Mirror Symmetry and Localization

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A. Physicists made conjectures based on physical principles, or formal mathematical arguments.

Localizations on infinite dimensional spaces: Path integrals. (Witten's proposed proof of mirror conjecture, 1990)

B. The mathematical proofs depend on **Localization Techniques** on various finite dimensional moduli spaces.

Both have interesting symmetry to work with.

Application of Atiyah-Bott formula to counting rational curves in the quintic threefold: Kontsevich, Ellingsrud-Stromme, Givental, Lian-Liu-Yau.

(1). Mirror Principle: a much more general principle to count curves of arbitrary genus.
Lian, K. Liu, Yau,
Asian J. Math. 1 (1997), no.4, 729–763
Asian J. Math. 3 (1999), no.1, 109–146
Asian J. Math. 3 (1999), no.4, 771–800
Surv. Differ. Geom. VII, 475–496

(2). Proof of the Hori-Vafa Formula
(Hori-Vafa, hep-th/0002222)
Lian, C.H. Liu, K. Liu, Yau, math.AG/0111256.

(3). Proof of the Mariño-Vafa Formula
(Mariño-Vafa, Contemp.Math.310,185–224)
C.-C. Liu, K. Liu, J. Zhou, math.AG/0306434.

(4). SYZ Conjecture and Duality:
Strominger, Yau, Zaslow, Nuclear Phys. B 479 (1996), no. 1-2, 243–259.

Common and Key technique:

Equivariant cohomology

$$H_T^*(X) = H^*(X \times_T ET)$$

where ET is the universal bundle of T.

Example: $ES^1 = \mathbf{P}^\infty$. If S^1 acts on $\mathbf{P}^n$ by

$$\lambda \cdot [Z_0, \ldots, Z_n] = [\lambda^{w_0} Z_0, \ldots, \lambda^{w_n} Z_n],$$

then

$$H_{S^1}(\mathbf{C}P^n; \mathbb{Q}) \cong \mathbb{Q}[H, \alpha]/\langle (H - w_0\alpha) \cdots (H - w_n\alpha) \rangle$$

Atiyah-Bott Localization Formula:
For $\omega \in H_T^*(X)$ an equivariant cohomology class, we have

$$\omega = \sum_E i_{E*} \left(\frac{i_E^* \omega}{e_T(E/X)} \right).$$

where E runs over all connected components of T fixed points set.

Functorial Localization Formula:

$f: X \to Y$ equivariant map. $F \subset Y$ a fixed component, $E \subset f^{-1}(F)$ fixed components in $f^{-1}(F)$. Let $f_0 = f|_E$, then

For $\omega \in H_T^*(X)$ an equivariant cohomology class, we have identity on F:

$$f_{0*}[\frac{i_E^*\omega}{e_T(E/X)}] = \frac{i_F^*(f_*\omega)}{e_T(F/Y)}.$$

This formula will be applied to various settings to prove the conjectures from physics:

It is used to push computations on complicated moduli space to simpler moduli space.

Remarks:

1. Consider the diagram:

$$\begin{array}{ccc} H^*_T(X) & \xrightarrow{f_*} & H^*_T(Y) \\ \downarrow\, {i_E}^* & & \downarrow\, {i_F}^* \\ H^*_T(E) & \xrightarrow{f_{0*}} & H^*_T(F)\,. \end{array}$$

   The functorial localization formula is like Riemann-Roch with the inverted equivariant Euler classes of the normal bundle as "weights", as the Todd class for the RR.

2. The mirror formulas, which are essentially given by integrals of the generating series of the left hand side of the functorial localization formula, is like a "graded" version of the index formula:

   The numerator of the right hand side replaced by a hypergeometric term, a simple equivariant class, through mirror transformations.

   Grading: the degrees of the moduli spaces of stable maps.

3. The main theme of mirror principle is to work out the numerator. The denominator is easily known if the linearized moduli is known.

   Otherwise more work needed: to be discussed more in the Hori-Vafa formula.

4. (Atiyah-Witten) Formally applied to loop spaces and the natural S^1-action, one gets the index formula: Chern characters are equivariant forms on loop space, and the $\widehat{A}$ genus is the inverse of the equivariant Euler class of the normal bundle of X in its loop space LX:

$$e_T(X/LX)^{-1} \sim \widehat{A}(X),$$

which follows from the infinite product

$$\left(\prod_{n\neq 0}(x+n)\right)^{-1} \sim \frac{x}{\sin x}.$$

5. (Kefeng Liu, Comm. Math. Phys. 174, 1995, no. 1, 29–42) The "normalized" product

$$\prod_{m,n}(x+m+n\tau) = 2q^{\frac{1}{8}}\sin(\pi x)$$
$$\cdot \prod_{j=1}^{\infty}(1-q^j)(1-e^{2\pi i x}q^j)(1-e^{-2\pi i x}q^j),$$

where $q = e^{2\pi i\tau}$. This is Eisenstein's formula, which is a double loop space analogue of the Atiyah-Witten observation. This is the basic Jacobi θ-function. Formally this gives the $\widehat{A}$ class of the loop space, and the Witten genus, the index of the Dirac operator on the loop space:

$$e_T(X/LLX) \sim \widehat{W}(X),$$

where LLX is the double loop space, the space of maps from $S^1 \times S^1$ into X.

