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Holomorphic equivariant cohomology

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Introduction

The deRham equivariant cohomology has drawn much attention in the last ten years. Many interesting applications in geometry and topology, especially in symplectic geometry, have been found. On the other hand, vector fields are the links between the topological properties and the geometric properties of manifolds. In the past thirty years, many leading mathematicians have studied problems related to vector fields. There are close relationships between equivariant cohomology and vector fields. For compact Lie group and Killing vector fields, a very good theory was developed by Atiyah-Bott in [2].

It is interesting to consider the corresponding problem for holomorphic vector fields. As pointed out by Atiyah and Bott [2], holomorphic vector field generally does not generate a compact action group, so in general there is no equivariant cohomology theory associated to the corresponding holomorphic actions. In this paper, motivated by Witten [17], we introduce a cohomology group related to a holomorphic vector field or a meromorphic vector field. The properties of this cohomology, such as localizations and the Duistermaat-Heckman type integral formula, are very similar to those of the deRham equivariant cohomology. We would like to call it the Dolbeault, or holomorphic equivariant cohomology. This cohomology is very useful. First it gives a systematic method to deal with many well-known theorems. Second it can be used to get some new results. We also have similar results for meromorphic vector fields which are more interesting, since there always exist many meromorphic vector fields on a projective manifold.

Our motivation is to develop a general equivariant cohomology theory for complex Lie groups acting on complex manifolds, and use it to study the topology of the orbit space in geometric invariant theory. Such kind of problem is being under active study in symplectic geometry. See Sect. 8 for further discussions.

1 Main results

Let M be a compact complex manifold of dimension n and V be a holomorphic vector field on it. We define

$$\bar{\partial}_s = \bar{\partial} + si(V),$$

where $\bar{\partial}$ is the (0, 1)-differential operator, $i(V)$ is the contraction operator with respect to V, and s is a complex parameter.

Let $A^{p,q}(M)$ be the smooth (p, q)-forms on M and define

$$A^{(r)}(M) = \bigoplus_{q-p=r} A^{p,q}(M).$$

Since $\bar{\partial}i(V) + i(V)\bar{\partial} = 0$, it is easy to verify that

$$0 \to A^{(-n)}(M) \xrightarrow{\bar{\partial}_s} A^{(-n+1)}(M) \xrightarrow{\bar{\partial}_s} \cdots \xrightarrow{\bar{\partial}_s} A^{(n-1)}(M) \xrightarrow{\bar{\partial}_s} A^{(n)}(M) \to 0,$$

forms a differential complex. We define its cohomology group by

$$H_s^{(r)}(M) \equiv \ker \bar{\partial}_s / \operatorname{Im} \bar{\partial}_s | A^{(r)}(M)$$

Let C^* denote the nonzero complex numbers. Then our starting point is the following

Lemma 1.1. *$H_s^{(r)}(M) \cong H_{s'}^{(r)}(M)$ for any $s, s' \in C^*$.*

Proof. Let $\exp \lambda(p+q)$ be the operator which multiplies (p, q) form by $e^{\lambda(p+q)}$. Then this is an invertible operator and does not change the type of (p, q)-forms. We have

$$\exp(-\lambda(p+q))\bar{\partial}_s \exp(\lambda(p+q)) = e^{-\lambda}\bar{\partial}_{s'}$$

where $s' = s \cdot e^{2\lambda}$. Since λ can be changed arbitrarily, we get the lemma.

Our main results contain the following theorems.

Theorem 1.2. *If V has no zero point, then*

$$H_s^{(r)}(M) = 0 \quad \text{for any } r \text{ and } s \neq 0.$$

Theorem 1.3. $\dim H_s^{(r)}(M) \leqq \dim H^{(r)}(M)$, *where $H^{(r)}(M) \equiv \bigoplus_{q-p=r} H^{p,q}(M)$. If M is furthermore Kahler, and V has non-empty zero points, we have*

$$\dim H_s^{(r)}(M) = \dim H^{(r)}(M) \quad \text{for any } s.$$

Theorem 1.4. *If V has only isolated zero points $\{p_\alpha\}$ and write $V = \sum_i v_i^\alpha(z)\frac{\partial}{\partial z_i}$ around p_α, we then have*

$$\dim H_s^{(r)}(M) = \begin{cases} 0, & r \neq 0,\ s \neq 0, \\ \sum_\alpha \operatorname{rank} A_\alpha, & r = 0,\ s \neq 0. \end{cases}$$

Here $A_\alpha = O/I_{p_\alpha}$ *with* O *the structure sheaf of* M *and* I_{p_α} *the ideal sheaf generated by* $\{v_i^\alpha(z)\}$ *in* O *around* p_α.

This theorem can be reformulated as

$$H_s^{(r)}(M) = \begin{cases} 0, & r \neq 0,\ s \neq 0\,, \\ H^0(M, O/I_Z), & r = 0,\ s \neq 0 \end{cases}$$

where I_Z is the ideal sheaf of $\{p_\alpha\}$.

Theorem 1.5. *For a compact complex surface* M *which admits a holomorphic vector field* V *with only isolated zero points, the map*

$$i(V) : H^2(M, O) \to H^1(M, O)$$

is an isomorphism.

We hope this theorem can be used to classify the complex surfaces with holomorphic vector fields.

Theorem 1.6. *Suppose the zero points of* V, *zero*(V), *consists of nondegenerate isolated zero points* $\{p_\alpha\}$, *then for any* $w = w_n/s^n + \cdots + w_0 \in H_s^{(0)}(M)$, *we have*

$$\frac{1}{(-2\pi i)^n}\int_M w_n = \sum_{p_\alpha} \frac{w_0(p_\alpha)}{\det B_\alpha}\,,$$

where $B_\alpha = (\partial v_i^\alpha/\partial z_j)_{n\times n}$ *evaluated at* p_α.

Here by a nondegenerate zero point p_α of V, we mean that B_{p_α} is a non-degenerate matrix near p_α.

This formula implies an interesting relation between the holomorphic Lefschetz fixed point formula [1] and the zeroes of a holomorphic vector field. Similar result for Dirac operator and Killing vector field was obtained by Berline-Vergne [4]. See Sect. 5. Our results are still true even when the zero points of V are degenerate. See Sect. 8 for further discussions. Section 4 contains a similar formula for meromorphic vector fields.

The following theorems deal with special vector fields (cf. [7]). We say a holomorphic vector field V on M is special, if V satisfies the following condition.

1) $Z(V)$ is a submanifold of M. Here $Z(V)$ denotes the zero points of V.

2) For any $z \in M_r$ which is a connected component of $Z(V)$, there exists a small neighborhood $U_z \ni z$, such that in U_z,

$$M_r \cap U_z = \{z_{r+1} = \cdots = z_n = 0\}\,,$$

$$V = \sum_{i=r+1}^{n} a_i(z)\frac{\partial}{\partial z_i}$$

where the $a_i(z)$'s are holomorphic functions.

3) $\det\left(\dfrac{\partial a_i}{\partial z_j}\right)_{r+1\le i,j\le n} \neq 0\,.$

These conditions may be equivalent to the condition that $Z(V)$ consists of nondegenerate submanifolds. See Sect. 5.

Theorem 1.7. *If V is a special holomorphic vector field on M, we have*

$$\dim H^{(r)}(Z(V)) \leqq \dim H^{(r)}(M).$$

For the corresponding results about meromorphic vector fields, see Sect. 6. In next section, we state some corollaries of the above theorems.

2 Some corollaries and examples

In this section we deduce some corollaries from the theorems in last section. Still let M be a compact complex manifold with a holomorphic vector field V.

