Asymptotic Differential Algebra

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Two success stories at the crossroads of algebra and model theory:

(1) Differential algebra (Ritt, Kolchin; A. Robinson, L. Blum ...)

- Basic objects are *differential fields*: fields K equipped with a *derivation* $f \mapsto f'$:

$$(f+g)' = f' + g', \quad (fg)' = f'g + fg'.$$

- "Universal domains": *differentially closed fields.*
- Fits in the model-theoretic framework of "ω-stable theories."
- Differential algebraic geometry: Nullstellensatz (Seidenberg).
- Applications: Diophantine questions (Buium, Hrushovski), integration in finite terms (Risch) ...

(2) Real algebra (Artin-Schreier, Krull; Tarski, A. Robinson . . .)

- Basic objects are *ordered fields*: fields K equipped with a total ordering $\leq$ such that

$$a \leq b,\ 0 \leq d \quad \Rightarrow \quad a + c \leq b + c,\ ad \leq bd.$$

- "Universal domains": *real closed fields* (such as $\mathbb{R}$).
- Real (semi-) algebraic geometry: Null- and Positivstellensätze (Dubois-Risler-Krivine).
- Fits in the wider framework of "o-minimal theories."
- Applications: Hilbert's 17^{th} problem (Artin), quantifier elimination, optimization . . .

Definition. (Bourbaki, 1961.) A **Hardy field** K is a set of germs at $+\infty$ of differentiable real-valued functions on half-lines $(a, +\infty)$, $a \in \mathbb{R}$, forming a differential field with respect to the usual operations on (germs of) functions.

Examples.

- $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}(x)$, where $x =$ germ of the identity function on $\mathbb{R}$
- $\mathbb{R}(x, e^x)$, $\mathbb{R}(x, \ln x)$, $\mathbb{R}(\Gamma, \Gamma', \Gamma'', \dots)$, $\dots$
- G. Hardy's field of *LE-functions:* constructed from $\mathbb{R}(x)$ by algebraic operations, exponentiation, logarithm, and composition.
- Every o-minimal expansion $\widetilde{\mathbb{R}}$ of the field of reals gives rise to a Hardy field $H(\widetilde{\mathbb{R}})$: the field of germs at $+\infty$ of functions $\mathbb{R} \to \mathbb{R}$ definable in $\widetilde{\mathbb{R}}$.

Any Hardy field K carries a natural ordering:

$$f > 0 \qquad :\Longleftrightarrow \qquad f(x) > 0 \text{ eventually.}$$

In particular, it follows that every $f \in K$ is eventually monotonic, and

$$\lim_{x\to\infty} f(x) \in \mathbb{R} \cup \{\pm\infty\}$$

exists.

History: du Bois-Reymond (1870s), Hardy (1910), Bourbaki, Rosenlicht, Boshernitzan, Shackell …

Any Hardy field K comes equipped with “dominance relations”

$$f \preceq g \quad \Longleftrightarrow \quad f = O(g) \quad \Longleftrightarrow \quad |f| \leq c|g| \text{ for some } c \in \mathbb{R},$$
$$f \prec g \quad \Longleftrightarrow \quad f = o(g) \quad \Longleftrightarrow \quad |f| \leq c|g| \text{ for all } c \in \mathbb{R},\ c > 0.$$

H-fields.

Definition. An H**-field** is an ordered differential field K (with constant field C) such that:

(H1) $f > C \Rightarrow f' > 0$;

(H2) $f \preceq 1 \Rightarrow f - c \prec 1$ for some $c \in C$.

In **(H2)**, we consider K as equipped with the dominance relations

$$f \preceq g \iff f = O(g) \iff |f| \leq c|g| \text{ for some } c \in C,$$
$$f \prec g \iff f = o(g) \iff f \preceq g \text{ and } g \not\preceq f.$$

Examples:

Every Hardy field $K \supseteq \mathbb{R}$ *is an* H*-field* (*with constant field* $\mathbb{R}$).

Properties of the dominance relation. For all elements f, g, h of an H-field K:

(D1) $f \preceq f$

(D2) $f \preceq g$ or $g \preceq f$

(D3) $f \preceq g,\ g \preceq h \ \Rightarrow\ f \preceq h$

(D4) $f \preceq g \ \Rightarrow\ fh \preceq gh$

(D5) $f \preceq h,\ g \preceq h \ \Rightarrow\ f + g \preceq h$

(A) If $f, g \prec 1$, then $f \preceq g \iff f' \preceq g'$.

Terminology:

$f \prec 1:$ f is *infinitesimal*

$f \succ 1:$ f is *infinite*

$f \preceq 1:$ f is *finite* (or *bounded*).