6. A K-theory version of the above formula also holds, interesting applications expected.

7. The proofs of such formulas depend on Atiyah-Bott localization, or the Atiyah-Bott-Segal-Singer fixed point formula.

(1). Mirror Principle:

Compute characteristic numbers on moduli spaces of stable maps $\Leftarrow$ Hypergeometric type series:

Counting curves: Euler numbers.

Hirzebruch multiplicative classes, total Chern classes.

Marked points and general GW invariants.

We hope to develop a "black-box" method which makes easy the computations of the characteristic numbers and the GW-invariants on the moduli space of stable maps:

Starting data $\Longrightarrow$ Mirror Principle $\Longrightarrow$ Closed Formulas for the invariants.

General set-up: X: Projective manifold.

$\mathcal{M}_{g,k}(d, X)$: moduli space of stable maps of genus g and degree d with k marked points into X, modulo the obvious equivalence.

Points in $\mathcal{M}_{g,k}(d, X)$ are triples: $(f; C; x_1, \cdots, x_k)$:

$f : C \to X$: degree d holomorphic map;

$x_1, \cdots, x_k$: k distinct smooth points on the genus g curve C.

$f_*([C]) = d \in H_2(X, \mathbb{Z})$: identified as integral index $(d_1, \cdots, d_n)$ by choosing a basis of $H_2(X, , \mathbf{Z})$ (dual to the Kahler classes).

Virtual fundamental cycle of Li-Tian, (Behrend-Fantechi): $[\mathcal{M}_{g,k}(d,X)]^v$, a homology class of the expected dimension

$$2\left(c_1(TX)[d] + (\dim_{\mathbb{C}} X - 3)(1-g) + k\right)$$

on $\mathcal{M}_{g,k}(d,X)$.

Consider the case $k = 0$ first. The expected dimension is 0 if X is a Calabi-Yau 3-fold.

Mirror Principle motivated by physics: (P. Candelas, X. de la Ossa, P. Green, L. Parkes) The A-model potential of a Calabi-Yau 3-fold M is give by

$$\mathcal{F}_0(T) = \sum_{d \in H_2(M;\mathbb{Z})} K_d^0 e^{d \cdot T},$$

where $T = (T^1, \ldots, T^n)$ are coordiates of Kahler moduli of M, and K_d^0 is the genus zero, degree d GW-invariant of M.

Mirror conjecture asserts that there exists a mirror Calabi-Yau 3-fold M' with B model potential $\mathcal{G}(T)$, which can be computed by period integrals, such that

$$\mathcal{F}(T) = \mathcal{G}(t),$$

where t accounts for coordinates of complex moduli of M'. The map $t \mapsto T$ is the mirror map.

In the toric case, the period integrals are solutions to **GKZ-system** (Gelfand-Kapranov-Zelevinsky hypergeometric series).

Starting data:

V: concavex bundle on X, direct sum of a positive and a negative bundle on X.

V induces sequence of vector bundles V_d^g on $\mathcal{M}_{g,0}(d,X)$: $H^0(C, f^*V) \oplus H^1(C, f^*V)$.

b: a multiplicative characteristic class.
(So far for all application in string theory, b is the Euler class.)

Problem: Compute $K_d^g = \int_{[\mathcal{M}_{g,0}(d,X)]^v} b(V_d^g)$.

Compute

$$F(T,\lambda) = \sum_{d,\,g} K_d^g\,\lambda^g\ e^{d\cdot T}$$

in terms of hypergeometric type series.

Here λ, $T = (T_1, \cdots, , T_n)$ formal variables.

Key ingredients for the proof of the Mirror Principle:

(1). Linear and non-linear moduli spaces;

(2). Euler data and Hypergeometric Euler data (HG Euler data).

Non-linear moduli: $M_d^g(X)$ = stable map moduli of degree $(1, d)$ and genus g into $\mathbf{P}^1 \times X$ $= \{(f, C) : f : C \to \mathbf{P}^1 \times X\}$ with C a genus g (nodal) curve, modulo obvious equivalence.

Linearized moduli: W_d for toric X. (Witten, Aspinwall-Morrison):

Example: $\mathbf{P}^n$, $[z_0, \cdots, z_n]$

W_d: $[f_0(w_0, w_1), \cdots, f_n(w_0, w_1)]$
$f_j(w_0, w_1)$: homogeneous polynomials of degree d.

Simplest compactification.

Lemma:(LLY+Li; c.f. Givental for $g = 0$) There exists an explicit equivariant collapsing map

$$\varphi : \; M_d^g(\mathbf{P}^n) \longrightarrow W_d.$$

$M_d^g(X)$, embedded into $M_d^g(\mathbf{P}^n)$, is very "singular" and complicated. But W_d smooth and simple. The embedding induces a map of $M_d^g(X)$ to W_d.