Corollary 2.1. *If $H^{(1)}(M) = 0$, then $H_s^{(0)}(M) \neq 0$ for $s \neq 0 \Rightarrow$ and V has no zero point.*

Proof. Suppose w is the volume form of M, then

$$\bar{\partial} i(V)w = -i(V)\bar{\partial} w = 0$$
$$\Rightarrow i(V)w \in H^{n-1,n}(M) = 0$$
$$\Rightarrow i(V)w = \bar{\partial} w_{n-1},$$

and

$$\bar{\partial} i(V)w_{n-1} = -i(V)\bar{\partial} w_{n-1} = -i(V)i(V)w_{n-1} = 0$$
$$\Rightarrow i(V)w_{n-1} \in H^{n-2,n-1}(M) = 0$$
$$\Rightarrow i(V)w_{n-1} = \bar{\partial} w_{n-1},$$

so on. We finally have $\bar{\partial}_s \tilde{w} = 0$, where

$$\tilde{w} = w - sw_{n-1} + s^2 w_{n-2} + \cdots + (-1)^n s^n w_0 .$$

So $\tilde{w} \in H_s^{(0)}(M)$.

If $H_s^{(0)}(M) = 0$ we have $\tilde{w} = \bar{\partial}_s \psi$. Comparing two sides, we have $w = \bar{\partial}\psi'$ a contradiction.

Remark. Corollary 2.1 implies all of the results in L. Karp [12].

Corollary 2.2. *If $\dim H^{p,q}(M) = 0$ for $p + q =$ odd (or even), we have $\dim H^{(r)}(M) = \dim H_s^{(r)}(M)$ for any r and s.*

Proof. By Theorem 1.3, we have

$$\dim H^{(r)}(M) = \dim H_s^{(r)}(M) = 0 \quad \text{for } r = \text{odd (or even)}.$$

By the proof of Theorem 1.3, (cf. Sect. 3), we have

$$\dim H_s^{(r)}(M) \leqq \dim H^{(r)}(M),$$

$$\sum_r \dim H_s^{(r)}(M) = \sum_r \dim H^{(r)}(M),$$

so we have $\dim H_s^{(r)}(M) = \dim H^{(r)}(M)$.

Corollary 2.3. (Carrell-Liebermann). *If M is Kahler, V a holomorphic vector field with isolated zero points, we have*

$$H^{p,q}(M) = 0 \quad for \ p \neq q .$$

Proof. A direct consequence of Theorem 1.3 and 1.4.

For a different proof of Corollary 2.3, see Witten [17]. Theorem 1.3 has the following analytic meaning.

Corollary 2.4. *Let*

$$\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial},$$

$$\Delta_s = \bar{\partial}_s\bar{\partial}_s^* + \bar{\partial}_s^*\bar{\partial}_s .$$

Then

$$\dim \ker \Delta_s \leqq \dim \ker \Delta .$$

If a holomorphic vector field comes from the complexification of a Killing vector field, that is its real part generates an isometry, we call it holomorphic Killing vector field.

Corollary 2.5. *Suppose M is a compact Kahler manifold, V is a holomorphic Killing vector field, then*

$$\dim H^{(r)}(Z(V)) = \dim H^{(r)}(M),$$

where $H^{(r)}(\,\cdot\,) = \bigoplus_{q-p=r} H^{p,q}(\,\cdot\,)$.

Proof. By Theorems 1.3 and 1.7,

$$\dim H^{(r)}(Z(V)) \leqq \dim H_s^{(r)}(M) \leqq \dim H^{(r)}(M),$$

so $\sum_r \dim H^{(r)}(Z(V)) \leqq \sum_r \dim H^{(r)}(M)$, i.e. $\sum_r \dim H^r(Z(V)) \leqq \sum_r \dim H^r(M)$. By Frankel [10], we know that the above should be equal. So

$$\sum_r \dim H^{(r)}(Z(V)) = \sum_r \dim H^{(r)}(M),$$

therefore

$$\dim H^{(r)}(Z(V)) = \dim H^{(r)}(M).$$

The method to prove Theorem 1.7 can also be used to prove the following well-known results in topology.

Corollary 2.6. *Suppose V is a Killing vector field on compact Riemannian manifold M. Then*

1) *if* $\dim M$ *is even, then*

$$\sum_{k \text{ even}} \dim H^k(M, \mathbf{R}) \geqq \sum_{k \text{ even}} \dim H^k(Z(V), \mathbf{R}),$$

$$\sum_{k \text{ odd}} \dim H^k(M, \mathbf{R}) \geqq \sum_{k \text{ odd}} \dim H^k(Z(V), \mathbf{R}),$$

2) *if* $\dim M$ *is odd, then*

$$\sum_k H^k(M, R) \geqq \sum_k \dim H^k(Z(V), R).$$

For Kahler manifolds, we have the following corollary. Similar result for symplectic manifolds and Killing vector fields was obtained by Atiyah-Bott [2].

Corollary 2.7. *Suppose M is a simply connected, compact Kahler manifold, then*

$$\dim H_s^{(r)}(M) = \dim H^{(r)}(M) \quad \textit{for any } r.$$

Especially there can not exist holomorphic vector field without zero points.

Proof. If M is simply connected, $H^{0,1}(M) = 0$, so the Deligne degeneracy criteria works (cf. [11]). The spectral sequence in the proof of Theorem 1.3 also degenerates. Therefore

$$H_s^{(r)}(M) \cong H^{(r)}(M).$$

If there exists a holomorphic vector field without zero point, then $H_s^{(0)}(M) = 0 = H^{(0)}(M)$ which is a contradiction.

Example 1. On Hermitian homogeneous spaces there exist many holomorphic vector fields.

Example 2. On the Calabi-Eckmann manifold $S^{2p-1} \times S^1$, we can easily construct a holomorphic vector field without zero.

Example 3. On the Grassmannian manifold $G(p,q) = SU(p+q)/SU(p) \times SU(q)$, we can construct explicit holomorphic vector fields. Suppose

$$\mathfrak{X} = \begin{pmatrix} z_{11} & \cdots & z_{1\,p+q} \\ \vdots & & \vdots \\ z_{p1} & \cdots & z_{p\,p+q} \end{pmatrix}$$

is the homogeneous coordinate of $\mathscr{Z} \in G(p,q)$, then

$$V = \mathrm{tr}\left(\mathscr{Z}\Lambda\frac{\partial}{\partial\mathscr{Z}}\right)$$

is a holomorphic vector field, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{p+q} \end{pmatrix}$$

is a diagonalized matrix, and

$$\frac{\partial}{\partial\mathscr{Z}} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{1\,p+q}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial z_{p1}} & \cdots & \frac{\partial}{\partial z_{p\,p+q}} \end{pmatrix} .$$

If $\lambda_i \neq \lambda_j$ for $i \neq j$, V has nondegenerate isolated zeroes.

If $\lambda_1 = \lambda_2$ and $\lambda_i \neq \lambda_j$ for $i,j \geqq 3$, $i \neq j$, then V has a submanifold as its zero sets. This submanifold is nondegenerate. See Sect. 4 for the definition of nondegenerate submanifold.

3 Proofs of main results

The following elementary proofs of Theorems 1.2, 1.4 are the revised versions of the author's original proofs in Sect. 6. The idea is motivated by Berline-Vergne [4] and by discussions with Zhang.

