_k\big(K(A,k);\mathbb{Z}\big).$$

Since $H_{k-1}\big(K(A,k);\mathbb{Z}\big) = 0$, the Universal Coefficients Theorem gives

$$H^k\big(K(A,k);M\big) \cong \mathrm{Hom}(A,M).$$

□

In particular, a choice of isomorphism

$$\pi_k\big(K(A,k),x\big) \cong A$$

gives a canonical class

$$u \in H^k\big(K(A,k);A\big)$$

corresponding to the identity map on A. Many texts include an isomorphism as part of the data for something to be *Eilenberg-MacLane*.

Theorem 23.3. *Let $K(A,k)$ be an Eilenberg-MacLane space, and let ϕ be an isomorphism $\pi_k K(A,k) \cong A$. Then the map*

$$\big[X, K(A,k)\big] \to H^k(X;A)$$

sending f to $f^*(u)$ is an isomorphism.

By the Yoneda Lemma, we then gain some huge information about the natural transformations of the ordinary homology functor.

Corollary 23.4. *Let A and B be abelian groups and k and n be natural numbers. Then we have a natural bijection*

$$\mathrm{Nat}\left(H^k(-;A), H^n(-;B)\right) \cong H^n\big(K(A,k);B\big).$$

Theorem 23.5. *For any k,*

$$H^*\big(K(\mathbb{Q},k);\mathbb{Q}\big) \cong \begin{cases} E_{\mathbb{Q}}(u_k) & k \text{ odd} \\ \mathbb{Q}[u_k] & k \text{ even.} \end{cases}$$

Exercise 23.1. *Using the Serre spectral sequence for the fibration*

$$K(\mathbb{Q},k-1) \to PK(\mathbb{Q},k) \to K(\mathbb{Q},k)$$

and induction on k, prove Theorem 23.5.

Since $\mathbb{Q}$ is a field, we have a Künneth isomorphism for finite type complexes, like these Eilenberg-MacLane spaces.

Corollary 23.6. *The only natural transformations of the rational cohomology functor are the cup products and polynomials.*

At a particular prime, we have many more operations.

Definition 23.7 (Steenrod Operations)**.** *For each $i \geq 0$, there are natural transformations of abelian groups*

$$Sq^i \colon H^n(X, A; \mathbb{F}_2) \to H^{n+i}(X, A; \mathbb{F}_2)$$

such that

(1) $Sq^0 = Id$
(2) *If δ is the connecting homomorphism is the long exact sequence for the pair, then $\delta \circ Sq^i = Sq^i \circ \delta$.*
(3) *(Cartan Formula)*
$$Sq^k(x \smile y) = \sum_{i+j=k} Sq^i(x) \smile Sq^j(y).$$
(4) *(Adem Relation) If $a < 2b$, then*
$$Sq^a Sq^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c.$$
(5) *(Unstable axiom)*
$$Sq^i(x) = \begin{cases} 0 & i > |x| \\ x^2 & i = |x|. \end{cases}$$

Definition 23.8. *The Steenrod algebra $\mathcal{A}$ is the graded, associative $\mathbb{F}_2$-algebra generated by symbols Sq^i and subject to the Adem Relations.*

The following proposition is immediate by its construction.

Proposition 23.9. *The Steenrod algebra acts on the cohomology of a space by natural transformations.*

We can also recast the Cartan formula here.

Definition 23.10. *Define the* ***total Steenrod operation*** *by*

$$Sq := \sum_{i \geq 0} Sq^i.$$

This is a non-homogenous operation, but by considering the particular homogeneous pieces, we can recover any of the individual squares. The Cartan formula now becomes

Proposition 23.11. *The total square is a ring homomorphism:*

$$Sq(x+y) = Sq(x) + Sq(y) \text{ and } Sq(x \smile y) = Sq(x) \smile Sq(y).$$

Example 23.12. *In $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x]$, we have*

$$Sq^i(x^k) = \binom{k}{i} x^{k+i}.$$

We start by observing that the unstable axiom describes the total Square on the class x:

$$Sq(x) = Sq^0 x + Sq^1 x + Sq^2(x) + \cdots = x + x^2.$$

This gives the total square on all of the higher powers of x:

$$Sq(x^k) = \big(Sq(x)\big)^k = (x + x^2)^k \sum_{j=0}^{k} \binom{k}{j} x^{2k-j}$$

Definition 23.13. *If $I = (i_1, i_2, \dots)$ is a finite sequence of positive integers, let*

$$Sq^I = Sq^{i_1} Sq^{i_2} \dots .$$

*We say that I is **admissible** if $i_j = \geq 2i_{j+1}$ for all j.*

*If I is admissible, then let $|I| = i_1 + \dots$ be the **degree** of I, and let*

$$e(I) = 2i_1 - |I| = (i_1 - 2i_2) + (i_2 - 2i_3) + \dots$$

*be the **excess**.*

Remark 23.14. *There are clearly only finitely many admissible sequences of a particular degree. In particular, there is a maximal excess for any particular degree.*

Proposition 23.15.