Functorial localization formula connects the computations of mathematicians and physicists. Push all computations on the nonlinear moduli to the linearized moduli.

Mirror symmetry formula = Comparison of computations on nonlinear and linearized moduli!?

Balloon manifold X: Projective manifold with torus action and isolated fixed points. (introduced by Goresky-Kottwitz-MacPherson)

$$H = (H_1, \cdots, H_k)$$

a basis of equivariant Kahler classes.

(1). $H(p) \neq H(q)$ for any two fixed points.

(2). T_pX has linearly independent weights for any fixed point p.

Theorem: Mirror principle holds for balloon manifolds for any concavex bundles.

Remarks: All homogeneous and toric manifolds are balloon manifolds. For $g = 0$ we can identify HG series explicitly. Higher genus needs more work.

1. For toric manifolds and $g = 0 \Longrightarrow$ all mirror conjectural formulas from physics.

2. Grassmannian: Hori-Vafa formula.

3. Direct sum of positive line bundles on $\mathbf{P}^n$ (including the Candelas formula): Two independent approaches: Givental, Lian-Liu-Yau.

Ideas of Proof:

Apply functorial localization formula to φ, the collapsing map and the pull-back class $\omega = \pi^* b(V_d^g)$, where

$$\pi : \quad M_d^g(X) \to M_{g,0}(d, X)$$

is the natural projection.

Introduce the notion of **Euler Data**, which naturally appears on the right hand side of the functorial localization formula:

$$Q_d = \varphi_!(\pi^* b(V_d^g))$$

which is a sequence of polynomials in equivariant cohomology rings of the linearized moduli spaces (or restricted to X) with simple quadratic relations.

From functorial localization formula we prove that:

Knowing Euler data $Q_d \equiv$ knowing the K_d^g.

There is another much simpler Euler data, the **HG Euler data** P_d which coincides with Q_d on the "generic" part of the nonlinear moduli.

The quadratic relations + coincidence determine the Euler data uniquely up to certain degree.

Q_d always have the right degree for $g = 0$.

Mirror transformation used to reduce the degrees of the HG Euler data P_d.

Remark: Both the denominator and the numerator in the HG series are equivariant Euler classes. Especially the denominator is exactly from the localization formula. Easily seen from the functorial localization formula.

Remark: The quadratic relation of Euler data, which naturally comes from gluing and functorial localization on the A-model side, is closely related to special geometry, and is similar to the Bershadsky-Cecotti-Ooguri-Vafa's **holomorphic anomaly** equation on the B-model side.

Such relation can determine the polynomial Euler data up to certain degree.

It is interesting task to use special geometry to understand the mirror principle computations, especially the mirror transformation as a coordinate change.

Mariño-Vafa formula is needed to determine the hypergeometric Euler data for higher genus computations in mirror principle.

Mariño-Vafa formula which comes from Chern-Simons and String theory duality, and the matrix models for Chern-Simons indicate that mirror principle may have matrix model description.

Example: CY quintic in $\mathbf{P}^4$,

$P_d = \prod_{m=0}^{5d}(5\kappa - m\alpha)$

with α weight from S^1 action on $\mathbf{P}^1$, and κ generator of equivariant cohomology ring of W_d.

Starting data: $V = \mathcal{O}(5)$ on $X = \mathbf{P}^4$ and the hypergeometric series (taking $\alpha = -1$) is:

$HG[B](t) = e^{H\,t} \sum_{d=0}^{\infty} \frac{\prod_{m=0}^{5d}(5H+m)}{\prod_{m=1}^{d}(H+m)^5}\, e^{d\,t},$

H: hyperplane class on $\mathbf{P}^4$; t: parameter.

Introduce

$\mathcal{F}(T) = \frac{5}{6}T^3 + \sum_{d>0} K_d^0 \, e^{dT}$.

Algorithm: Take expansion in H:

$HG[B](t) = H\{f_0(t) + f_1(t)H + f_2(t)H^2 + f_3(t)H^3\}$.

Candelas Formula: With $T = f_1/f_0$,

$$\mathcal{F}(T) = \frac{5}{2}\left(\frac{f_1}{f_0}\frac{f_2}{f_0} - \frac{f_3}{f_0}\right).$$

(proved by Givental, Lian-Liu-Yau)

Example: X, toric manifold; $g = 0$.

$D_1, .., D_N$: T-invariant divisors

$V = \oplus_i L_i, \quad c_1(L_i) \geq 0$ and $c_1(X) = c_1(V)$.

$b(V) = e(V)$

$A(T) = \sum K_d^0 \, e^{d \cdot T}$.

