Effective Descriptive Set Theory

Andrew Marks

December 14, 2019

These notes introduce the effective (lightface) Borel, Σ^1_1 and Π^1_1 sets. This study uses ideas and tools from descriptive set theory and computability theory. Our central motivation is in applications of the effective theory to theorems of classical (boldface) descriptive set theory, especially techniques which have no classical analogues. These notes have many errors and are very incomplete. Some important topics not covered include:

- The Harrington-Shore-Slaman theorem [HSS] which implies many of the theorems of Section 3.
- Steel forcing (see [BD, N, Mo, St78])
- Nonstandard model arguments
- Barwise compactness, Jensen's model existence theorem
- α-recursion theory
- Recent beautiful work of the "French School": Debs, Saint-Raymond, Lecompte, Louveau, etc.

These notes are from a class I taught in spring 2019. Thanks to Adam Day, Thomas Gilton, Kirill Gura, Alexander Kastner, Alexander Kechris, Derek Levinson, Antonio Montalbán, Dean Menezes and Riley Thornton, for helpful conversations and comments on earlier versions of these notes.

Contents

Notation/Conventions

- i, j, k, n, m will stand for elements of ω.
- s, t will stand for elements of $\omega^{<\omega}$. The length of s is denoted $|s|$ and $s^\frown t$ notes their concatenation.
- σ, τ will typically be finite binary strings in $2^{<\omega}$. The set of all binary strings of length $\leq n$ is denoted $2^{\leq n}$.
- N_s is the basic open neighborhood of ω^ω determined by s: $N_s = \{x \in \omega^\omega \colon x \supseteq s\}$.
- x, y, z will stand for elements of ω^ω which we call reals.
- An overline $\overline{x}$ or $\overline{n}$ will stand for a finite tuple of such elements, so $\overline{n}$ stands for a tuple of numbers and $\overline{x}$ stands for a tuple of reals.
- e will typically stand for a program for a partial computable function.
- φ_e denotes the eth partial computable function from ω to ω, and φ_e^x denotes the eth partial computable function relative to x. We will use Φ_e to denote the eth partial computable function from $\omega^\omega \to \omega^\omega$, so $\Phi_e(x)(n) = \varphi_e^x(n)$.
- A, B, C will stand for subsets of ω or ω^ω.
- α, β, λ will stand for countable ordinals.
- A tree on a set X is a nonempty subset T of $X^{<\omega}$ that is closed downward if $t \in T$, then for all $t' \subseteq t$, we have $t' \in T$. Hence, every tree contains the empty string. The letters S, T will typically stand for trees. If T is a tree on X, then $[T]$ denotes the set of infinite paths through T, the set of $x \colon \omega \to X$ such that for every n, $x \upharpoonright n \in T$.
- A tree on a product $X \times Y$ is a nonempty subset T of $X^{<\omega} \times Y^{<\omega}$ such that $(s, t) \in T$ implies $|s| = |t|$, and for all $s' \subseteq s$ and $t' \subseteq t$ with $|s'| = |t'|$, $(s', t') \in T$.
- $\leq_T$ denotes Turing reducibility.
- If $x \in \omega^\omega$, then x' denotes the Turing jump of x.
- π_k denotes the projection of an n-tuple onto its kth coordinate. We let $\pi = \pi_0$ be the projection onto the 0th coordinate.
- We write $A \subseteq^* B$ if $A \setminus B$ is finite.

1 Characterizing Σ^1_1, Δ^1_1, and Π^1_1 sets

1.1 Σ^1_n formulas, closure properties, and universal sets

We briefly recall the definition of computable and arithmetic formulas and relations before defining Σ^1_n formulas and discussing their closure properties and universal sets.

A relation $R(x_1, \ldots, x_i, n_1, \ldots, n_j)$ on $(\omega^\omega)^i \times \omega^j$ is **computable** if there is a single computer program φ_e so that $\varphi_e^{x_1 \oplus \cdots \oplus x_i}(n_1, \ldots, n_j)$ always halts, and accepts its input if $R(x_1, \ldots, x_i, n_1, \ldots, n_j)$ is true, and rejects its input if $R(x_1, \ldots, x_i, n_1, \ldots, n_j)$ is false.

A formula is Σ^0_k if it is of the form

$$\exists n_1 \forall n_2 \exists n_3 \ldots Q n_k R(\overline{x}, \overline{m}, n_1, \ldots, n_k).$$

where these quantifiers alternate between $\exists$ and $\forall$ and are quantifiers over ω. A formula is Π^0_k if it is of the form

$$\forall n_1 \exists n_2 \forall n_3 \ldots Q n_k R(\overline{x}, \overline{m}, n_1, \ldots, n_k).$$

A formula is **arithmetic** if it is Σ^0_k or Π^0_k for some k. We say a set or relation is Σ^0_k (resp. Π^0_k) if it is defined by a Σ^0_k (resp. Π^0_k, arithmetic) formula. The Σ^1_n formulas are defined analogously, but allowing quantification over real numbers.

The following are standard closure properties of arithmetical formulas:

Exercise 1.1.

1. *If φ and ψ are Σ^0_k formulas, then $\varphi \vee \psi$ and $\varphi \wedge \psi$ are equivalent to Σ^0_k formulas, and $\neg\varphi$ is equivalent to a Π^0_k formula.*

2. *If φ is a Σ^0_k formula which includes a free variable m, then $(\exists m)\varphi$ and $(\forall m < n)\varphi$ are equivalent to Σ^0_k formulas.*

A formula is Σ^1_k if it is of the form

$$\exists x_1 \forall x_2 \exists x_3 \ldots Q x_k A(x_1, \ldots, x_k, \overline{y}, \overline{n})$$

where the quantifiers are over elements of ω^ω, alternate between $\exists$ and $\forall$, and A is an arithmetical relation. A formula is Π^1_k if it is of the form

$$\forall x_1 \exists x_2 \forall x_3 \ldots Q x_k A(x_1, \ldots, x_k, \overline{y}, \overline{n})$$

where A is an arithmetical relation. We say a set or relation is Σ^1_k (resp. Π^1_k) if it is defined by a Σ^1_k (resp. Π^1_k) formula.

We may also relativize all the above definitions to some real $z \in \omega^\omega$ by relativizing the definition of computable relation to relation computable from z.

We have the following obvious closure properties for Σ^1_n formulas.

Exercise 1.2.

1. *If φ and ψ are Σ^1_k formulas, then $\varphi \vee \psi$ and $\varphi \wedge \psi$ are equivalent to Σ^1_k formulas, and $\neg\varphi$ is equivalent to a Π^1_k formula.*

2. *If φ is a Σ^1_k formula which includes a free variable n, then $\forall n \varphi$ and $\exists n \varphi$ are equivalent to Σ^1_k formulas.*

1.2 Boldface vs lightface sets and relativization

The definitions in Section 1.1 relativize to a real parameter. For example, a formula is Σ^1_n **relative to** $x \in \omega^\omega$ or $\Sigma^{1,x}_n$ if it is of the form $\exists x_1 \forall x_2 \dots Q_k A(\overline{x}, \overline{y}, \overline{n})$, where A is a relation that is arithmetic relative to x. We will use a superscript x to denote definitions relativized to x.

We use boldface fonts $\boldsymbol{\Sigma}^1_n/\boldsymbol{\Pi}^1_n$ and $\boldsymbol{\Sigma}^0_\alpha/\boldsymbol{\Pi}^0_\alpha$ to denote formulas/sets that are $\Sigma^{1,x}_n/\Pi^{1,x}_n$ and $\Sigma^{0,x}_\alpha/\Pi^{0,x}_\alpha$ relative to some real parameter x.

These boldface definitions agree with the usual definitions in classical descriptive set theory. For example, $\boldsymbol{\Sigma}^0_1$ sets are the open sets, $\boldsymbol{\Pi}^0_\alpha$ sets are complements of $\boldsymbol{\Sigma}^0_\alpha$ sets, and a set A is $\boldsymbol{\Sigma}^0_\beta$ if $A = \bigcup_n A_n$ where each A_n is $\boldsymbol{\Pi}^0_\alpha$ for some $\alpha < \beta$.

All of our lightface proofs relativize to yield boldface versions. For example, we prove in Theorem 1.27 that a set is Δ^1_1 iff it effectively Borel. The relativized result here is Suslin's theorem that a set is $\boldsymbol{\Delta}^1_1$ iff it is Borel.

Many results in classical descriptive set theory have effective analogues since their proofs only use computably describable constructions. However, the lightface version of the result often gives more information and additional tools. For example, Harrison's effective perfect set theory tells us that every $\boldsymbol{\Sigma}^{1,x}_1$ set either is countable, or has a perfect subset. But furthermore, if it is countable, every element is $\leq_{\mathsf{HYP}} x$. It is this extra power and information we are interested in when studying effective descriptive set theory.

1.3 Normal forms for Σ^1_1 formulas

We begin with the following normal form theorem for Σ^1_1 formulas.

Exercise 1.3. *Every Σ^1_1 formula with free variables $\overline{y}$ and $\overline{m}$ is equivalent to a formula of the form*

$$\exists x \forall n R(x, \overline{y}, n, \overline{m})$$

where R is computable. (In particular the arithmetical relation R above can always be taken to be Π^0_1.)

One possible solution to this exercise goes as follows. If we think of an arithmetic formula as a game, with two players (one corresponding to $\forall$ quantifiers and the other to $\exists$ quantifiers), then an arithmetical formula $\exists x A(x, \overline{y}, \overline{n})$ is equivalent to the formula "there exists x and there exists a strategy for winning the game associated to the formula R". This formula has the required form by Exercise 1.2.(2).

One consequence of the normal form theorem is the existence of universal Σ^1_1 subsets of ω and ω^ω. These follow from the existence of a universal Turing machine. From the existence of universal sets it follows that there are Σ^1_1 sets that are not Π^1_1.

Exercise 1.4.

1. *There is a universal Σ^1_1 set $U \subseteq \omega \times \omega$ so $A \subseteq \omega$ is Σ^1_1 iff there is an m so that $n \in A \leftrightarrow (n, m) \in U$. Hence, there is a Σ^1_1 set that is not Π^1_1. [Hint:*

$A = \{n\colon (n,n) \in U\}$ is Σ^1_1. It is not Π^1_1 since if it were there would be some m so that $n \in A \leftrightarrow (n,m) \notin U$. But then $m \in A \leftrightarrow (m,m) \notin U \leftrightarrow m \notin A$ contradiction.]

2. *There is a universal Σ^1_1 set $U \subseteq \omega^\omega \times \omega$ so that $A \subseteq \omega^\omega$ is Σ^1_1 iff there is an m so that $x \in A \leftrightarrow (x,m) \in U$. Conclude there is a Σ^1_1 subset of ω^ω that is not Π^1_1.*

3. *Finally, there is a universal Σ^1_1 set $U \subseteq \omega^\omega \times \omega^\omega$ so that $A \subseteq \omega^\omega$ is Σ^1_1 relative to some real parameter iff there is a y so that $x \in A \leftrightarrow (x,y) \in U$. Conclude there is a Σ^1_1 subset of ω^ω that is not Π^1_1 relative to any real parameter.*

From the normal form theorem we also get the following way of associating trees with points to determine membership in Σ^1_1 sets. Recall a tree $T \subseteq \omega^{<\omega}$ is a nonempty set that is closed downwards, so $t \in T \to (\forall s \subseteq t) s \in T$. A tree is **illfounded** if it has an infinite branch. That is, there is an $x \in \omega^\omega$ such that $(\forall n) x \restriction n \in T$.

Lemma 1.5. *A set $A \subseteq \omega^\omega$ is Σ^1_1 if and only if there is a computable map $y \mapsto T_y$ so that $y \in A$ iff T_y is illfounded.*

Proof. The direction $\Leftarrow$ is clear. $\Rightarrow$ follows from the normal form theorem Exercise 1.3. If $A \subseteq \omega^\omega$ is Σ^1_1 then it has a definition of the form

$$y \in A \leftrightarrow \exists x \forall n R(x,y,n)$$

where R is a computable relation. Let T_y be the tree of t such that for all $s \subseteq t$ and $n < |t|$, $R(s,y,n)$ has not halted rejecting its input in $\leq |t|$ steps. (Where $R(s,y,n)$ is undefined if the computation asks for a bit of s not in its domain). □

Exercise 1.6. *A set $A \subseteq \omega$ is Σ^1_1 if and only if there is a computable map $n \mapsto T_n$ so that $n \in A$ iff T_n is illfounded.*

Hence, by Lemma 1.5 and Exercise 1.6, the set of illfounded trees, and illfounded computable trees are Σ^1_1 complete subsets of ω^ω and ω respectively.

Exercise 1.7.

1. *Show that if A is Σ^1_1, then $A \leq_m \{n\colon$ the nth program φ_n computes an illfounded subtree of $\omega^{<\omega}$.*

2. *By identifying $\omega^{<\omega}$ with ω, we can regard the set of trees as a closed subset of 2^ω. Show that the set I of illfounded trees is Σ^1_1 complete in the sense that if $A \subseteq \omega^\omega$ is Σ^1_1, then there is a computable continuous function $f\colon \omega^\omega \to \omega^\omega$ so that $x \in A \leftrightarrow f(x) \in I$.*

Exercise 1.8. *Show that $A \subseteq \omega^\omega$ is Σ^1_1 iff there is a computable tree T on $\omega^{<\omega} \times \omega^{<\omega}$ (so $[T] \subseteq \omega^\omega \times \omega^\omega$) such that A is the projection of $[T]$. That is, $A = \pi[T] = \{x\colon \exists y (x,y) \in [T]\}$.*

Our next goal is to characterize Δ^1_1 sets in a number of ways. We will begin by taking the form of a Σ^1_1 set given by Lemma 1.5 and bounding the ranks of the trees it gives in definitions of Δ^1_1 sets. Doing this will first require some basic lemmas on ranking trees.

1.4 Ranking trees and Spector boundedness

Then we can analyze the wellfoundedness of the tree by ranking its elements as follows.

Definition 1.9. *Suppose $T \subseteq \omega^{<\omega}$ is a tree. We define an ordinal-indexed decreasing sequence of subtrees of T as follows:*

- *Let $T_0 = T$*
- *For all α, $T_{\alpha+1} = T_\alpha \setminus \{s \colon \neg(\exists t \supsetneq s) t \in T_\alpha\}$, where we remove all leaves of T_α.*
- *If λ is a limit, then $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$.*

Then define the rank function $\mathrm{rank}_T \colon T \to \mathsf{ORD} \cup \{\infty\}$ *for elements of T as follows:*

$$\mathrm{rank}_T(t) = \begin{cases} \alpha & \text{if } \alpha \text{ is least such that } t \notin T_{\alpha+1}. \\ \infty & \text{if } t \in T_\alpha \text{ for all } \alpha. \end{cases}$$

Finally, define $\mathrm{rank}(T) = \mathrm{rank}_T(\emptyset)$.

Note that since there are only countably many elements of the tree T to remove, the sequence of T_α must stabilize at some countable ordinal (which will be $\mathrm{rank}(T)$).

Exercise 1.10. *Prove* $\mathrm{rank}_T(s) = \sup_{t \supsetneq s} \mathrm{rank}_T(t) + 1$.

We will often use the following definition of the tree T above a node s. If T is a tree, and $s \in T$, then $T_s = \{t \in \omega^{<\omega} \colon s^\frown t \in T\}$.

Exercise 1.11. $\mathrm{rank}(T) = \sup\limits_{s \in T, |s|=1} \mathrm{rank}(T_s) + 1$.

Exercise 1.12. *Show that for trees S, T we have* $\mathrm{rank}(S) \leq \mathrm{rank}(T)$ *iff for every s with $|s| = 1$, there is a t with $|t| = 1$ such that* $\mathrm{rank}(S_s) \leq \mathrm{rank}(T_t)$.

Exercise 1.13. *If T is wellfounded, then for every $\beta <$* $\mathrm{rank}(T)$ *there exists some s so that $\beta =$* $\mathrm{rank}(T_s)$.

Definition 1.14. *If T is a tree, let Letting $T^+ = \{\emptyset\} \cup \{(0)^\frown s \colon s \in T\}$.*

Exercise 1.15. *If T is illfounded T^+ is illfounded. If T is wellfounded,* $\mathrm{rank}(T^+) = \mathrm{rank}(T) + 1$.

Ranking trees provides a way of understanding whether the tree is wellfounded.

Lemma 1.16. *T is illfounded iff* $\mathrm{rank}(T) = \infty$*.*

Proof. If $\mathrm{rank}(T) = \infty$, then we can find an increasing sequence $s_0 \subseteq s_1 \subseteq \dots$ where $\mathrm{rank}_T(s_n) = \infty$ by recursion. Then $x = \bigcup_n S_n$ is an infinite branch in $[T]$. Conversely, if $\mathrm{rank}(T) < \infty$, then T is wellfounded since there is no infinite descending sequence of ordinals. □

We have the following convenient way of comparing ranks of trees. If $T, T' \subseteq \omega^{<\omega}$ are trees, then a function $f \colon T \to T'$ is **monotone** if $s \subsetneq t \to f(s) \subsetneq f(t)$.

Lemma 1.17. *If T, T' are trees,* $\mathrm{rank}(T) \leq \mathrm{rank}(T')$ *iff there is a monotone function from T to T'.*

Proof. The lemma is clear if T' is illfounded; take an infinite branch x of T' and let $f(t) = x \upharpoonright |t|$.

We prove the remaining case by transfinite induction on $\mathrm{rank}(T')$. To construct a monotone function $f \colon T \to T'$ note that for each sequence $\langle n \rangle \in T$ of length 1, there is some $\langle m(n) \rangle \in T'$ such that $\mathrm{rank}(T_{\langle n \rangle}) \leq \mathrm{rank}(T'_{\langle m(n) \rangle})$. Hence, by our induction hypothesis, there is a monotone function f_n from each such $T_{\langle n \rangle}$ to $T'_{\langle m(n) \rangle}$. To finish the theorem, let $f(\emptyset) = \emptyset$, and then $f(\langle n \rangle^\frown s) = \langle m(n) \rangle^\frown f_n(s)$. □

Remark 1.18. *In the proof of the above lemma, our monotone function has the property that for all t, $|f(t)| = |t|$.*

Definition 1.19. *A countable ordinal α is **computable** if it is the rank of a computable tree.*

Lemma 1.20 (Spector's Boundedness Lemma)**.** *If $y \mapsto T_y$ is a uniformly computable function assigning a* wellfounded *tree to each $y \in \omega^\omega$, then there is a computable ordinal α such that for all y,* $\mathrm{rank}(T_y) \leq \alpha$*.*

Proof. We will construct a computable wellfounded tree T such that $\mathrm{rank}(T_y) \leq \mathrm{rank}(T)$ for all y. Let e be the program computing T_y. Then let T be the set of (s, t) with $|s| = |t|$ so that φ_e^s does not halt in $\leq |s|$ steps rejecting any initial segment of t. Then the following is a monotone function from T_y to T: $t \mapsto \langle y \upharpoonright |t|, t \rangle$, so $\mathrm{rank}(T_y) \leq \mathrm{rank}(T)$ for all y. T is wellfounded since any infinite branch (y, z) in T would have z be an infinite branch in T_y. □

Exercise 1.21. *Suppose A is a Σ^1_1 set of wellfounded trees. Then there is a computable ordinal α such that for all y,* $\mathrm{rank}(T_y) \leq \alpha$*.*

Our next goal is a normal form for Δ^1_1 sets. To get this normal form, we'll first use the following way of combining trees:

If $T, T' \subseteq \omega^{<\omega}$ are trees, then let

$$T \times T' = \{(t, t') \colon |t| = |t'| \land t \in T \land t' \in T'\}$$

(we will often work with trees on $\{(t, t') \in \omega^{<\omega} \times \omega^{<\omega} \colon |t| = |t'|\}$ which is computably isomorphic to $\omega^{<\omega}$).