Let $\langle\, , \rangle$ be an Hermitian structure on M. We then have a dual $(1,0)$-form $\tilde{V}$ to V, s.t.

$$i(V)\tilde{V} = \langle V, V\rangle .$$

Proof of Theorem 1.2. Let

$$\alpha = \tilde{V}/\langle V,V\rangle, \quad \beta = \alpha(s + \bar{\partial}\alpha)^{-1} .$$

If V has no zero, α, β are well defined on M. It is easy to verify that $(\bar{\partial} + si(V))\beta = 1$, since

$$\bar{\partial}_s(s + \bar{\partial}\alpha) = -\bar{\partial}i(V)\alpha = -\bar{\partial}1 = 0 ,$$

we have

$$(\bar{\partial} + si(V))(s + \bar{\partial}\alpha)^{-1} = 0 .$$

Which tells us that, for any φ satisfying

$$(\bar{\partial} + si(V))\varphi = 0 ,$$

we have

$$\varphi = (\bar{\partial} + si(V))(\beta \wedge \varphi) .$$

We remark that the construction of β is due to Bott [5].

Proof of Theorem 1.4. This proof consists of Lemmas 1.4.1 and 1.4.2.

Lemma 1.4.1. *For any small neighborhood U of $\{p_\alpha\}$, the restriction map*

$$i^* : H_s^{(r)}(M) \to H_s^{(r)}(U), \quad s \neq 0$$

is an isomorphism.

Proof. Let

$$g(t) = \begin{cases} 1, & t \leqq \varepsilon^2 , \\ 0, & t \geqq (2\varepsilon)^2 , \end{cases}$$

be a smooth function with $\operatorname{supp}(g) \subset U$.

If $\mu \in H_s^{(r)}(U)$, then $\mu - \bar{\partial}_s((1-g)\beta \wedge \mu)$ is extended by zero to M and compactly supported in U. It represents the same element as μ in $H_s^{(r)}(U)$, so i^* is surjective.

Conversely, if $\mu \in H_s^{(r)}(M)$ s.t.

$$\mu = \bar{\partial}_s v \quad \text{on } U ,$$

the form $\mu' = \mu - \bar{\partial}_s(gv)$ is a form on M which is zero in a neighborhood in U, so

$$\mu' = \bar{\partial}_s(\beta \wedge \mu') \quad \text{is well-defined on } M$$

$$\Rightarrow \mu = \bar{\partial}_s(gv + \beta \wedge \mu') \quad \text{on } M .$$

We can suppose $U = \bigcup_\alpha U_\alpha$, where U_α is a small ball around p_α, and $U_\alpha \cap U_\beta = \emptyset$ for $\alpha \neq \beta$.

By the proof, we know that for any $U' \subset U$, restriction map

$$i^* : H_s^{(r)}(U) \to H_s^{(r)}(U')$$

is an isomorphism.

Lemma 1.4.2. *For $s \neq 0$,*

$$H_s^{(r)}(U_\alpha) = \begin{cases} O/I_{p_\alpha}, & r = 0 , \\ 0, & r \neq 0 . \end{cases}$$

Proof. Let $\varphi \in H_s^{(r)}(M)$.
i) $r > 0$, $\varphi = \varphi^{p,p+r}/s^p + \cdots + \varphi^{0,r}$. Then

$$\begin{aligned} &\bar{\partial}_s \varphi = 0 \\ \Rightarrow &\bar{\partial}\varphi^{0,r} + i(V)\varphi^{1,r+1} = 0 \\ &\cdots \\ &\bar{\partial}\varphi^{p-1,p-1+r} + i(V)\varphi^{p,p+r} = 0 \\ &\bar{\partial}\varphi^{p,p+r} = 0 \,. \end{aligned}$$

By Poincare lemma, we have on U_α

$$\begin{aligned} &\varphi^{p,p+r} = \bar{\partial}\psi^{p,p+r-1} \\ &\varphi^{p-1,p-1+r} = \bar{\partial}\psi^{p-1,p+r-2} + i(V)\psi^{p,p+r-1} \\ &\cdots \end{aligned}$$

and so on. We finally have $\varphi = \bar{\partial}_s \psi$, where

$$\psi = \psi^{p,p+r-1}/s^p + \cdots + \psi^{0,r-1} \,.$$

ii) $r < 0$, $\varphi = \varphi^{p-r,p}/s^p + \cdots + \varphi^{-r,0}$. Then

$$\begin{aligned} &\bar{\partial}_s \varphi = 0 \\ \Rightarrow &\bar{\partial}\varphi^{-r,0} + i(V)\varphi^{-r+1,1} = 0 \\ &\cdots \\ &\bar{\partial}\varphi^{p-r-1,p-1} + i(V)\varphi^{p-r,p} = 0 \\ &\bar{\partial}\varphi^{p-r,p} = 0 \,. \end{aligned}$$

Also by Poincare lemma, we can get

$$\begin{aligned} &\varphi^{p-r,p} = \bar{\partial}\psi^{p-r,p-1} \\ &\varphi^{p-r-1,p-1} = \bar{\partial}\psi^{p-r-1,p-2} + i(V)\psi^{p-r,p-1} \\ &\cdots \\ &\bar{\partial}(\varphi^{-r,0} - i(V)\psi^{-r+1,0}) = 0 \\ &i(V)\varphi^{-r,0} = 0 \,. \end{aligned}$$

So $\varphi^{-r,0} - i(V)\psi^{-r+1,0}$ is a holomorphic $(-r)$-form on U_α and $i(V)(\varphi^{-r,0} - i(V)\psi^{-r+1,0}) = i(V)\varphi^{-r,0} = 0$.

By using the Koszul complex [11], we know that on $U'_\alpha \subset U_\alpha$,

$$\varphi^{-r,0} - i(V)\psi^{-r+1,0} = i(V)\psi_r \,,$$

so

$$\varphi^{-r,0} = i(V)(\psi^{-r+1,0} + \psi_r) \quad \text{on } U'_\alpha$$

$$\Rightarrow \varphi = \bar\partial_s \psi \quad \text{on } U'_\alpha,$$

where

$$\psi = \psi^{p-r,p-1}/s^{p-1} + \cdots + \psi^{-r+2,1}/s + (\psi^{-r+1,0} + \psi_r).$$

By the remark following Lemma 1.4.1, we know that

$$\varphi = \bar\partial_s \psi' \quad \text{on } U_\alpha.$$

So $H_s^{(r)}(U_\alpha) = 0$.
iii) $r = 0$, $\varphi = \varphi^{p,p}/s^p + \cdots + \varphi^{0,0}$, $\bar\partial_s \varphi = 0$. Completely the same discussion as above gives

$$\varphi = \bar\partial_s \psi + (\varphi^{0,0} - i(V)\psi^{1,0}),$$

where

$$\psi = \psi^{p,p-1}/s^p + \cdots + \psi^{1,0}/s,$$

$$\bar\partial(\varphi^{0,0} - i(V)\psi^{1,0}) = 0.$$

Also by Koszul complex, we have that

$$\varphi^{0,0} - i(V)\psi^{1,0} = i(V)\psi_0 \quad \text{on } U'_\alpha \subset U_\alpha$$

$$\Leftrightarrow \varphi^{0,0} - i(V)\psi^{1,0} \in I_{p_\alpha}.$$

So $\varphi = \bar\partial_s \psi_1 \Leftrightarrow \varphi^{0,0} - i(V)\psi^{1,0} \in I_{p_\alpha}$, this tells us that $H_s^{(0)}(U_\alpha) \cong O/I_{p_\alpha}$

Proof of Theorem 1.3. Consider the double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \bar\partial \uparrow & & \bar\partial \uparrow & & \\
0 & \to & A^{n,1}(M) & \xrightarrow{i(V)} & A^{n-1,1}(M) & \xrightarrow{i(V)} & \cdots \\
 & & \bar\partial \uparrow & & \bar\partial \uparrow & & \\
0 & \to & A^{n,0}(M) & \xrightarrow{i(V)} & A^{n-1,0}(M) & \xrightarrow{i(V)} & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