HG Euler series: generating series of the HG Euler data.

$$B(t) = e^{-H \cdot t} \sum_d \prod_i \prod_{k=0}^{\langle c_1(L_i), d\rangle} (c_1(L_i) - k)$$

$$\times \frac{\prod_{\langle D_a, d\rangle < 0} \prod_{k=0}^{-\langle D_a, d\rangle - 1} (D_a + k)}{\prod_{\langle D_a, d\rangle \geq 0} \prod_{k=1}^{\langle D_a, d\rangle} (D_a - k)} \; e^{d \cdot t}.$$

Mirror Principle $\Longrightarrow$ There are explicitly computable functions $f(t), g(t)$, which define the mirror map, such that

$$\int_X \left(e^f B(t) - e^{-H\cdot T} e(V)\right) = 2A(T) - \sum T_i \frac{\partial A(T)}{\partial T_i}$$

where $T = t + g(t)$.

Easily solved for $A(T)$.

Note the (virtual) equivariant Euler classes in the HG series $B(t)$.

In general we want to compute:

$K_{d,k}^{g} = \int_{LT_{g,k}(d,X)} \prod_{j=1}^{k} ev_j^* \omega_j \cdot b(V_d^g)$

where $\omega_j \in H^*(X)$.

Form a generating series

$F(t, \lambda, \nu) = \sum_{d,g,k} K_{d,k}^{g} e^{dt} \lambda^{2g} \nu^k$.

Ultimate Mirror Principle: Compute this series in terms of explicit HG series!

The classes induce Euler data: Euler data encode the geometric structure of the stable map moduli.

Example: Consider open toric CY, say $\mathcal{O}(-3) \to \mathbf{P}^2$. $V = \mathcal{O}(-3)$. Let

$Q_d = \sum_{g\geq 0} \varphi_!(\pi^* e_T(V_d^g)\, \lambda^{2g}$.

The corresponding HG Euler data is given explicitly by

$P_d\, J(\kappa, \alpha, \lambda) J(\kappa - d\alpha, -\alpha, \lambda)$.

Where P_d is exactly the **genus** 0 HG Euler data and J is generating series of Hodge integrals (sum over all genus): **degree** 0 Euler data.

Euler data includes computations of all Gromov-Witten invariants and more general. Some closed formulas can be obtained....

Mirror principle holds!

Recent Results:

(1) **Refined linearized moduli** for higher genus: *A-twisted moduli stack* $\mathcal{AM}_g(X)$ of genus g curves associated to a smooth toric variety X, induced from the gauge linear sigma model (GLSM) of Witten.

A morphism from a curve of genus g into X corresponds a triple $(L_\rho, u_\rho, c_m)_{\rho,m}$, where each L_ρ is a line bundle pulled back from X, u_ρ is a section of L_ρ satifying a nondegeneracy condition, and c_m gives condition to compare the sections u_ρ in different line bundles L_ρ. (David Cox, *The functor of a smooth toric variety*, Tohoku Math. J. (2) 47 (1995), no. 2, 251–262.)

$\mathcal{AM}_d^g(X)$ is the moduli of such data. It is an Artin stack, fibered over the moduli space of quasi-stable curves. (C.-H. Liu, K. Liu, Yau, *On A-twisted moduli stack for curves from Witten's gauged linear sigma models*, math.AG/0212316.)

(2) **Open mirror principle**: Open string theory: counting holomorphic discs with boundary inside a Lagrangian submanifold; more generally counting number of open Riemann surfaces with boundary in Lagrangian submanifold. Linearized moduli space being constructed which gives a new compactification of the moduli. (C.-H. Liu-K. Liu-Yau).

(2). **Proof of the Hori-Vafa Formula**.

This is refined mirror principle for Grassmannian.

Problem: No known good linearized moduli.

Solution: We use the Grothendieck quot scheme to play the role of the linearized moduli. The method gives a proof of the Hori-Vafa formula. (Lian-C.-H. Liu-K. Liu-Yau, 2001, Bertram et al 2003.)

The existence of linearized moduli made easy the computations for toric manifolds. Let

$$ev: \ \mathcal{M}_{0,1}(d, X) \to X$$

be evaluation map, and c the first Chern class of the tangent line at the marked point. One of the key ingredients for mirror formula is to compute:

$$ev_*[\frac{1}{\alpha(\alpha-c)}] \in H^*(X)$$

or more precisely the generating series

$$HG[1]^X(t) = e^{-tH/\alpha} \sum_{d=0}^{\infty} ev_*[\frac{1}{\alpha(\alpha-c)}]\, e^{dt}.$$

Remark: Assume the linearized moduli exists. Then functorial localization formula applied to the collpasing map: $\varphi: \ M_d \to N_d$, immediately gives the expression as hypergeometric denominator.

Example: $X = \mathbf{P}^n$, then we have $\varphi_*(1) = 1$, functorial localization:

$$ev_*[\frac{1}{\alpha(\alpha-c)}] = \frac{1}{\prod_{m=1}^d (x-m\alpha)^{n+1}}$$

where the denominators of both sides are equivariant Euler classes of normal bundles of fixed points. Here x denotes the hyperplane class.