Lemma 1.22. $\mathrm{rank}(T \times T') = \min(\mathrm{rank}(T), \mathrm{rank}(T'))$.

Proof. The projection function $(t, t') \mapsto t$ is clearly a monotone function from $T \times T'$ to T. Similarly, the other projection is a monotone function to T', so the direction $\mathrm{rank}(T \times T') \leq \min(\mathrm{rank}(T), \mathrm{rank}(T'))$ is clear. WLOG assume $\mathrm{rank}(T) \leq \mathrm{rank}(T')$. Then there is a monotone function $g \colon T \to T'$ with the property that $|g(t)| = |t|$ by Remark 1.18. Then $t \mapsto \langle t, g(t) \rangle$ is a monotone function from T to $T \times T'$. □

1.5 Δ^1_1 = effectively Borel

A set $A \subseteq \omega^\omega$ is Δ^1_1 if both A and $\omega^\omega \setminus A$ are Σ^1_1. We can now prove the normal form for Δ^1_1 sets.

Theorem 1.23. *$A \subseteq \omega^\omega$ is Δ^1_1 iff there is a computable ordinal α and a computable map $y \mapsto T_y$ from ω^ω to trees so that $y \in A$ iff $\mathrm{rank}(T_y) > \alpha$ iff there is a computable ordinal α and a computable map $y \mapsto S_y$ so $y \in A$ iff $\mathrm{rank}(T_y) \leq \alpha$.*

Proof. First assume A is Δ^1_1. Then since A and its complement are Σ^1_1, there are computable maps $y \mapsto T_y$ and $y \mapsto S_y$ to trees such that $y \in A$ iff T_y is illfounded iff S_y is wellfounded. Thus, for each y, exactly one of T_y and S_y is wellfounded, and so $T_y \times S_y$ is wellfounded for all y. Hence, by Spector's boundedness lemma, there is a computable ordinal α so that $\mathrm{rank}(T_y \times S_y) = \min(\mathrm{rank}(T_y), \mathrm{rank}(S_y)) \leq \alpha$. Thus, T_y (resp. S_y) is illfounded iff $\mathrm{rank}(T_y) > \alpha$ (resp. $\mathrm{rank}(S_y) > \alpha$). So $y \in A$ iff T_y is wellfounded iff $\mathrm{rank}(T_y) > \alpha$ iff S_y is wellfounded iff $\mathrm{rank}(S_y) \leq \alpha$.

Suppose now we have a computable ordinal α and computable map $y \mapsto T_y$ such that $y \in A$ iff $\mathrm{rank}(T_y) > \alpha$. Let T be a computable tree with $\mathrm{rank}(T) = \alpha$. Letting $T^+ = \{\emptyset\} \cup \{(0)^\frown s \colon s \in T\}$, we see that $\mathrm{rank}(T^+) = \mathrm{rank}(T) + 1$. Then $y \in A$ iff there is no monotone function from T_y to T iff there is a monotone function from T^+ to T_y. Hence A is Π^1_1 and Σ^1_1. □

Our next goal is proving the Suslin-Kleene theorem that the Δ^1_1 sets are exactly the effectively Borel sets.

Definition 1.24. *An* ***effective Borel code*** *is a pair (T, l) where T is a computable wellfounded tree and l is a computable function*

$$l \colon T \to \omega^{<\omega} \times \{\text{“}\cup\text{”}, \text{“}\cap\text{”}, \text{“}\neg\text{”}\}$$

such that $l(t) \in \omega^{<\omega}$ iff t is a leaf in T, and if $l(t) = \text{“}\neg\text{”}$, then t has exactly one successor in T.

Now if (T, l) is an effective Borel code, then the set of leaves of T is computable. We note that this does not restrict possible rank of T among computable trees.

Exercise 1.25. *Show that if T is a computable tree, then there is a computable tree of the same rank where the set of leaves of T is computable. [Hint: given*

$s \in \omega^{<\omega}$, let s^+ be a sequence of the same length where $s^+(n) = s(n)+1$. Then consider $T' = \{s^+ \colon s \in T\} \cup \{s^{+\frown}(0) \colon s \in T\}$. Show T' is computable, has a computable set of leaves, and $\operatorname{rank}(T') \geq \operatorname{rank}(T)$. Finish by showing there is some $s \in T'$ such that T'_s is as required.]

We define the interpretation of a Borel code inductively.

Definition 1.26. *If (T, l) is a Borel code, then its interpretation is the Borel set $B_{(T,l)}$ defined as follows.*

- *if $\operatorname{rank}(T) = 0$, then $B_{(T,l)} = N_s$ where $s = l(\emptyset)$. Otherwise,*
- *if $l(\emptyset) =$ "$\cup$", $B_{(T,l)} = \bigcup_{s \in T \wedge |s|=1} B_{(T_s, l_s)}$.*
- *if $l(\emptyset) =$ "$\cap$", $B_{(T,l)} = \bigcap_{s \in T \wedge |s|=1} B_{(T_s, l_s)}$.*
- *if $l(\emptyset) =$ "$\neg$", $B_{(T,l)} = \omega^\omega \setminus B_{(T_s, l_s)}$ where s is the unique successor of $\emptyset$ (so $|s| = 1$).*

*where l_s is the function on T_s where $l_s(t) = l(s^\frown t)$. A Borel set is **effectively Borel** if it is the interpretation of an effective Borel code.*

Now we prove Δ^1_1 = effectively Borel.

Theorem 1.27 (Suslin-Kleene)**.** *$A \subseteq \omega^\omega$ is Δ^1_1 iff it is effectively Borel.*

Proof. To begin, suppose B is effectively Borel with Borel code (T, l). Then B has the following Σ^1_1 definition. $y \in B$ iff there exists $f \colon T \to \{0, 1\}$ with $f(\emptyset) = 1$ such that for all $t \in T$,

- if t is a leaf of T, $f(t) = 1$ iff $l(t) \subseteq y$, and
- if $l(t) =$ "$\cup$", $f(t) = 1$ iff there exists $s \in T$ where $s \supsetneq t$ and $|s| = |t| + 1$, and $f(s) = 1$.
- if $l(t) =$ "$\cap$", $f(t) = 1$ iff for all $s \in T$ where $s \supsetneq t$ and $|s| = |t| + 1$, $f(s) = 1$.
- if $l(t) =$ "$\neg$", $f(t) = 1$ iff $f(s) = 0$ where $s \in T$ is the unique successor of t with $|s| = |t| + 1$.

(The above bulleted conditions ensure that f "Skolemizes" whether y is in each subtree of the Borel code, where 1 represents yes, and 0 represents no. So the condition $f(\emptyset) = 1$ corresponds to y actually being in the full set $B_{(T,l)}$). An almost identical definition (except saying there does not exists such an $f \colon T \to \{0, 1\}$ with $f(\emptyset) = 0$) gives a Π^1_1 definition of A.

Next, we show that every Δ^1_1 set is effectively Borel. Suppose $A \subseteq \omega^\omega$ is Δ^1_1 and hence by Theorem 1.23 there is a computable map $y \mapsto T_y$ and a computable ordinal α such that $y \in A \leftrightarrow \operatorname{rank}(T_y) \leq \alpha$. By Exercise 1.25 we can find a computable tree T with a computable set of leaves such that $\operatorname{rank}(T) = \alpha$.

Now we uniformly recursively define an effective Borel set $B(s,t)$ where $t \in T$ and $|s| = |t|$ such that $B(s,t) = \{y \colon \operatorname{rank}((T_y)_s) \leq \operatorname{rank}(T_t)\}$. First, if t is a leaf of T, then

$$\begin{aligned} B(s,t) &= \{y \in A \colon \operatorname{rank}((T_y)_s) = 0\} \\ &= \{y \in A \colon \neg(\exists s' \supsetneq s) s' \in T_y\} \\ &= \omega^\omega \setminus \{y \in A \colon (\exists s' \supsetneq s) s' \in T_y\} \end{aligned}$$

Now $\{y \colon (\exists s' \supsetneq s) s' \in T_y\}$ is clearly a computable union of basic open sets, since if the program defining T_y halts accepting the string s', then this computation only uses a finite initial segment r of y. So the set of such y is the union of all basic open neighborhoods N_r, which is effectively Borel. If t is not a leaf of T, then by Exercise 1.12

$$\begin{aligned} B(s,t) &= \{y \colon \operatorname{rank}((T_y)_s) \leq \operatorname{rank}(T_t)\} \\ &= \{y \colon (\forall s' \supsetneq s)(\exists t' \supsetneq t) \operatorname{rank}((T_y)_{s'}) \leq \operatorname{rank}(T_{t'})\} \\ &= \bigcap_{s' \supsetneq s} \bigcup_{t' \supsetneq t} B(s', t') \end{aligned}$$

Which gives an effective Borel code, since the $B(s',t')$ for (s',t') extending (s,t) are effective Borel codes.

To finish, note that $B(\emptyset, \emptyset) = A$ is effectively Borel. □

1.6 Computable ordinals, hyperarithmetic sets

In order to develop the hyperarithmetic hierarchy, we need to introduce a different way of representing computable ordinals rather than just as ranks of computable trees.

Definition 1.28. *A* ***computable wellorder*** *is a computable linear ordering $(\leq_L, L)$ where L is a computable subset of ω and $\leq_L$ is a computable linear ordering on L which is a wellorder.*

It is an important exercise that the ranks of computable trees are precisely the same ordinals as the ordertypes of computable wellorderings. To see this, we first have the following connection between linear orderings and wellfounded trees:

Definition 1.29. *The Kleene-Brouwer order on $\omega^{<\omega}$ is the ordering where $s \leq_{KB} t$ iff s and t are compatible and $s \supseteq t$, or s and t are incompatible and s is lex-less than t. (Recall s is lex-less than t if n is least such that $s(n) \neq t(n)$ implies $s(n) < t(n)$).*

Exercise 1.30. *Show that T is wellfounded iff the Kleene-Brouwer restricted to T, $\leq_{KB} \restriction T$ is a wellorder.*

Now we have the exact correspondence:

Exercise 1.31. *Show that α is the rank of a computable tree iff α is the ordertype of a computable linear order. [Hint: show that if T is a computable tree, then $\operatorname{rank}(T) \leq \operatorname{ot}(\leq_{KB}\restriction T)$, the order type of $\leq_{KB}\restriction T$, which is a computable linear order. Hence, by restricting this order to a computable subset, we can find a computable wellorder of exactly the same ordertype as $\operatorname{rank}(T)$. Similarly, show if $\leq_L$ is a computable wellorder, the tree of $\leq_L$-descending sequences is a computable tree of rank at least $\operatorname{ot}(\leq_L)$.]*

In order to better represent ordinals, we will in addition demand that certain data on a computable wellorder is computable. Note: in most texts on effective descriptive set theory, a (computable isomorphic) notion of "ordinal notations" is used. The set of ordinal notations is denoted $\mathcal{O}$.

Definition 1.32. *A **labeled computable wellorder** or **computable ordinal code** is a tuple $a = ((\leq_L, L), m, l, s, t)$ where m is the $\leq_L$-minimal element of L, l is a computable subset of L giving the set of elements of L that are limits, $s\colon L \to L$ is the successor function; $s(n)$ is the $\leq_L$-successor of n in L except if n is the maximal element of n in which case $s(n) = n$. Finally $t \in \{$"zero", "successor", "limit"$\}$ is the **type** of the wellorder. By abuse of notation, we write $n \in a$ to mean n is an element of the set on which the computable wellorder of a is defined. We write $|a|$ for the ordinal giving its ordertype.*

Exercise 1.33. *If a is a computable ordinal code, the predecessor function (which is defined on the computable set of elements which are not limits), is computable.*

The restriction of a to its initial segments is a uniformly computable operation:

Definition 1.34. *Given a computable ordinal code $a = ((\leq_L, L), m, l, s, t)$, write $a_{<n}$ for the computable ordinal code for the order $\leq_L\restriction \{m\colon m <_L n\}$. $a_{<n}$ is uniformly computable from a and n. Finally if a is a successor ordinal, then write a^- for $a_{<n}$ where n is the greatest element of x. We call a^- the predecessor of a.*

Exercise 1.35. *If a is a computable ordinal code for a limit ordinal, then $\{a_{<n}\colon n \in a\}$ are unbounded in a, and $|a| = \sup(|a_{<n}|)$. Hence show there is a computable function taking limits to an increasing subsequence that limit to them.*

Similarly to how Exercise 1.25 shows that we can always find a computable tree of a given computable ordinal rank whose set of leaves is computable, for every computable wellorder, we can find a computable ordinal code having the same ordertype.

Exercise 1.36. *Show that if $\leq_L$ is a computable wellordering, there is a computable ordinal code of the same ordertype. [Hint: begin by replacing every element of L with a copy of ω to get a computable ordinal code a where $|a|$ is greater than or equal to the ordertype of $\leq_L$.]*

Recall that if $x \in \omega^\omega$, we use x' to denote the Turing jump of x. Now we define how to iterate the Turing jump along a computable ordinal. If a is a computable ordinal code, then define

$$x^{(a)} = \begin{cases} x & \text{if } a \text{ represents } 0 \\ (x^{(a^-)})' & \text{if } a \text{ is a successor} \\ \{\langle n, m\rangle \colon n \in a \wedge m \in x^{(a_{<n})}\} & \text{if } a \text{ is a limit} \end{cases}$$

Definition 1.37. *$x \in \omega^\omega$ is hyperarithmetic if $x \leq_T \emptyset^{(a)}$ for some computable ordinal code a.*

In dealing with hyperarithmetic sets, we'll often use the recursion theorem to define programs which compute from them.

Lemma 1.38. *If a is a computable ordinal code, $\left(\emptyset^{(a)}\right)'' \geq_T \{e \colon \varphi_e$ computes a wellfounded tree T with $\mathrm{rank}(T) \leq |a|\}$.*

Proof. By the recursion theorem, we define a program $e(a)$ which takes an ordinal code a as a parameter and computes the given set from $\left(\emptyset^{(a)}\right)''$. We define $e(a)$ as follows:

- If a represents 0, $\{e \colon \varphi_e$ computes a tree of rank $0\}$ is a Π^0_2 set. Let $e(a)$ be a program computing this set from $\emptyset''$.

- If a represents a successor ordinal

  $$\begin{aligned} &\{e \colon \varphi_e \text{ computes a wellfounded tree } T \text{ with } \mathrm{rank}(T) \leq |a|\} \\ =&\{e \colon \varphi_e \text{ computes a wellfounded tree } T \text{ and } (\forall s, |s| = 1)\, \mathrm{rank}(T_s) \leq |a^-|\} \end{aligned}$$

  since the trees of rank $\leq |a^-|$ are computable from $\left(\emptyset^{(a^-)}\right)''$ via the program $e(a^-)$, $\forall a, |a| = 1\, \mathrm{rank}(T_s) \leq |a^-|$ is a Π^0_1 fact relative to $\left(\emptyset^{(a^-)}\right)''$. Let $e(a)$ be the program computing this $\mathbf{\Pi}^0_1$ fact from another Turing jump $\left(\emptyset^{(a^-)}\right)''' = \left(\emptyset^{(a)}\right)''$.

- If a represents a limit ordinal

  $$\begin{aligned} &\{e \colon \varphi_e \text{ computes a wellfounded tree } T \text{ with } \mathrm{rank}(T) \leq |a|\} \\ =&\{e \colon \varphi_e \text{ computes a wellfounded tree } T \text{ and } (\forall s, |s| = 1)(\exists n \in a)\, \mathrm{rank}(T_s) \leq |a_{<n}|\} \end{aligned}$$

  The set of trees T such that $\mathrm{rank}(T_s) \leq |a_{<n}|$ is uniformly computable from $\left(\emptyset^{(a_{<n})}\right)''$ via the program $e(a_{<n})$. $\left(\emptyset^{(a_{<n})}\right)''$ is uniformly computable from $\emptyset^{(a)}$, since $\left(\emptyset^{(a_{<n})}\right)'' = \left(\emptyset^{(a_{<m})}\right)$ where m is the double successor of n in a, which has a computable successor function. So the trees T such that $(\forall s, |s| = 1)(\exists n \in a)\, \mathrm{rank}(T_s) \leq |a_{<n}|$ are Π^0_2 relative to $\emptyset^{(a)}$. Let $e(a)$ be the program computing this set from $\left(\emptyset^{(a)}\right)''$

□

Similar proofs using a program defined via the recursion theorem where zero, successor, and limit cases are defined recursively in terms of the program at previous steps can be used to show that:

Exercise 1.39. *If a is a computable ordinal code, $\left(\emptyset^{(a)}\right)'' \geq_T \{b\colon b$ is a computable ordinal code with $|a| = |b|\}$.*

Exercise 1.40. *If $x \geq_T y$ and a is a computable ordinal code, $x^{(a)} \geq_T y^{(a)}$.*

Finally, we have the following theorem which shows that though there are many ordinal codes for a given computable ordinal, $x^{(a)}$ is well-defined up to Turing degree.

Theorem 1.41. *If a and b are computable ordinal codes with $|a| = |b|$, then $x^{(a)} \equiv_T x^{(b)}$.*

Proof. By the recursion theorem, we define a program $e(a, b)$ which takes a and b are parameters and witnesses $x^{(a)} \geq_T x^{(b)}$ for all x.

- If a, b represent 0, $e(a, b)$ is the identity.
- If a, b are successors, $x^{(a^-)} \geq_T x^{(b^-)}$ via $e(a^-, b^-)$. By a fact of computability theory, uniformly in $e(a^-, b^-)$ there is a program $e(a, b)$ witnessing. $\left(x^{(a^-)}\right)' \geq_T \left(x^{(b^-)}\right)'$.
- If a, b are limits, by Exercise 1.40, $x^{(a)} \geq_T \emptyset^{(a)}$. By Exercise 1.39, we can hence compute (uniformly in a and b) the set $\{(n, m)\colon |a_{<n}| = |b_{<m}|\}$ from $x^{(a)}$. Then to compute $x^{(b)}$ from $x^{(a)}$, for each $m \in b$ we compute the corresponding $n \in a$ such that $|a_{<n}| = |b_{<m}|$ and then use $e(a_{<n}, b_{<m})$ to compute $x^{(b_{<m})}$ from $x^{(a_{<n})}$.