1. *Sq^I raises degree by $|I|$*
2. *If $|x| < e(I)$, then $Sq^I(x) = 0$*
3. *If $|x| = e(I)$, then*
$$Sq^I(x) = \big(Sq^{I'}(x)\big)^2,$$
where $I' = (i_2, i_3, \dots)$.

Exercise 23.2. *Prove Proposition 23.15.*

Theorem 23.16. *The set*

$$\{Sq^I \mid I \text{ admissible}\}$$

is a basis for $\mathcal{A}$.

This will also follow from our analysis of the cohomology of Eilenberg-MacLane spaces.

Remark 23.17. *Iterative use of the Adem relations always take a sequence in a non-admissible form to an admissible sequence. This shows that the admissibles span. We can show directly that they are linearly independent by evaluating the admissibles of some degree on $(\mathbb{R}P^\infty)^{\times k}$ for k sufficiently large.*

The dual Steenrod algebra. Since $\mathcal{A}$ is finite dimensional in each degree, we can consider the degree-wise dual A_*.

Proposition 23.18. *The map*

$$Sq^k \mapsto \sum_{i+j=k} Sq^i \otimes Sq^j$$

induces a coproduct on $\mathcal{A}$ that is an algebra homomorphism.

This makes $\mathcal{A}$ into a cocommutative Hopf algebra. In particular, A_* is a commutative Hopf algebra.

Theorem 23.19 (Milnor)**.** *We have*

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \dots],$$

where $|\xi_i| = 2^i - 1$*. The coproduct is given by*

$$\Delta(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j.$$

The class ξ_k is dual to Sq^{I_k}, where $I_k = (2^{k-1}, 2^{k-2}, \dots, 1)$.

We have an evaluation pairing (or "cap product") for every $p \in \mathcal{A}_*$

$$\langle p, - \rangle \colon \mathcal{A}_k \to \mathcal{A}_{k-|p|}.$$

Taking p to be primitive in the sense that

$$\Delta(p) = p \otimes 1 + 1 \otimes p,$$

we get a derivation when capping with p.

Exercise 23.3. *Let $\mathcal{H}$ be a Hopf algebra, and let p be a primitive element in the dual $\mathcal{H}_*$. Show that the map*

$$\mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{1 \otimes p} \mathcal{H}$$

is a derivation.

We can use this to get another form of the Adem relations.

Proposition 23.20. *For any n, we have*

$$Sq^{2n-1} Sq^n = 0.$$

Proposition 23.21. *We have* $\langle \xi_1, Sq^i \rangle = Sq^{i-1}$.

This gives us other relations. These generate all the Adem relations.

24. Homology and Cohomology with Twisted Coefficients

Our formulation of the Serre spectral sequence had a very strong assumption: that the fundamental group of the base acted trivially on the homology or cohomology. We first review twisted homology and cohomology.

24.1. **Twisted Coefficients.** Here, let X be a CW complex and let $p \colon \tilde{X} \to X$ be the universal cover. Let $G = \pi_1(X, x)$. There are close connections between the chains on the universal cover and those of X.

Proposition 24.1.

(1) $\mathrm{Map}(\Delta^n, \tilde{X})$ *is a free G-set, and*

(2) *the projection map p_* induces an isomorphism*

$$C_*\big(\tilde{X}; \mathbb{Z}\big) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong C_*\big(X; \mathbb{Z}\big).$$

Proof. Since the action of G is free on $\tilde{X}$ and since $\Delta^n \neq \emptyset$, the action of G on $\mathrm{Map}(\Delta^n, \tilde{X})$ is also free. The projection map $\tilde{X} \to X$ coincides with the quotient map

$$\tilde{X} \to \tilde{X}/G = X,$$

and it sends a singular n-simplex in $\tilde{X}$ to the orbit it generates. The second part will follow from showing that

$$\big(\mathrm{Map}(\Delta^n, \tilde{X})\big)/G \cong \mathrm{Map}(\Delta^n, X).$$

For this, let $\sigma\colon \Delta^n \to X$, and let $x_0 = \sigma(v_0)$. Choose any $\tilde{x}_0$ which maps to x_0 under p. Since Δ^n is simply connected, the Lifting Criterion shows that the solid diagram below completes:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \Delta^n \arrow[r, "\sigma"] \arrow[ur, dashed, "\tilde{\sigma}"] & X, \end{tikzcd}$$

where $\tilde{\sigma}(v_0) = \tilde{x}_0$. In particular, this shows that the map is surjective.

For injectivity, we consider two lifts $\tilde{\sigma}$ and $\tilde{\sigma}'$ of σ. Let $\tilde{x}_0$ and $\tilde{x}_0'$ be the respective values at v_0. Since $\tilde{X}$ is the universal cover, there is an element $g \in G$ (necessarily unique) such that $g \cdot \tilde{x}_0 = \tilde{x}_0'$. Then $g\tilde{\sigma}$ and $\tilde{\sigma}'$ are two lifts of σ which agree at a point, and hence coincide. $\square$

This is a really powerful observation: the singular simplices for the universal cover record all of the information for the singular simplices of the base. By the standard construction of the intermediate covers, we also see that the simplices of these are also recorded.

