INDESTRUCTIBLE SUSLIN TREES

NEEMAN, ITAY

Abstract. We show how to construct an $\aleph_1$-Suslin tree which is indestructible under forcing with a given c.c.c. poset of size $\aleph_1$, in $L(x)$ for any real x. This answers a recent question of Woodin. More generally we do this at any regular uncountable cardinal which is not weakly compact, and the construction can be carried out in any model satisfying standard condensation properties.

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Dedicated to Ronald B. Jensen (1936–2025), a great mathematician and a wonderful mentor.

1. Introduction

Say that a κ-Suslin tree is *$\mathbb{P}$-indestructible* if it remains Suslin in the extension by $\mathbb{P}$. Say that a model M has the *indestructible Suslin tree property* at κ if it satisfies that for every κ-c.c. poset $\mathbb{P}$ of size κ, there is a $\mathbb{P}$-indestructible κ-Suslin tree.

In his talk at the *2023 Very Informal Gathering of Logicians at UCLA*, Woodin [3] raised the following question: Is it true that for every real x, the model $L(x)$ has the indestructible Suslin tree property at $\aleph_1$?

This question came up as part of the work in Woodin [4] on generic MA-models. Woodin proved the indestructible Suslin tree property at $\aleph_1$ in $L(x)$ for a *cone* of reals x, and this was sufficient for his work in [4]. He noted that the property likely holds in more models, including in particular L, and that this would give additional information on generic MA-models, for example implying that generic MA-models over L satisfy "$V = \text{HOD}$."

Given enough determinacy, obtaining the indestructible Suslin tree property at $\aleph_1$ on a cone of reals x reduces to showing that for every real x, there is a real $y \geq_T x$ so that the property holds in $L(y)$. Woodin does this by forcing over $L(x)$, first to add ω_2 generic $\aleph_1$-Suslin trees, and then to code these by a single real y.

In this short paper we construct indestructible Suslin trees using an adaptation of the classical construction of Jensen [1]. This allows us to show that every model that satisfies condensation, satisfies the indestructible Suslin tree property at $\aleph_1$. In particular this applies to the models L and $L(x)$ for all reals x, answering Woodin's question in the positive.

A key ingredient in the construction is a new diamond principle, that requires guessing some sets on a club while simultaneously guessing that club. We call this principle *coherent diamond*, and we prove it from condensation. This is done in Section 2.

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In Section 3 we use coherent diamond to construct indestructible Suslin trees. The general structure of the construction follows the classic construction of a Suslin tree in [1]. The key new ingredient here is folding in a construction of nodes that are outright forced to belong to guessed names for antichains. The proof that this works makes use of coherent diamond.

We work in greater generality than trees on $\aleph_1$, handling all regular uncountable cardinals in Section 2, and regular uncountable cardinals which satisfy a certain square principle in Section 3. In L and in $L(x)$ the needed square principle is equivalent to not being weakly compact, by a classical result of Jensen [1]. This equivalence generalizes to higher Jensen-indexed fine structural inner models by work of Schimmerling-Zeman [2] and Zeman [5], leading to our final result that in all such models, at regular uncountable cardinals, the indestructible Suslin tree property holds when the cardinal is not weakly compact, and is therefore equivalent to the existence of a Suslin tree.

2. Coherent diamond

Fix a regular uncountable cardinal κ. Say that the sequences $\vec{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ and $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ *coherently guess* a set $A \subseteq \kappa$ on a club $C \subseteq \kappa$ if for every $\alpha \in C$ we have that $A_\alpha = A \cap \alpha$ and $C_\alpha = C \cap \alpha$. Note that C, the guessing club, must itself be guessed correctly at α; it is not enough to guess just the set A. This will be key to the construction of indestructible Suslin trees later on.

Below we view a given sequence $\langle X_\alpha \mid \alpha < \kappa \rangle$ of subsets of κ as a binary predicate on κ, namely the predicate which holds of $\langle \alpha, \xi \rangle$ iff $\xi \in X_\alpha$.