□

Using this theorem, we will abuse notation and write $x^{(\alpha)}$ for the Turing degree of $x^{(a)}$ where $|a| = \alpha$.

1.7 Δ^1_1 = hyperarithmetic

Theorem 1.42 (Kleene)**.** *$A \subseteq \omega$ is Δ^1_1 iff it is hyperarithmetic.*

Proof. If $A \subseteq \omega$ is Δ^1_1, then there is a computable ordinal α and a computable map $n \mapsto T_n$ such that $n \in A \leftrightarrow \operatorname{rank}(T_n) \leq \alpha$. Let a be a computable ordinal code representing a. Then by Lemma 1.38, $\left(\emptyset^{(a)}\right)''$ computes the set of computable trees of rank $\leq |a|$, and hence computes A.

In the other direction, if a is a computable ordinal code and $A \leq_T \emptyset^{(a)}$, then A is Σ^1_1; $n \in A$ iff there exists a real representing $\emptyset^{(a)}$, a Skolem function witnessing that it is truly $\emptyset^{(a)}$ by checking the conditions in the definition of $\emptyset^{(a)}$, and a computation from this real which gives $n \in A$. Since the complement of A is thus Σ^1_1, A is also Π^1_1. □

1.8 The hyperjump, ω_1^x, and the analogy between c.e. and Π^1_1

There is a deep analogy between computable sets and hyperarithmetic sets. This analogy extends to one between Σ^0_1 and Π^1_1 sets. If we think as a Σ^0_1 subset of ω as an c.e. set which is enumerated via a computable procedure lasting ω many steps, we can similarly think of a Π^1_1 set $A \subseteq \omega$ as being "enumerated" via a transfinite procedure of length ω_1^{ck} (defined below) where $n \in A$ is enumerated at stage α once we see some corresponding computable tree T_n has rank α.

computability	hypercomputability
computable/Δ^0_1	hyperarithmetic/Δ^1_1
Σ^0_1	Π^1_1

This isn't just an analogy; we will discover in Section 5 that there is a precise connection. Classical computability and hypercomputability are examples of what is called admissible computability. In this setting, we have so called "admissible structure" (which is in particular a transitive set satisfying a weak set of axioms for set theory called KP). In this setting, "computable" becomes Δ_1 definability over this structure, and c.e. becomes Σ_1 definability over this structure. Computable sets and hyperarithmetic sets are the smallest two such notations of computability over the smallest two admissible structure: H_ω, the hereditary finite sets, and $L_{\omega_1^{\mathrm{ck}}}$.

We give an example of a theorem whose proof is guided by this analogy.

Theorem 1.43. *If $A, B \subseteq \omega$ are disjoint Σ^1_1 sets, then there is a Δ^1_1 set C separating them: $A \subseteq C$, and $C \cap B = \emptyset$.*

The analogous fact is classical computability is that if A, B are co-c.e. there is a computable set C separating them. We quickly sketch a proof of this classical fact. Run enumerations of $\omega \setminus A$ and $\omega \setminus B$ simultaneously. Note that since A, B are disjoint, every n must be enumerated into at least one of $\omega \setminus A$ and $\omega \setminus B$. The computable separating set is the set C of n that are enumerated into $\omega \setminus A$ before they are enumerated into $\omega \setminus B$.

Proof of Theorem 1.43. Fix computable maps $n \mapsto T_n$ and $n \mapsto S_n$ so that $n \notin A$ iff T_n is wellfounded and $n \notin B$ iff T_n is wellfounded. In our analogy, if we think of this as "enumerating" $\omega \setminus A$ and $\omega \setminus B$, then n is enumerated into $\omega \setminus A$ before it is enumerated into $\omega \setminus B$ if $\operatorname{rank}(T_n) \leq \operatorname{rank}(S_n)$. So let $C = \{n \colon \operatorname{rank}(T_n) \leq \operatorname{rank}(S_n)\}$. Then C is clearly a Σ^1_1 set ($\operatorname{rank}(T_n) \leq \operatorname{rank}(S_n)$ iff there is a monotone function from T_n to S_n. It is also a Π^1_1 set since $C = \{n \colon \neg \operatorname{rank}(S_n^+) \leq \operatorname{rank}(T_n)\}$. (Where S^+ is defined in Section 1.5). $\square$

Exercise 1.44. *If $A, B \subseteq \omega^\omega$ are disjoint Σ^1_1 sets, show there is a Δ^1_1 set C separating them. [Hint: let $y \mapsto T_y$ and $y \mapsto S_y$ be computable maps so $y \in A$ iff T_y is illfounded, and $y \in B$ iff S_y is illfounded. Let $C = \{y \colon \operatorname{rank}(T_y) \leq \operatorname{rank}(S_y)\}$.*

Next, we'll pursue this connection between computable and hyperarithmetic a little more, defining notions analogous to classical notions. We begin with the analogue of the Turing jump:

Definition 1.45 (The hyperjump)**.** *Let $\mathcal{O} = \{n\colon$ the nth program φ_n computes a wellfounded subtree of $\omega^{<\omega}\}$. By Exercise 1.7, this is a Π^1_1 complete subset of ω. The relativized version of this set is $\mathcal{O}^x = \{n\colon$ the nth program φ^x_n relative to x computes a wellfounded subtree of $\omega^{<\omega}\}$. This is a complete set among those sets that are Π^1_1 relative to x.*

Next, we have the analogue of Turing reducibility.

Definition 1.46. *If $x, y \in \omega^\omega$, then write $x \leq_{\mathsf{HYP}} y$ and say x is hyperarithmetically reducible to y if there is a Δ^1_1 definition of x relative to y. Equivalently, by the relativized version of Theorem 1.42, $x \leq_{HYP} y$ iff there is a computable-relative-to-y ordinal code a so that $x \leq_T y^{(a)}$. The set $\{y\colon y \equiv_{\mathsf{HYP}} x\}$ is called the hyperdegree of x.*

computability	hypercomputability
$\leq_T$	$\leq_{\mathsf{HYP}}$
Turing degree	hyperdegree
Turing jump: x'	hyperjump: $\mathcal{O}^x$

Now we have the following in analogy with facts from classical computability that $x \leq_T x'$ and $x \leq_T y$ implies $x' \leq_T y'$:

Exercise 1.47. *For all $x \in \omega^\omega$, we have $x <_{\mathsf{HYP}} \mathcal{O}^x$.*

Exercise 1.48. *If $x \leq_{\mathsf{HYP}} y$, then $\mathcal{O}^x \leq_{\mathsf{HYP}} \mathcal{O}^y$.*

To each hyperdegree, we can associate the least ordinal which is not computable relative to x.

Definition 1.49. *If $x \in \omega^\omega$, let ω^x_1 be the least ordinal α such that there is no tree computable from x of rank α. $\omega^\emptyset_1$ is called the Church-Kleene ordinal and denoted ω^{ck}_1.*

This ordinal is the same for every y in the hyperdegree of x.

Exercise 1.50. *If $x \geq_{\mathsf{HYP}} y$, then $\omega^x_1 \geq \omega^y_1$.*

After taking the hyperjump of x, this ordinal increases.

Proposition 1.51. *For all $x \in \omega^\omega$, $\omega^{\mathcal{O}^x}_1 > \omega^x_1$.*

Proof. The tree $\{\emptyset\} \cup \{n^\frown s\colon$ the nth program φ^x_n relative to x computes a wellfounded tree T^x_n, and $s \in T^x_n\}$ clearly has rank sup of all $\text{rank}(T) + 1$ where T is a wellfounded tree computable from x. This is equal to ω^x_1. Hence $\omega^{\mathcal{O}^x}_1 \geq \omega^x_1 + 1$. □

By this proposition, if $x \geq_{\mathsf{HYP}} \mathcal{O}$, then $\omega^x_1 > \omega^{\text{ck}}_1$. In fact, the converse of this is true.

Exercise 1.52. *For all $x \in \omega^\omega$, $\omega_1^x > \omega_1^{ck}$ implies $x \geq_{\mathsf{HYP}} \mathcal{O}$. Hence, $x \not\geq_{\mathsf{HYP}} \mathcal{O}$ implies $\omega_1^x = \omega_1^{ck}$. [Hint: let a be a computable-relative-to-x ordinal notation where $|a| = \omega_1^{ck}$. Show that $(x^{(a)})'' \geq_T \{n\colon$ the nth program φ_n computes a wellfounded tree with $\operatorname{rank}(T) \leq |a|\} = \mathcal{O}$. Hence $x \geq_{\mathsf{HYP}} \mathcal{O}$.]*

Definition 1.53. *Say that $x \in \omega^\omega$ is **hyperlow** if $\omega_1^x = \omega_1^{ck}$.*

2 Basic tools

2.1 Existence proofs via completeness results

One way to prove that two sets A, B are not equal is to prove that they have different complexities. For example, if A is Σ^1_1 complete, and B is Π^1_1, then $A \neq B$. We illustrate with an example:

Say that $x_0, x_1, \ldots$ is a **descending jump sequence** if $x_n \geq_T x'_{n+1}$, where x' is the Turing jump of x.

Theorem 2.1. *There exists an infinite descending jump sequence $(x_n)_{n\in\omega}$.*

Proof. Consider the set $A = \{T \subseteq \omega^{<\omega}$ such that there exists a map $f\colon T \to \omega^\omega$ such that if $s \subsetneq t$, then $f(s) \geq f(t)'\}$. This is a Σ^1_1 set of trees. It is easy to prove by transfinite induction that every wellfounded tree is in A. However, the set of wellfounded trees is $\mathbf{\Pi}^1_1$ complete, while A is Σ^1_1. Hence, there is an illfounded tree in A. An infinite branch in such an illfounded tree gives an infinite descending jump sequence. $\square$

It is not so easy to construct an infinite descending jump sequence explicitly. It is easy to see that in an infinite descending jump sequence can have no $x_n \in \mathsf{HYP}$. Further, Steel has shown [St75] that there is no infinite uniformly descending jump sequence, where there is a single program e so that $\Phi_e(x_n) = x'_{n+1}$.

Another nice example of an existence theorem proved by such a complexity result is the Theorem of Wesolek and Williams [WW] that the set of elementary groups is Π^1_1 complete. Hence, there is an elementary amenable group that is not amenable, since the set of amenable groups is arithmetic.

2.2 The effective perfect set theorem

One of the themes of these notes will be the relationship between the definability of a set of reals vs reals in the set. For example, if $\{x\} \subseteq \omega^\omega$ has a simple definition as a subset of ω^ω, does x necessarily have a simple definition as a function from $\omega \to \omega$. Here is a pair of exercises illustrating this type of connection.

Exercise 2.2. *Show that if $\{x\} \subseteq \omega^\omega$ is Π^0_1, then x is hyperarithmetic.*

Exercise 2.3. *Show that the Π^0_1 singletons are unbounded in the hyperarithmetic hierarchy. In particular, for every computable α, there is Π^0_1 set $\{x\} \subseteq \omega^\omega$ such that $x \not\leq_T \emptyset^\alpha$.*

It is a standard fact of classical descriptive set theory that analytic sets have the perfect set property. Our next theorem is the effective perfect set theorem which gives us more information of the type discussed above.

Theorem 2.4 (Harrison)**.** *Suppose $A \subseteq \omega^\omega$ is Σ^1_1. Then either*

1. *A contains a perfect subset.*
2. *There is a computable ordinal α such that $\emptyset^{(\alpha)}$ computes every element of A.*

Proof. By Exercise 1.8, there is a computable tree T on $\omega^{<\omega} \times \omega^\omega$ so that $A = \pi[T]$. If $(s,t) \in T$, say that there is a splitting above (s,t) in T if there exists $(s_0,t_0),(s_1,t_1) \in T$ such that $(s_0,t_0),(s_1,t_1)$ extend (s,t) and s_0, s_1 are incompatible. We now define a transfinite derivative of T. Let

$$
\begin{aligned}
T_0 &= T \\
T_{\alpha+1} &= T_\alpha \setminus \{(s,t) \in T_\alpha \colon \text{there is no splitting above } (s,t) \text{ in } T\} \\
T_\lambda &= \bigcap_{\alpha<\lambda} T_\alpha
\end{aligned}
$$

There must be an ordinal β such that $T_\beta = T_{\beta+1}$. Now we break into two cases.

Case 1: if T_β is nonempty, then we can construct a map $2^{<\omega} \to T_\beta$ where we associate to each string $\sigma \in 2^{<\omega}$ a pair $(s_\sigma, t_\sigma) \in T_\beta$ such that for every σ, $(s_{\sigma^\frown 0}, t_{\sigma^\frown 0}), (s_{\sigma^\frown 1}, t_{\sigma^\frown 1})$ extend (s_σ, t_σ) and $s_{\sigma^\frown 0}, s_{\sigma^\frown 1}$ are incompatible. Finally, let $T^* \subseteq T_\beta$ be the closure of these strings under initial segments. $T^* = \{(s,t) \in T_\beta \colon (\exists \sigma \in 2^{<\omega})(s_\sigma, t_\sigma) \text{ extends } (s,t)\}$. Then $\pi[T^*]$ is a perfect closed set contained in A.

Case 2: If T_β is empty, then it is clear that A is countable. This is because $\pi[T_\alpha] \setminus \pi[T_{\alpha+1}]$ is countable, since if x is in the difference, then there is some $(s,t) \in T$ and $y \in \omega^\omega$ so (x,y) extends (s,t) and there is no splitting in T above (s,t). Note that this also means that x must be computable from T_α. Given such an (s,t), we can search for any extension $(s',t') \in T_\alpha$ such that (s',t') extends (s,t), and we must have that s' is an initial segment of x. There exist such extensions of arbitrary length. So to finish, it is enough to show there is some computable β so that $\emptyset^{(\beta)}$ computes every T_α.

Say that a function $p \colon 2^{\leq n} \to T$ is a *splitting map into* T, if for all $\sigma \in 2^{\leq n}$ with $|\sigma| < n$, $(s_{\sigma^\frown 0}, t_{\sigma^\frown 0}), (s_{\sigma^\frown 1}, t_{\sigma^\frown 1})$ extend (s_σ, t_σ) and $s_{\sigma^\frown 0}, s_{\sigma^\frown 1}$ are incompatible. The set of all splitting maps into T forms a tree S by ordering these maps under extension. S is a computable tree since T is computable, and it is wellfounded since otherwise T would have a perfect set as in the above case.

Now if we perform the usual derivative process on S where

$$
\begin{aligned}
S_0 &= S \\
S_{\alpha+1} &= S_\alpha \setminus \{p \in S_\alpha \colon \text{there is no extension of } p \text{ in } S_\alpha\} \\
S_\lambda &= \bigcap_{\alpha<\lambda} S_\alpha
\end{aligned}
$$

then it is easy to check by transfinite induction that S_α is the set of splitting maps into T_α. This is because a splitting map $p \colon 2^{\leq n} \to T$ has no extensions to a splitting map defined on $2^{\leq n+1}$ iff there exists $(s_\sigma, t_\sigma) \in \operatorname{ran}(p)$ so that there is no splitting above (s_σ, t_σ) in T. Hence, it follows that the least ordinal β such that $T_\beta = \emptyset$ is $\operatorname{rank}(S)$.

Now it is an easy exercise to show that $\emptyset^\beta$ computes T_α for all $\alpha < \beta$. We've already noted that from T_α we can compute each x such that $x \in \pi[T_\alpha] \setminus \pi[T_{\alpha+1}]$. $\square$

Exercise 2.5. *Suppose $x \in \omega^\omega$. Then x is hyperarithmetic iff $\{x\}$ is Δ^1_1 iff $\{x\}$ is Σ^1_1.*

2.3 Harrison linear orders, Π^0_1 sets with no HYP elements

We've shown above that every element of a countable Π^0_1 set (and more generally Σ^1_1 set) must be hyperarithmetic. In contrast, there are uncountable Π^0_1 sets with no HYP branches.

Exercise 2.6. *The set of computable ordinal codes is a Π^1_1 complete subset of ω.*

Exercise 2.7. $\{\emptyset^{(a)} : a$ *is a computable ordinal code*$\}$ *is* Π^1_1.

Exercise 2.8. $\{x \in \omega^\omega : x$ *is hyperarithmetic*$\}$ *is* Π^1_1.

Lemma 2.9. *There is a computable illfounded tree $T \subseteq \omega^{<\omega}$ so that $[T]$ contains no hyperarithmetic branches.*

Proof. Consider the Σ^1_1 set of reals that are not hyperarithmetic. Let T be the tree so that $T = \pi[T]$. $\square$

Theorem 2.10 (Harrison)**.** *There is a computable illfounded linear ordering with no hyperarithmetic descending sequence.*

Proof. Consider the Kleene-Brouwer order on the tree T in Lemma 2.9. $\square$

2.4 Π^1_1 ranks

Suppose A is a Π^1_1 set. By our normal form in Lemma 1.5 there is a map $y \mapsto T_y$ so that $y \in A$ iff T_y is illfounded. This map is key to our understanding of A. However, often we use it a particular way as in our proof of Theorem 1.43, relying heavily on the relations $\operatorname{rank}(T_x) \leq \operatorname{rank}(T_y)$ and $\operatorname{rank}((T_x)^+) \leq \operatorname{rank}(T_y)$. We formalize these two relations in terms of notions of Π^1_1 ranks and prewellorderings.

Definition 2.11. *A **prewellordering** on a set A is a symmetric, transitive relation $\leq$ such that for all $x, y \in A$ either $x \leq y$ or $y \leq x$, and such that the associated strict ordering $<$ is wellfounded, where $x < y \leftrightarrow x \leq y \wedge \neg y \leq x$.*

Definition 2.12. *A **rank** on a set $A \subseteq \omega^\omega$ is a function $\varphi \colon A \to \mathsf{ORD}$. Every rank $\varphi \colon A \to \mathsf{ORD}$ on A gives rise to the prewellordering $x \leq_\varphi y$ iff $\varphi(x) \leq \varphi(y)$. We write $\varphi(x) = \infty$ if $x \notin A$, and extend $\leq_\varphi$ to the whole space ω^ω by*

$$x \leq^*_\varphi y \leftrightarrow x \in A \wedge (\varphi(y) = \infty \vee x \leq_\varphi y)$$

$$x <^*_\varphi y \leftrightarrow x \in A \wedge (\varphi(y) = \infty \vee x <_\varphi y)$$

We say the rank $\varphi\colon A \to \mathsf{ORD}$ is a Π^1_1 rank iff the relations $\leq^*_\varphi$ and $<^*_\varphi$ are both Π^1_1.

