Residual Properties of 3-Manifold Groups

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UCLA

The authors back in 1979 . . .

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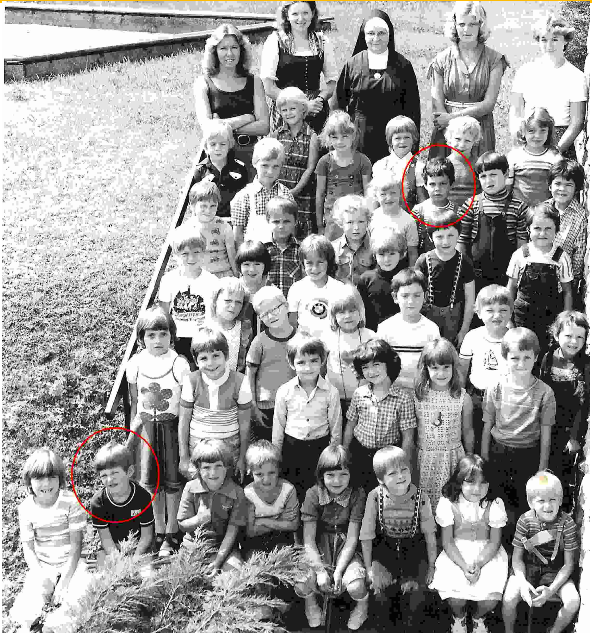

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The group $\mathbb{Z}$ is residually p for every prime p: any non-zero $k \in \mathbb{Z}$ is non-zero in $\mathbb{Z}/p^e\mathbb{Z}$, where $e > 0$ is such that $p^e \nmid k$.

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Non-example

The group $G = \langle a, b : a^{-1}b^2a = b^3 \rangle$ is not residually finite.

Properties of residually finite groups G

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Residually p is a strong property; e.g.:

Every non-abelian subgroup of a residually p group has a quotient isomorphic to $\mathbb{F}_p \times \mathbb{F}_p$.

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Theorem (Thurston & Hempel 1987, + Perelman)

$\pi_1(N)$ *is residually finite:* for every homotopically non-trivial loop $\gamma \colon [0,1] \to N$ there is a finite-sheeted covering $\widetilde{N} \to N$ and some lifting $[0,1] \to \widetilde{N}$ of γ which is not a loop.

Remarks

- Let S be a surface. Then $\pi_1(S)$ is residually finite (Baumslag, 1962), in fact, residually p for every prime p.

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- Given any finitely presented group G there exists a smooth 4-manifold M with $\pi_1(M) = G$.

Question

Is $\pi_1(N)$ always residually p? residually nilpotent? residually solvable?

$\pi_1(N) = \langle x, y : x^2 = y^3 \rangle$

Non-example

Suppose $N = S^3 \setminus \nu(K)$ where $K \subseteq S^3$ is a knot. Then

$$\pi_1(N) \text{ residually } p \iff \pi_1(N) \cong \mathbb{Z} \underset{\text{Dehn's Theorem}}{\iff} K \text{ trivial.}$$

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so $P = \mathbb{Z}/p^k\mathbb{Z}$ for some k. Thus α factors through

$$\pi_1(N) \to \pi_1(N)_{ab} = \mathbb{Z}.$$

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Example

Every finitely generated abelian group contains a free abelian subgroup of finite index, hence is virtually [residually p for all p].

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1. *For all but finitely many primes p,* $\text{Aut}(\pi_1(N))$ *is virtually residually p.* (Hence $\text{Aut}(\pi_1(N))$ is virtually torsion-free, so there is a bound on the size of its finite subgroups.)

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More substantial application of our main theorem in the recent proof by Friedl & Vidussi of a conjecture of Taubes.

Our theorem can also be seen as evidence of the following:

Conjecture (Thurston?)

$\pi_1(N)$ is *linear*, i.e., there is an embedding

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This is due to the following theorem:

Theorem (Malcev)

Let $G \leqslant \mathrm{GL}(n, \mathbb{C})$ be finitely generated. Then G is virtually residually p for all but finitely many primes p.