Definition 2.1. *Coherent diamond at* κ, denoted $\diamond^c_\kappa$, states that for any finitely many predicates $\Delta_1, \dots, \Delta_n$ on κ, and any first order sentence φ, there are sequences $\vec{A} = \langle A_\alpha \subseteq \alpha \mid \alpha < \kappa \rangle$ and $\vec{C} = \langle C_\alpha \subseteq \alpha \mid \alpha < \kappa \rangle$, with the property that if there is $A \subseteq \kappa$ so that $(\kappa; \Delta_1, \dots, \Delta_n, \vec{A}, \vec{C}, A) \models \varphi$, then one can find such A, and a club $C \subseteq \kappa$, so that A is coherently guessed by $\vec{A}$ and $\vec{C}$ on C. Moreover one can do this in such a way that for every $\alpha \in C$, $(\alpha; \Delta_1 \restriction \alpha, \dots, \Delta_n \restriction \alpha, \vec{A} \restriction \alpha, \vec{C} \restriction \alpha, A \cap \alpha) \models \varphi$.

When using this principle we will take A to code a pair $\langle U, Y \rangle$ of subsets of κ, and will be interested primarily in guessing the first coordinate, U. In such a situation one can view φ as specifying a Σ^1_1 property of U in parameters $\Delta_1, \dots, \Delta_n, \vec{A}, \vec{C}$, namely the property that there exists Y so that $(\kappa; \Delta_1, \dots, \Delta_n, \vec{A}, \vec{C}, \langle U, Y \rangle) \models \varphi$. Coherent diamond in this context implies that if this Σ^1_1 property is satisfiable, then it is satisfied by a set which is guessed coherently (by the first coordinates of the A_αs) on a club.

Claim 2.2. *$\diamond^c_\kappa$ implies $\diamond_\kappa(E)$ for every stationary $E \subseteq \kappa$.*

Proof. For a set of ordinals X, let $(X)_0 = \{\xi \mid 2\xi \in X\}$ and $(X)_1 = \{\xi \mid 2\xi{+}1 \in X\}$. This allows us to view X as coding the pair of sets $(X)_0$ and $(X)_1$.

Let $E \subseteq \kappa$ be stationary. Recall that $\diamond_\kappa(E)$ states that there is a sequence $\vec{U}$ so that for every $U \subseteq \kappa$, the set $\{\alpha \in E \mid U_\alpha = U \cap \alpha\}$ is stationary. Let φ be the sentence which states that $U = (A)_0$ and a club $Y = (A)_1$ witness that the sequence $\langle (A_\alpha)_0 \mid \alpha < \kappa \rangle$ is *not* a $\diamond_\kappa(E)$ sequence, meaning that $(\forall \alpha \in E \cap (A)_1)(A_\alpha)_0 \neq (A)_0 \cap \alpha$, phrased in the language of structures $(\kappa; \in, E, \Delta, \vec{A}, \vec{C}, A)$, where Δ is the graph of ordinal multiplication.

Let $\vec{A}, \vec{C}$ witness coherent diamond for this sentence φ. We claim that $\langle U_\alpha = (A_\alpha)_0 \mid \alpha < \kappa\rangle$ is a $\diamond_\kappa(E)$-sequence.

Suppose otherwise. Then there is $U \subseteq \kappa$ and a club $Y \subseteq \kappa$ so that U is not guessed correctly by U_α for any $\alpha \in E \cap Y$. Letting A be such that $(A)_0 = U$ and $(A)_1 = Y$ we have that $(\kappa; \in, E, \Delta, \vec{A}, \vec{C}, A) \models \varphi$. By $\diamond^c_\kappa$, and replacing A by another witness for φ if needed, we may assume that A is coherently guessed by $\vec{A}$ and $\vec{C}$ on some club C. This implies that for every $\alpha \in C$, $A_\alpha = A \cap \alpha$, and hence, if α is closed under ordinal multiplication, $(A_\alpha)_0 = (A)_0 \cap \alpha$. At the same time, since $(\kappa; \in, E, \Delta, \vec{A}, \vec{C}, A) \models \varphi$, for every $\alpha \in E \cap (A)_1$, $(A_\alpha)_0 \neq (A)_0 \cap \alpha$. Since $E \cap (A)_1 \cap C$ includes ordinals which are closed under ordinal multiplication, this is a contradiction. $\square$

We will not actually use Claim 2.2. It is included as a justification for viewing $\diamond^c_\kappa$ as a *diamond* principle.