Proposition 24.2. *For any subgroup $H \subset G$, let $\tilde{X}_H \to X$ be the cover corresponding to the (conjugacy class of the) subgroup H. Then we have a natural isomorphism*

$$C_*\big(\tilde{X};\mathbb{Z}\big) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/H] \cong C_*\big(\tilde{X};\mathbb{Z}\big) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \cong C_*\big(\tilde{X}_H;\mathbb{Z}\big).$$

In Proposition 24.1, we used $\mathbb{Z}$ with a trivial action to recover the chain on the base, and in Proposition 24.2, we used the induced module $\mathbb{Z}[G/H]$ to instead recover the chains on an intermediate cover. Homology with twisted coefficients handles the general case.

Definition 24.3. *If M is a G-module, then let*

$$H_*(X;\mathcal{M}) := H_*\big(C_*(\tilde{X};\mathbb{Z}) \otimes_{\mathbb{Z}[G]} M\big).$$

This recovers the classical construction of homology with coefficients in some abelian group.

Proposition 24.4. *Let M be an abelian group, endowed with the trivial G-module action. Then*

$$H_*(X;\mathcal{M}) \cong H_*(X;M).$$

Proof. Proposition 24.1 shows that

$$C_*(\tilde{X};\mathbb{Z}) \otimes_{\mathbb{Z}[G]} M \cong C_*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} M = C_*(X;M).$$

$\square$

For cohomology, we have the standard dual construction.

Definition 24.5. *If M is a G-module, then let*

$$H^*(X;\mathcal{M}) := H^*\Big(\operatorname{Hom}_{\mathbb{Z}[G]}\big(C_*(\tilde{X};\mathbb{Z}), M\big)\Big).$$

Again, when M has a trivial action, then this recovers the ordinary cohomology of the base with coefficients in M.

Remark 24.6. *There is a sheaf-theoretic interpretation of homology with twisted coefficients. We replace X be a cover $\tilde{X}$, and then consider a sheaf with descent data for this cover: a G-module M. This sheaf then descends to a sheaf on X, and we are taking the cohomology of X with coefficients in this sheaf. This is the reason for replacing M with $\mathcal{M}$.*

There is a cellular version of these results as well. Any cell structure of X can be lifted to a cell structure of $\tilde{X}$ in which $\pi_1 X$ acts by just permuting the cells. In other words, the set of k-cells is again a free $\pi_1(X)$-set. We can therefore use cellular chains for this lifted cell structure to compute the twisted homology and cohomology groups.

Example 24.7. *Let $X = \mathbb{R}P^2$, and let $M = \mathbb{Q}_-$ be $\mathbb{Q}$ with the sign representation of $\pi_1 X = \mathbb{Z}/2$. A cell structure for $\tilde{X} = S^2$ as a $\mathbb{Z}/2$-space is given by*

$$S^2 = (\mathbb{Z}/2) \cup (\mathbb{Z}/2 \times D^1) \cup (\mathbb{Z}/2 \times D^2).$$

This is just the usual cell decomposition with 2 k-cells for $k = 0, 1, 2$. The cellular chains are then

$$\mathbb{Z}[\mathbb{Z}/2] \xleftarrow{1-\gamma} \mathbb{Z}[\mathbb{Z}/2] \xleftarrow{1+\gamma} \mathbb{Z}[\mathbb{Z}/2],$$

where γ is the non-zero element of $\mathbb{Z}/2$. Tensoring this with $\mathbb{Q}_-$ gives

$$\mathbb{Q} \xleftarrow{2} \mathbb{Q} \xleftarrow{0} \mathbb{Q},$$

so we deduce that

$$H_*(\mathbb{R}P^2; \mathcal{M}) \cong \begin{cases} \mathbb{Q} & * = 2 \\ 0 & \textit{otherwise.} \end{cases}$$

In the cohomological case, we have via standard homological algebra a pairing in cohomology.

Proposition 24.8. *Let M and N be G-modules. Then we have a twisted cup-product*

$$H^k(X; \mathcal{M}) \otimes H^\ell(X; \mathcal{N}) \to H^{k+\ell}(X; \mathcal{M} \otimes \mathcal{N}).$$

There is an extremely important example of twisted coefficients: when X has a contractible universal cover.

Proposition 24.9. *Let X be a space with $\tilde{X} \simeq *$. Then $C_*(\tilde{X}; \mathbb{Z})$ is a resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules.*

Proof. Since $\tilde{X}$ is contractible, augmentation map $C_*(\tilde{X}; \mathbb{Z}) \to \mathbb{Z}$ induces an isomorphism in homology. In particular, the chains on $\tilde{X}$ forms a resolution of $\mathbb{Z}$. Proposition 24.1 then gives the freeness result. □

This is a categorical description, not one that's particular to the space: we have formed a free resolution of $\mathbb{Z}$, which is a purely algebraic construction. In particular, the twisted homology and cohomology are familiar algebraic constructions.

Proposition 24.10. *Let M be a G-module, and let X have contractible universal cover. Then*

$$H_*(X; \mathcal{M}) \cong H_*(G; M) := \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

and

$$H^*(X; \mathcal{M}) \cong H^*(G; M) := \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Proof. Since $C_*(\tilde{X};\mathbb{Z})$ is a projective resolution of $\mathbb{Z}$ in $\mathbb{Z}[G]$-modules, the result follows from the standard homological algebra techniques. $\square$

We can use this to deduce other, more surprising results.