It induces a spectral sequence with total complex (K^*, D), where

$$K^r = \bigoplus_{q-p=r} A^{p,q}(M) \quad \text{and} \quad D = \bar\partial + i(V).$$

Filter (K^*, D) by

$$F^p K^r = \bigoplus_{\substack{q-p'=r \\ p' \leq p}} A^{p',q}(M).$$

Then

$$E_0^{p,q} = F^p K^{q-p} / F^{p-1} K^{q-p} \cong A^{p,q}(M)$$

$$d_0 = \bar{\partial} .$$

Therefore

$$E_1^{p,q} = H_{\bar{\partial}}^{p,q}(M) \quad \text{with } d_1 = i(V) .$$

Obviously the total cohomology is given by $H_s^{(r)}(M)$. Hence

$$\dim H_s^{(r)}(M) \leqq \dim \bigoplus_{q-p=r} E_1^{p,q} = \dim H^{(r)}(M) .$$

Furthermore, by standard discussion

$$\sum_r (-1)^r \dim H_s^{(r)}(M) = \sum_r (-1)^r \dim H^{(r)}(M) .$$

Suppose M is in addition Kahler, V has non-empty zero sets, then $i(V)$ is a zero action on $H_{\bar{\partial}}^{(r)}(M)$ by the Deligne degeneracy criteria. Thus $E_1^{p,q}$ converges to $E_\infty^{p,q}$. We have

$$H_s^{(r)}(M) \cong \bigoplus_{q-p=r} E_\infty^{p,q} = \bigoplus_{q-p=r} E_1^{p,q}$$

$$= \bigoplus_{q-p=r} H^{p,q}(M) .$$

Proof of Theorem 1.5. For any $\varphi \in H^{1,0}(M)$, $i(V)\varphi$ is a holomorphic function on M, so $i(V)\varphi = 0$ and

$$\bar{\partial}_s \varphi = \bar{\partial}\varphi + s i(V)\varphi = 0$$

$$\Rightarrow \varphi \in H_s^{(-1)}(M) = 0 \quad \text{by Theorem 1.4}$$

we get

$$\varphi = \bar{\partial}_s \psi \quad \text{with } \psi \in A^{(-2)}(M) ,$$

comparing two sides gives

$$\varphi = i(V)\psi_1 \quad \text{with } \psi_1 \in A^{2,0}(M) ,$$

but $0 = \bar{\partial}\varphi = \bar{\partial} i(V)\psi_1 = -i(V)(\bar{\partial}\psi_1)$, since $\bar{\partial}\psi_1 \in A^{2,1}(M) \Rightarrow \bar{\partial}\psi_1 = 0$.

So $\psi_1 \in H^{2,0}(M)$, hence

$$i(V) : H^{2,0}(M) \to H^{1,0}(M) \quad \text{is surjective} .$$

If $\psi \in H^{2,0}(M)$ and $i(V)\psi = 0$, we have $\bar{\partial}_s \psi = 0$, then

$$\psi \in H_s^{(-2)}(M) = 0 \Rightarrow \psi = \bar{\partial}_s \psi_2 .$$

Comparing two sides, we get $\psi = 0$, therefore $i(V)$ is injective.

Proof of Theorem 1.7. By Theorem 1.3, we only need to prove

$$\dim H^{(r)}(Z(V)) \leqq \dim H_s^{(r)}(M) .$$

Our idea is to construct injective maps

$$\alpha_r : H^{(r)}(Z(V)) \to H_s^{(r)}(M) .$$

Let $U \subset \bigcup_{z \in \text{zero } V} U_z$, where U_z is the neighborhood in the definition of special vector field. Let

$$g(t) = \begin{cases} 1, & t \leqq \varepsilon^2 \\ 0, & t \geqq (2\varepsilon)^2 , \end{cases}$$

be a smooth function, where ε is a small positive number, s.t.

$$U_\varepsilon = \{x \in M | \langle V, V \rangle \leqq 4\varepsilon^2\} \subset U .$$

Obviously there exists a projection

$$\pi : U_\varepsilon \to Z(V) .$$

In fact, for any $z \in Z(V)$, we can choose coordinate chart

$$(z, z_{r+1}, \ldots, z_n)$$

in $U_z \cap Z(V)$, where z denotes the tangent coordinate of $Z(V)$. Then we have

$$\pi : (z, z_{r+1}, \ldots, z_n) \to (z, 0) .$$

For any $w \in H^{(r)}(Z(V))$, $\pi^* w$ is an extension of w to U_ε.

Obviously $i(V)(\pi^* w) = 0$, on U_ε. So $(\bar{\partial} + si(V))(\pi^* w) = 0$.

Motivated by Witten [17], we define

$$\sigma = g(|V|^2) + \frac{1}{s} g'(|V|^2) \bar{\partial} \tilde{V} + \cdots + \frac{1}{ns^n} g^{(n)}(|V|)^2 (\bar{\partial} \tilde{V})^{\wedge n} .$$

Since $i(V) \bar{\partial} \tilde{V} = -\bar{\partial} |V|^2$, it is easy to verify that

$$(\bar{\partial} + si(V))\sigma = 0$$

and σ is zero outside U, thus.

$$\tilde{w} = \pi^* w \wedge \sigma$$

is a form on M and

$$(\bar{\partial} + si(V))\tilde{w} = 0 .$$

Define

$$\alpha_r : \; H^{(r)}(Z(V)) \to H_s^{(r)}(M) , \qquad w \to \tilde{w} .$$

Then α_r is injective. In fact, if

$$\tilde{w} = (\bar{\partial} + si(V))\psi ,$$
$$\Rightarrow w = \bar{\partial}\psi | Z(V) \quad \text{on } Z(V) .$$

4 Integral formulas

In this section, we first prove that Theorem 1.3 implies the integral formula in Theorem 1.6. As corollaries, we get the Bott residue formula and the Duistermaat-Heckman integral formula of complex type. Note that we do not need the vector field to be Killing, therefore our formula are quite different from that of Duistermaat-Heckman [2]. All of the results in this section can be generalized to meromorphic vector field and to the case when the zero points are degenerate. See Sect. 8 for the detail.

Proof of Theorem 1.6. Suppose g is a smooth function of real variable t, s.t.

$$g(t) = \begin{cases} 1, & t \leqq \varepsilon^2 , \\ 0, & t \geqq (2\varepsilon)^2 , \end{cases}$$

around p_α.

Similar to last section, let us define a form σ_{p_α} around p_α

$$\sigma_{p_\alpha} = g(|V|^2) + \frac{1}{s} g'(|V|^2)\bar{\partial}\tilde{V} + \cdots \frac{1}{n}\frac{1}{s^n} g^{(n)}(|V|^2)(\bar{\partial}\tilde{V})^{\wedge n} .$$

Obviously $\bar{\partial}_s \sigma_{p_\alpha} = 0$.

From Theorem 1.3 and the proof of Theorem 1.6 we know that

$$\bigoplus_{p_\alpha} A_{p_\alpha} \cong \bigoplus_{p_\alpha}^{N} \mathbf{C} \cdot 1_{p_\alpha} \to H_s^{(0)}(M)$$

$$(c_{p_1}, \ldots, c_{p_N}) \to \sum_{p_\alpha} c_{p_\alpha} \cdot \sigma_{p_\alpha}$$

is an isomorphism. Here N is the number of zero points of V. In fact this map is first injective, also note that the two spaces have the same dimension.