For $X = \mathrm{Gr}(k,n)$, no explicit linearized moduli known. Hori-Vafa conjectured a formula for $HG[1]^X(t)$ by which we can compute this series in terms of those of projective spaces:

Hori-Vafa Conjecture:

$$HG[1]^{\mathrm{Gr}(k,n)}(t) = e^{(k-1)\pi\sqrt{-1}\sigma/\alpha} \frac{1}{\prod_{i<j}(x_i-x_j)}.$$

$$\prod_{i<j}(\alpha\frac{\partial}{\partial x_i} - \alpha\frac{\partial}{\partial x_j})|_{t_i = t+(k-1)\pi\sqrt{-1}} HG[1]^{\mathbf{P}}(t_1, \cdots, t_k)$$

where $\mathbf{P} = \mathbf{P}^{n-1} \times \cdots \times \mathbf{P}^{n-1}$ is product of k copies of the projective spaces. Here σ is the generator of the divisor classes on $\mathrm{Gr}(k, n)$ and x_i the hyperplane class of the i-th copy $\mathbf{P}^{n-1}$:

$$HG[1]^{\mathbf{P}}(t_1, \cdots, t_k) = \prod_{i=1}^{k} HG[1]^{\mathbf{P^{n-1}}}(t_i).$$

Idea of Proof:

We use another smooth moduli, the Grothendieck quot-scheme Q_d to play the role of the linearized moduli, and apply the functorial localization formula, and a general set-up.

Step 1: Plücker embedding: $\tau: \ \mathrm{Gr}(k, n) \rightarrow \mathbf{P}^N$ induces embedding of the nonlinear moduli M_d of $\mathrm{Gr}(k, n)$ into that of $\mathbf{P}^N$. Composite with the collapsing map gives us a map

$$\varphi: \quad M_d \rightarrow W_d$$

into the linearized moduli space W_d of $\mathbf{P}^N$.

On the other hand the Plücker embedding also induces a map:

$\psi: \quad Q_d \to W_d$.

Lemma: The above two maps have the same image in W_d: $\mathrm{Im}\,\psi = \mathrm{Im}\,\varphi$.

And all the maps are equivariant with respect to the induced circle action from $\mathbf{P}^1$.

Step 2: Analyze the fixed points of the circle action induced from $\mathbf{P}^1$. In particular we need the distinguished fixed point set to get the equivariant Euler class of its normal bundle.

The distinguished fixed point set in M_d is:

$\mathcal{M}_{0,1}(d, \mathrm{Gr}(k,n))$

with equivariant Euler class of its normal bundle: $\alpha(\alpha - c)$, and φ restricted to ev.

Lemma: The distinguished fixed point set in Q_d is a union: $\cup_s E_{0s}$, where each E_{0s} is a fiber bundle over $\mathrm{Gr}(k,n)$ with fiber given by flag manifold.

It is a complicated work to determine the fixed point sets E_{0s} and the weights of the circle action on their normal bundles.

Step 3: Let p denote the projection from E_{0s} onto $\mathrm{Gr}(k,n)$. Functorial localization formula, applied to φ and ψ, gives us

Lemma: We have equality on $\mathrm{Gr}(k,N)$:

$ev_*[\frac{1}{\alpha(\alpha-c)}] = \sum_s p_*[\frac{1}{e_T(E_{0s}/Q_d)}]$

where $e_T(E_{0s}/Q_d)$ is the equivariant Euler class of the normal bundle of E_{0s} in Q_d.

Step 4: Compute $p_*[\frac{1}{e_T(E_{0s}/Q_d)}]$. (LLLY 2001, Bertram, Ciocan-Fontanine, B. Kim 2003)

Note that

$$e_T(TQ|_{E_{0s}} - TE_{0s}) = e_T(TQ|_{E_{0s}})/e_T(TE_{0s}).$$

Both $e_T(TQ|_{E_{0s}})$ and $e_T(TE_{0s})$ can be written down explicitly in terms of the universal bundles on the flag bundle $E_{0s} = Fl(m_1, \cdots, m_k, S)$ over $\mathrm{Gr}(r,n)$, here S is the universal bundle on the Grassmannian.

The push-forward by p from $Fl(m_1, \cdots, m_k, S)$ to $\mathrm{Gr}(r,n)$ is done by a family localization formula of Atiyah-Bott: sum over Weyl groups along the fiber which labels the fixed point sets.

The final formula of degree d is given by

$$p_*[\frac{1}{e_T(E_{0s}/Q_d)}] = (-1)^{(r-1)d}$$

$$\sum_{\substack{(d_1,\ldots,d_r)\\ d_1+\ldots+d_r=d}} \frac{\prod_{1\leq i<j\leq r}(x_i-x_j+(d_i-d_j)\alpha)}{\prod_{1\leq i<j\leq r}(x_i-x_j)\prod_{i=1}^{r}\prod_{l=1}^{d_i}(x_i+l\alpha)^n}.$$

$x_1, \ldots x_r$ are the Chern roots of S^*.

Remark: Similar formula for general Flag manifolds can be worked out along the same line (LLLY).