Proposition 24.11. *Let R be a ring in which $|G|$ is a unit, and let X be a space such that $\tilde{X} \simeq *$. Then for any $R[G]$-module M, we have*
$$H^*(X;\mathcal{M}) \cong \begin{cases} M^G & * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The conditions given ensure that the category of $R[G]$-modules is semisimple. In particular, every representation is projective, and hence the Ext groups above Ext^0 vanish. $\square$

24.2. **The Serre E_2-term.** With twisted coefficients, we can handle the general case of a fibration.

Theorem 24.12. *Let $F \to E \to B$ be a fibration. Then we have a spectral sequence of algebras with*
$$E_2^{p,q} = H^p\big(B;\mathcal{H}^q(F;R)\big)$$
converging to $H^{p+q}(E;R)$.

We have a spectral sequence with
$$E^2_{p,q} = H_p\big(B;\mathcal{H}_q(F;R)\big)$$
converging to $H_{p+q}(E;R)$.

Proof. A short exact sequence of groups gives a fibration
$$BN \to BG \to B(G/N).$$
The spectral sequence given is the spectral sequence for this fibration. $\square$

We will need several examples where we have an interesting non-trivial action on the fiber. One of the most important from algebra is the "Lyndon-Hochschild-Serre" spectral sequence.

Corollary 24.13 (Lyndon-Hochschild-Serre)**.** *Let*
$$\{e\} \to N \to G \to G/N \to \{e\}$$
be a short exact sequence of groups. Then for any G-module M, we have a spectral sequence with
$$E_2^{p,q} = H^p\big(G/N;H^q(N;M)\big)$$
and converging to $H^{p+q}(G;M)$.

This is refining the statement about H^0 which is the "fixed points" functor for a G-module:
$$H^0\big(G/N;H^0(N;M)\big) \cong \big(M^N\big)^{G/N} \cong M^G = H^0(G;M).$$

Example 24.14. *Consider the short exact sequence*
$$\{e\} \to \mathbb{Z}/3 \to \Sigma_3 \to \mathbb{Z}/2 \to \{e\}.$$
This is a split exact sequence, and the homomorphism defining the semi-direct product is
$$\mathbb{Z}/2 \xrightarrow{\cong} \mathrm{Aut}(\mathbb{Z}/3) = (\mathbb{Z}/3)^\times.$$

This gives a spectral sequence

$$E_2^{p,q} = H^p\big(\mathbb{Z}/2; H^q(\mathbb{Z}/3; R)\big) \Rightarrow H^{p+q}(\Sigma_3; R).$$

Now let $R = \mathbb{F}_3$. Then since 2 is a unit in $\mathbb{F}_3$, we have

$$H^p\big(\mathbb{Z}/2; H^q(\mathbb{Z}/3; \mathbb{F}_3)\big) \cong \begin{cases} \big(H^q(\mathbb{Z}/3; \mathbb{F}_3)\big)^{\mathbb{Z}/2} & p = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

For degree reasons, the Serre spectral sequence collapses.

Now we need only understand the action of $\mathbb{Z}/2$ on

$$H^*(\mathbb{Z}/3; \mathbb{F}_3) \cong E(x_1) \otimes \mathbb{F}_3[y_2].$$

The action preserves skeleta and gives a ring homomorphism, so it will suffice to understand what happens on the 1-skeleton. Here, the we have just a copy of S^1. The inversion map on $\mathbb{Z}/3 = \{e^{\frac{2\pi i k}{3}}\}$ comes from complex conjugation on S^1, which is a map of degree -1. Thus $x_1 \mapsto -x_1$. Since y_2 and x_1 are connected by a mod 3 Bockstein, we also deduce $y_2 \mapsto -y_2$. This gives the action.

Remark 24.15. *We can play the same game for the exact sequence*

$$\{e\} \to \mathbb{Z}/p \to G \to \mathbb{Z}/p^\times \to \{e\}.$$

We see that for mod p coefficients, the action of $\mathbb{Z}/p^\times$ is give by the ordinary multiplication of $\mathbb{Z}/p^\times$ on $\mathbb{Z}/p$. The group $\mathbb{Z}/p$ is the p-Sylow subgroup of Σ_p, and G is the normalizer of it in Σ_p. A general argument shows that the cohomology of G and Σ_p with coefficients in F_p agree.

25. Constructing the Squares

We will give a geometric construction of the Steenrod squares. There is also a purely algebraic construction, and we can recover this one from thinking of the *chains* at each stage, rather than the actual spaces. We fix some notation for this section.

Notation 25.1. *Let X be a pointed space, pointed at x_0, and let $\pi \subset \Sigma_n$ be a subgroup. Let $\mathbb{F}$ be a field, and let $K_n = K(\mathbb{F}, n)$. All cohomology in this section will be taken with coefficients in $\mathbb{F}$.*

Since X is pointed, the Cartesian powers are filtered, just as in Definition 7.8.