Now for any

$$\varphi = \varphi_n / s^n + \cdots + \varphi_0 \in H_s^{(0)}(M) .$$

We have

$$\varphi = \sum_{p_\alpha} c_\alpha \sigma_{p_\alpha} + \bar{\partial}_s \psi ,$$

by the above isomorphism. Comparing two sides, we can easily get

$$\varphi_n = \frac{1}{n} \sum_{p_\alpha} \varphi_0(p_\alpha) g^{(n)}(|V|^2)(\bar{\partial}\tilde{V})^n + \bar{\partial}\psi' .$$

Theorem 1.6 follows from the following lemma.

Lemma 1.6.1.

$$\frac{1}{n} \int_M g_{p_\alpha}^{(n)}(|V|^2)(\bar{\partial}\tilde{V})^n = \frac{(-2\pi i)^n}{\det B_{p_\alpha}} .$$

Proof. We choose an Hermitian metric $\langle\, ,\rangle$, s.t. $\langle\, ,\rangle$ is Euclidean around p_α. Changing variable by

$$w_i = v_i^\alpha(z) \quad \text{where } V = \sum_i v_i^\alpha \frac{\partial}{\partial z_i}$$

around p_α, we have

$$\frac{1}{n}\int_M g_{p_\alpha}^{(n)}(|V|^2)(\bar\partial\tilde V)^n = \frac{n!}{\det B_{p_\alpha}} \int_{\varepsilon^2 \leqq |w|^2 \leqq 4\varepsilon^2} g^{(n)}(|w|^2) \wedge_{i=1}^n dw_i \wedge d\bar w_i\,.$$

Changing w_i's into polar variables and using integration by part, we easily get

$$\frac{1}{n}\int_M g_{p_\alpha}^{(n)}(|V|^2)(\bar\partial\tilde V)^n = \frac{2^n i^n n! S}{\det B_{p_\alpha}}\int_\varepsilon^{2\varepsilon} g^{(n)}(r^2) r^{2n-1}\,dr$$
$$= \frac{(-2\pi i)^n}{\det B_{p_\alpha}}\,.$$

In the above procedure, S is the area of the unit sphere $S^{2n-1}(1)$.

Corollary 4.1. *Let M, V be as in Theorem* 1.6, *and $P(X)$ be an invariant polynomial of degree $k \leqq n$. Then*

$$\int_M P(\Omega) = (-2\pi i)^n \sum_{p_\alpha} \frac{P(B_\alpha)}{\det B_{p_\alpha}}\,.$$

Especially if $k < n$,

$$\sum_{p_\alpha} \frac{P(B_\alpha)}{\det B_{p_\alpha}} = 0\,.$$

This is the Bott residue formula.

Proof. Let Ω be the curvature matrix of TM with respect to an Hermitian metric. Then we can find a tensor matrix J_M, s.t.

$$i(V)\Omega = \bar\partial J_M\,, \quad \text{(cf. [11])}\,.$$

Suppose P' is the polarization of P, we let

$$P_r(\Omega) = \binom{k}{r} P'(\underbrace{J_M \ldots J_M}_{k-r}, \Omega \ldots \Omega)\,.$$

Then

$$\widetilde{P(\Omega)} = P(\Omega)/s^k - P_{k-1}(\Omega)/s^{k-1} + \cdots + (-1)^k P_0(\Omega)$$

satisfies

$$(\bar\partial + s i(V))\widetilde{P(\Omega)} = 0\,.$$

In fact

$$P_0(\Omega)_{p_\alpha} = P(B_{p_\alpha})\,. \quad \text{(cf. [8])}$$

By our integral formula, we can easily get the result.

As to $k < n$, we note that

$$\widetilde{P(\Omega)} = \frac{1}{n}\sum_{p_\alpha} P(B_{p_\alpha})\sigma_{p_\alpha} + \bar{\partial}_s\psi\,.$$

Comparing two sides, we have

$$\frac{1}{n}\sum_{p_\alpha} P(B_{p_\alpha})g^{(n)}_{p_\alpha}(|V|^2)(\bar{\partial}\tilde{V})^n = \bar{\partial}\psi_{n-1}\,.$$

The result follows from integration.

Corollary 4.2. *If V has no zero, then for any*

$$\varphi = \varphi_n/s^n + \cdots + \varphi_0 \in H^{(0)}_s(M)$$

we have $\varphi_n = \bar{\partial}\eta$. Especially $\int_M \varphi_n = 0$.

Proof. In this case, by our Theorem 1.2

$$\varphi \in H^{(0)}_s(M) = 0$$
$$\Rightarrow \varphi = \bar{\partial}_s\psi\,.$$

Comparing two sides, we get $\varphi_n = \bar{\partial}\eta$.

We have especially the following Duistermaat-Heckman integral formula for holomorphic vector field.

Corollary 4.3. *Suppose M is Kahler, V is as in Theorem 1.6. Let ω be the Kahler form on M, then there exists a function f s.t. $i(V)\omega = \bar{\partial} f$ and*

$$\frac{1}{(2\pi i)^n}\int_M e^{-sf}\frac{\omega^n}{n!} = \frac{1}{s^n}\sum_{p_\alpha}\frac{e^{-sf(p_\alpha)}}{\det B_{p_\alpha}}\,.$$

Proof. We can always find a function f by Deligne degeneracy criteria, such that $\tilde{\omega} = \omega - sf \in H^{(0)}_s(M)$. Then

$$e^{\tilde{\omega}} = e^{\omega}\cdot e^{-sf} \in H^{(0)}_s(M)\,.$$

The result follows from our integral formula.

Corollary 4.4. *Let M, V be as in Corollary 4.3. Then for any $\varphi \in H^{n,n}(M)$, there exists a function φ_0, such that*

$$\frac{1}{(-2\pi i)^n}\int_M \varphi = \sum_{p_\alpha}\frac{\varphi_0(p_\alpha)}{\det B_{p_\alpha}}$$

and for any $\varphi \in H^{k,k}(M)$ with $k < n$, there exists a function φ_0 such that

$$\sum_{p_\alpha}\frac{\varphi_0(p_\alpha)}{\det B_{p_\alpha}} = 0\,.$$

Proof. By Corollary 2.3, we have

$$H^{p,q}(M) = 0 \quad \text{for } p \neq q .$$

So for any $\varphi \in H^{k,k}(M)$

$$\begin{aligned} &\bar{\partial} i(V)\varphi = -i(V)\bar{\partial}\varphi = 0 \\ &\Rightarrow i(V)\varphi \in H^{k-1,k}(M) = 0 \\ &\Rightarrow i(V)\varphi = \bar{\partial}\varphi_{k-1} \end{aligned}$$

and

$$\bar{\partial} i(V)\varphi_{k-1} = -i(V)\bar{\partial}\varphi_{k-1} = -i(V)i(V)\varphi = 0$$

so on. We finally have

$$\tilde{\varphi} = \varphi/s^k + \varphi_{k-1}/s^{k-1} + \cdots + \varphi_0$$

satisfying $\bar{\partial}_s\tilde{\varphi} = 0$.

The same discussion as the proof of Corollary 4.2 gives the result.