(3). **Proof of the Mariño-Vafa formula**.

To compute mirror formula for higher genus, we need to compute Hodge integrals (i.e. intersection numbers of λ classes and ψ classes) on the moduli space of stable curves $\overline{\mathcal{M}}_{g,h}$.

The Hodge bundle $\mathbb{E}$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over $[(C, x_1, \ldots, x_h)]$ is $H^0(C, \omega_C)$. The λ classes are defined by

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

The cotangent line $T^*_{x_i}C$ of C at the i-th marked point x_i gives a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The ψ classes are defined by

$$\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

Define

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

Mariño-Vafa formula: Generating series of triple Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^h (1-\mu_i\psi_i)},$$

given by close formulas of **finite** expression in terms of representations of symmetric groups:

Conjectured from large N duality between Chern-Simons and string theory:

Conifold transition:

conifold X

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = 0 \right\}$$

deformed conifold T^*S^3

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = \epsilon \right\}$$

(ϵ real positive number)

resolved conifold $\tilde{X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$

$$\left\{ ([Z_0, Z_1], \begin{pmatrix} x & y \\ z & w \end{pmatrix}) \in \mathbf{P}^1 \times \mathbf{C}^4 : \begin{array}{l} (x, y) \in [Z_0, Z_1] \\ (z, w) \in [Z_0, Z_1] \end{array} \right\}$$

$$\begin{array}{ccc} \tilde{X} & \subset & \mathbf{P}^1 \times \mathbf{C}^4 \\ \downarrow & & \downarrow \\ X & \subset & \mathbf{C}^4 \end{array}$$

Witten 92: The open topological string theory on the N D-branes on S^3 of T^*S^3 is equivalent to $U(N)$ Chern-Simons gauge thoery on S^3.

Gopakumar-Vafa 98, Ooguri-Vafa 00: The open topological string theory on the N D-branes on S^3 of the deformed conifold is equivalent to the closed topological string theory on the resolved conifold $\tilde{X}$.

Mathematical Consequence:

$\langle Z(U,V)\rangle = \exp(-F(\lambda,t,V))$

U: holonomy of the $U(N)$ Chern-Simons gauge field around the a knot $K \subset S^3$; V: $U(M)$ matrix

$\langle Z(U,V)\rangle$: Chern-Simons knot invariants of K.

$F(\lambda,t,V)$: Generating series of the open Gromov-Witten invariants of $(\tilde{X}, L_K)$, where L_K is a Lagrangian submanifold of the resolved conifold $\tilde{X}$ "canonically associated to" the knot K.

t'Hooft large N expansion, and canonical identifications of parameters similar to mirror formula.

Special case: When K is the unknot, $\langle Z(U,V)\rangle$ was computed in the zero framing by Ooguri-Vafa and in any framing $\tau \in \mathbb{Z}$ by Mariño-Vafa.

Comparing with Katz-Liu's computations of $F(\lambda, t, V)$, Mariño-Vafa conjectured a striking formula about triple Hodge integrals in terms of representations and combinatorics of symmetric groups. The framing in Mariño-Vafa's computations corresponds to choice of the circle action on the pair $(\tilde{X}, L_{\text{unknot}})$ in Katz-Liu's localization computations. Both choices are parametrized by an integer.

Mariño-Vafa formula:

Geometric side:

For every partition $\mu = (\mu_1 \geq \cdots \mu_{l(\mu)} \geq 0)$, define triple Hodge integral:

$$G_{g,\mu}(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\operatorname{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1}$$

$$\prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^{\vee}(1)\Lambda_g^{\vee}(-\tau-1)\Lambda_g^{\vee}(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

and generating series

$G_\mu(\lambda;\tau) = \sum_{g\geq 0} \lambda^{2g-2+l(\mu)} G_{g,\mu}(\tau)$.

Special case when $g=0$:

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{\Lambda_0^\vee(1)\Lambda_0^\vee(-\tau-1)\Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

which is equal to $|\mu|^{l(\mu)-3}$ for $l(\mu)\geq 3$, and we use this expression to extend the definition to the case $l(\mu)<3$.

Introduce formal variables $p=(p_1,p_2,\ldots,p_n,\ldots)$, and define

$$p_\mu = p_{\mu_1}\cdots p_{\mu_{l(\mu)}}$$

for any partition μ.

Generating series for all genera and all possible marked point:

$G(\lambda;\tau;p) = \sum_{|\mu|\geq 1} G_\mu(\lambda;\tau)p_\mu$.

Representation side:

χ_μ: the character of the irreducible representation of symmetric group $S_{|\mu|}$ indexed by μ with $|\mu| = \sum_j \mu_j$,

$C(\mu)$: the conjugacy class of $S_{|\mu|}$ indexed by μ.