Definition 25.2. *For each $0 \le i \le n$, let*

$$F_i(X) = \bigcup_{|I|=n-i} F_n^I(X)$$

be the subspace of all points of X^n, at most i of which are possibly not the basepoint.

The group π acts on X^n via

$$\sigma(x_1, \dots, x_n) = (x_{\sigma^{-1}1}, \dots, x_{\sigma^{-1}n}),$$

and this action restricts to actions on the filtered pieces $F_i(X)$, since they are determined by a count of non-basepoint coordinates and this is independent of the ordering. We conclude that the maps

$$F_i(X) \hookrightarrow F_{i+1}(X)$$

are π-equivariant, and hence the mapping cones inherit an action of π. The most important of these is again the smash powers $F_n(X)/F_{n-1}(X)$. It is important to note here that the π-action is not free: for all $\sigma \in \pi$,

$$\sigma(x, \dots, x) = (x, \dots, x).$$

This means that we expect bad behavior upon passage to orbits, since we are losing information here. We can fix this via a more homotopically robust construction.

Definition 25.3. *If G is a group, then let EG be a contractible space on which G acts freely, and let BG be the quotient EG/G.*

Remark 25.4. *There are several ways to build such a space. If G is a discrete group, then we can first build BG by producing a connected CW complex with $\pi_1 = G$, and then killing all of the higher homotopy groups. The universal cover then works as EG. When G has a topology, then Milnor produced a good EG as the infinite join of G with itself, and for nice topological groups G, Segal produced an explict CW complex.*

Definition 25.5. *If X is a G-space, then the* ***Borel construction*** *or* ***homotopy orbits*** *is the orbit space*

$$X_{hG} := EG \times_G X = (EG \times X)/G,$$

where G acts diagonally on $EG \times X$.

Exercise 25.1.

(1) *If G acts freely on X, then $X_{hG} \simeq X_G$.*
(2) *If G acts trivially on X, then $X_{hG} \simeq BG \times X$.*

The idea for the first part is that if G acts freely, then the map $EG \times X \to X$ is a G-equivariant homotopy equivalence.

We can realize X_{hG} as the total space of a fibration over BG. There are two ways to think of this:

(1) We have a map $X \to *$, and this is G-equivariant for any G that acts on X. The Borel construction is functorial, so this gives us a canonical map
$$X_{hG} \to (*)_{hG} = BG.$$
(2) We have a fibration $G \to EG \to BG$. We can then form X_{hG} by fiberwise replacing G with X using the G-action. This gives us a fibration
$$X \to X_{hG} \to BG.$$

Everything we have done is natural in X and equivariant maps, so we can apply it to $X^{\wedge n}$, the filtered pieces, and $G = \pi$. In particular, we have bundles

$$E\pi \times_\pi F_{n-1} \subset E\pi \times_\pi X^n,$$

both bundles over $B\pi$.

Definition 25.6. *The π-extended power of X is*

$$D_\pi X = \big(E\pi \times_\pi X^n\big)/\big(E\pi \times_\pi F_{n-1}\big).$$

The standard properties of the quotient allow us to rewrite the right-hand side:

$$D_\pi X \cong E\pi_+ \wedge_\pi X^{\wedge n}.$$

Working in the relative case is slightly easier.

The π-extended power construction is fundamental in algebraic topology, and it is the source of all Steenrod operations. We need to understand this in cohomology, especially in universal cases.

Theorem 25.7. *If $\tilde{H}^q(X) = 0$ for $q < r$, then*

$$\tilde{H}^s(D_\pi X) = \begin{cases} 0 & s < nq \\ \big(\tilde{H}^r(X)^{\otimes n}\big)^\pi & s = nr. \end{cases}$$

The intuition for this is as follows. If X is simply connected, then by cellular approximation and the Hurewicz theorem, we may assume that the q-skeleton of X is a wedge of S^q. This forces the nq-skeleton of $X^{\wedge n}$ to be a wedge of S^{nq}, and hence the same is true for the Borel construction. To prove this, however, we need a relative Serre spectral sequence.

Theorem 25.8. *Let $F \to E \to B$ be a fibration and let $E' \subset E$ be a subspace such that $E' \to B$ is also a fibration (with fiber F'). Then we have a spectral sequence of algebras with*

$$E_2^{p,q} = H^p\big(B; \mathcal{H}^q(F, F'; R)\big) \Rightarrow H^{p+q}(E, E'; R).$$