We can actually prove the same result when the zero set of V contains submanifolds. For simplicity we take $s = 1$. Write $\bar{\partial}_s$ as $\bar{\partial}_1$ and the corresponding equivariant cohomology group as $H_1^{(r)}(M)$. First let us recall the notion of nondegenerate submanifold. Given any Hermitian metric on M, let Ω_α be the curvature matrix of the normal bundle $N_\alpha = TM/TM_\alpha$, r_α be the codimension of M_α, and L_α be the following endomorphism induced by V,

$$\begin{aligned} L_\alpha|M_\alpha : N_\alpha &\to N_\alpha \\ Y &\to [V, Y] . \end{aligned}$$

We say M_α is a nondegenerate submanifold, if L_α is a nondegenerate homomorphism. Denote by i_α the inclusion of M_α in M. Then

Theorem 4.1. *Suppose V is a holomorphic vector field on M with nondegenerate zero set $\bigcup M_\alpha$. Then for any $\varphi = \varphi_n + \cdots + \varphi_0 \in H_1^{(0)}(M)$, we have*

$$\int_M \varphi_n = \sum_{M_\alpha} (2\pi i)^{r_\alpha} \int_M \frac{i_\alpha^* \varphi}{\det(L_\alpha - \Omega_\alpha)}$$

This theorem can also be proved by adapting Bott's method in [5]. In fact using the β constructed in Sect. 3 with $s = 1$, we have $\varphi = \bar{\partial}_1(\varphi \wedge \beta)$ outside the zero points of V. This gives

$$\varphi_n = d\sum_j \varphi_j \wedge \alpha \wedge (\bar{\partial}\alpha)^{n-j-1} .$$

Let N_α^ε be a neighborhood of M_α. Any point N_α^ε has geodesic distance from M_α less than ε. Then

$$\int_M \varphi_n = -\sum_{p_\alpha} \int_{\partial N_\alpha^\varepsilon} \sum_j \varphi_j \wedge \alpha \wedge (\bar{\partial}\alpha)^{n-j-1} .$$

Let ε go to zero, Following Bott [5], we can evaluate the limit which is exactly the left hand side of the formula in Theorem 4.1. We leave the detail to the reader.

Motivated by our work, Zhang [19] has given a different proof of this result using Bismut's idea. Similar formula also holds for meromorphic vector field.

5 Fixed point and localization formula

In this section, we relate our integral formulae to the Lefschetz fixed point formula. This can be viewed as an extension of the formula of Berline-Vergne [4] to non-compact group action case. Similar formula also holds for meromorphic vector fields. See Sect. 6.

Let V be a holomorphic vector field on a compact complex manifold M with $\dim M = n$. Then V generates a holomorphic one parameter group g_t. That is $\frac{d}{dt} g_t|_{t=0} = V$. Assume that V has only isolated non-degenerate zero point $\{p_\alpha\}$ and g_t lifts to an action on a holomorphic vector bundle E. Put Hermitian metrics on TM and E. Then from [8], we have tensor matrices J_M and J_E, such that

$$i(V)\Omega_M = -\bar{\partial} J_M, \quad i(V)\Omega_E = -\bar{\partial} J_E$$

where Ω_M, Ω_E are the curvature matrices of TM and E respectively. For convenience we omit the factor $2\pi i$ in the following

Let

$$Td(M) = \det \frac{\Omega_M}{1 - \exp(-\Omega_M)}$$

be the Todd class of M. Define its equivariant deformation by

$$Td^V(M) = \det \frac{\Omega_M + tJ_M}{1 - \exp(-\Omega_M - tJ_M)} .$$

Similarly define the equivariant deformation of the Chern character of E by

$$ch^V E = Tr \exp(\Omega_E + tJ_E) .$$

Obviously both $Td^V(M)$ and $ch^V E$ belong to $H_t^{(0)}(M)$. As a corollary of Theorem 1.6, we have the following localization formula

$$\int_M Td^V(M) ch^V E = \sum_{p_\alpha} \frac{Tr \exp(tJ_E)}{\det(1 - \exp(-tJ_M))} (p_\alpha) .$$

Consider the differential complex

$$0 \to A^{0,0}(E) \xrightarrow{\bar{\partial}} A^{0,1}(E) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} A^{0,n}(E) \to 0 .$$

Denote its Lefschetz number under the action of g_t by $L_{g_t}(E)$. Recall that

$$L_{g_t}(E) = \sum_{j=1}^{n}(-1)^j Tr_{g_t} H^j(M,E) .$$

Combine with the fixed point formula in [1], we get

Theorem 5.1. *Let V be a holomorphic vector field with nondegenerate isolated zero points, then*

$$L_{g_t}(E) = \int_M Td^V(M) ch^V E .$$

It is easy to see that this formula holds in the cases of meromorphic vector field and the non-degenerate zero submanifolds. See Sect. 8 for the case when the zero points are isolated and degenerate.

6 Meromorphic vector fields and sheaf cohomology

In this section, we describe some extensions of our results to meromorphic vector field and analytic sheaf cohomology. Since the proofs are basically the same as before, we omit the details.

Suppose M is a compact complex manifold and

$$L \to M$$

is a holomorphic line bundle. Let

$$V \in \Gamma(TM \otimes L)$$

be a holomorphic section of $TM \otimes L$. Note that if L is ample, such section always exists for $TM \otimes L^m$ for some positive integer m.

In local coordinate chart, one has

$$V = \sum_i v_i \frac{\partial}{\partial z_i} \quad \text{with } v_i\text{'s the sections of } L .$$

Write $L^r = L^{\otimes r}$ and let $A^{p,q}(L^r)$ be L^r-valued (p,q)-forms, and

$$A^{(r)}(L_k) = \bigoplus_{q-p=r} A^{p,q}(L^{-p+k})$$

$$\bar{\partial}_s = \bar{\partial} + si(V) .$$

Since we still have $\bar{\partial}_s^2 = 0$, we get a differential complex

$$0 \to A^{(-n)}(L_k) \xrightarrow{\bar{\partial}_s} A^{(-n+1)}(L_k) \xrightarrow{\bar{\partial}_s} \cdots \xrightarrow{\bar{\partial}_s} A^{(n)}(L_k) \to 0$$

and define its cohomology group by

$$H_s^{(r)}(M, L_k) = \ker \bar{\partial}_s / \operatorname{Im} \bar{\partial}_s | A^{(r)}(L_k)$$

Lemma 6.1. *For any* $s, s' \in C^*$, *we have*

$$H_s^{(r)}(M, L_k) \cong H_{s'}^{(r)}(M, L_k).$$

Theorem 6.2. *If V has no zero points*

$$H_s^{(r)}(M, L_k) = 0 \quad \text{for any } r \text{ and } s \neq 0.$$

Theorem 6.3. *Suppose V has only isolated zero points* $\{p_\alpha\}$. *Then for* $s \neq 0$

$$\dim H_s^{(r)}(M, L_k) = \begin{cases} 0 & r \neq 0 \\ \sum_{p_\alpha} \text{rank } O/I_{p_\alpha}, & r = 0. \end{cases}$$

Here I_{p_α} is still generated by $\{v_j^\alpha\}$ around p_α.

Theorem 6.4.

$$\dim H_s^{(r)}(M, L_k) \leqq \dim H^{(r)}(M, L_k),$$

where

$$H^{(r)}(M, L_k) = \bigoplus_{q-p=r} H^{p,q}(M, L^{-p+k}).$$

Theorem 6.5. *For a compact complex surface M, suppose V has only isolated zero points. Then*

$$i(V) : H^{2,0}(M, L^{-2}) \to H^{1,0}(M, L^{-1})$$

is an isomorphism.

Theorem 6.6. *Suppose the zero points of V are all nondegenerate. For any* $\varphi = \varphi_n + \varphi_{n-1} + \cdots + \varphi_0 \in A^{(0)}(M, L)$ *where* $A^{(0)}(M, L) = \bigoplus_p A^{p,p}(L^{n-p})$ *such that* $(\bar{\partial} + i(V))\varphi = 0$, *we have*

$$\frac{1}{(-2\pi i)^n} \int_M \varphi_n = \sum_{p_\alpha} \frac{\varphi_0(p_\alpha)}{\det B_{p_\alpha}}.$$

Here p_α *and* B_{p_α} *are defined as before.*

This formula obviously contains the Baum-Bott integral formula for meromorphic vector field, (cf. [8]). In fact, Chern's proof can also be used here to prove this theorem. A Duistermaat-Heckman integral formula can be easily deduced from this formula.