Lemma 2.13. *If A is Π^1_1, then it admits a Π^1_1 rank.*

Proof. Suppose $A \subseteq \omega^\omega$ is Π^1_1. Let $y \mapsto T_y$ be a computable map such that $y \in A$ iff T_y is wellfounded. Then let $\varphi\colon A \to \mathsf{ORD}$ be defined by $\varphi(y) = \text{rank}(T_y)$. Then φ is a Π^1_1 rank since $x <^*_\varphi y$ if there is no monotone function from T_y to T_x, and $x \leq^* y$ if there is no monotone function from T_y to T^+_x, by Exercise 1.15. $\square$

Exercise 2.14. *If A admits a Π^1_1 rank, then A is Π^1_1. [Hint: $x \in A \leftrightarrow x \leq^*_\varphi x$]*

There are many Π^1_1 ranks which arise naturally from transfinite mathematical analyses, and not just from our normal form for Π^1_1 sets. See [K, Section 34] for many examples. For instance, in the space of compact subsets of ω^ω, the set of countable compact sets is a complete Π^1_1, and has a natural Π^1_1 rank arising from the Cantor-Bendixson derivative. The set of everywhere differentiable functions in $C([0,1])$ is a complete Π^1_1 set and Kechris and Woodin [KW] have associated a natural rank to this set. Wesolek and Williams show that the set of elementary amenable groups is Π^1_1 complete and that the elementary amenability rank is a Π^1_1 rank.

2.5 Number Uniformization

Suppose $A \subseteq X \times Y$. Then we say $A' \subseteq A$ **uniformizes** A if $\forall x \in X(\exists y \in Y(x,y) \in A \leftrightarrow \exists! y \in Y(x,y) \in A'$. So A' is the graph of a (partial) function $f\colon \pi_0(A) \to Y$ such that $(x, f(x)) \in A$.

We'll use the formalism of Π^1_1 ranks to prove the theorem.

Theorem 2.15 (Number uniformization for Π^1_1)**.** *Suppose $A \subseteq \omega^\omega \times \omega$ is Π^1_1. Then A has a Π^1_1 uniformization.*

Proof. Fix a Π^1_1 rank φ on A. Let $A' = \{(x,n)\colon \forall m < n (x,n) <^*_\varphi (x,m) \wedge \forall m (x,n) \leq^*_\varphi (x,m)\}$. That is, $(x,n) \in A$ iff n is minimal among all m such that (x,m) has minimal rank $< \infty$. $\square$

We mention the following property of functions:

Lemma 2.16. *If $f\colon \omega^\omega \to \omega^\omega$ is a Σ^1_1 total function then it is Δ^1_1.*

Proof. $f(x) = y$ iff $\forall y' \in \omega^\omega (y' \neq y \to f(x) \neq y')$. $\square$

An identical proofs give the following:

Exercise 2.17. *$f\colon \omega^\omega \to \omega$ is Σ^1_1 iff it is Δ^1_1 iff it is Π^1_1.*

The analogue of Lemma 2.16 is false for Π^1_1 functions.

Exercise 2.18. *Show there is a Π^1_1 function $f\colon \omega^\omega \to \omega^\omega$ that is not Δ^1_1.*

2.6 Π^1_1 scales and Π^1_1 uniformization

Before defining scales, we'll briefly discuss Suslin representations of sets, which are closely related. Recall $A \subseteq \omega^\omega$ is **κ-Suslin** if there is a tree T on $\omega \times \kappa$ so that $A = \pi[T]$. Hence, every Σ^1_1 set is ω-Suslin.

Many basic properties and proofs concerning Σ^1_1 sets have generalizations to κ-Suslin sets. For example

Exercise 2.19 (Mansfield). *If $A \subseteq \omega^\omega$ is κ-Suslin, then $|A| \leq \kappa$, or A contains a perfect closed set.*

In the same way that the ω-Suslin representation of a Σ^1_1 set is key to understanding it, key to understanding Π^1_1 sets are their Suslin representations.

Theorem 2.20 (Shoenfield). *Every Π^1_1 set $A \subseteq \omega^\omega$ is ω_1-Suslin.*

Proof. Fix a computable map $x \mapsto T_x$ so that $x \in A$ iff T_x is wellfounded. For $s \in \omega^{<\omega}$, define a tree $T_s \subseteq \omega^{<\omega}$ by $t \in T_s$ if the program computing T_x run with oracle s halts accepting t. Let $(t_n)_{n\in\omega}$ be an enumeration of $\omega^{<\omega}$.

Define a tree $T \subseteq \omega^{<\omega} \times \omega_1^{<\omega}$ as follows. $(s, t) \in T$ provided for all $i, j < |t|$ if $t_i \subsetneq t_j \in T_s$, then $t(i) > t(j)$. Then if there is an infinite branch $(x, y) \in [T]$, if $t_i \subsetneq t_j \in T_x$, then $y(i) > y(j)$. Hence the map then the map $t_i \mapsto y(t_i)$ witnesses that T_x is wellfounded. Conversely, if T_x is wellfounded, then the function $y(i) = 0$ if $t_i \notin T_x$, otherwise $y(i) = \mathrm{rank}_T(t_i)$, has $(x, y) \in [T]$. $\square$

The ranks on A used in Shoenfield's proof have the following nice properties, when paired with a representation of x itself.

Definition 2.21. *A* ***very good scale*** *on a set $A \subseteq \omega^\omega$ is a sequence $\varphi_n \colon A \to \mathsf{ORD}$ of ranks on A such that the following holds. If $x_i \in A$ and $\varphi_n(x_i) \to \alpha_n$ for all n, then $x_i \to x$ for some $x \in A$. Furthermore, $\varphi_n(x) \leq \varphi_n(y) \to \forall m \leq n\varphi_m(x) \leq \varphi_m(y))$. We say a very good scale on A is Π^1_1 if and only if the ranks φ_n are uniformly Π^1_1.*

Here by $\varphi_n(x_i) \to \alpha_n$ we mean that for sufficiently large i, $\varphi_n(x_i) = \alpha_n$. That is, we're taking the limit in the discrete topology.

If α is an ordinal, the lex ordering on α^n is defined by $(\alpha_0, \dots, \alpha_{n-1}) <_{lex} (\beta_0, \dots, \beta_{n-1})$ iff $(\exists i)(\alpha_i < \beta_i \wedge (\forall j < i)\alpha_j = \beta_j)$, and it is a wellordering. We use $\langle \alpha_0, \dots, \alpha_{n-1} \rangle$ to denote the rank of $\alpha_0, \dots, \alpha_{n-1}$ in the lex ordering.

Lemma 2.22. *Let A be a Π^1_1 set, and $x \mapsto T_x$ be such that $x \in A$ iff T_x is wellfounded. Let $(t_i)_{i\in\omega}$ be a computable enumeration of $\omega^{<\omega}$ so that $t_0 = \emptyset$. Let $\varphi_n \colon A \to \omega_1^n$ be defined by $\varphi_n(x) = \langle \mathrm{rank}((T_x)_{t_0}), x(0), \mathrm{rank}((T_x)_{t_1}, x(1), \dots, x(n) \rangle$. Then (φ_n) is a very good Π^1_1 scale on A.*

Proof. Clearly if $\varphi_n(x_i)$ converge for each n, then $x_i \to x$ for some real x, and for every t, $\mathrm{rank}((T_{x_i})_t)$ converges to β_t for some β_t. Since the map $x \mapsto T_x$ is continuous, we have $T_x = \lim T_{x_i}$, and clearly $t \mapsto \beta_t$ has the property that if $s \subsetneq t \in T_x$, then for sufficiently large i, $s, t \in T_{x_i}$, hence $\beta_s > \beta_t$, and the map

$t \mapsto \beta_t$ witnesses that T_x is wellfounded, and hence $x \in A$. It is straightforward to check that φ_n are uniformly Π^1_1 ranks.

Let ψ_n be the rank $\psi_n(x) = \text{rank}((T_x)_{t_n})$ on the Π^1_1 set $A_n = \{x : (T_x)_{t_n}$ is wellfounded$\}$. Note that ψ_n is not a Π^1_1 rank on A in general, however, $A = A_0$, and $A_n \supseteq A$ for all n. Then for example,

$$x \leq^*_{\varphi_0} y \leftrightarrow x \leq^*_{\psi_0} y \wedge (x <^*_{\psi_0} y \vee (y \leq^*_{\psi_0} x \wedge x(0) \leq y(0)))$$

□

We'll use scales to select a canonical element of a set by picking the element which minimizes all of the ranks in the scale. We give an easy example of this idea:

Theorem 2.23. *Let $A \subseteq \omega^\omega$ be a Π^1_1 set. Then there is some $x \in A$ such that $\{x\}$ is Π^1_1.*

Proof. Let $(\varphi_n)_{n\in\omega}$ be a very good Π^1_1 scale on A, and let $A_n = \{x : \varphi_n(x)$ is minimal$\}$ = $\{x : \forall y x \leq^*_{\varphi_n} y\}$. Then by the properties of a very good Π^1_1 scale, $\bigcap A_n = \{x\}$ for some x. This is a Π^1_1 set, $\{x : \forall n \forall y x \leq_{\varphi_n} y\}$. □

Exercise 2.24. *Suppose $A \subseteq \omega^\omega$ is Π^1_1 then there is some $x \in A$ such that $x \in L$.*

Using the same idea as Theorem 2.23, we can prove Π^1_1 uniformization, by taking the y minimizing the scale in each section A_x.

Theorem 2.25 (Π^1_1 uniformization)**.** *If $A \subseteq \omega^\omega \times \omega^\omega$ is Π^1_1, then A has a Π^1_1 uniformization $A' \subseteq A$.*

Proof. Let $(\varphi_n)_{n\in\omega}$ be a very good Π^1_1 scale on A. Then let

$$A' = \{(x, y) : \forall n \forall z (x, y) \leq^*_{\varphi_n} (x, z)\}.$$

□

2.7 Reflection

Definition 2.26. *Say that a collection of Σ^1_1 sets Φ is Π^1_1 on Σ^1_1 if and only if $n : U_n \in \Phi$ is Π^1_1, where $U \subseteq \omega^\omega \times \omega^\omega$ is a universal Σ^1_1 set so that $A \subseteq \omega^\omega$ is Σ^1_1 iff $(\exists n) A = U_n$.*

Theorem 2.27 (The first reflection theorem)**.** *If Φ is Π^1_1 on Σ^1_1, then and $A \in \Phi$, then there is some $B \supseteq A$ that is Δ^1_1 such that $B \in \Phi$.*

Proof. Let $n \mapsto S_n$ be a computable map so that $n \in \mathcal{O}$ iff S_n is wellfounded. Let $y \mapsto T_y$ be a computable map so that $y \in A$ iff T_y is illfounded. Consider the sets $A_n = \{y : \text{rank}(T_y) \geq \text{rank}(S_n)\}$. If S_n is wellfounded, then A_n is a Δ^1_1 set with $A_n \supseteq A$. So if there is some n such that S_n is wellfounded and $A_n \in \Phi$, then we are done. Otherwise, since if S_n is illfounded then $A_n = A$, we have S_n is wellfounded iff $A_n \notin \Phi$. But since Φ is Π^1_1 on Σ^1_1 this would give a Σ^1_1 definition of $\mathcal{O}$, which is a contradiction. □

Exercise 2.28. *Prove that every Σ^1_1 singleton is Δ^1_1.*

Exercise 2.29. *Prove that every countable Σ^1_1 set is contained in a countable Δ^1_1 set.*

Exercise 2.30. *Prove the separation theorem for Σ^1_1 sets (Theorem 1.43) using the first reflection theorem.*

Definition 2.31. *Let $U \subseteq \omega \times \omega^\omega$ be a universal Π^1_1 set. Let Φ be a collection of sets of the form $A \times B$ where $A, B \subseteq \omega^\omega$ are Π^1_1. Then we say Φ is Π^1_1 on Π^1_1 if $\{(n,m)\colon U_n \times U_m \in \Phi\}$ is Π^1_1. Say that Φ is monotone if $A \times B \in \Phi$ and $A \subseteq A'$ and $B \subseteq B'$ implies $A' \times B' \in \Phi$. Finally, say Φ is continuous downward in the second variable if whenever $A \times B_n \in \Phi$ for $B_0 \supseteq B_1 \supseteq \ldots$, then $A \times \bigcap_n B_n \in \Phi$.*

One natural way such a Π^1_1 on Π^1_1 property arises is when $P \subseteq \omega^\omega \times \omega^\omega$ is a Π^1_1 relation, and $A \times B \in \Phi \leftrightarrow \forall x \notin A \forall y \notin Y P(x,y)$.

Exercise 2.32 (The second reflection theorem)**.** *If Φ is Π^1_1 on Π^1_1 is monotone, and continuous downward in the second variable, then if there is a Π^1_1 set A such that $A \times \omega^\omega \setminus A \in \Phi$, then there is a Δ^1_1 set $B \subseteq A$ so that $B \times \omega^\omega \setminus B \in Phi$.*

3 Gandy-Harrington forcing

Gandy-Harrington forcing was invented by Gandy to prove the following theorem.

Theorem 3.1 (Gandy basis theorem)**.** *If $A \subseteq \omega^\omega$ is Σ^1_1 and nonempty, there exists $x \in A$ such that $\mathcal{O}^x \equiv_T \mathcal{O}$, and hence x is hyperlow.*

Note that we have already showed that there are nonempty Σ^1_1 subset of ω^ω (indeed, Π^0_1 sets) which contain no hyperarithmetic elements in Lemma 2.9.

Theorem 3.1 is proved by forcing with Σ^1_1 sets, in analogy to how the low basis theorem in classical computability is proved by forcing with Π^0_1 sets. Approximating a real using Σ^1_1 sets has an additional complication though. There is no reason a decreasing sequence of Σ^1_1 sets of decreasing diameter need intersect to a single real. We will address this by using a winning strategy for player II in the associated Choquet game to ensure that the real we build is in the intersection of the sets we use to approximate it.

3.1 The Choquet game on Σ^1_1 sets.

Definition 3.2. *If X is a space and $\mathcal{A} \subseteq X$ is a collection of sets, then the Choquet game on $\mathcal{A}$ is the infinite two player game where the players alternate playing elements of $\mathcal{A}$ which are decreasing:*

$$\begin{array}{llllll} I & A_0 & & A_1 & & A_2 \ldots \\ II & & B_0 & & B_1 & \ldots \end{array}$$

where $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1$. Then II wins the game if and only if $\bigcap_i A_i = \bigcap B_i$ is nonempty.

Exercise 3.3 (Oxtoby)**.** *If X is a space and $\mathcal{A}$ is its collection of open subsets, then player I has no winning strategy in the Choquet game on $\mathcal{A}$ iff X is a Baire space iff every comeager subset of X is dense.*

We'll begin by showing that player II has a winning strategy in the Choquet game on Σ^1_1 sets. Recall our notation that if T is a subtree of $\omega^{<\omega} \times \omega^{<\omega}$, then $T \upharpoonright (s, t)$ is all nodes in T compatible with (s, t).

Lemma 3.4. *There is a winning strategy for player II in the Choquet game on nonempty Σ^1_1 subsets of ω^ω.*

Proof. By Exercise 1.8, every Σ^1_1 set is the projection of the paths through a computable tree. Let T_i be a computable tree so that $A_i = \pi[T_i]$. We recursively define pairs (s^n_i, t^n_i) for $i \leq n$ such that $|s^n_i| = |t^n_i| = n$, and (s^{n+1}_i, t^{n+1}_i) extends (s^n_i, t^n_i), and the move B_n for player II is

$$B_n = \pi[T_0 \upharpoonright (s^n_0, t^n_0)] \cap \ldots \cap \pi[T_n \upharpoonright (s^n_n, t^n_n)]$$

In particular, on move n, since

$$B_{n-1} \cap A_n = \pi[T_0 \upharpoonright (s^{n-1}_0, t^{n-1}_0)] \cap \ldots \cap \pi[T_{n-1} \upharpoonright (s^{n-1}_{n-1}, t^{n-1}_{n-1})] \cap \pi[T_n]$$

is nonempty, we can find length n extensions (s_i^n, t_i^n) of (s_i^{n-1}, t^{n-1}, i) for $i \leq n-1$ and some (s_n^n, t_n^n) of length n such that

$$\pi[T_0 \upharpoonright (s_0^n, t_0^n)] \cap \ldots \cap \pi[T_n \upharpoonright (s_n^n, t_n^n)]$$

is nonempty. Let B_n be this set.

Now having defined this strategy for player II, we show that it is a winning. For each n and $i \leq n$, since $\pi[T_0 \upharpoonright (s_0^n, t_0^n)] \cap \ldots \cap \pi[T_n \upharpoonright (s_n^n, t_n^n)]$ is nonempty we must have $s_i^n = s_j^n$ for all i, j. Let $x \in \omega^\omega$ be the real $x = \bigcup_n s_i^n$. We claim $\bigcap_i A_i = \bigcap_i B_i = \{x\}$. This is because letting $y_i = \bigcup_n t_i^n$, we have $(s_i^n, t_i^n) \in T_i$ for all n and hence $(x, y_i) \in [T_i]$ and $x \in \pi[T_i] = A_i$. $\square$

The computability of this winning strategy is important in some of our applications; it is computable from Kleene's $\mathcal{O}$.

Lemma 3.5. *Consider the game associated to the Choquet game on Σ_1^1 sets where instead of playing a Σ_1^1 set, each player plays an index for a program which computes a subtree T of $\omega^{<\omega} \times \omega^{<\omega}$ which projects to the desired Σ_1^1 set. $\mathcal{O}$ can compute a winning strategy for player II in this game.*

Proof. In our strategy defined above, choose $(s_0^n, t_0^n), \ldots, (s_n^n, t_n^n)$ to be the lex-least sequence extending $(s_0^{n-1}, t_0^{n-1}), \ldots, (s_{n-1}^{n-1}, t_{n-1}^{n-1})$ such that $\pi[T_0 \upharpoonright (s_0^n, t_0^n)] \cap \ldots \cap \pi[T_n \upharpoonright (s_n^n, t_n^n)]$ is nonempty. Then $\mathcal{O}$ can compute $(s_0^n, t_0^n), \ldots, (s_n^n, t_n^n)$ since it can compute which Σ_1^1 sets are nonempty. $\square$

Exercise 3.6. *Show that $\{x \in \omega^\omega : x \in \mathsf{HYP}\}$ is Π_1^1. Show that its complement $\{xin\omega^\omega : x \notin \mathsf{HYP}\}$ is a Σ_1^1 set that does not contain any Σ_1^1 singleton.*

Exercise 3.7. *Show that there is a winning strategy for player II in the Choquet game on Δ_1^1 subsets of ω^ω.*

The following strengthening of the Choquet game is useful in many applications:

Definition 3.8. *If X is a space and $\mathcal{A} \subseteq X$ is a collection of sets, then the strong Choquet game on $\mathcal{A}$ is the infinite two player game where the players alternate playing elements of $\mathcal{A}$ which are decreasing:*

$$\begin{array}{lccccc} I & x_0, A_0 & & x_1, A_1 & & x_, A_2 \ldots \\ II & & B_0 & & B_1 & \ldots \end{array}$$

where $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1$, and for all i, $x_i \in A_i$ and $x_i \in B_i$. Then II wins the game if and only if $\bigcap_i A_i = \bigcap B_i$ is nonempty.