Proof of Theorem 25.7. We apply Theorem 25.8 to $E = E\pi \times_\pi X^n$, and $E' = E\pi \times_\pi F_{n-1}$, so

$$H^{p+q}(E, E') = H^{p+q}(D\pi X).$$

Now $B = B\pi$ has $\pi_1 B = \pi$, and the action on the fibers (which are F_{n-1} and $X^{\times n}$ respectively) is very non-trivial. Note that

$$H^*(F, F') \cong \tilde{H}^*(F/F') = \tilde{H}^*(X^{\wedge n}).$$

By the Künneth theorem (since $\mathbb{F}$ is a field), we have that this is

$$\big(\tilde{H}^*(X)\big)^{\otimes n},$$

and π, which rotated the factors of X, just has the natural action rotating the tensor factors. Again, since $B = B\pi$, the twisted cohomology groups are just group cohomology:

$$E_2^{p,q} = H^p\big(\pi; \tilde{H}^q(X)^{\otimes n}\big).$$

This means that for $q < nr$, we have that

$$E_2^{p,q} = 0.$$

Moreover, the group

$$E_2^{0,nr} = H^0\big(\pi; \tilde{H}^r(X)^{\otimes n}\big) = \big(\tilde{H}^r(X)^{\otimes n}\big)^\pi$$

are all non-bounding permanent cycles. Since nothing else can contribute to this degree, and since this is the non-zero group of smallest degree, we conclude the theorem. □

Corollary 25.9. *For any r, we have*

$$\tilde{H}^{nr}(D_\pi K_r) \cong R,$$

generated by a class $P_\pi(\iota_r)$ such that under the restriction map

$$\tilde{H}^{nr}(D_\pi K_r) \to \tilde{H}^{nr}(K_r^{\wedge n}),$$

we have

$$P_\pi \iota_r \mapsto \iota_r^{\otimes n}.$$

Since ordinary cohomology is represented by maps to Eilenberg-MacLane spaces, this gives us a map

$$P_\pi \iota_r \colon D_\pi K_r \to K_{nr}.$$

Definition 25.10. *Let $u \in H^r(X)$. Then the* ***total Steenrod power*** *on u is the composite*

$$D_\pi X \xrightarrow{D_\pi u} D_\pi K_r \xrightarrow{P_\pi \iota_r} K_{nr}.$$

To get the Steenrod operations in their usual form, we pull back along the diagonal map

$$\Delta \colon X \to X^{\wedge n}.$$

This map is π-equivariant, and hence induces a map on Borel constructions:

$$B\pi_+ \wedge X \cong E\pi_+ \wedge_\pi X \xrightarrow{\Delta_{h\pi}} D_\pi X.$$

Since we are working over a field, the Künneth isomorphism shows

$$\tilde{H}^*(B\pi_+ \wedge X) \cong H^*(B\pi) \otimes \tilde{H}^*(X),$$

so the composite

$$B\pi_+ \wedge X \xrightarrow{\Delta_{h\pi}} D_\pi X \xrightarrow{P_\pi u} K_{rn}$$

is a sum

$$\sum b_i \otimes x_{rn-i},$$

where $b_i \in H^i(B\pi)$ and $x_{rn-i} \in \tilde{H}^{rn-i}(X)$.

Restricting attention to $n = 2$, $\pi = \Sigma_2$ and $\mathbb{F} = \mathbb{F}_2$, we have that $B\pi = \mathbb{R}P^\infty$, and so $H^*(B\pi) = \mathbb{F}_2[x]$.

Definition 25.11. *If $u \in H^r(X)$, then define classes $Sq^i(u)$ for all i by*

$$\Delta^*_{h\Sigma_2} P_{\Sigma_2} u = \sum x^{q-i} \otimes Sq^i(u).$$

Every map we have used is described by a universal property, and every construction we have used is functorial. This then defines natural operations

$$Sq^i \colon H^r(X) \to H^{r+i}(X).$$

Proposition 25.12. *For any $u \in H^r(X)$, we have*

$$Sq^r(u) = u^2.$$

Proof. We prove this via considering the universal case. The diagonal map $K_r \to K_r^{\wedge 2}$ fits into a commutative square

$$\begin{tikzcd}
E\pi_+ \wedge_\pi K_r \arrow[r, "\Delta"] & D_{\Sigma_2} K_r \\
K_r \arrow[u] \arrow[r, "\Delta"'] & K_r^{\wedge 2}, \arrow[u]
\end{tikzcd}$$

where the unlabeled maps are the "inclusions of the fibers". These inclusions of the fibers correspond to the inclusion of zero cell in $B\Sigma_2$, and hence pulling back along them exactly gives the class $Sq^r \iota_r$. By assumption, the pullback of $P_{\Sigma_2}\iota_r$ along the inclusion of the fiber from $K_r^{\wedge 2}$ is exactly $\iota_r^{\otimes 2}$, and by the definition of the cup product, that pulls back to ι_r^2 under the diagonal map. $\square$

26. The cohomology of Eilenberg-MacLane spaces

The squares actually give complete information about the cohomology of Eilenberg-MacLane spaces. This is a beautiful theorem of Serre and a fantastic application of the Serre spectral sequence.