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There exists $\alpha \in \mathbb{N}$ such that for every field K,

$$1 \in (f_1, \ldots, f_n)K[X] \Rightarrow$$
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Then a non-principal ultraproduct $K = \prod_\alpha K_\alpha/\mathcal{U}$ is an algebraically closed field such that $V_K(f_1, \ldots, f_n) = \emptyset$ and $1 \notin (f_1, \ldots, f_n)K[X]$, contradicting Hilbert's Nullstellensatz.

Using similar techniques we show an approximation result which plays an important role in our proof:

Theorem

Let R be a finitely generated subring of $\mathbb{C}$. For all but finitely many p there exists a maximal ideal $\mathfrak{m} \subseteq R$ such that $R_{\mathfrak{m}}$ is unramified regular of mixed characteristic p.

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- $p \equiv \pm 1 \mod 8 \Rightarrow$ there is $\alpha \in \mathbb{Z}$ with $\overline{\alpha}^2 = 2$, and $\mathfrak{m} = (p, \alpha - \sqrt{2})$ or $\mathfrak{m} = (p, \alpha + \sqrt{2})$. ✓

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- $p \equiv \pm 3 \bmod 8 \Rightarrow R/pR$ is a field and $\mathfrak{m} = pR$. ✓

The theorem is a consequence of the fact that a reduced algebra over a field whose regular locus is open has a regular point, and the following:

Proposition

Let $R \subseteq \mathbb{C}$ be finitely generated. Then for all but finitely many primes p, the ring R/pR is reduced.

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(Uses that the property of $f_1, \ldots, f_n \in \mathbb{Z}[X_1, \ldots, X_N]$ to generate a radical ideal in $K[X_1, \ldots, X_n]$ is constructible, in the coefficients of the f_i, uniformly for each perfect field K; van den Dries & Schmidt.)

The theorem is a consequence of the fact that a reduced algebra over a field whose regular locus is open has a regular point, and the following:

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The analogous statement for being an integral domain is false: $R = \mathbb{Z}[X]/(X^4 + 1)$ is a domain, but $R/pR = \mathbb{F}_p[X]/(X^4 + 1)$ is never a domain.

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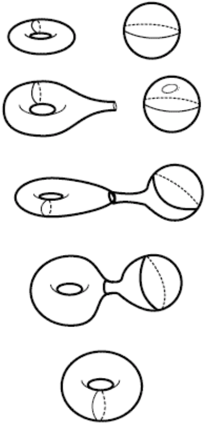

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Prime Decomposition

Every closed orientable 3-manifold N can be decomposed as a connected sum

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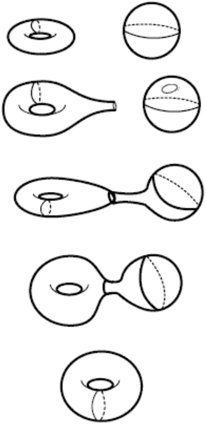
of prime 3-manifolds N_i (H. Kneser), and the N_i are unique up to permutation (Milnor).



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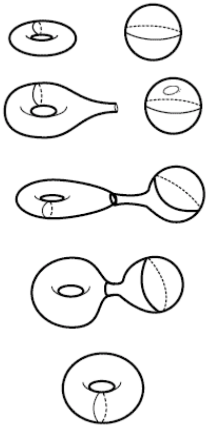

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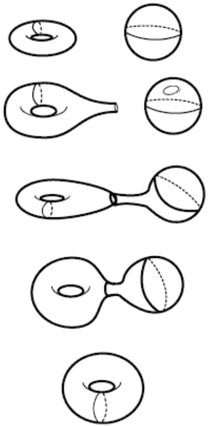

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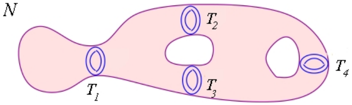
$N = S^2 \times S^1$ is prime, but for technical reasons it's best to exclude it among prime 3-manifolds, which we do from now on.

JSJ (or Torus) Decomposition

Suppose N is closed, prime, and orientable.