We also define an equivalence relation $\asymp$ (*asymptotic*) on K:

$$f \asymp g \quad :\Longleftrightarrow \quad f \preceq g \text{ and } g \preceq f.$$

The equivalence classes $v(f)$, where $f \in K^\times = K \setminus \{0\}$, are the elements of an ordered abelian group $\Gamma = v(K^\times)$:

$$v(f) + v(g) = v(fg), \quad v(f) \geq v(g) \Longleftrightarrow f \preceq g.$$

We have a (Krull) *valuation*

$$v\colon K \to \Gamma_\infty = \Gamma \cup \{\infty\} \qquad (v(0) := \infty)$$

with *value group* Γ. (By **(D1)–(D5)**.)

By property **(A)**, we have, for $f, g \in K^\times$ with $v(f), v(g) \neq 0$:

$$v(f) \leq v(g) \qquad \Longleftrightarrow \qquad v(f') \leq v(g').$$

In particular, $v(f')$ *depends only on* $v(f)$, provided $v(f) \neq 0$.

So the derivation induces a function

$$\psi \colon \Gamma^* = \Gamma \setminus \{0\} \to \Gamma$$

by

$$\psi\big(v(f)\big) := v\left(\frac{f'}{f}\right) = v(f') - v(f).$$

The pair (Γ, ψ) is called the **asymptotic couple** of K. (Rosenlicht.)

The field of logarithmic-exponential series.

Let Γ be a (multiplicative) ordered abelian group. Then

$$\mathbb{R}(\!(\Gamma)\!) := \left\{ f = \sum_{\gamma\in\Gamma} c_\gamma \gamma : c_\gamma \in \mathbb{R}, \text{supp } f \text{ anti-wellordered} \right\},$$

where $\text{supp } f := \{\gamma \in \Gamma : c_\gamma \neq 0\}$, is called the **field of formal series with coefficients in** $\mathbb{R}$ **and monomials in** Γ.

The field $\mathbb{R}(\!(\Gamma)\!)$ carries a natural ordering:

$$f > 0 \quad :\Longleftrightarrow \quad c_{\mathrm{Lm}(f)} > 0, \qquad \text{where } \mathrm{Lm}(f) := \max \text{supp } f.$$

Example. $\mathbb{R}(\!(x^{-1})\!) = \mathbb{R}(\!(x^{\mathbb{Z}})\!)$, the field of **formal Laurent series in x^{-1}.**

Problem:

For $\Gamma \neq \{1\}$, the ordered field $K = \mathbb{R}(\!(\Gamma)\!)$ does not admit an **exponential function** $(K, 0, +, <) \xrightarrow{\cong} (K^{>0}, 1, \cdot, <)$.

Solution:

Construction of the field $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ of **logarithmic-exponential series** with real coefficients, as a subfield of some $\mathbb{R}(\!(\Gamma)\!)$ for a "very big" Γ (the group of **LE-monomials**).

Its elements are formal series with real coefficients:

$$\underbrace{e^{e^x} - \sqrt{2}e^{x^5} - \log x}_{\text{infinite part}} + \underbrace{42}_{\text{constant}} + \underbrace{x^{-1} + x^{-2} + \cdots + e^{-x} + e^{-x^2} + \cdots}_{\text{infinitesimal part}}$$

LE-series may be viewed as *asymptotic expansions* (often divergent) of germs at $+\infty$ of (non-oscillatory) real-valued functions, in terms of LE-monomials.

Examples.

- Stirling expansion for $\Gamma(x)$:

$$\log\Gamma(x) \sim \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}x^{1-2k}$$

- Formal solutions to algebraic ODE's (Écalle, van der Hoeven).

History: Hahn (1907), Higman (1950s), Dahn, Göring, Écalle, Macintyre, Marker, van den Dries, van der Hoeven ...

Some properties of $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$:

- $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}} \supseteq \mathbb{R}(\!(x^{-1})\!)$;
- has a natural *ordering* (with $x > \mathbb{R}$);
- has a natural *exponential function* $f \mapsto e^f$;
- has a natural *derivation* (with $x' = 1$, constant field $\mathbb{R}$);
- has a natural *composition operation* $(f, g) \mapsto f(g)$, for $g > \mathbb{R}$;
- the iterates of $\exp$ $e^x, e^{e^x}, e^{e^{e^x}}, \ldots$ are *cofinal* in $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$;
- is an *elementary extension* of $\mathbb{R}$ equipped with lots of further (analytic) structure: exp, analytic functions on compact cubes, Γ-function on $(0, \infty)$ …

There is positive evidence that $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ plays the role of a “universal domain” for the theory of H-fields:

- $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ is a Liouville closed H-field: an H-field K is **Liouville closed** if it is real closed and solves all differential equations

$$y' = a, \quad \frac{z'}{z} = b \qquad (a, b \in K).$$

- A fragment of the theory of $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ is “completely” understood, namely the theory of its asymptotic couple. (A., van den Dries 1999; A. 2000.)