We can go further to consider the following situation. Let V be a holomorphic vector field on M and

$$F \to M$$

be a holomorphic vector bundle. We consider differential complex

$$0 \to A^{(-n)}(F) \xrightarrow{\bar{\partial}_s} A^{(-n+1)}(F) \xrightarrow{\bar{\partial}_s} \cdots \xrightarrow{\bar{\partial}_s} A^{(n)}(F) \to 0$$

where

$$A^{(r)}(F) = \bigoplus_{q-p=r} A^{p,q}(F).$$

Some similar results to our main results can also be obtained in the same way for this complex. One can also take V to be a meromorphic vector field or to be a section of $TM \otimes \operatorname{End} F$. We omit the details here.

7 Analytic discussion

The original proof of Theorem 1.4 is by analytic method, i.e. the so-called semi-classical approximation developed by Helffer-Sjostrand, (cf. [16]). This proof has its independent interest. We would like to give a sketch of it. Let $M, V, \bar\partial_s$ be as above and $\langle\, ,\, \rangle$ be an Hermitian metric on M. We can define the dual operator $\bar\partial^*$ of $\bar\partial_s$ by using this metric. Obviously

$$\bar\partial_s^* = \bar\partial^* + s\tilde V,$$

where $\tilde V$ is the wedge product by the dual $(1,0)$-form of V.

We define $\Delta_s = (\bar\partial_s + \bar\partial_s^*)^2$. It is easy to calculate that

$$\begin{aligned}\Delta_s &= \bar\partial_s\bar\partial_s^* + \bar\partial_s^*\bar\partial_s\\ &= \Delta_{\bar\partial} + s^2|V|^2 + s(\bar\partial\tilde V + i(\bar\partial\tilde V))\end{aligned}$$

where $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$, and $i(\bar\partial\tilde V)$ is the dual operator of $\bar\partial\tilde V$ which is the wedge product by $\bar\partial\tilde V$.

Δ_s is a linear elliptic operator of second order. By standard Hodge theoretic discussion, we have

1) $\dim H_s^{(r)}(M) < \infty$,

2) $H_s^{(r)}(M) \cong \ker \Delta_s | A^{(r)}(M)$.

Analytic proof of Theorem 1.2. Suppose $\lambda^{(r)}$ is the minimal eigenvalue of Δ_s acting on $A^{(r)}(M)$. Then $\lambda^{(r)} \geqq 0$. Let $\varphi \in A^{(r)}(M)$ be the eigenvalue of $\lambda^{(r)}$, then

$$\Delta_s\varphi = \lambda^{(r)}\varphi.$$

Let $m = \min|V|$; $\bar M = \sup|L|$ where $L = i(\bar\partial\tilde V) + \bar\partial\tilde V$. Since V has no zero, $m > 0$. We have

$$\lambda^{(r)} = \frac{(\Delta\varphi,\varphi)}{(\varphi,\varphi)} + s^2\frac{(|V|^2\varphi,\varphi)}{(\varphi,\varphi)} + s\frac{(L\varphi,\varphi)}{(\varphi,\varphi)}$$

Here $(\, ,\,)$ is the induced metric on $A^{(r)}(M)$ from the Hermitian structure $\langle\, ,\, \rangle$.

When $s > 0$ is large enough, we have

$$\begin{aligned}\lambda^{(r)} &\geqq 0 + s^2 m - s\bar{M} \\ &\geqq \frac{s^2}{2} m^2 > 0 .\end{aligned}$$

So $\ker \Delta_s | A^{(r)}(M) = 0 \cong H_s^{(r)}(M)$.

Now we suppose that V has only isolated zero points $\{p_\alpha\}$, all of which are nondegenerate, i.e. around p_α

$$V = \sum_{i,j} A_{ij}^{\alpha} z_i \frac{\partial}{\partial z_j} + O(|z|^2)$$

and the matrix $(A_{ij}^{\alpha})_{n \times n}$ is nonsingular.

Sketch of the analytic proof of theorem 1.4. For simplicity, we assume that on the small balls $\{U_\alpha \ni p_\alpha\}$,

$$V = \sum_{i,j} \lambda_i^{\alpha} z_i \frac{\partial}{\partial z_i} + O(|z|^2)$$

and $\bar{U}_\alpha \cap \bar{U}_\beta = \emptyset$ for $\alpha \neq \beta$.

Let

$$\begin{aligned}h &= \frac{1}{s} \\ P(h) &= h^2 \Delta_{h-1} = \frac{1}{s} \Delta_s \\ P_\alpha(h) &= P(h) | U_\alpha \\ \overline{P_\alpha(h)} &= -h^2 \sum_i \frac{\partial^2}{\partial z_i \partial \bar{z}_i} + \sum_i |\lambda_i^{\alpha}|^2 |z_i|^2 \\ &\quad + h \left[\sum_i \left(\overline{\lambda_i^{\alpha} d z_i} \wedge dz_i + \lambda_i^{\alpha} i \left(\frac{\partial}{\partial z_i} \right) i \left(\frac{\partial}{\partial \bar{z}_i} \right) \right) \right] .\end{aligned}$$

Note that $\lambda_i^{\alpha} \neq 0$ by the nondegeneracy. Let

$$\begin{aligned}P(h)_r &= P(h) | A^{(r)}(M) \text{ and} \\ P_\alpha(h)_r &= P_\alpha(h) | A^{(r)}(U_\alpha)\end{aligned}$$

with Dirichlet boundary condition. Let

$$\overline{P_\alpha(h)_r} = \overline{P_\alpha(h)} | A^{(r)}(U_\alpha)$$

also with Dirichlet boundary condition.

Then using semi-classical approximation we can prove that

1)

$$\begin{aligned}&\dim \ker \overline{P_\alpha(h)_0} = 1 \text{ and} \\ &\dim \ker \overline{P_\alpha(h)_r} = 0, \text{ for } r \neq 0 .\end{aligned}$$

2)

$$\# \operatorname{sp}\{P_\alpha(h)_r\} \cap [0, c_\alpha h] = \dim \ker \overline{P_\alpha(h)}_r$$

when the constant $c_\alpha h$ is small enough. Here sp denotes the spectrum.

3)

$$\dim \ker \overline{P_\alpha(h)}_0 = \# \operatorname{sp}\{P_\alpha(h)_0\} \cap [0, ch]$$
$$= \text{ the number of the zero points of } V$$

and

$$\# \operatorname{sp}\{P_\alpha(h)_r\} \cap [0, ch] = 0 \quad \text{for } r \neq 0 .$$

Then if we take c, h small enough, 3) obviously gives the theorem.

The semi-classical approximation method is very useful, we can use it to get analytic proofs of other interesting topological results.

8 Generalizations

In this section we describe some further generalizations of our results in the above sections. We will only give sketches and postpone the detail to a forthcoming paper.