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Suppose N is closed, prime, and orientable. Then there are pairwise disjoint, incompressible 2-tori

$$T_1, \ldots, T_n \subseteq N$$

that cut N into pieces which are either

- Seifert fibered, or
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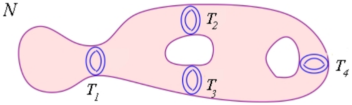

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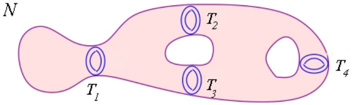

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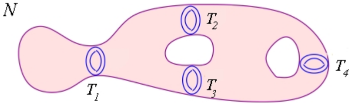

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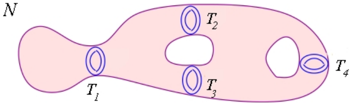

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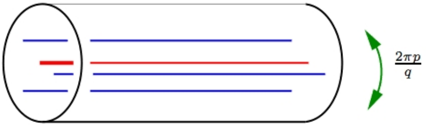

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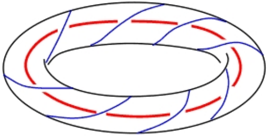

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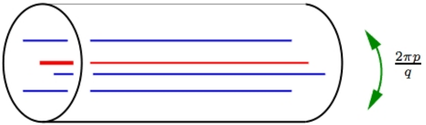

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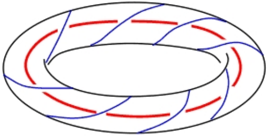

Well-known: N Seifert fibered $\Rightarrow \pi_1(N)$ linear (over $\mathbb{Z}$).

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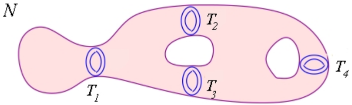

Part of the Geometrization Conjecture (proved by Perelman)

Suppose N is prime and atoroidal with $\pi_1(N)$ infinite. Then N is *hyperbolic* (its interior admits a complete Riemannian metric of constant curvature -1), so

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Some 3-manifold topology

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There is a lifting

$$\begin{array}{ccc} & & \mathrm{SL}(2,\mathbb{C}) \\ & \nearrow & \downarrow \\ \pi_1(N) \cong \Gamma & \longrightarrow & \mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/(\text{center}) \end{array}$$

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Graphs of groups

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It is much easier to be residually finite than residually p

The amalgamated product $G_1 *_H G_2$ of finite groups G_1, G_2 over a common subgroup H is always residually finite. If G_1, G_2 are p-groups, $G_1 *_H G_2$ might not be residually p. (Higman)

Useful criterion for $\pi_1(\mathcal{G})$ with finite vertex groups to be residually p:

There exists a morphism

$$\pi_1(\mathcal{G}) \xrightarrow{\text{injective on vertex groups}} P \text{ (p-group).}$$

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$\mathcal{G}_p := \mathcal{G}/\gamma_2^p(\mathcal{G})$ has vertex groups $H_1(G_v; \mathbb{F}_p)$ and edge groups $\mathbb{F}_p \oplus \mathbb{F}_p$ which are $\mathbb{F}_p$-linear spaces.

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Show:

$$\begin{aligned} \pi_1^*(\mathcal{G}_p, T) \text{ residually } p \quad &\Rightarrow \quad \textit{all } \pi_1^*(\gamma_n^p(\mathcal{G})/\gamma_{n+1}^p(\mathcal{G}), T) \text{ residually } p \\ &\Rightarrow \quad \textit{all } \pi_1(\mathcal{G}/\gamma_n^p(\mathcal{G})) \text{ residually } p. \end{aligned}$$

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[Main ingredients:

- a refinement of an amalgamation theorem by Higman, for which we need some facts on group rings;
- a criterion for HNN extensions to be residually p by Chatzidakis.]

5. Unfolding $\mathcal{G}$. Sufficient condition for $\pi_1^*(\mathcal{G}_p, T)$ residually p: all identification isomorphisms between associated subgroups in the iterated HNN extension $\pi_1^*(\mathcal{G}_p, T)$ of Σ are the identity.

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Show that one can pass to finite cover of N to achieve this, using kernel of

$$\pi_1(N) = \pi_1(\mathcal{G}) \to \mathrm{Aut}(\Sigma)$$

obtained by extending each identification map in $\mathcal{G}_p$ to an automorphism of the $\mathbb{F}_p$-linear space Σ.

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