- The *intermediate value property* (IVP) for differential polynomials holds in $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$. (van der Hoeven, 1999.)

- Many Hardy fields can be embedded into $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ (as ordered differential fields), e.g., $H(\mathbb{R}_{\mathrm{an,exp}})$.

Conjecture. (van den Dries) An H-field K is *existentially closed* (i.e., is a “universal domain” for the theory of H-fields) if and only if

- K is Liouville closed, and
- K satisfies the IVP for differential polynomials over K.

(So in particular, $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ is a universal domain for H-fields.)

We do know:

Theorem. (A., van den Dries, 2000)

$$K \text{ existentially closed} \Rightarrow \begin{cases} \text{Liouville closed,} \\ \text{IVP for differential polynomials} \\ \text{of order } 1. \end{cases}$$

Differential Equations over H-Fields.

Let K be an H-field, with asymptotic couple (Γ, ψ). Put

$$\Psi := \{\psi(\gamma) : 0 \neq \gamma \in \Gamma\} = \{v(f'/f) : f > C\}.$$

Let $P(Y) \in K\{Y\} = K[Y, Y', Y'', \ldots]$ be a non-zero differential polynomial.

Then $y \mapsto P(y)$ defines a *continuous* function $K \to K$ which is not identically zero on any non-empty open subset of K.

Question 1: What are the zeros of P? (In K, or an H-field extension.)

Some basic facts:

Theorem. (A., van den Dries 2000) *Suppose K is Liouville closed. Then there exists $f \in K^{>0}$ such that $P(y)$ has constant sign > 0 or < 0, for all y in an H-field extension of K with $y > f$.*

(In particular, the zero set of P is discrete.)

Theorem. (A., van den Dries 2000) *Suppose K is Liouville closed, and the coefficients of $P(Y)$ lie in some H-subfield E of K with Ψ_E having a largest element. Then there exists $a > C$ in K such that $P(y) \neq 0$ for all y in all H-field extensions L of K with $C_L < y < a$.*

The hypothesis is always satisfied for $K = \mathbb{R}((x^{-1}))^{\mathrm{LE}}$. It can be omitted if P is of order 1. It *cannot* be omitted if P is of order ≥ 2. (A., van der Hoeven, 2001.)

Question 2: How does $v\big(P(y)\big)$ behave as y varies? (In K, or an H-field extension of K.)

First we study a somewhat better behaved quantity. Let

$$v(P) := \text{minimum of the valuations of the coefficients of } P$$

(This defines a valuation on $K\{Y\}$.)

Now put

$$P_{\times h} := P(hY) \text{ for } h \in K \qquad (\textit{multiplicative conjugation}).$$

Fact: $v(P_{\times h})$ only depends on $v(h)$.

We get an induced function

$$v_P \colon \Gamma \to \Gamma, \quad v_P\big(v(h)\big) := v(P_{\times h}).$$

We have a good understanding of v_L for

$$L(Y) = a_0 Y + a_1 Y' + \cdots + a_n Y^{(n)} \in K\{Y\}.$$

Functoriality: $v_{L_1 \circ L_2} = v_{L_1} \circ v_{L_2}$;

Definability: if K is Liouville closed, then v_L is definable in (Γ, ψ);

Bijectivity: if K is real closed, then $v_L \colon \Gamma \to \Gamma$ is an order-preserving bijection;

Relation to $v\big(L(h)\big)$: if Ψ has a maximum, then $v\big(L(h)\big) = v_L(\gamma)$ for all but finitely many $\gamma = v(h)$.

Main ideas in the proofs: *Newton diagrams* and *Ricatti polynomials*. (à la Ramis, Malgrange, van der Hoeven.)

Suppose $P(Z) = a_0 + a_1 Z + \cdots + a_n Z^n \in K[Z]$ is an ordinary polynomial over K, $a_n \neq 0$. The **Newton diagram** of P is

$$\mathcal{N}(P) := \big\{ \big(i, v(a_i)\big) : 0 \leq i \leq n, a_i \neq 0 \big\} \subseteq \mathbb{Z} \times \Gamma.$$

An **approximate zero** of P is an element $z \in K$ such that

$$P(z) \prec a_i z^i \qquad \text{for all } i.$$

Studying how $\mathcal{N}(P)$ changes when passing from $P(Z)$ to

$$P(Z + \phi) = P_{+\phi}(Z),$$

where ϕ is an approximate zero of P, one obtains a piecewise uniform description of $z \mapsto v\big(P(z)\big)$ in terms of functions of the form

$$z \mapsto v(z - \theta), \qquad \theta \in K.$$

(Provided K is *henselian* as valued field.)