Lemma 2.3. *L satisfies $\diamond^c_\kappa$ for every regular uncountable cardinal κ. Similarly for $L(x)$ for any real x.*

Proof. We prove the lemma for L. The proof for $L(x)$ is similar.

Let $H^{L_\gamma}(X)$ denote the Skolem hull of X in L_γ.

Fix κ. Fix predicates $\Delta_1, \dots, \Delta_n$ on κ, and a sentence φ.

For notational simplicity below, we will use P to refer to the sequence of predicates $\Delta_1, \dots, \Delta_n, \vec{A}, \vec{C}$, and will use $P{\restriction}\alpha$ to refer to $\Delta_1{\restriction}\alpha, \dots, \Delta_n{\restriction}\alpha, \vec{A}{\restriction}\alpha, \vec{C}{\restriction}\alpha$.

Define A_α and C_α by recursion on α as follows: If there is $A \subseteq \alpha$ so that $(\alpha; P{\restriction}\alpha, A) \models \varphi$, then let A_α be the $<_L$-least such set, let δ_α be least so that $\alpha, P{\restriction}\alpha, A_\alpha \in L_{\delta_\alpha+1}$, and let C_α be the set of $\beta < \alpha$ such that $H^{L_{\delta_\alpha+1}}(\beta \cup \{\alpha, P{\restriction}\alpha, A_\alpha\}) \cap \alpha = \beta$.

This recursive definition determines the sequences $\vec{A} = \langle A_\alpha \mid \alpha < \kappa\rangle$ and $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$. It remains to show that these sequences satisfy the requirements of coherent diamond for φ.

Suppose that there is $A \subseteq \kappa$ so that $(\kappa; P, A) \models \varphi$, and fix the $<_L$-least one. Let δ be least so that $\kappa, P, A \in L_{\delta+1}$. Let C be the set of $\beta < \kappa$ so that $H^{L_{\delta+1}}(\beta \cup \{\kappa, P, A\}) \cap \kappa = \beta$.

It is clear that C is club in κ. For $\alpha \in C$, let $H_\alpha = H^{L_{\delta+1}}(\alpha \cup \{\kappa, P, A\})$, let M_α be the transitive collapse of H_α, and let $\pi_\alpha \colon H_\alpha \to M_\alpha$ be the collapse embedding. Since $H_\alpha \cap \kappa = \alpha$, π_α maps κ to α. It is clear by condensation and elementarity that M_α is precisely equal to $L_{\delta_\alpha+1}$, that $(\alpha; P{\restriction}\alpha, A \cap \alpha) \models \varphi$, that $A \cap \alpha = \pi_\alpha(A) = A_\alpha$, and consequently that $C_\alpha = C \cap \alpha$. $\square$

Remark 2.4. We phrased Lemma 2.3 for L and $L(x)$, but the only properties of L used in the proof are the fact that it has a definable wellordering, the absoluteness of this wellordering to sufficiently closed initial segments, and condensation of the Skolem hulls used in the proof to initial segments. These properties hold in all standard fine structural inner models, and the proof of Lemma 2.3 can be adapted to show that coherent diamond holds at all regular uncountable cardinals in all these models.

3. INDESTRUCTIBLE SUSLIN TREES

Recall that $\square(\kappa)$ is the statement that there is a sequence of clubs $G_\alpha \subseteq \alpha$, for limit $\alpha < \kappa$, which is: (a) *coherent*, meaning that $\alpha \in \lim(G_\beta) \to G_\alpha = G_\beta \cap \alpha$, and (b) *not threadable*, meaning that there is no club $G \subseteq \kappa$ so that $G \cap \alpha = G_\alpha$ for all $\alpha \in \lim(G)$.