Introduce:

$$V_\mu(\lambda) = \prod_{1\leq a<b\leq l(\mu)} \frac{\sin[(\mu_a-\mu_b+b-a)\lambda/2]}{\sin[(b-a)\lambda/2]}$$

$$\cdot\frac{1}{\prod_{i=1}^{l(\nu)}\prod_{v=1}^{\mu_i} 2\sin[(v-i+l(\mu))\lambda/2]}.$$

This has an interpretation in terms of **quantum dimension** in Chern-Simons knot theory.

Define

$$R(\lambda;\tau;p)=\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sum_{\mu}[(\sum_{\cup_{i=1}^{n}\mu^i=\mu}$$

$$\prod_{i=1}^{n}\sum_{|\nu^i|=|\mu^i|}\frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}}e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2}V_{\nu^i}(\lambda)]p_{\mu}$$

where μ^i are sub-partitions of μ, $z_\mu=\prod_j \mu_j! j^{\mu_j}$ and $\kappa_\mu=|\mu|+\sum_i(\mu_i^2-2i\mu_i)$ for a partition μ: standard for representations.

Mariño-Vafa Conjecture:

$$G(\lambda; \tau; p) = R(\lambda; \tau; p).$$

Remark: (1). This is a formula:

G: Geometry = R: Representations

Representations of symmetric groups are essentially combinatorics.

(2). Each $G_\mu(\lambda, \tau)$ is given by a **finite and closed** expression in terms of representations of symmetric groups:

$G_\mu(\lambda, \tau) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (\sum_{\cup_{i=1}^n \mu^i = \mu} \prod_{i=1}^n$

$\sum_{|\nu^i| = |\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau + \frac{1}{2})\kappa_{\nu^i}\lambda/2} V_{\nu^i}(\lambda)$

$G_\mu(\lambda, \tau)$ gives triple Hodge integrals for moduli spaces of curves of all genera with $l(\mu)$ marked points.

(3). Mariño-Vafa: this formula gives values for all Hodge integrals up to three Hodge classes.

Taking Taylor expansion in τ on both sides, various Hodge integral identities have been derived by C.-C. Liu, K. Liu and Zhou.

Idea of Proof:(Chiu-Chu Liu, Kefeng Liu, Jian Zhou)

The proof is based on the **Cut-and-Join** equation: a beautiful match of Combinatorics and Geometry.

Cut-and-Join: The combinatorics and geometry:

Combinatorics: Denote by $[s_1, \cdots, s_k]$ a k-cycle in the permutation group:

Cut: a k-cycle is cut into an i-cycle and a j-cycle:

$$[s, t] \cdot [s, s_2, \cdots, s_i, t, t_2, \cdots t_j]$$

$$= [s, s_2, \cdots, s_i][t, t_2, \cdots t_j].$$

Join: an i-cycle and a j-cycle are joined to an $(i + j)$-cycle:

$$[s, t] \cdot [s, s_2, \cdots, s_i][t, t_2, \cdots t_j]$$

$$= [s, s_2, \cdots, s_i, t, t_2, \cdots t_j].$$

Geometry:

Cut: One curve split into two lower degree or lower genus curves.

Join: Two curves joined together to give a higher genus or higher degree curve.

The combinatorics and geometry of cut-and-join are reflected in the following two differential equations, like heat equation:

proved either by direct computations in combinatorics or by localizations on moduli spaces of relative stable maps (Jun Li):

Combinatorics: Computation:

Theorem 1:

$$\begin{aligned}\frac{\partial R}{\partial \tau} &= \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty}\Bigg((i+j)p_ip_j\frac{\partial R}{\partial p_{i+j}} \\ &\quad + ijp_{i+j}\left(\frac{\partial R}{\partial p_i}\frac{\partial R}{\partial p_j} + \frac{\partial^2 R}{\partial p_i \partial p_j}\right)\Bigg)\end{aligned}$$

Geometry: Localization:

Theorem 2:

$$\begin{aligned}\frac{\partial G}{\partial \tau} &= \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty}\Bigg((i+j)p_ip_j\frac{\partial G}{\partial p_{i+j}} \\ &\quad + ijp_{i+j}\left(\frac{\partial G}{\partial p_i}\frac{\partial G}{\partial p_j} + \frac{\partial^2 G}{\partial p_i \partial p_j}\right)\Bigg)\end{aligned}$$

Initial Value: Ooguri-Vafa formula

$$G(\lambda, 0, p) = \sum_{d=1}^{\infty} \frac{p_d}{2d \sin\left(\frac{\lambda d}{2}\right)} = R(\lambda, 0, p).$$

The solution is unique! Series of homogeneous ODE:

$G(\lambda, \tau, p) = R(\lambda, \tau, p)$

Remark: (1). Cut-and-join equation is more fundamental: encodes both geometry and combinatorics: Vafa: Virasoro operators come out of the cut-and-join.

(2). Witten conjecture is about KdV equations. But the Marinõ-Vafa formula gives **closed formula!**

The proof of the combinatorial cut-and-join formula is based on Burnside formula and various results in symmetric functions.

The proof of the geometric cut-and-join formula used functorial localization formula.