1. Theorem 1.6 can be generalized to the situation that the zero points $\{p_\alpha\}$ of V are isolated and degenerate. Using the Grothendieck residue which we denote by $\operatorname{Res}_{p_\alpha}$ at p_α (cf. P. Baum-R. Bott: Singularities of holomorphic foliations, *J. Diff. Geometry* 7 (1972) 279–342), we have

Theorem 8.1. *Let M be a compact complex manifold of dimension n and V be a holomorphic vector field on it with isolated zero points $\{p_\alpha\}$. Let*

$$w = w_n/s^n + \cdots w_0 \in H_s^{(0)}(M),$$

then

$$\int_M w_n = \sum_{p_\alpha} w_0(p_\alpha) \operatorname{Res}_{p_\alpha} \begin{pmatrix} dz_1 \ldots dz_n \\ v_1, \ldots, v_n \end{pmatrix} .$$

Here the z_i's are the local coordinates and $V = \sum_{j=1}^n v_j \frac{\partial}{\partial z_j}$ is the local expression around p_α. Note that by the construction of σ_{p_α} in Sect. 4, we can obviously choose w_0 to be constant around each p_α.

An algorithm to compute the residue was given in the paper of Baum-Bott. First around p_α, there exist integers $\{a_i\}$ such that $\{z_i^{a_i}\}$ generates the ideal generated by the $\{v_1, \ldots, v_n\}$. By Hilbert Nullstallensatz, we can find holomorphic functions $\{b_{ij}\}$ such that

$$z_i^{a_i} = \sum_{j=1}^n b_{ij} v_j .$$

Then

$$\operatorname{Res}_{p_\alpha} \begin{pmatrix} dz_1 \ldots dz_n \\ v_1, \ldots, v_n \end{pmatrix} = \operatorname{Res}_{p_\alpha} \begin{pmatrix} \det[b_{ij}] dz_1 \ldots dz_n \\ z^{a_1}, \ldots, z^{a_n} \end{pmatrix} .$$

Plug in the Taylor expansion of deb$[b_{ij}]$. The residue is just the coefficient of the term $\frac{dz_1 \ldots dz_n}{z_1, \ldots, z_n}$.

The proof of Theorem 8.1 can also be completed by using the β constructed in Sect. 3. Actually we find $w = \bar{\partial}_s(\beta \wedge w)$ outside $\{p_\alpha\}$, which gives $w_n = \sum_{j=1}^n w_j \wedge \alpha \wedge (\bar{\partial}\alpha)^{n-j-1}$. Therefore let B_α^ε be a small ball of radius ε around p_α, then

$$\int_M w_n = -\sum_{p_\alpha} \int_{\partial B_\alpha^\varepsilon} \sum_{j=1}^n w_j \wedge \alpha \wedge (\bar{\partial}\alpha)^{n-j-1} .$$

Let ε go to zero, it is an easy exercise to show that on the right hand side, all of the terms go to zero except the term of $j = 0$ which is equal to

$$\sum_{p_\alpha} w_0(p_\alpha) \mathrm{Res}_{p_\alpha} \left(\frac{dz_1 \ldots dz_n}{v_1, \ldots, v_n} \right) .$$

All of the results in Sect. 4, except Theorem 4.1, can be generalized correspondingly. Similar results hold also for meromorphic vector field. Recently we find that a similar formula was also discussed by J. Carrell (A remark on the Grothendieck residue map, *Proc. of the AMS*, Vol. 70, No. 1 (1978) 43–48). Note that our motivation is quite different from Carrell's.

2. Combining Theorem 8.1 with the result of Toledo-Tong (Holomorphic Lefschetz fixed point formula, *Bulletin of the AMS* 81(1975) No. 6, 1133–1135) which expresses the Lefschetz number in terms of the Grothendieck residue, we get the generalization of Theorem 5.1 to the case that the zero points of V are isolated and degenerate. Let us use the same notation as in Sect. 5

Theorem 8.2. *Suppose the zero points of V are isolated or nondegenerate submanifolds, then*

$$L_{g_t}(E) = \int_M Td^V(M) ch^V E .$$

Theorem 4.1 and the Atiyah-Bott Lefschetz fixed point formula combined together give the above identity when the zero points of V are nondegenerate submanifolds.

3. Let G be a complex Lie group which acts on M holomorphically. Let g be its Lie algebra. Consider the $A^{(*)}(M)$-valued polynomials on g,

$$A^{(*)}(M, g) = A^{(*)}(M) \otimes S(g^*)$$

where $S(g^*)$ denotes the symmetric polynomials on the dual g^* of g. Define a differential D on $A^{(*)}(M, g)$ in the following way. Let $\alpha \otimes f \in A^{(*)}(M, g)$ and $X \in g$, then

$$D(\alpha \otimes f)(X) = f(X)(\bar{\partial}\alpha + i(\bar{X})\alpha)$$

where $\bar{X}$ is the holomorphic vector field generated by X on M. This makes $A^{(*)}(M, g)$ into a differential complex. We can define its cohomology and study its localization. A residue formula in the spirit of Baum-Bott can then be proved. We will continue our discussion in this general setting in a forthcoming paper.

As remarked by the referee, this becomes especially interesting if $G = K \otimes C$ where K is a compact Lie group. For Kahler manifold, this connects the deRham equivariant cohomology to our holomorphic equivariant cohomology. Combining with the above integral formulas, this allows us to study the cohomology of symplectic reduced space (L. Jeffrey-F. Kirwan, *Localization and nonabelian group actions*, preprint 1993) by directly using geometric invariant theory instead of symplectic quotient.

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References

1. Atiyah, M.F. and Bott, R.: A Lefschetz fixed points formula for elliptic complexes, I, Ann. of Math., **86** (1967) 374–407; II, Ann. of Math., **88** (1968) 415–441
2. Atiyah, M.F. and Bott, R.: The moment map and equivariant cohomology, Top., Vol. 23, No. 1 (1984) 1–23
3. Atiyah, M.F. and Singer, I.M.: The index formula for elliptic operator, III, Ann. of Math., **88** (1968) 546–604
4. Berligne and Vergne, M.: The equivariant index and Kirillov character formula, Amer. J. of Math., (1985) 1159–1190
5. Bott, R.: A residue formula for holomorphic vector fields, J. of Diff. Geom., Vol 1, No. 1 (1967) 311–330
6. Carrell, J. and Liebermann, D.: Holomorphic vector fields and Kahler manifolds, Invent. Math., vol. 21 (1973) 303–309
7. Carrell, J. and Sommese, A.J.: Some topological aspects of C^*-actions on compact Kahler manifolds, Comment. Math. Helv., **54** (1979) 567–582
8. Chern, S.S.: Meromorphic vector fields and characteristic numbers. Scripta. Math., Vol. XXIX, No. 3–4 (1973) 243–251
9. Crew, R. and Fried, D.: Nonsingular holomorphic flows, Topology Vol. 25, No. 4 (1986) 471–473
10. Frankel, T.: Fixed points and torsion on Kahler manifold, Ann. of Math., **70** (1959) 1–8
11. Griffiths, P.A. and Harris, J.: Principles of Algebraic Geometry. John Wiley 1978
12. Karp, L.: Holomorphic vector fields on complex manifold, The Michigan Math. J., No. 1 (1987) 31–38
13. Kirwan, F.: Morse functions for which the stationary phase approximation is exact, Topology, Vol. 26, No. 1 (1987) 37–40
14. Kobayashi, S.: Transformation Groups, In Differential Geometry, Springer-Verlag 1972
15. Kodaira, K.: Complex Manifold, Springer 1986
16. Wang, X.P.: Thesis, Nante Univ. 1985
17. Witten, E.: Supersymmetry and Morse theory, J. Diff. Geom., **17** (1982)
18. Witten, E.: Holomorphic Morse inequality, Taubner-texte, Zur Math. 70, Algebraic and Differential Topology
19. Zhang, W.P.: A remark on the Bott residue formula, Acta Math. Sinica, New series, Vol. 6 No. 4 (1990) 306–314