One way to ensure that $\langle G_\alpha \mid \alpha < \kappa \rangle$ is not threadable is to require the existence of a stationary $E \subseteq \kappa$ so that the sets $\lim(G_\alpha)$ are all disjoint from E.

It is standard to use $\square(\kappa, E)$ to denote the resulting principle, namely the principle asserting that there is a coherent sequence of clubs $G_\alpha \subseteq \alpha$, for limit $\alpha < \kappa$, with the sets $\lim(G_\alpha)$ all disjoints from E.

The principle $\square(\kappa, E)$ was isolated by Jensen [1], and the following result is among the first applications of fine structure. Jensen proved the result in L, but the same proof relativizes to $L(x)$ for every real x. That (3) implies (2) implies (1) is clear. The main work is to show that (1) implies (3), and this is done in Theorems 5.2 and 6.1 of [1].

Lemma 3.1 (Jensen [1])**.** *In L, and similarly in $L(x)$ for every real x, the following are equivalent for every uncountable regular cardinal κ:*

1. *κ is not weakly compact.*
2. *There is a stationary non-reflecting subset of κ.*
3. *There is a stationary $E \subseteq \kappa$ so that $\square(\kappa, E)$.*

Jensen [1, Theorem 6.2] relies on this characterization to construct κ-Suslin trees in L, for regular uncountable κ which are not weakly compact. The construction uses $\square(\kappa, E)$ and $\diamond_\kappa(E)$ for a stationary $E \subseteq \kappa$. The next lemma provides a parallel of this construction, for indestructible Suslin trees, and using coherent diamond.

Lemma 3.2. *Let κ be a regular uncountable cardinal and let $\mathbb{P}$ be a κ-c.c. poset of size κ. Suppose $\diamond^c_\kappa$ holds, and suppose there is a stationary $E \subseteq \kappa$ so that $\square(\kappa, E)$ holds. Then there is a κ-Suslin tree which is $\mathbb{P}$-indestructible.*

Proof. Readers familiar with the construction of a Suslin tree T in Jensen [1] will recall that it hinges on using $\diamond_\kappa(E)$ to predict at each stage $\alpha \in E$ an initial segment A_α of a potential maximal antichain, and (if the initial segment is a maximal antichain in $T{\restriction}\alpha$) *sealing* it, meaning constructing level α of the tree so that all nodes on that level extend nodes in A_α. This implies that all nodes on levels α or higher are compatible with nodes in A_α, so that A_α cannot be contained in an antichain of size κ.

We wish to follow a similar strategy, but predicting initial segments of $\mathbb{P}$-names for antichains, so that we can create a $\mathbb{P}$-indestructible Suslin tree T. To do this in the ground model we need to generate nodes of T which are outright forced to extend nodes in the predicted antichain name. We will incorporate a process to produce such nodes into the construction of T.

Without loss of generality assume that $\mathbb{P}$ is a poset on κ.

Fix a $\square(\kappa, E)$ sequence of clubs $G_\alpha \subseteq \alpha$, for limit $\alpha < \kappa$.

Work with sequences $\vec{A} = \langle A_\alpha \subseteq \alpha \mid \alpha < \kappa \rangle$ and $\vec{C} = \langle C_\alpha \subseteq \alpha \mid \alpha < \kappa \rangle$. We will define a κ-tree T, using these sequences.

Let $(X)_i = \{\xi \mid 4\xi + i \in X\}$ for $i \in \{0, 1, 2, 3\}$. This allows us to view X as coding four sets. We will use $(A_\alpha)_1$ as our predicted $\mathbb{P}$-names for antichains, and $(A_\alpha)_0$ as conditions forcing this. We will use $(A_\alpha)_2$ and $(A_\alpha)_3$ to witness some Σ^1_1 statements.

As we construct T, we make sure that $(\forall \beta < \alpha < \kappa)$ every node on level β of T extends to a node on level α of T.