Definition 26.1. *The differential*

$$d_{n+1}\colon E_{n+1}^{0,n} \to E_{n+1}^{n+1,0}$$

is called the ***transgression*** τ*. If $\phi \in E_2^{0,n}$ survives to E_{n+1} and $d_{n+1}(\phi) \neq 0$, then we say ϕ is* ***transgressive****.*

This is the last possible differential from $E^{0,n}$, and it is related to actual geometric content. Consider the coboundary map

$$H^n(E) \to H^n(F) \xrightarrow{\delta} H^{n+1}(E,F) \to H^{n+1}(E).$$

The projection map p also gives a map of pairs $(E,F) \to (B,b)$, so we have a diagram:

$$\begin{tikzcd} H^n(F) \arrow[r, "\delta"] & H^{n+1}(E,F) & \\ & H^{n+1}(B,b) \arrow[u, "p^*"] & H^{n+1}(B). \arrow[l, "\cong"'] \end{tikzcd}$$

Proposition 26.2. *The transgression is*

$$\tau = (p^*)^{-1} \circ \delta.$$

This is obviously not entirely well-defined, since we have no reason to believe that $Im(\delta) \subset Im(p^*)$. One way to restate this proposition is

$$\delta(\phi) = p^*(\beta) \Leftrightarrow d_{n+1}(\phi) = \beta.$$

So really, we have that $d_{n+1}(\phi)$ hits a coset of β.

Corollary 26.3. *If ϕ is transgressive, then for all i, $Sq^i\phi$ is also transgressive and*

$$\tau(Sq^i\phi) = Sq^i\tau(\phi).$$

Proof. By naturality and the commuting of the squares with the coboundary, we have

$$\delta(Sq^i\phi) = Sq^i\delta(\phi) = Sq^i p^*(\beta) = p^*(Sq^i\beta).$$

□

The second important piece is a theorem of Borel.

Definition 26.4. *A collection of elements $\{x_1, \dots\}$ is a* ***simply system of generators*** *for $H^*(X)$ if the simple products*

$$x_{i_1} \smile \cdots \smile x_{i_k}$$

forms a basis.

Example 26.5.

(1) *The polynomial ring $\mathbb{F}_2[x]$ has $\{x, x^2, x^4, \dots\}$ as a simple system of generators.*
(2) *If $\{x_1, \dots\}$ is a simple system for A and $\{y_1, \dots\}$ is a simple system for B, then $\{x_1, \dots, y_1, \dots\}$ is a simple system for $A \otimes B$.*

Theorem 26.6 (Borel)**.** *Let $F \to E \xrightarrow{p} B$ be a fibration with $E \simeq *$. If $H^*(F)$ has a simple system of transgressive generators, then $H^*(B)$ is polynomial on the transgressions.*

Theorem 26.7 (Serre)**.** *For any n,*

$$H^*\big(K(\mathbb{Z}/2, n); \mathbb{F}_2\big) \cong \mathbb{F}_2\big[Sq^I u_n \mid e(I) < n\big].$$

Example 26.8. *If $n = 1$, then the condition $e(I) = 0$ forces $I = 0$. We recover that*

$$H^*\big(K(\mathbb{Z}/2, 1); \mathbb{F}_2\big) = H^*\big(\mathbb{R}P^\infty; \mathbb{F}_2\big) \cong \mathbb{F}_2[u_1].$$

When $n = 2$, the only strings of excess 1 are the strings

$$I_k = (2^k, 2^{k-1}, \dots, 1),$$

so we deduce

$$H^*\big(K(\mathbb{Z}/2, 2); \mathbb{F}_2\big) \cong \mathbb{F}_2\big[u_2, Sq^1 u_2, Sq^2 Sq^1 u_2, \dots\big].$$

Proof of Theorem 26.7. The proof will be by induction on n. Recall also that if $e(I) > n$, then Proposition 23.15 shows that $Sq^I u_n = 0$ and if $e(I) = n$, then

$$Sq^I u_n = \big(Sq^J u_n\big)^{2^k},$$

for some subsequence J with $e(J) < n$. We will proceed by induction on n, the base case of which is $\mathbb{R}P^\infty$. The inductive hypothesis says that

$$H^*\big(K(\mathbb{Z}/2, n); \mathbb{F}_2\big) \cong \mathbb{F}_2[Sq^I u_n \mid e(I) < n].$$

Example 26.5 shows that this has a simple system of generators

$$\Big\{\big(Sq^I u_n\big)^{2^k} \mid e(I) < n\Big\}.$$

We can then rewrite this as

$$\Big\{Sq^I u_n \mid e(I) \le n.\Big\}$$

The element u_n is transgressive for degree reasons, and it must therefore hit u_{n+1}. Corollary 26.3 then shows that all of the squares on this are transgressive, and hence $H^*\big(K(\mathbb{Z}/2, n); \mathbb{F}_2\big)$ has a simple system of transgressive generators. Borel's theorem then gives the result. □

Corollary 26.9. *The admissible sequences for a basis for $\mathcal{A}$.*

Proof. The map of $\mathcal{A}$-modules

$$\mathcal{A} \mapsto H^*\big(K(\mathbb{Z}/2, n); \mathbb{F}_2\big)$$

which sends 1 to u_n sends classes of excess less than n to linearly independent elements. Since linear independence is a finite condition, choose n sufficiently large (larger than the excess of any element in a chosen finite set) shows that these are linearly independent in $\mathcal{A}$. □

The same arguments apply to give the cohomology of $K(\mathbb{Z}, n)$ and $K(\mathbb{Z}/2^k, n)$.