Exercise 3.9. *Show that there is a winning strategy computable from $\mathcal{O}$ in the strong Choquet game on Σ_1^1 sets.*

3.2 The Gandy basis theorem

We're now ready to prove the Gandy basis theorem:

Proof of Theorem 3.1: Fix a Σ^1_1 set $A \subseteq \omega^\omega$. We will construct $x \in A$ such that $\mathcal{O} \geq_T \mathcal{O}^x$. We will do this by constructing a decreasing sequence $A_0 \supseteq A_1 \ldots$ of Σ^1_1 sets such that our desired real x has $\{x\} = \bigcap_i A_i$. We'll choose A_{n+1} so that it decides the nth bit of $\mathcal{O}^x$. We will also play an instance of the Choquet game on Σ^1_1 sets to insure $\cap_i A_i$ is nonempty. Let $A_0 = A$. Let B_n be the response of the winning strategy computable from $\mathcal{O}$ of player II in the Choquet game on Σ^1_1 sets.

Let $A_{n+1} = \{x \in B_n \colon$ the nth program φ^x_n relative to x does not compute a subtree of $\omega^{<\omega}$ or it computes an illfounded subtree of $\omega^{<\omega}\}$ if this set is nonempty, otherwise let $A_{n+1} = B_n$. In the first case we have ensured that if $x \in A_{n+1}$, then $n \notin \mathcal{O}^x$. In the second case we have insured that $n \in \mathcal{O}^x$. From $\mathcal{O}$ we can compute if this set is nonempty, and hence, then nth bit of $\mathcal{O}^x$. From $\mathcal{O}$ we can also compute player II's response in the Choquet game. Hence, $\mathcal{O} \geq_T \mathcal{O}^x$. □

By relativizing the Gandy basis theorem, we obtain the following corollary

Corollary 3.10. *If $\{x\}$ is Σ^1_1 relative to y, then $\omega^x_1 \leq \omega^y_1$.*

This generalizes Exercise 1.50.

Exercise 3.11. $\{x \colon \omega^x_1 = \omega^{ck}_x\}$ *is Σ^1_1 and not Δ^1_1.*

Lemma 3.12 (Cone avoidance in Σ^1_1 sets)**.** *Suppose $A \subseteq \omega^\omega$ is Σ^1_1 and nonempty, $B \subseteq \omega$ is not Δ^1_1, and $\varphi^x(n)$ and $\psi^x(n)$ are Σ^1_1 formulas relative to a real parameter x with a single free variable n. Then there is some nonempty Σ^1_1 set $A' \subseteq A$ so that for all $x \in A$, either $\varphi^x(n)$ is not a definition of B or $\neg\psi^x(n)$ is not a definition of B, or $\varphi^x(n)$ and $\neg\psi^x(n)$ do not define the same set.*

Proof. For each n, let

$$A_{n,\varphi} = \{x \in A \colon \varphi^x(n)\}$$
$$A_{n,\psi} = \{x \in A \colon \psi^x(n)\}$$

Case 1: Suppose that there is some n such that $\forall x \in A(\neg\varphi^x(n) \wedge \neg\psi^x(n))$. Then forall $x \in A$, φ^x and $\neg\psi^x$ do not define the same set.

Case 2: there is some $n \notin B$ such that $A_{n,\varphi} \neq \emptyset$. Then for all $x \in A_{n,\varphi}$ φ^x does not give a Σ^1_1 definition of B relative to x.

Case 3: there is some $n \in B$ such that $A_{n,\psi} \neq \emptyset$, then for all $x \in A_{n,\psi}$ the formula $\neg\psi^x$ does not give a Π^1_1 definition of B relative to x.

Finally, if none of the previous cases hold, then for all n

$$n \notin B \to A_{n,\varphi} = \emptyset \to (\forall x \in A)\neg\varphi^x(n)$$

Since Case 2 does not hold, and

$$n \in B \to A_{n,\psi} = \emptyset \to (\forall x \in A)\neg\psi^x(n)$$

Since Case 3 does not hold. Finally, since Case 1 does not hold, for each n there must be some $x \in A$ such that $\varphi^x(n) \vee \psi^x(n)$. Hence $n \in B \leftrightarrow (\forall x \in A)\neg\psi^x(n)$ and $n \notin B \leftrightarrow (\forall x \in A)\neg\varphi^x(n)$. These are a Π^1_1 and Σ^1_1 definition of B, which is a contradiction. $\square$

Exercise 3.13. *Suppose $A \subseteq \omega^\omega$ is Σ^1_1 and nonempty and $B \subseteq \omega$ is not Δ^1_1. Then show there is some $x \in A$ such that $x \not\geq_{\mathsf{HYP}} B$.*

Exercise 3.14. *Show that $x \in \mathsf{HYP}$ if and only if every countable ω-model of ZFC contains x. [Hint: every ω-model of ZFC contains every HYP real by absoluteness. For the other direction, use Exercise 3.13]*

Exercise 3.15. *Suppose $A \subseteq \omega^\omega$ is Σ^1_1 and nonempty and $B \subseteq \omega$ is not Δ^1_1. Then show there is some $x \in A$ such that $x \not\geq_{\mathsf{HYP}} B$ and $\omega_1^x = \omega_1^{ck}$.*

Exercise 3.16. *Show that $\omega_1^x = \omega_1^{ck}$ iff for every Σ^1_1 set $A \subseteq \omega^\omega$, either $x \in A$ or there exists a Σ^1_1 set B disjoint from A so that $x \in B$. [Hint: To prove $\leftarrow$, note that for each e, $x \in \{x \colon \varphi_e^x$ computes an illfounded subtree of $\omega^{<\omega}\}\}$ or x is in a Σ^1_1 set disjoint from this set. Then apply Spector boundedness.]*

3.3 The G_0 dichotomy

In this section, we'll study the problem of graph coloring. Recall that if G is a graph (symmetric irreflexive relation) on a vertex set X, then a **Y-coloring** of G is a function $c \colon X \to Y$ such that if $x_0, x_1 \in X$ are G-adjacent, then $c(x_0) \neq c(x_1)$. A G-independent set is a set $A \subseteq X$ so that A contains no two adjacent points. Note that A is **independent** iff we could assign every element of A to be the same color in a coloring of G.

In particular, we'll prove the G_0 dichotomy of Kechris, Solecki, and Todorcevic which characterizes when a Σ^1_1 graph has a Δ^1_1 ω-coloring. We begin with an example of a class of graphs which do not admit ω-colorings.

Definition 3.17. *Suppose $S \subseteq 2^{<\omega}$. Then let G_S be the graph with vertex set 2^ω where $x, y \in 2^\omega$ are adjacent if there exists some $s \in S$ such that $x = s^\frown i^\frown z$ and $y = s^\frown 1 - i^\frown z$ for some i, z. That is, x and y differ by exactly one bit, which occurs immediately after s, which is an initial segment of x and y.*

Say that S is **dense** if for every $s \in 2^{<\omega}$ there exists $t \in S$ such that t extends s.

Lemma 3.18. *If S is dense, There is no Baire measurable ω-coloring of G_S.*

Proof. Let $C_n = \{x \in 2^\omega \colon c(x) = n\}$. Then by the Baire category theorem, there is some n such that C_n is nonmeager. Hence, there is some nonempty open set U such that C_n is comeager in U. It now suffices to prove the following claim, which contradicts c being a coloring:

Claim: if A is comeager in a basic open set N_s, then A contains two G_S-adjacent points.

To prove the claim, we begin by noting that since S is dense, by extending N_s we may assume $s \in S$. Now let $f\colon N_{s^\frown 0} \to N_{s^\frown 1}$ be the function where $f(s^\frown 0^\frown z) = s^\frown 1^\frown z$. Note that f maps each $x \in N_s$ to a point it is adjacent to. f is a homeomorphism, so since A is comeager in $N_{s^\frown 0}$, $f(A \cap N_{s^\frown 0}$ is comeager in $N_{s^\frown 1}$. But A is also comeager in $N_{s^\frown 1}$, so $f(A \cap N_{s^\frown 0}$ and A intersect. Any x in this intersection has x, and $f^{-1}(x)$ are in A and are G_S-adjacent. $\square$

Suppose now $S = \{s_n\}_{n\in\omega}$ is dense and $|s_n| = n$, so S contains one string of each length. By abuse of notation we use G_0 to denote the graph G_S (though it depends on the particular sequence S we have chosen). It will turn out that all such G_S are bi-embeddable.

In this case there is an inductive way of understanding the graph G_S as a sort of inverse limit. Let G_S^m be the graph on 2^m where $tt' \in 2^n$ are adjacent if $t = s^\frown i^\frown r$ and $t' = s^\frown 1-i^\frown r$ for some i, r, s_m. So x, y are G_S-adjacent iff there exists some m so that $x \restriction m$ and $y \restriction m$ are G_S^m-adjacent. Then G_S^0 is the graph with one vertex (the empty string), and G_S^{m+1} is the graph obtained by taking two copies of G_S^m and adding a single edge between to corresponding vertices ($s_n{}^\frown 0$ and $s_n{}^\frown 1$). For example, this inductive characterization can be used to show the following:

Exercise 3.19. *Suppose $S \subseteq 2^{<\omega}$ has exactly one string of each length. Then for every m, G_S^m is acyclic. Hence, G_S is acyclic.*

If G is a graph on the vertex set X and H is a graph on the vertex set Y, then a **homomorphism** from G to H is a map $f\colon X \to Y$ such that if $x_0, x_1 \in X$ are G-adjacent, then $f(x_0), f(x_1)$ are H-adjacent. Note that this implies that if $c\colon Y \to Z$ is a Z-coloring of H, then $c \circ f$ is a Z-coloring of G.

Theorem 3.20 (Kechris, Solecki, Todorcevic, the G_0 dichotomy [KST])**.** *Suppose G is a Σ_1^1 graph on ω^ω. Then exactly one of the following holds.*

1. *G has a Δ_1^1 ω-coloring*
2. *There is a continuous homomorphism (computable from $\mathcal{O}$) from G_0 to G.*

Proof. By Lemma 3.18 options (1) and (2) are mutually exclusive.

Consider the set of Σ_1^1 sets A that are G-independent. This collection is Π_1^1 on Σ_1^1. Hence, every Σ_1^1 G-independent set is contained in a Δ_1^1 G-independent set.

Let $C = \bigcup\{A\colon A \text{ is } \Delta_1^1 \text{ and } G\text{-independent}\}$. Then C is Π_1^1 since $\Delta_1^1 =$ effectively Borel. Now we break into two cases:

Case 1: $C = \omega^\omega$.

Exercise 3.21. *In this case, G has a Δ_1^1 coloring.*

Case 2: Fix a sequence $S = \{s_n\}_{n\in\omega}$ so $G_0 = G_S$. We will construct a continuous homomorphism $f\colon 2^\omega \to \omega^\omega$ from G_0 to G. By abuse of notation we will use $G \subseteq \omega^\omega \times \omega^\omega$ to indicate the edge relation of the graph.

Let $A_\emptyset = \omega^\omega \setminus C$. Note that for every Σ^1_1 set $A' \subseteq A_\emptyset$, we have that $A' \times A' \cap G$ is nonempty.

For each m, we associate to each $s \in 2^m$ a Σ^1_1 set A_s where if $s \subseteq t$, then $A_s \subseteq A_t$. Our homomorphism $f\colon 2^\omega \to \omega^\omega$ will be $f(x) = y$ where $\{y\} = \bigcap_m A_{y \restriction m}$. To ensure that if x_0, x_1 are G_0-adjacent, then $f(x_0)$ and $f(x_1)$ are G-adjacent, we will also associate to each edge (s,t) of G^m_S a Σ^1_1 set $A_{(s,t)} \subseteq \omega^\omega \times \omega^\omega$ where $A_{(s,t)} \subseteq G$ consists only of G-related points. Finally, we will have that

$$\pi_0(A_{(s,t)}) = A_s \text{ and } \pi_1(A_{(s,t)}) = A_t \tag{*}$$

where π_0 and π_1 are the projections onto the 0th and 1st coordinates respectively. We will also ensure that if (x_0, x_1) is an edge in G_0, then $\bigcap_m A_{x_0 \restriction m, x_1 \restriction m} = \{(f(x_0), f(x_1)\}$, and hence $f(x_0), f(x_1)$ are G-related since $A_{(s,t)} \subseteq G$.

Inductively, suppose we have define A_s and $A_{s,t}$ for all $s \in 2^m$ and edges (s,t) in G^m_S. Now we proceed as follows. Let $A'_{(s_m {}^\frown 0, s_m {}^\frown 1)} = A_{s_m} \times A_{s_m} \cap G$. For every $s \in 2^m$, let $A'_{s^\frown 0} = A'_{s^\frown 1} = A_s$. For every edge (s,t) in G^m_S, let $A'_{(s^\frown 0, t^\frown 0)} = A'_{(s^\frown 1, t^\frown 1)} = A_{(s,t)}$. Note that (*) does not hold here because the projections of the set $A'_{(s_m {}^\frown 0, s_m {}^\frown 1)}$ are not necessarily $A_{s_m {}^\frown 0}$ and $A_{s_m {}^\frown 1}$.

However, if we refine any set A'_s, then to make (*) hold we can replace any adjacent $A_{(s,t)}$ with $A_s \times \omega^\omega \cap A_{(s,t)}$ and any adjacent $A_{(t,s)}$ with $\omega^\omega \times A_s \cap A_{(t,s)}$. Similarly if we refine any set $A'_{(s,t)}$ we can replace A'_s with $\pi_0(A'_{(s,t)})$ and A'_t with $\pi_1(A'_{(s,t)})$. Since G^{m+1}_S is acyclic, this process will finish, having refined each set associated to each set or vertex once, ending with an assignment satisfying (*). Hence, we can begin by fixing the projections of the set $A'_{(s_m {}^\frown 0, s_m {}^\frown 1)}$. Then for each A'_s and $A'_{(s,t)}$, we play a move in the Choquet game as player I, replace the set with the response of player II, and then refine again to ensure (*) holds. It is clear that the resulting f will be a homomorphism from G_0 to G. $\square$

Corollary 3.22. *If G is a Σ^1_1 graph on ω^ω, then if there is a $\boldsymbol{\Delta}^1_1$ ω-coloring of G, there must be a Δ^1_1 coloring of G.*

Proof. Suppose not. Then there would be a Borel homomorphism from G_0 to G and also a Borel ω-coloring of G. But this is a contradiction, since the composition would be a Borel (and hence Baire measurable) coloring of G_0. $\square$

3.4 Silver's theorem

Perfect set-type properties occur for many structures more complex than just sets. For example,

Theorem 3.23 (Harrington, Marker, Shelah [HMS])**.** *Every Δ^1_1 partial order either is a union of countable many Borel chains, or has a perfect set of incomparable elements.*

Our focus in this section is on Silver's theorem

Theorem 3.24 (Silver)**.** *Suppose E is a Π^1_1 equivalence relation on ω^ω. Then either E has countably many equivalence classes, or there is a perfect set of E-inequivalent elements.*

We'll prove this theorem in several ways. Our first proof is due to Ben Miller. This proof uses the G_0 dichotomy to isolate a closed subset of ω^ω on which E is meager, and then applies Mycielski's theorem (Exercise A.5).

Miller has shown that a huge number of dichotomies in descriptive set theory can be proved this way, by using graph-theoretic dichotomies to isolate the correct setting for running a Baire category argument to prove the theorem. For more see Miller's Paris lectures [Mi].

Miller's Proof of Theorem 3.24: Consider the graph G on ω^ω where $x\ G\ y$ if $x \not{E} y$. Note that E is $\mathbf{\Pi}^1_1$ so has the Baire property. If G has a countable Borel coloring, then clearly E has countably many classes, since E-unrelated points must be assigned different colors.

Suppose now there is a continuous homomorphism $f\colon 2^\omega \to \omega^\omega$ from G_0 to G. Then let $x\ E'\ y$ iff $f(x)\ E\ f(y)$.

We claim that for each x, $[x]_{E'} = \{y\colon x\ E'\ y\}$ is meager. Otherwise, $[x]_{E'}$ would be comeager in some basic open set N_s which contains two G_0 related points by the claim in Lemma 3.18, which is a contradiction since $x\ G_0\ y \to f(x)\ G\ f(y) \to f(x) \not{E} f(y) \to x \not{E'} y$.

Hence, by the Kuratowski-Ulam theorem (Exercise A.6), E' is meager, and so by Mycielski's theorem (Exercise A.5), there is a perfect closed set $C \subseteq 2^\omega$ of E'-unrelated points. $f \restriction C$ must be an injection since $f(x) = f(y) \to x\ E'\ y$. Hence $f(C)$ is the injective continuous image of a perfect set which is therefore perfect. $\square$

Next, we give a forcing proof of Silver's theorem. This was Harrington's first application of Gandy-Harrington forcing. We begin with an exercise, that every Gandy-Harrington generic filter intersects to a single real:

Exercise 3.25. *Consider the forcing partial order $\mathbb{P}$ of Σ^1_1 sets under inclusion. There are countably many dense sets $D_n \subseteq \mathbb{P}$ so that if $G \subseteq \mathbb{P}$ is a generic filter which meets every D_n, then $\bigcap G$ is a singleton $\{g\}$.*

We will use Mostowski's Absoluteness theorem in Harrington's proof.

Exercise 3.26. *Suppose M is a transitive model of a sufficiently large fragment of* ZFC*, and $\varphi(x)$ is a Σ^1_1 formula. Then $M \models \varphi(x) \leftrightarrow V \models \varphi(x)$. [Hint: use the absoluteness of wellfoundedness]*

Harrington's Proof of Theorem 3.24: By the first reflection theorem, every Σ^1_1 set A that is contained in a single E-class has some Δ^1_1 $B \supseteq A$ where B is contained in a single E-class. Now let $C = \bigcup\{A\colon A \text{ is } \Delta^1_1 \text{ and contained in a single } E\text{-class}\}$. Then C is Π^1_1. We now have two cases.

Case 1: $C = \omega^\omega$. Then clearly E has countably many classes.

Case 2: Otherwise, let $A = \omega^\omega \setminus C$. Fix a countable transitive model M of a sufficiently large fragment of ZFC. Let $\mathbb{P}$ be Gandy-Harrington forcing.

We claim $A \times A \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{x_0} \not{E} \dot{x_1}$, where $\dot{x_0}$ and $\dot{x_1}$ are names for the first and second coordinates of the generic real. Suppose otherwise. Then it must be that $A_0 \times A_1 \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{x_0}\ E\ \dot{x_1}$ for some $A_0 \times A_1$ extending $A \times A$. We will build (x_0, x_1) and (x_0, x_1') (with the same first coordinate) which are $\mathbb{P}\times\mathbb{P}$ generic over M and extend $A_0 \times A_1$, but where $x_1 \not{E} x_1'$. This will contradict the fact that we have forced $x_0 E x_1$ and $x_0 E x_1'$ combined with Σ^1_1 absoluteness.