Let $\mathcal{M}_g(\mathbf{P}^1, \mu)$ denote the moduli space of relative stable maps from a genus g curve to $\mathbf{P}^1$ with fixed ramification type μ at ∞, where μ is a partition.

Apply the functorial localization formula to the divisor morphism from the relative stable map moduli space to projective space:

$$\mathrm{Br}: \quad \mathcal{M}_g(\mathbf{P}^1, \mu) \to \mathbf{P}^r,$$

where r denotes the dimension of $\mathcal{M}_g(\mathbf{P}^1, \mu)$.

This is similar to the set-up of mirror principle, with a different linearized moduli.

The fixed points of the target $\mathbf{P}^r$ **precisely labels** the cut-and-join of the triple Hodge integrals.

Applications: Computing GW invariants on Toric Calabi-Yau:

Physical approaches: Aganagic-Mariño-Vafa (2002): BPS numbers for toric Calabi-Yau by using large N dulaity and Chern-Simons invariants.

Aganagic-Klemm-Mariño-Vafa (2003): Topological vertex. Complete formula for computations of GW invariants and BPS numbers: Chern-Simons. (BPS numbers are related to GW invariants by Gopakumar-Vafa formula.)

Iqbal's instanton counting in terms of Chern-Simons.

Mathematical approach (Jian Zhou): Mariño-Vafa formula can be used to compute BPS numbers (which are conjectured to be **integers** by Gopakumar-Vafa) for toric Calabi-Yau 3-fold.

Re-organize contributions of fixed points as combinations of Mariño-Vafa formula.

Recovered the formula of Iqbal (based on a two partition generalization of the MV formula, proved by them).

The topological vertex is the three partition generalization.

The physical and mathematical approaches should be equivalent:

Bridge: The Mariño-Vafa formula: which can be viewed as a duality

Chern-Simons $\Longleftrightarrow$ **Calabi-Yau.**

(4). **SYZ Conjecture and Duality**.

Strominger-Yau-Zaslow proposed the construction of the **mirror manifold** by looking at special Lagrangian torus fibration. The mirror manifold is obtained by taking duality along the torus. (There is a parallel development by Kontsevich and Fukaya on homological mirror conjecture).

Similar construction is proposed for other manifolds with special holonomy group. (Acharya, Vafa, Gukov-Yau-Zaslow, Leung...)

The SYZ construction is based on the newly developed M-theory. Therefore the geometric construction has support from intuition of physics. The complicated question of singularities should be solvable.

$$
\begin{tikzcd}
T^3 \arrow[d] & M \times_{S^3} M' \arrow[dl] \arrow[dr] & (T^3)^* \arrow[d] \\
M \arrow[dr] & & M' \arrow[dl] \\
& S^3 &
\end{tikzcd}
$$

The above diagram allows us to transfer objects from M to M' and vice-versa. This is a kind of Fourier-Mukai transformation.

Special Lagrangian cycles from M should be mapped to stable holomorphic bundles over M', and coupling should be preserved. (Vafa, Leung-Yau-Zaslow for the case with no instanton correction)

The map sends odd cohomology of M to even cohomology of M':

$$\begin{tikzcd}[column sep=small] & M \times_{S^3} M' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M & & M' \end{tikzcd}$$

$$\begin{array}{rcl} H^3(M) & \to & H^3(M \times_{S^3} M') \\ \Omega & \mapsto & \pi_1^*\Omega \end{array}$$

$$\begin{array}{rcl} H^3(M \times_{S^3} M') & \to & H^3(M \times_{S^3} M') \\ \omega & \mapsto & \omega \exp(c_1(L)) \end{array}$$

$$\begin{array}{rcl} H^3(M \times_{S^3} M') & \to & H^{\text{even}}(M') \\ \omega & \mapsto & (\pi_2)_*\omega \end{array}$$

Many interesting questions arise in the SYZ construction. Most of the geometric quantities including complex structure and Ricci flat metrics will require **quantum corrections** from disk instantons. (There is a background semi-flat Ricci flat metric given by cosmological string construction.)

How to compute such instantons are nontrivial. Works by Katz-Liu, Li-Song, Graber-Zaslow and others are making progresses on the important questions. Generalizations to G_2 and $Spin(7)$ are interesting.

The important question is to interpret the localization results in terms of the picture of T-duality along the fibers.

In the fibration

$$\begin{array}{ccc} T^3 & \longrightarrow & M \\ & & \downarrow \\ & & S^3 \end{array}$$

there is a singular locus in S^3 where T^3 collapses. The singular locus is expected to look like a graph Γ. In $S^3 \setminus \Gamma$, there exists a locally real affine structure (introduced by Cheng-Yau, 1989). In each affine coordinate, there is a convex potential u so that the metric is given by

$$g_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

We require $\det(g_{ij})$ =const. There is prescribed singular behaviour of g_{ij} along Γ. In general Γ is trivalent and the detailed singular structure at the vertices should be related to the **topological vertex** (Aganagic-Klemm-Mariñ0-Vafa, hep-th/0305132). Zaslow-Yau is working on this singular structure.