Ricatti polynomials. For each $n \in \mathbb{N}$ there exists a differential polynomial $R_n(Z)$ (with coefficients in $\mathbb{N}$) such that

$$y^{(n)}/y = R_n(z) \qquad \text{for all } y \in K^\times,\ z = y'/y.$$

Examples. $R_0(Z) = 1$, $R_1(Z) = Z$, $R_2(Z) = Z^2 + Z'$, $\ldots$

To a linear homogeneous differential polynomial

$$L(Y) = a_0 Y + a_1 Y' + \cdots + a_n Y^{(n)} \in K\{Y\}$$

we associate its **Ricatti polynomial**

$$\mathrm{Ric}(L) := a_0 R_0(Z) + a_1 R_1(Z) + \cdots + a_n R_n(Z) \in K\{Z\}$$

and its **Newton diagram** $\mathcal{N}(L) := \mathcal{N}(P)$, where

$$P(Z) := a_0 + a_1 Z + \cdots + a_n Z^n \in K[Z].$$

We have, for $y \in K^\times$, $z = y'/y$:

- $L(y)/y = \mathrm{Ric}(L)(z)$;
- $\mathrm{Ric}(L_{\times y}) = y\,\mathrm{Ric}(L)_{+z}$;
- $v(L) = v\big(\mathrm{Ric}(L)\big)$.

An element $z \succeq 1$ of K is an **approximate non-infinitesimal zero** of $\mathrm{Ric}(L)$ if

$$\mathrm{Ric}(L)(z) \prec a_i R_i(z) \qquad \text{for all } i.$$

Fact:

$$z \text{ is an approximate non-infinitesimal zero of } \mathrm{Ric}(L) \iff z \text{ is an approximate zero of } P.$$

This leads to a piecewise uniform description of $z \mapsto v\big(\mathrm{Ric}(L)_{+z}\big)$ in terms of functions of the form

$$z \mapsto v(z-\theta), \qquad \theta \in K.$$

Consequences for solving $L(y) = g$: Suppose that K is a real closed H-field such that

(1) Ψ has a *largest element* $\max \Psi = 0$;

(2) K has *no immediate H-field extension.*

Put $K(i) =$ algebraic closure of K (where $i^2 = -1$) and $K(i)^{\mathrm{dc}} =$ differential closure of $K(i)$. Let $x \in K(i)^{\mathrm{dc}}$ with $x' = 1$.

(I) The map

$$y \mapsto L(y)\colon K(i)[x] \to K(i)[x]$$

is *surjective.*

(II) There is a non-zero $y \in K(i)^{\mathrm{dc}}$ with

$$L(y) = 0 \quad \text{and} \quad y'/y \in K(i).$$

(Think of y as $\exp(\int f + ig)$, $f, g \in K$.)

Corollary. Suppose K is a Liouville closed H-field which can be written as a directed union

$$K = \bigcup_{i \in I} K_i$$

of H-fields K_i satisfying conditions (1) and (2) from before. *Then any linear homogeneous differential polynomial over K is a composition of 1^{st} and 2^{nd} order linear homogeneous polynomials over K.*

The H-field $K = \mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ satisfies (1) & (2):

$$\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}} = \bigcup_i \mathbb{R}(\!(1/\log_i x)\!)^{\mathrm{E}}$$

Question: Can every existentially closed H-field be written as a directed union of H-fields satisfying (1)?

The corollary follows from (II) and:

Observation. Let K be a Liouville closed H-field. Any non-zero element y of $K(i)^{\mathrm{dc}}$ with $y'/y \in K(i)$ satisfies a 1st or 2nd order linear homogeneous ODE over K.

Proof. Suppose $y'/y = a + ib$ with $a, b \in K$. If $b = 0$, then $y' = ay$. Assume $b \neq 0$. Write $a = -\frac{\lambda'}{\lambda}$ with $\lambda \in K^\times$; then $\frac{(\lambda y)'}{\lambda y} = ib$. So we may assume as well that $a = 0$. Now differentiate $y' = iby$ to get

$$y'' = ib'y + iby' = \frac{b'}{b}y - b^2 y'.$$

$\square$