To build these generics, fix an enumeration of the countably many dense sets in $\mathbb{P}\times\mathbb{P}$ contained in M. We define $B_n \in P$ and Σ^1_1 sets $C_n \subseteq \omega^\omega \times \omega^\omega$ (with $C_n \subseteq \overline{E}$) so that $\{(x_0, x_1)\} = \bigcap_n B_0 \times \pi_0(C_n)$ and $\{(x_0, x_1')\} = \bigcap_n B_0 \times \pi_1(C_n)$.

Let $B_0 = A_0$, and $C_0 = A_1 \times A_1 \cap \overline{E}$. Since every condition A' extending A has $A' \times A'$ meets $\overline{E}$, we have that C_0 is nonempty. To ensure that (x_0, x_1) and (x_0, x_1') are M-generic, we let $B_n^* \times D_n^*$ extend $B_n \times \pi_0(C_n)$ meet the nth dense set in $\mathbb{P}\times\mathbb{P}$ contained in M. Then let $C_n^* = D_n^* \times \omega^\omega \cap C_n$. Next, let $B_n^{**} \times D_n^{**}$ extend $B_n^* \times \pi_0(C_n^*)$ meet the nth dense set in $\mathbb{P}\times\mathbb{P}$ contained in M. Then let $C_n^{**} = \omega^\omega \times D_n^{**} \cap C_n^*$. Let $B_{n+1} = B_n^{**}$, and $C_{n+1} = C_n^{**}$. This finishes the proof of our claim.

Now let $A_\emptyset = A$. We build a map $\sigma \mapsto A_\sigma$ from $2^{<\omega}$ to $\mathbb{P}$, ensuring that for any $\sigma \neq \tau$ with $|\sigma| = |\tau|$, we have $A_\sigma \times A_\tau$ meets the nth dense set in $\mathbb{P}\times\mathbb{P}$ contained in M. Then we have a corresponding function $f\colon 2^\omega \to \omega^\omega$ defined by $f(x) = y$ if $\{y\} = \bigcap_n A_{x\restriction n}$. Clearly for every $x \in 2^\omega$, $f(x)$ is $\mathbb{P}$-generic over M, and for every $x_0, x_1 \in 2^\omega$, $(f(x_0), f(x_1))$ is $\mathbb{P}\times\mathbb{P}$ generic, and hence $f(x_0) \not{E} f(x_1)$, since $A \times A \Vdash \dot{x_0} \not{E} \dot{x_1}$. Our desired perfect set of E-inequivalent elements is $f(2^\omega)$. $\square$

In contrast to the situation for Π^1_1 equivalence relations, Σ^1_1 equivalence relations may have ω_1 many classes, but no perfect set of inequivalent elements:

Exercise 3.27. *Consider the equivalence relation where $x\ E\ y$ if $\omega_1^x = \omega_1^y$. Show that E is Σ^1_1 but has no perfect set of inequivalent elements.*

Burgess has used Harrington's ideas to prove the following:

Theorem 3.28 (Burgess)**.** *Suppose E is a Σ^1_1 equivalence relation. Then E has either at most ω_1 many classes, or there is a perfect set of E-inequivalent points.*

Fix a complete first-order theory T in the language $\mathcal{L}$. It is a famous conjecture of Vaught that T has either countably many or continuum many countable models. Note here that the equivalence relation of isomorphism of models of T with universe ω is a Σ^1_1 equivalence relation. However, this equivalence relation has the special property that it is generated by a continuous action of the Polish group S_∞ of permutations of ω. More generally, the following is an open question:

Open Problem 3.29 (The topological Vaught conjecture)**.** *Suppose a Polish group G acts continuously on a Polish space X. Then either this action has countably many orbits, or there is a perfect set a points that are pairwise in different orbits.*

3.5 The Polish space of hyperlow reals with basis of Σ^1_1 sets

Anther way of formalizing Harrington's proof is by doing a genuine Baire category argument, but on a Polish subspace of ω^ω where the Σ^1_1 sets form a basis.

Exercise 3.30. *Show that the set $X = \{x \in \omega^\omega : \omega_1^x = \omega_1^{ck}\}$ is a Polish space when equipped with the topology generated by the Σ^1_1 sets. [Hint: let S be the set of Σ^1_1 subsets of ω^ω. Show that $f : X \to 2^\omega$ defined by $f(x)(A) = 1$ if $x \in A$ and $f(x)(A) = 0$ if $x \notin A$ is a continuous injection onto its image and hence a homeomorphism onto its image. This is because $\{x : f(x)(A) = 0\} = \bigcup\{B : B$ is Σ^1_1 and $B \cap A = \emptyset\}$ is Σ^1_1. Finally, show* $\mathrm{ran}(f)$ *is a G_δ subset of 2^ω, and is hence Polish. Use Exercise 3.16 and the same idea as the winning strategy for player II in Choquet game for Σ^1_1 sets.*

We can then use a genuine Baire category argument mirroring Harrington's proof to replace the use of forcing and Mostowski absoluteness. Let $X_2 = \{(x_0, x_1) \in \omega^\omega \times \omega^\omega : \omega_1^{x_0 \oplus x_1} = \omega_1^{\mathrm{ck}}\}$ equipped with the topology of Σ^1_1 sets. Then letting $A = \omega^\omega \setminus C$ be as in Harrington's proof, we can mirror Harrington's proof to show that E is meager in $A \times A \cap X_2$, then use Mycielski's theorem.

3.6 Louveau's theorem

In this section, we will prove Louveau's characterization of lightface Δ^1_1 that are boldface $\boldsymbol{\Sigma}^0_\alpha$. We will prove a stronger version of this theorem which is based on Σ^1_1 separation.

Theorem 3.31 (Louveau)**.** *Suppose $A_0, A_1 \subseteq \omega^\omega$ are disjoint Σ^1_1 sets, and there is a set A separating A_0, A_1 which is Σ^1_1 and also $\boldsymbol{\Sigma}^0_\alpha$ for $\alpha < \omega_1$. Then there is a $\Sigma^{0,x}_\alpha$ set A' separating A_0, A_1 where $x \in \mathsf{HYP}$.*

Corollary 3.32. *If $A \subseteq \omega^\omega$ is Δ^1_1 and $\boldsymbol{\Sigma}^0_\alpha$ for $\alpha < \omega_1^{ck}$, then $\mathcal{A}$ is $\Sigma^{0,x}_\alpha$ for some $x \in \mathsf{HYP}$.*

Proof. Let τ be the Gandy-Harrington topology on ω^ω, let τ_1 be the usual Polish topology, and let τ_α for $\alpha > 1$ be the topology on ω^ω generated by the sets that are Σ^1_1 and $\boldsymbol{\Pi}^0_\alpha$. Note that all these topologies have a countable basis.

We will prove the following by induction on $\alpha < \omega_1^{\mathrm{ck}}$.

(*) If A is Σ^1_1, then $\overline{A}^\alpha$, the closure of A in τ_α, is Σ^1_1.

(**) If A is $\boldsymbol{\Sigma}^0_\alpha$, then there is a τ_α-open set A^* so that $A \triangle A^*$ is τ-meager.

(***) If A_0, and A_1 are disjoint Σ^1_1 sets, and A is a $\boldsymbol{\Sigma}^0_\alpha$ set separating them, then A_0 and $\overline{A_1}^\alpha$ are disjoint, and there is a $\Sigma^{0,x}_\alpha$ set separating them for some $x \in \mathsf{HYP}$.

We begin with the case $\alpha = 1$. Here (*) and (**) are clear. (Note that $x \notin \overline{A}$ iff $\exists s(x \in N_s \wedge N_s \cap A = \emptyset)$. To prove (***), consider $\{(x, s) : x \in N_s \wedge N_s \cap A_1 =$

$\emptyset\}$. By Theorem 2.15 there is a Π^1_1 function $f\colon \omega^\omega \to \omega^{<\omega}$ such that if $x \notin A_1$, then $N_{f(x)} \cap A_1 = \emptyset$. Now

$$B_0 = \{s \in \omega^{<\omega}\colon N_s \cap A_1 = \emptyset\} \text{ is } \Pi^1_1,$$

and

$$B_1 = \{s \in \omega^{<\omega}\colon (\exists x \in A_0) f(x) = s\} = \text{ is } \Sigma^1_1.$$

(since $B_1 = \{s \in \omega^{<\omega}\colon (\exists x \in A_0)\forall t \neq s f(x) \neq t\}$). Clearly $B_1 \subseteq B_0$. So by Σ^1_1 separation, there is a Δ^1_1 set C so that $B_1 \subseteq C \subseteq B_0$. Our separating set is $\bigcup\{N_s\colon s \in C\}$ which is $\boldsymbol{\Sigma}^{0,C}_1$.

The inductive step is left as an exercise. □

3.7 Further results

Gandy-Harrington forcing has been remarkably useful for proving dichotomy theorems in descriptive set theory. For example, it is used in the proof of dichotomies about the structure of Borel partial orders [HMS], and Solecki's dichotomy characterizing when a Borel function is piecewise continuous [So].

Many of the most spectacular uses of Gandy-Harrington forcing have been in the theory of Borel equivalence relations. For example, Harrington, Kechris, Louveau's Glimm-Effros dichotomy [HKL], Kechris and Louveau's classification of hypersmooth Borel equivalence relations [KL], and Hjorth's turbulence dichotomy [H].

4 Effective analysis of forcing and ideals

4.1 Hechler forcing; computation from fast-growing functions

Solovay has shown that $x \in \omega^\omega$ can be computed from sufficiently fast-growing functions iff x is Δ^1_1. In this section, we'll give a proof of this fact using Hechler forcing.

Definition 4.1. *Say that $y \in \omega^\omega$ is a **modulus** for $x \in \omega^\omega$ if for all $z \geq y$ (i.e. $(\forall n) z(n) \geq y(n)$, we have $z \geq_T x$. Say that y is a **uniform modulus** for x if there is a program e so that for all $z \geq y$ we have $z \geq_T x$ via e. That is, $\Phi_e(z) = x$.*

First, we will prove that if x has a modulus, then x has a uniform modulus. We will prove this using Hechler forcing.

Definition 4.2. *Hechler forcing is the forcing where conditions are pairs (s, x) where $s \in \omega^{<\omega}$ and $x \in \omega^\omega$, and $(s^*, x^*) \leq (s, x)$ iff*

- $s^* \supseteq s$
- $(\forall n \in \mathrm{dom}(s^*) \setminus \mathrm{dom}(s)) s^*(n) \geq x(n)$.
- $\forall n x^*(n) \geq x(n)$.

A Hechler generic filter G is in bijective correspondence with the associated Hechler generic real $g = \bigcup \{s \colon (s, x) \in G\}$, where $g \in \omega^\omega$. We think of a condition (s, x) as specifying an initial segment s of the generic real, and a function x that the remaining values of g must grow at least as fast as.

We'll prove that if x has a modulus, then x has a uniform modulus using Mostowski absoluteness.

Lemma 4.3 (Groszek-Slaman)**.** *If x has a modulus, then x has a uniform modulus.*

Proof. Let y be a modulus for x. Let g be a Hechler generic real over V extending the condition $(\emptyset, y)$. The sentence that y is a modulus for x is Π^1_1 and hence absolute and true in $V[g]$ by Exercise 3.26. Since $g \geq y$, we must have that there is some e so that $\Phi_e(g) = x$. Hence, some condition $(s, y') \Vdash \Phi_e(g) = x$.

Consider the oracle Turing machine program e' so that $\varphi^z_{e'}(n)$ searches for any $s^* \in \omega^{<\omega}$ with $s* \supseteq s$ and $(\forall n \in \mathrm{dom}(s^*) \setminus \mathrm{dom}(s)) s^*(n) \geq z(n)$ so that $\varphi^{s^*}_e(n)\downarrow$, and then outputs the value $\varphi^{s^*}_e(n)$. We claim that y' is a uniform modulus for x witnessed by the program e'.

First, if $z \geq y'$ and $\varphi^z_{e'}(n)\downarrow$, then we must show $\varphi^z_{e'}(n) = x(n)$. This is because if s^* is the string found by this program making $\varphi^{s^*}_e(n)\downarrow$, then $(s^*, z) \leq (s, y)$ is a condition, and so $(s^*, z) \Vdash \Phi_e(g) = x$, so $\varphi^{s*}_e(n) = x(n)$ by absoluteness.

Now, we must show that $\varphi^z_{e'}(n)$ halts for every n provided $z \geq y'$. Since $z \geq y'$, we have $(s, z) \Vdash \Phi_e(g)$. But then taking a Hechler generic real g

extending (s, z), we must have that $\varphi_e^g(n)\downarrow$ and so it halts relative to a finite initial segment $s^* \subseteq z$. So $\varphi_{e'}^z$ must eventually halt, since we have found at least one such string s^*. $\square$

Now we show that any real with a uniform modulus has a Δ^1_1 definition.

Lemma 4.4. *If x has a uniform modulus, then x is Δ^1_1.*

Proof. Fix a uniform modulus y for x witnessed by the program φ_e. Then x has a Σ^1_1 definition:

$$x(n) = m \leftrightarrow \forall s \in (\omega^{<\omega}) s \geq y \varphi_e^s(n)\downarrow \rightarrow \varphi_e^s(n) = m.$$

so x is Δ^1_1 by Exercise 2.5. $\square$

Finally, we can conclude Solovay's theorem.

Theorem 4.5 (Solovay)**.** *$x \in \omega^\omega$ is Δ^1_1 iff it has a modulus.*

Proof. If x has a modulus, it has a uniform modulus by Lemma 4.3, and hence is Δ^1_1 by Lemma 4.4.

In the other direction, it is clear that if x has a modulus, and $y \leq_T x$, then y has a modulus. So it suffices to show that for each computable oracle code a, $\emptyset^{(a)}$ has a uniform modulus witnessed by the program $e(a)$, where $a \mapsto e(a)$ is uniformly computable. This is an easy transfinite induction. $\square$

4.2 The Ramsey property

Solovay's original proof of Theorem 4.5 used an effective analysis of the Ramsey property. We give this effective analysis in this section.

Definition 4.6. *If $A \subseteq \omega$, let $[A]^\omega$ be the collection of infinite subsets of A and $[A]^{<\omega}$ be the set of all finite subset of ω. We can identify $[\omega]^\omega$ with the closed set of increasing elements $\{x \in \omega^\omega : (\forall n) x(n) < x(n+1)\}$ by identifying an element of $[\omega]^\omega$ with its increasing enumeration. We endow $[\omega]^\omega$ with this Polish topology. Say $X \subseteq [\omega]^\omega$ has the **Ramsey property** if there exists an infinite $A \subseteq \omega$ such that $[A]^\omega \subseteq X$, or $[A]^\omega \cap X = \emptyset$.*

The Ramsey property is connected with Ramsey's theorem in the following way. Suppose $f : [\omega]^2 \to 2$. Then to f we can associate the open set $X_f = \{A \in [\omega]^\omega : f(\{A(0), A(1)\}) = 0\}$. (Here by $A(0)$ we mean the least element of A, and by $A(1)$ we mean the least element of $A \setminus \min(A)$.) Then if $[A]^\omega \subseteq X_f$ or $[A]^\omega \cap X_f = \emptyset$, then A is f-homogeneous.

Ramsey's theorem asserts that certain open subset of $[\omega]^\omega$ have the Ramsey property. However, the collection of sets with the Ramsey property is much larger:

Theorem 4.7 (Galvin-Prikry, Silver)**.** *Every $\mathbf{\Sigma}^1_1$ set has the Ramsey property.*

We will prove a pair of theorems due to Solovay. Our first will given an example of a closed set so that no witness to the fact that it has the Ramsey property can be in HYP. Our proof will rely on Kőnig's lemma:

Exercise 4.8 (Kőnig's lemma). *Let $T \subseteq \omega^{<\omega}$ be finitely branching, so each $t \in T$ has finitely many successor in T. Then T has an infinite branch iff T is infinite. Furthermore, T' can compute an infinite branch in T if it has one.*

Theorem 4.9 (Solovay). *There is a lightface Π^0_1 set $X \subseteq [\omega]^\omega$ such that if $A \in \mathsf{HYP}$, then neither $[A]^\omega \subseteq X$ nor $[A]^\omega \cap X = \emptyset$.*

Proof. Let $T \subseteq \omega^{<\omega}$ be a computable illfounded tree with no HYP branches. Let $T' = \{t \in \omega^{<\omega} \colon \exists(s \in T)|s| = |t| \wedge (\forall n \in \mathrm{dom}(t))s(n) \leq t(n)\}$. It is easy to see that T' is also a computable illfounded tree, and $[T'] = \{x \in \omega^\omega \colon \exists y \in [T](\forall n)(y(n) \leq x(n)\}$ by Kőnig's lemma.

Using the bijection between increasing elements of ω^ω and $[\omega]^\omega$, let $X = [T'] \cap [\omega]^\omega$. Now since $[T']$ is closed upward under $\leq$, it is clear there is no A such that $[A]^\omega \cap X = \emptyset$. However, no $x \in X$ is in HYP. This is because if $x \in \mathsf{HYP}$, then $S = \{s \in \omega^{<\omega} \colon (\forall n \in \mathrm{dom}(s))s(n) \leq x(n)\}$, is a finitely branching tree. If there was an infinite branch in $S \cap T$, then it would be computable in x' by Kőnig's lemma. ☐

Solovay's proof of Theorem 4.5 was based on the following contrasting result:

Theorem 4.10 (Solovay). *Suppose $X \subseteq [\omega]^\omega$ is open, and $\forall A \in [\omega]^\omega([A]^\omega \cap X \neq \emptyset)$. Then there is exists $A \in \mathsf{HYP}$ such that $[A]^\omega \subseteq X$.*

Our proof of this Theorem is due to Avigad [A], and is based on the following proof that all open sets have the Ramsey property:

Lemma 4.11. *If $X \subseteq [\omega]^\omega$ is open, then X has the Ramsey property.*

Proof. For this proof, we view X as a subset of the increasing sequences in ω^ω, and only work with $s \in \omega^{<\omega}$ and $x \in \omega^\omega$, that are increasing. Let U be a nonprincipal ultrafilter on ω.

Let S be an upwards closed set determining X, so $X = \{x \colon (\exists s \in S)s \subseteq x\}$, and if $s \in S$ and $s \subseteq s'$, then $s' \in S$. Say that an increasing sequence $s \in \omega^{<\omega}$ is **0-good** if $s \in S$. Say that s is **α-good** if $\{n \colon s^\frown n \text{ is } \beta\text{-good for some } \beta < \alpha\} \in U$. Say that s is **bad** if s is not α-good for any alpha.