Theorem 26.10. *For each n, let $\mathcal{I}_n^{\mathbb{Z}}$ be the set of admissible sequences I such that $e(I) < n$ and the last term in I is not 1. Then*

$$H^*\big(K(\mathbb{Z}, n); \mathbb{F}_2\big) \cong \mathbb{F}_2\big[Sq^I u_n \mid I \in \mathcal{I}_n^{\mathbb{Z}}\big].$$

Definition 26.11. *Let β_k denote the connecting homomorphism for the coefficient sequence*

$$\mathbb{Z}/2 \to \mathbb{Z}/2^{k+1} \to \mathbb{Z}/2^k.$$

This gives a natural transformation

$$H^*(-;\mathbb{Z}/2^k) \Rightarrow H^{*+1}(-;\mathbb{Z}/2).$$

Theorem 26.12. *For each n and each k,*

$$H^*\big(K(\mathbb{Z}/2^k, n);\mathbb{F}_2\big) \cong \mathbb{F}_2\big[\tilde{Sq}^I u_n \mid e(I) < n\big],$$

where $\tilde{Sq}^I = Sq^I$ if I does not end in 1, and if I does end in 1, then $\tilde{Sq}^I$ is Sq^I with the final Sq^1 replaced with β_k.

Part 5. Extra Topics

Appendix A. Postnikov Towers

This procedure for killing the homotopy groups above some point seems very *ad hoc*. In fact, we can do so functorially by building a *much* bigger space. This gives the Postnikov tower.

Definition A.1. *Let X be a pointed space. For each $k \geq 0$, we define a sequence of spaces $\left(P^k X\right)_i$ via the pushout squares*

$$\begin{array}{ccc} \bigvee_{m \geq k+1} \bigvee_{f \in \mathrm{Map}\left(S^m, (P^k X)_{i-1}\right)} S^m & \xrightarrow{F} & (P^k X)_{i-1} \\ \downarrow & & \downarrow \\ \bigvee_{m \geq k+1} \bigvee_{f \in \mathrm{Map}\left(S^m, (P^k X)_{i-1}\right)} D^{m+1} & \longrightarrow & (P^k X)_i, \end{array}$$

where F is the map which on the summand corresponding to f is just f.

Let

$$P^k X = \varinjlim (P^k X)_i.$$

*This is the k**th Postnikov section functor**.*

Proposition A.2. *The assignment $X \mapsto P^k X$ is functorial in X.*

Proof. Let $g\colon X \to Y$. We will show that the cofiber squares defining $P^k X$ are actually functors. Assume that we have shown this for $(i-1)$. This gives us a map

$$(P^k g)_{i-1}\colon (P^k X)_{i-1} \to (P^k Y)_{i-1}.$$

For the other pieces, note that if $f\colon S^m \to X$, then $g \circ f\colon S^m \to Y$. In other words, we have a map on indexing sets

$$(P^k g)_{i-1*}\colon \mathrm{Map}\left(S^m, (P^k X)_{i-1}\right) \to \mathrm{Map}\left(S^m, (P^k Y)_{i-1}\right),$$

and if we use the identity map on the corresponding summands, then we have a map of the left vertical columns. This gives a map of the pushouts, as desired. □

Remark A.3. *We have chosen here functorial models of the pushout and of the colimit.*

The Postnikov section has two key features:

Theorem A.4. *Let X be a pointed space, and let $f_k\colon X \to P^k X$ be the inclusion of X into the colimit. Then*

(1)
$$\pi_j(P^k X, x) \cong \begin{cases} \pi_j(X, x) & j \leq k \\ 0 & j > k, \end{cases}$$
and the isomorphism is induced by f_k.

(2) *The map f_k is initial among all maps from X to a space with $\pi_{>k} = 0$.*

Proof. Note that in forming the stages of the space P^kX, we attached cells of dimensions at least $(k+2)$. In particular, by Theorem 22.7, we know that the homotopy groups through dimension k are unchanged. Now given any

$$[f] \in \pi_j\big((P^kX)_i, x\big),$$

with $j > k$, any representative f gives a map of a sphere into $(P^kX)_i$. This is exactly what we cone off to form $(P^kX)_{i+1}$, so we deduce that the map

$$\pi_j\big((P^kX)_i, x\big) \to \pi_j\big((P^kX)_{i+1}, x\big)$$

is an isomorphism for $j \leq k$ and is zero for $j > k$. Since the homotopy groups of a direct limit are the direct limit of the homotopy groups, this gives the result. □

This universal property of the Postnikov spaces also guarantees that we have natural transformations

$$P^k(-) \Rightarrow P^{k-1}(-).$$

Thus to any space, we have a tower

$$\begin{tikzcd} & \vdots \arrow[d] \\ & P^k(X) \arrow[d] \\ X \arrow[ur, "f_k"] \arrow[r, "f_{k-1}"'] & P^{k-1}(X) \arrow[d] \\ & \vdots \end{tikzcd} \tag{13}$$

Definition A.5. *The tower of Equation 13 is the **Postnikov tower**.*

By considering the long exact sequence is homotopy, we see that the homotopy fiber of the map

$$P^k(X) \to P^{k-1}(X)$$

is the Eilenberg-MacLane space $K(\pi_k(X,x),k)$. Thus we can think of the Postnikov tower as a way to reassemble X by putting the homotopy groups in one at a time.