Case 1: $\emptyset$ is α-good for some α. Then we build an infinite set $A \subseteq \omega$ by recursion as follows. Let $s_0 = \emptyset$. Suppose we have determined the first n elements s_n of A, where every subsequence of s_n is β-good for some β. Now let $s^0_n, \ldots, s^k_n$ be all subsequences of s_n. For each $i \leq k$, let $V^i_k = \{n \colon \alpha \text{ is least such that } s^i_n \text{ is } \alpha\text{-good, and either } \alpha = 0, \text{ or } s^i_n{}^\frown n \text{ is } \beta\text{-good for } \beta < \alpha\}$. So $V^i_n \in U$. Let $V_n = \bigcap_{i \leq k} V^i_n$, so $V_n \in U$ is nonempty. Finally, let $s_{n+1} = s_n{}^\frown m$ where $m \in V_n$.

We claim any subset of A is in X. Suppose $B = \bigcup_n t_n \in [A]^\omega$ where t_n is the finite initial segment of B of length n. Then by construction, t_n is α_n-good for some ordinal α, and for every n, either $\alpha_n > \alpha_{n+1}$, or $\alpha_n = 0$. Because there is no infinite descending sequence of ordinals, there must therefore be some n so t_n is 0-good. So $B \in X$.

Case 2: $\emptyset$ is bad. The we build s_n by recursion as in Case 1 where every subsequence of s_n is bad. Let $A = \bigcup_n s_n$. Then $[A]^\omega \cap X = \emptyset$. This is because if t is an initial segment of some $B \in [A]^\omega$, then t is bad, and hence t does not witness $B \in X$. $\square$

Exercise 4.12. *Every Borel set has the Ramsey property.*

To prove Theorem 4.10, we will effectivize Lemma 4.11. We will use the fact that U does not need to be an ultrafilter; it can be a countable filter which decides the countably many sets used in the definition of goodness. We further use the fact that if no $[A]^\omega$ is disjoint from X, then in this analysis, we must have that $\emptyset$ must be 0-good, and since this is a computable transfinite process, it must terminate in $< \omega_1^{\mathrm{ck}}$ many steps.

Proof of Theorem 4.10: Let $S \subseteq \omega^{<\omega}$ be so that $X = \{x \colon (\exists s \in S) s \subseteq x\}$. Since X is Σ^0_1, S is computable. We may assume that S is closed upwards.

Let T be the tree of attempts to build some infinite $A \subseteq \omega$ so that $[A]^\omega \cap X = \emptyset$. That is, $T = \{s \colon$ s is increasing and for all subsequences t of s, $t \notin S$. T is a computable tree. It is wellfounded since an infinite branch would yield an infinite $A \subseteq \omega$ such that $[A]^\omega \subseteq X$.

Say s is 0-good if $s \in S$. Say that s is bad if $s \notin S$ and $s \notin T$. Now we build infinite sets $B_\alpha \subseteq \omega$ which are decreasing mod finite. We think of these sets as generating an increasing sequence of filters $U_\alpha = \{B \colon B \supseteq^* B_\alpha\}$ on ω. We classify each $s \in T$ as good or bad as follows. We begin with $B_0 = \omega$ and proceed by transfinite induction. At stage α we consider the element s of rank α in the Kleene-Brouwer order $\leq_{KB} \restriction T$ (so in particular we have already classified all extensions of s as good or bad). Consider $A_s = \{n \colon s^\frown n$ is β-good for $\beta < \alpha\}$. If $A_s \cap B_\alpha$ is finite (so $\overline{A_s} \in U_\alpha$), then let $B_{\alpha+1} = B_\alpha$, and say s is bad. Otherwise, say s is α-good, and let $B_{\alpha+1} = B_\alpha \cap A_s$. At limit stages, let B_α be the diagonal intersection of $B_\beta \colon \beta < \alpha$, so $U_\alpha \supseteq U_\beta$ for $\beta < \alpha$.

We finish as in Lemma 4.11. It must be that $\emptyset$ is α-good for some α, otherwise we can construct some infinite $A \subseteq \omega$ so that $[A]^\omega \cap X = \emptyset$. It is an easy exercise that we can carry out the above construction and the construction in Case 1 of Lemma 4.11 computably from $\emptyset^{\alpha+3}$ where α is the ordertype of $\leq_{KB} \restriction T$. $\square$

4.3 Coloring graphs generated by single Borel functions

A consequence of many dichotomies in descriptive set theory is that they lower the complexity of the concepts involved from the obvious upper bounds. For example, the set of closed sets $C \subseteq \omega^\omega$ such that $\pi[C]$ is a $\mathbf{\Sigma}^1_1$ graph on ω^ω that admit a Borel countable coloring is naively Σ^1_2, but since the set of graphs which admit a continuous homomorphism from G_0 is also Σ^1_2, the set of analytic graphs which admit a countable Borel coloring is Δ^1_2 by the G_0 dichotomy (Theorem 3.20).

In contrast then, a proof that some concept is Σ^1_2 complete is often a strong *anti*-dichotomy result.

Some recent results of this type concerns graphs generated by single functions. If $f\colon X \to X$ is a Borel function on a Polish space X, let G_f be the graph where $x_0, x_1 \in X$ are adjacent if $x_0 \neq x_1$, and $f(x_0) = x_1$ or $f(x_1) = x_0$.

Consider the shift function $f\colon [\omega]^\omega \to [\omega]^\omega$ on Ramsey space defined by:

$$S(A) = A \setminus \min(A)$$

The associated graph G_S has no countable Borel coloring.

Exercise 4.13. *G_S has no countable Borel coloring. [Hint: use the fact that every Borel set has the Ramsey property]*

For a long time, it was an open question whether for any Borel function $f\colon X \to X$ on a Polish space X, either G_f has a finite coloring, or there is a Borel homomorphism from G_S to G_f. This was answered negatively by Pequignot [P], who used a result of Marcone to show that the following:

Theorem 4.14 ([P])**.** *The set of codes for Borel functions f such that there is a Borel homomorphism from G_S to G_f is Σ^1_2 complete.*

Shortly afterward, Todorcevic and Vidnyánszky ruled out any sort of dichotomy for countable colorability of graphs generated by a single function with the following result.

Theorem 4.15 ([TV])**.** *The collection of closed set $C \subseteq [\omega]^\omega$ such that $G_S \restriction C$ is finitely Borel colorable is Σ^1_2 complete.*

Their proof uses ideas from the previous section, and has at its core the following construction of a Δ^1_1 set $C \subseteq [\omega]^\omega$ so that $G_s \restriction C$ admits a finite Borel coloring, but no finite Δ^1_1 coloring.

Exercise 4.16. *Suppose $f\colon X \to X$ is a Borel function on a Polish space. The following are equivalent*

1. *There is finite Borel coloring of G_f.*
2. *There is a G_f-independent set $A \subseteq X$ which is* ***forward-recurrent*** *that is, $\forall x \in X \exists n > 0 f^n(x) \in A$. [Hint: Let $A = \{x\colon c(x)$ is minimal such that $\forall n \exists m > n c(f^m(x)) = c(x)\}$.]*
3. *There is a Borel 3-coloring of G_f [Hint: let $c(x) = 0$ if $x \in A$, otherwise, if n is least such that $f^n(x) \in A$, then $c(x) = 1$ if n is odd, and $c(x) = 2$ if n is even.*

Exercise 4.17 (DiPrisco, Todorcevic)**.** *Identify $[\omega]^\omega$ with increasing functions in ω^ω as usual. For every x, $G_S \restriction \{y\colon (\exists n) y(n) \leq x(n)\}$ has a Borel 3-coloring.*

Exercise 4.18. *Let $A \subseteq \omega$ be Σ^1_1 complete (and hence not $\mathbf{\Pi}^1_1$).*

1. *Show that $A = \pi[C]$ where $C \subseteq \omega \times \omega^\omega$ is a Π^0_1 set such that if $(n, x) \in C$ and $(\forall n) y(n) \geq x(n)$, then $(n, y) \in C$. [Hint: use the idea in Theorem 4.9]*
2. *Let $C_n = \{x \colon (n, x) \in C\}$. Show that $G_S \restriction [\omega]^\omega \setminus C_n\}$ has a Borel finite coloring iff $n \in A$.*
3. *Show that $\{n \colon G_S \restriction [\omega]^\omega \setminus C_n$ has a Δ^1_1 finite coloring is Π^1_1.*
4. *Conclude there is some n such that $G_S \restriction [\omega]^\omega \setminus C_n\}$ has a Borel 3-coloring, but no finite Δ^1_1 coloring.*

4.4 Π^0_1 games

The perfect set property, the Baire property, and Lebesgue measurability can be proved for definable sets using games. In this section we effectively analyze games with Π^0_1 payoff sets. We will use this analysis in the next few sections to effectively analyze the Baire property, and Lebesgue measurability.

Definition 4.19. *If $T \subseteq \omega^{<\omega}$, let $G(T)$ be the two-player game:*

$$\begin{array}{cccccc} I & n_0 & & n_2 & & n_4 \\ II & & n_1 & & n_3 & \end{array}$$

where the players alternate playing inters k_k, and letting $s_k = (n_0, \ldots, n_{k-1})$, player I wins iff player I wins if $\forall n s_k \in T$.

Definition 4.20. *A **strategy for player I** is a map $\sigma \colon \omega^{<\omega} \to \omega^{<\omega}$ such that $|\sigma(s)| = |s| + 1$, and $s \subseteq t \to \sigma(s) \subseteq \sigma(t)$. Here if $\sigma((n_1, n_3, \ldots, n_{2k-1})) = (n_0, n_2, \ldots, n_2 k)$, then $n_0, n_2, \ldots, n_{2k}$ are player I's moves when player II plays $n_1, n_3, \ldots n_{2k-1}$. We say this strategy is a **winning strategy** if for possible player of player II $n_0, n_2, \ldots$, if player I plays according to σ, then player I wins. We define a strategy for player II similarly.*

Theorem 4.21. *Suppose T is a computable tree. Then either player I or player II has a winning strategy in $G(T)$. Furthermore,*

1. *If II wins, then there is a* HYP *winning strategy.*
2. *$\mathcal{O}$ can uniformly compute whether player I or player II has a winning strategy in the game $G(T)$, and a strategy for this player.*

Proof. Assuming determinacy for closed sets, (1) is trivial. Using a computable bijection between ω and $\omega^{<\omega}$, we can view a real $\sigma \in \omega^\omega$ as a strategy for I. Then it is easy to see that the set of winning strategies for I is a Π^0_1 set. Similarly for II. $\mathcal{O}$ can compute which of these sets is nonempty, and a strategy for the associated player.

To prove (2) we will prove determinacy for closed sets using an ordinal analysis, and then effectivize this proof. Define the following notion of rank for $s \in \omega^\omega$ even length in T as follows. Let $T_0 = \{s \in T \colon (\exists k)|s| = 2k\}$.

- $T_{\alpha+1} = T_\alpha \setminus (n_0, \ldots, n_{2k-1})\forall n_{2k} \exists n_{2k+1}(n_0, \ldots, n_{2k-1}, n_{2k}, n_{2k+1}) \notin T_\alpha\}$
- For limit λ, $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$.

Let $\operatorname{rank}(s)$ be the least α such that $s \notin T_{\alpha+1}$, if such as α exists, and $\operatorname{rank}(s) = \infty$ otherwise.

If $\operatorname{rank}(\emptyset) = \infty$, then I has a winning strategy; they should play moves n_{2k} so that $\forall n_{2k+1}$, the node $(n_0, \ldots, n_{2k+1})$ has rank ∞. The definition of our rank ensure that if $(n_0, \ldots, n_{2k-1})$ has rank ∞, then $(\exists n_{2k})(\forall n_{2k+1})$ so that $(n_0, \ldots, n_{2k+1})$ has rank ∞.

Now suppose $\operatorname{rank}(\emptyset) = \alpha$ for some countable ordinal α. Then we claim II has a winning strategy. By the definition of our rank, we can ensure that each of their moves n_{2k+1} produces a player $(n_0, \ldots, n_{2k+1})$ which is either not in T, or has smaller rank than their previous move. Since there is not an infinite descending sequence of ordinals, II will eventually win.

Now if II has a winning strategy, we claim that we can find a HYP winning strategy for II. Consider the computable wellfounded tree S of attempts to build a winning strategy for player II. By an argument very similar to our proof of the effective perfect set theorem, Theorem 2.4 we can consider the usual derivative process on this tree S of strategies, and can relate it to the rank on T described above, so that we will have ranked all the even length nodes of T once we have finished the computable ordinal length derivative process on S. Thus, the ranking described above stabilizes at some computable ordinal α, and by an easy effective transfinite recursion, $\emptyset^{(\alpha+2)}$ can compute a winning strategy for II. $\square$

Exercise 4.22. *Suppose T is a Δ^1_1 tree, and II wins the game $G(T)$. Then show that II has a hyperarithmetic winning strategy.*

4.5 Effective analysis via games: Baire category

Now we can apply the analysis of Section 4.4 to the unfolded Banach Mazur game to analyze Baire category.

Definition 4.23. *Suppose $A = \pi[T]$ is a Σ^1_1 set, where $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ is a computable tree. The unfolded Banach Mazur game $G^*(T)$ is the game:*

$$\begin{array}{lccccc} I & (s_0, t_0) & & (s_1, t_1) & & (s_2, t_2) \\ II & & s_0^* & & s_1^* & \end{array}$$

where $s_0 \subseteq s_0^* \subseteq s_1 \subseteq s_1^* \subseteq \ldots$ and $t_0 \subseteq t_1 \subseteq \ldots$, and I wins if $\forall n (s_n, t_n) \in T$.

Exercise 4.24. *If II has a winning strategy in the unfolded Banach-Mazur game $G^*(T)$, then $A = \pi[T]$ is meager. If I has a winning strategy in the unfolded Banach-Mazur game $G^*(T)$, then A is comeager in some basic open set N_s. [Hint: use the winning strategies to define countably many dense open sets]*

From the above, by a transfinite process pruning away basic open sets inside which A is comeager, we get the following:

Exercise 4.25. *Every Σ^1_1 set has the Baire property.*

We can code the unfolded Banach-Mazur by a standard game $G(T)$ of the sort considered in the previous section. From this we obtain the following:

Lemma 4.26. *Let $U \subseteq \omega \times \omega^\omega$ be a universal Σ^1_1 set. Then $\{n : U_n \text{ is meager}\}$ is Π^1_1.*

Proof. U_n is meager iff II has no winning strategy in the unfolded Banach-Mazur game iff if the tree of strategies for player II described in Theorem 4.21 is wellfounded. □

Theorem 4.27 (Thomason-Hinman's basis theorem)**.** *If $A \subseteq \omega^\omega$ is Π^1_1 and comeager, then there is an $x \in A$ so that $x \in \mathsf{HYP}$.*

Proof. Since the complement of A is Σ^1_1 and meager, there must be a winning strategy for II in the corresponding unfolded Banach-Mazur game. By coding this game into a standard game of the form $G(T)$, we have that since II has a winning strategy in this game, it has a HYP winning strategy. Now playing this winning strategy against any computable sequence of moves for player I produces a HYP real in A. □

Corollary 4.28. *$\{x : \omega_1^x = \omega_1^{ck}\}$ is comeager.*

Proof. Since $\{x : \omega_1^x = \omega_1^{\mathrm{ck}}\}$ is a tailset, it is either meager or comeager. If it was meager, then its complement would have a HYP element by Theorem 4.27. □

Exercise 4.29 (Hyperjump inversion)**.** *$x \geq_h \mathcal{O}$ iff $\exists y \mathcal{O}^y \equiv_h x$. [Hint: follow the proof of Friedberg jump inversion in classical recursion theory, using the fact that $\mathcal{O}$ can compute winning strategies in the Banach-Mazur game for Σ^1_1 sets to replace the classical fact that $0'$ can compute the strong forcing relation for Σ^0_1 sentences]*

We can use similar game-based techniques to analyze the effectivity of measurability. Instead, we'll give an alternate approach using scales, which also works for Baire category.

4.6 Effective analysis via scales: measure

To begin, we have the following important characterization of measure for analytic sets, which follows from the fact that the measure of a set is the sup of the measures of its compact subsets (see Exercise B.2).

Exercise 4.30. *Suppose μ is a Borel probability measure on ω^ω, and $A \subseteq \omega^\omega$ is a Σ^1_1 set; the projection of a computable tree T, so $A = \pi[T]$. Show that $\mu(A)$ is the sup of $\mu(\pi[T'])$ over all finitely branching $T' \subseteq T$. [Hint: first uniformize $[T]$ with a function f, use the measurability of the function $f : A \to \omega^\omega$, and then argue that for every $\epsilon > 0$, we can find a compact set $A_\epsilon \subseteq A$ so that $\mu(A_\epsilon) > \mu(A) - \epsilon$, and $f \restriction A_\epsilon$ is continuous.]*

The importance of this exercise is that its lets us compute the complexity of measurability.

Exercise 4.31. *For every rational number r, and computable measure μ on ω^ω, if $U \subseteq \omega^\omega \times \omega^\omega$ is Σ^1_1, then $\{x \colon \mu(U_x) > r\}$ is Σ^1_1. Hence $\{x \colon \mu(U_x) = 1\}$ is Σ^1_1 and $\{x \colon \mu(U_x) = 0\}$ is Π^1_1. [Hint: begin by showing that if $A \subseteq \omega^\omega \times \omega^\omega$ is Π^0_1, then $\{x \colon \mu(U_x) > r\}$ is arithmetic. Then use Exercise 4.30]*

From Exercise 4.31 we have the following important property of the null ideal:

Corollary 4.32. *Let μ be a a computable Borel probability measure, and $\mathcal{I}_\mu$ be the σ-ideals of nullsets of μ. Then if $A \subseteq \omega^\omega \times \omega^\omega$ is Π^1_1, then $\{x \colon A_x \notin I_\mu\}$ and $\{x \colon \omega^\omega \setminus A_x \in I_\mu\}$ is Π^1_1.*

Definition 4.33. *Say that an ideals $\mathcal{I}$ is Π^1_1 additive if for any transfinite sequence $(A_\alpha)_{\alpha<\lambda}$ of sets $A_\alpha \in I$, if the relation $\leq$ on $A = \bigcup_{\alpha<\lambda} A_\alpha$ defined by $x \leq y$ iff (the least α such that $x \in A_\alpha$) $\leq$ (the least β such that $y \in A_\beta$)) is Π^1_1, then $A \in \mathcal{I}$.*

So for example, if μ is a computable measure on ω^ω, then I_μ is Π^1_1 additive by Exercise B.5.

Exercise 4.34. *Show that the ideal of measure sets in ω^ω is Π^1_1 additive. [Hint: use the Kuratowski-Ulam theorem]*

Theorem 4.35. *Suppose $\mathcal{I}$ is a Π^1_1-additive ideal of subset of ω^ω such that for every Π^1_1 set $C \subseteq \omega^\omega \times \omega^\omega$, we have $\{x \colon C_x \notin \mathcal{I}\}$ and $\{x \colon \omega^\omega \setminus C_x \in \mathcal{I}\}$ are Π^1_1. (For example, $\mathcal{I} = \mathcal{I}_\mu$ for a computable Borel probability measure μ). Then if $A \subseteq \omega^\omega$ is Π^1_1 and $A \notin \mathcal{I}$, then there is some $x \in A$ so that $x \in \mathsf{HYP}$.*

Proof. Fix the very good Π^1_1 scale (φ_n) on C from Lemma 2.22. Then for each n, let $A_{n,\alpha} = \{x \in A \colon \varphi_n(x) = \alpha\}$ so $A = \bigcup_\alpha A_\alpha$. Since $A \notin \mathcal{I}$, there must be some α such that $A_{n,\alpha} \in \mathcal{I}$. Let $A_n = A_{n,\alpha}$ where α is least such that $A_{n,\alpha} \notin I$. So $A_{n,\alpha} = \{x \colon \{y \colon y \leq^*_{\varphi_n} x \wedge x \leq^*_{\varphi_n} y\} \notin I \wedge (\omega^\omega \setminus \{z \colon x \leq^*_{\varphi_n} z\}) \in \mathcal{I}\}$ is Π^1_1. Then $\bigcap_n A_n = \{x\}$, and we claim $x \in \mathsf{HYP}$.

The big difference between this theorem and the basis Theorem 2.23 for Π^1_1 sets in general is that determining whether a Π^1_1 set is nonempty is Σ^1_2. However, determining whether a Π^1_1 set is not in $\mathcal{I}$ is much simpler; it is Π^1_1. We have that $x \in N_s$ iff $N_s \cap A_n \notin \mathcal{I}$ (which is Π^1_1) iff for all t incompatible with s, $N_t \cap A_n \in \mathcal{I}$ (which is Σ^1_1). So x is Δ^1_1. □

Exercise 4.36. *If μ is a computable Borel probability measure on ω^ω, then $\mu(\{x \colon \omega^x_1 = \omega^{ck}\}) = 1$. [Hint: the complement of $\{x \colon \omega^x_1 = \omega^{ck}\}$ is Π^1_1, and if this set has positive measure, it would have a HYP element.]*

We note that the meager ideal also satisfies the hypothesis of Theorem 4.35.

Using the same idea, we can prove the following uniformization theorem for sets with large sections:

Exercise 4.37. *Suppose $\mathcal{I}$ is an ideal as in Theorem 4.35. Then if $A \subseteq \omega^\omega \times \omega^\omega$ is Π^1_1 and for every x, $A_x \notin \mathcal{I}$, then A has a Δ^1_1 uniformization.*

5 Admissible sets, admissible computability, KP

5.1 ω-models of KP

Kripke-Platek set theory, or KP, is the following system of axioms in the language of set theory, consisting roughly of ZF without the powerset axiom or infinity, and with only Δ_0 instances of separation and collection.

Definition 5.1. *The axioms of* KP *are*

1. *Extensionality:* $(\forall x)(\forall y)(x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y))$.
2. *Foundation:* $(\forall x)\,[((\exists y)y \in x) \to (\exists y \in x)(\forall z \in x)(z \notin y)]$
3. *Pairing:* $\forall x \forall y \exists z(x \in z \wedge y \in z)$
4. *Union:* $\forall x \exists y \forall z \in x \forall u \in z(u \in y)$
5. Δ_0*-separation. For every* Δ_0 *formula* φ*,* $\forall x \exists y \forall z(z \in x \leftrightarrow z \in y \wedge \varphi(z))$
6. Δ_0*-collection. For every* Δ_0 *formula* $\varphi(x, y)$*.* $\forall u(\forall x \in u \exists y \varphi(x, y)) \to (\exists v)(\forall x \in u)(\exists y \in v)(\varphi(x, y))$.

INF is the axiom of infinity: $(\exists x)[\emptyset \in x \wedge (\forall y \in x) y \cup \{y\} \in x]$. We'll often work with the system $\mathsf{KP} + \mathsf{INF}$.

KP is sufficient to prove a large amount of standard set theory. For example, KP proves that the ordinals (i.e. transitive sets wellordered by ϵ) are linearly ordered, and to define the rank function on sets. Recall that every linear order has a maximal wellordered initial segment, and if M is a model of KP, we let $s(M)$ be the ordertype of the maximal wellordered initial segment of M. We say that an ordinal of M is **standard** if it is in this wellordered initial segment.

The **standard part** of a model (M, E) is the set M' of $x \in M$ such that $M \models \operatorname{rank}(x) = \alpha$, and α is a standard ordinal. This is true if and only if the tree of E-descending sequences $(x_0, \ldots, x_k)$ where $x_k E x_{k-1} \ldots E x_0$ is wellfounded in the real universe and has rank α. We identify the standard part of M with its Mostowski collapse, and the relation E with ϵ. So for example, model M of $\mathsf{KP} + \mathsf{INF}$ is an ω-model iff $s(M) > \omega$ iff ω^M is the standard ω.

However, KP is still too weak to prove some basic facts, such as the following important example. We'll show below that $L_{\omega_1^{\mathrm{ck}}}$ is a model of KP, and that $\omega^\omega \cap L_{\omega_1^{\mathrm{ck}}} = \mathsf{HYP}$. Now there is an illfounded computable tree T. Since T is computable, $T \in L_{\omega_1^{\mathrm{ck}}}$. However, $L_{\omega_1^{\mathrm{ck}}} \models T$ is wellfounded, since $L_{\omega_1^{\mathrm{ck}}}$ contains only HYP reals and hence no infinite descending sequence in T.

In contrast, we have the following lemma.

Lemma 5.2. *If* $T \subseteq \omega^{<\omega}$ *is a wellfounded tree (in the real universe), and* M *is an* ω*-model of* $\mathsf{KP} + \mathsf{INF}$*, then* $\operatorname{rank}(T) \in M$*.*

Proof. By transfinite induction. Suppose that for all $s \in T$ with $|s| \geq 1$, the function $\operatorname{rank}_{T_s} \colon T_s \to \mathsf{ORD}$ is in M. This function is Δ_0 definable from s, hence by Δ_0 collection, there is a set of all such rank functions. But then by Δ_0

collection and the union axiom, M contains the set $\{\mathrm{rank}(T_s)\colon s \in T \wedge |s| \geq 1\}$. But this set is the ordinal $\mathrm{rank}(T)$. So the rank function $\mathrm{rank}_T\colon T \to \mathsf{ORD}$ is in M. □

Corollary 5.3. *If M is an ω-model of $\mathsf{KP} + \mathsf{INF}$, then $s(M) \geq \omega_1^{ck}$.*

The analysis of L can be developed in KP, which is powerful enough to prove that $L \models V = L$.

Exercise 5.4. *The function $\alpha \mapsto L_\alpha$ from ordinals to sets L_α is a Σ_1 (in fact Δ_1) definable function in KP. If M is a model of KP, then $L^M = \{x \in M\colon M \models x \in L\}$ is a model of $\mathsf{KP} + \mathsf{V} = L$. If M is transitive, then $L^M = L_\alpha$ for some α.*

Now from KP, we can prove stronger forms of the separation and collection axioms.

Exercise 5.5. *Δ_1 separation is provable from KP.*

Exercise 5.6. *Σ_1 collection is provable from KP.*

Exercise 5.7. *If M is a ω-model of $\mathsf{KP} + \mathsf{INF}$, then the standard part of M is also an ω-model of KP. [Hint: this is trivial for all the axioms except Δ_0 separation.]*

Exercise 5.8. *For every $x \notin HYP$, there is an ω-model of $\mathsf{KP} + \mathsf{INF}$ that does not contain X. [Hint: the set of countable ω-models of $\mathsf{KP} + \mathsf{INF}$ form a Δ^1_1 set. Then use Theorem 3.13]*

5.2 The Spector-Gandy theorem

Theorem 5.9. *$L_{\omega_1^{ck}}$ is a model of $\mathsf{KP} + \mathsf{INF}$. It is the minimal ω-model of $\mathsf{KP} + \mathsf{INF}$.*

Proof. There is a transitive model M of $\mathsf{KP} + \mathsf{INF}$ that does not contain $\mathcal{O}$. (Use Exercise 5.8, then take the standard part of an ω-model of $\mathsf{KP} + \mathsf{INF}$ not contain $\mathcal{O}$.). We claim ω_1^{ck} is not in this model. If it was, then $L_{\omega_1^{\mathrm{ck}}}$ would also be an element of M. But then $\mathcal{O}$ would in M; the tree T_n computable by φ_n is wellfounded iff there is a function $f \in L_{\omega_1^{\mathrm{ck}}}$ such that f ranks the tree T_n. Hence $s(M) = \omega_1^{\mathrm{ck}}$. Finally, this implies $L^M = L_{\omega_1^{\mathrm{ck}}}$.

Given any ω-model M of $\mathsf{KP} + \mathsf{INF}$, if M' is the standard part of M, then $L^{M'}$ is also a model of KP by Exercises 5.7 and 5.4. Finally, $L_{\omega_1^{\mathrm{ck}}} \subseteq L^{M'}$ by Lemma 5.2. □

Corollary 5.10. *If $\varphi(n)$ is a Σ_1 formula, then for all $n \in \omega$, $L_{\omega_1^{ck}} \models \varphi(n)$ iff for every ω-model M of $\mathsf{KP} + \mathsf{INF}$, $M \models \varphi(n)$.*

Proof. If $M \models \neg\varphi(n)$, then by downwards absoluteness of Π_1 formulas, $L_{\omega_1^{\mathrm{ck}}} \models \neg\varphi(n)$, since $L_{\omega_1^{\mathrm{ck}}}$ is the minimal model. □

Theorem 5.11 (Spector-Gandy). *$A \subseteq \omega$ is Π^1_1 iff there is a Σ_1 formula φ so that $n \in A \leftrightarrow L_{\omega_1^{ck}} \models \varphi(n)$.*

Proof. Fix a computable map $n \mapsto T_n$ so that $n \in A$ iff T_n is wellfounded. Then $n \in A$ iff in $L_{\omega_1^{\mathrm{ck}}}$ there is a function $f\colon T_n \to \mathsf{ORD}$ so that $s \subsetneq t$ implies $f(s) > f(t)$.

Conversely, suppose $\varphi(n)$ is Σ_1, and $n \in A \leftrightarrow L_{\omega_1^{\mathrm{ck}}} \models \varphi(n)$. Then $n \in A$ iff for every *omega*-model of $\mathsf{KP} + \mathsf{INF}$, $\varphi(n)$ is true. This is Π^1_1, since the set of ω-models of $\mathsf{KP} + \mathsf{INF}$ is Δ^1_1. $\square$

We mention another variant of the Spector-Gandy theorem.

Exercise 5.12. *$A \subseteq \omega$ is Π^1_1 iff there is a arithmetical formula $\varphi(x, n)$ so that $n \in A \leftrightarrow \exists x \in \mathsf{HYP}\varphi(x, n)$. [Hint: $\leftarrow$ is trivial. For $\rightarrow$. Fix a computable map $n \mapsto T_n$ so $n \in A$ iff T_n is wellfounded. Then T_n is wellfounded iff there exists a map $x\colon T_n \to \omega^\omega$ so that for all $s, t \in T_n$, if $s \subsetneq t$, then $x(s) \geq_T x(t)'$, where $x(t)'$ is the Turing jump of $x(t)$. Show that if x is such a function, then if $\rho(s)$ is the least α such that $x(s) \leq_T \emptyset^{(\alpha)}$, then $s \subsetneq t$ implies $\rho(s) > \rho(t)$.]*

A Baire category

In this section we very briefly give an overview of Baire category. For more, see [K].

The notion of Baire category concerns topological smallness notions. Recall that if X is a topological space, then a subset $A \subseteq X$ is **nowhere dense** if for every open U there exists an open $V \subseteq U$ such that $A \cap V = \emptyset$. This is a natural notion of "topological smallness," but it has a defect of not being closed under countable unions. To remedy this we say a set $A \subseteq X$ is **meager** if it a countable union of nowhere dense set. This is the type of topological which defines Baire category. The Baire category theorem says for nice X, the whole space is not small in this sense.

Theorem A.1 (Baire category theorem)**.** *Suppose X is a complete metric space. Then X is not meager, and hence X is not a countable union of meager sets.*

We say $A \subseteq X$ is **comeager** if $X \setminus A$ is meager. We say that at set $A \subseteq X$ is **Baire measurable** if A differs from an open set by a meager set; there is an open U such that $A \triangle U$ is meager.

Exercise A.2. *Suppose $A \subseteq \omega^\omega$ is Σ^1_1. Then A is Baire measurable.*

Exercise A.3. *Show that $\{x \in \omega^\omega : \omega_1^x = \omega_1^{ck}\}$ is comeager.*

If $A \subseteq X$, and $U \subseteq X$ is open, we say that A is **comeager inside** U and we write $U \Vdash A$ if $U \setminus A$ is meager. Note that this does not require A to be a subset U. Indeed if $U \Vdash A$, then for all open $V \subseteq U$, $V \Vdash A$.

Exercise A.4. *If X is Polish and $A \subseteq X$ is Baire measurable, then A is nonmeager iff there is some nonempty open set U such that A is comeager in U.*

Note that if U and V are open, and $f : U \to V$ is a homeomorphism, then since f preserves notions of density, nowhere density, and meagerness, $A \subseteq U$ is meager (resp. comeager) in U iff $F(A)$ is meager (resp. comeager) in V.

Exercise A.5 (Mycielski)**.** *If X is a perfect Polish space and R is a meager relation, then there is a perfect closed set $C \subseteq X$ of R-inequivalent elements.*

Exercise A.6 (Kuratowski-Ulam)**.** *Suppose X, Y are Polish spaces and $A \subseteq X \times Y$ has the Baire property. Then A is meager iff for a comeager set of x, A_x is meager.*

B Measure

A **Borel probability measure** μ on a Polish space X is a countably additive measure μ defined on the Borel subset of X, and such that $\mu(X) = 1$. Se say that $A \subseteq X$ is a nullset if $A \subseteq B$ for some Borel set such that $\mu(B) = 0$. We say that a set $A \subseteq X$ is μ-measurable if it differs from a Borel set (equivalently a G_δ set) by a nullset.

Exercise B.1. *If μ is a Borel probability measure, then μ is determined by its values on basic open sets.*

Exercise B.2. *If μ is a Borel probability measure on X and $A \subseteq X$ is μ-measurable then $\mu(A) = \sup_{K \text{ compact}} \mu(K) = \inf_{U \text{ open}} \mu(U)$.*

Exercise B.3. *If μ is a Borel probability measure on X, and $A \subseteq X$ is $\mathbf{\Sigma}^1_1$, then A is μ-measurable.*

Exercise B.4. *Suppose X, Y are Polish spaces, μ is a Borel probability measure on X, and $f\colon X \to Y$ is μ-measurable. Then show that for every $\epsilon > 0$, there is a Borel set $A \subseteq X$ so that $\mu(A) > 1 - \epsilon$, and $f \restriction A$ is continuous.*

Exercise B.5. *Suppose X is a Polish space, μ is a Borel probability measure on X, and $(A_\alpha)_{\alpha<\lambda}$ is a sequence of sets $A_\alpha \subseteq X$, where each A_α is a μ-nullset. Let $A = \bigcup_{\alpha<\lambda} A_\alpha$ and define the relation $\leq$ on A by $x \leq y$ iff α is least such that $x \in A_\alpha$, β is least such that $y \in A_\beta$, and $\alpha \leq \beta$. Then if $\leq$ is $\mu \times \mu$ measurable, then A is a nullset. [Hint: use Fubini's theorem]*

We'll often deal with computable measure.

Definition B.6. *Say that a Borel probability measure μ on ω^ω is* ***computable*** *if there is a computable function f from $\omega^{<\omega} \times \omega \to Q \times Q$ so that if $f(s, n) = (a_{s,n}, b_{s,n})$, then $\mu(N_s) \in [a_{s,n}, b_{s,n}]$, and $|b_{s,n} - a_{s,n}| \leq 1/2^n|$. That is $[a_{s,n}, b_{s,n}]$ is a sequence of closed intervals of length at most $1/2^n$ containing the measure of the basic open set $\mu(N_s)$.*

So for example, Lebesgue measure λ on 2^ω is a computable measure.

References

[A] J. Avigad, An effective proof that open sets are Ramsey, *Archive for Math. Logic*, **37**, No. 4, (1998), 235–240.

[BD] H. Becker and R. Dougherty, On disjoint Borel uniformizations. *Adv. Math.* **146** (1999), No. 2, 167–174.

[H] G. Hjorth, A dichotomy theorem for turbulence, *J. Symbolic Logic*, **67**, No. 4, (2002), 1520–1540.

[HKL] L. Harrington, A.S. Kechris, A. Louveau, A Glimm-Effros dichotomy for Borel equivalence relations, *J. American Math. Soc.*, **3** No. 4, (1990), 903–928.

[HMS] L. Harrington, D. Marker, and S. Shelah, Borel orderings, *Trans. American Math. Soc.*, **310** No. 1, (1988), 293–302.

[HSS] L. Harrington, R. Shore, and T. Slaman Σ^1_1 in every real in a Σ^1_1 class of reals is Σ^1_1 Computability and complexity, 455–466, *Lecture Notes in Comput. Sci.*, 10010, Springer, Cham, 2017.

[N] I. Neeman, Necessary use of Σ^1_1 induction in a reversal. *J. Symbolic Logic* **76** (2011), No. 2, 561–574.

[K] A.S. Kechris, *Classical descriptive set theory*, Springer, 1995.

[KL] A.S. Kechris and A. Louveau, The classification of hypersmooth Borel equivalence relations, *J. American Math. Soc*,**10**, No. 1, (1997), 215–242.

[KST] A.S. Kechris, S. Solecki and S. Todorcevic, *Borel chromatic numbers*, Adv. Math., **141** (1999), 1–44.

[KW] A.S. Kechris, and W.H. Woodin, Ranks of differentiable functions, *Mathematika*, **33** (1986), 252–278.

[Ma] D. A. Martin, Proof of a conjecture of Friedman, *Proc. American Math. Soc*, **55**, No. 1 (1976) 129.

[Mi] B. Miller, *Paris lectures*, `http://www.logic.univie.ac.at/ millerb45/otherwork.html`.

[Mo] A. Montalbán, On the Π^1_1-separation principle *Math. Log. Q.* **54** (2008), No. 6, 563–578.

[P] Y. Pequignot, Finite versus infinite: an insufficient shift, *Advances in Mathematics*, **320** No. 7, (2017) , 244–249.

[St78] J. Steel, Forcing with tagged trees, *Ann. Math. Logic* **15** (1978), No. 1, 55–74.

[St75] J. Steel, Descending Sequences of Degrees, *J. Symbolic Logic*, **40**, No. 1 (1975) 59–61.

[So] S. Solecki, Decomposing Borel sets and functions and the structure of Baire class 1 functions, *J. American Math. Soc.*, **11**, (1998), 521–550.

[TV] S. Todorcevic, Z. Vidnyánszky, A complexity problem for Borel graphs. *arXiv:1710.05079.*

[WW] P. Wesolek and J. Williams, Chain conditions, elementary amenable groups, and descriptive set theory, *Groups Geom. Dyn.* **11** (2017), no. 2, 649–684.