We use $T\restriction\alpha$ to denote the restriction of T to nodes on levels below α. We will construct T so that its αth level is a subset of $\{\alpha\} \times \alpha$ for infinite α, and a subset of $\{n\} \times 2^n$ for finite n. The construction is by recursion on α. At stage α of the construction we determine the $\leq_T$-predecessors of the nodes $\langle \alpha, \xi\rangle$. If α is a successor, we do this in such a way that each node on level $\alpha - 1$ has two successors on level α. As usual with Suslin tree constructions, this ensures that any chain of size κ in T gives rise to an antichain of size κ, so that T is Suslin provided it has no antichains of size κ.

To be specific: Let $\langle 0, 0\rangle$ be the only node of T on level 0. For $m \leq n < \omega$, $k < 2^m$, and $l < 2^n$, set $\langle m, k\rangle \leq_T \langle n, l\rangle$ iff the remainder of l divided by 2^m equals k. For infinite α, set $\langle \alpha, \xi\rangle \leq_T \langle \alpha + 1, \eta\rangle$ for $\eta \in \{f_\alpha(0, \xi), f_\alpha(1, \xi)\}$, where $f_\alpha\colon 2 \times \alpha \to \alpha + 1$ is a fixed bijection.

At limit α, we will attempt to produce for each node $u \in T\restriction\alpha$ two (or fewer) cofinal branches $b(\alpha, u)$ and $e(\alpha, u)$ in $T\restriction\alpha$. Then using fixed bijections $g_\alpha\colon 2\times(\alpha\times\alpha) \to \alpha$, for each $u \in T\restriction\alpha \subseteq \alpha\times\alpha$, if $b(\alpha, u)$ is defined then we put $\langle \alpha, g_\alpha(0, u)\rangle$ into T and set its $\leq_T$-predecessors to be the nodes in $b(\alpha, u)$, and similarly if $e(\alpha, u)$ is defined then we put $\langle \alpha, g_\alpha(1, u)\rangle$ into T and set its $\leq_T$-predecessors to be the nodes in $e(\alpha, u)$. We refer to $\langle \alpha, g_\alpha(0, u)\rangle$ as a *cap* for $b(\alpha, u)$, and similarly with $g_\alpha(1, u)$ and $e(\alpha, u)$. At least one of $b(\alpha, u)$ and $e(\alpha, u)$ will be defined, and capping the defined branch(es) ensures, among other things, that every node on a level $\beta < \alpha$ extends to a node on level α.

The above structural specifications are all standard for a Suslin tree construction from a diamond principle, with the slight variation that we are attempting to construct two cofinal extensions of u, instead of one. What is new here is that we construct $b(\alpha, u)$ below in a way that produces nodes which are outright forced to extend nodes in predicted antichain names.

Recall that $\mathbb{P} \subseteq \kappa$. When $X \subseteq \kappa$ is a singleton $\{p\}$ for a condition $p \in \mathbb{P}$, then abusing notation we will refer to X as if it were the condition p, for example saying that X forces a statement θ in the $\mathbb{P}$ forcing language if $p \Vdash \theta$.

Fix a bijection $h\colon \kappa \to \kappa^{<\omega}$. Abusing notation we can view a canonical $\mathbb{P}$-name for an antichain in T as a subset of $T \times \mathbb{P} \subseteq \kappa \times \kappa \times \kappa$. Hence every canonical $\mathbb{P}$-name for an antichain in T can be viewed as $h''X$ for some $X \subseteq \kappa$.

By recursion attempt to construct nodes $v_\xi^{\alpha,u}$ (v_ξ for short when α, u are clear from the context) forming a cofinal chain in $T\restriction\alpha$ extending u.

Set $v_0 = u$.

At limit γ, if $\{v_\xi \mid \xi < \gamma\}$ has a cap in $T\restriction\alpha$, set v_γ to be the least one. If $\{v_\xi \mid \xi < \gamma\}$ is not yet cofinal in $T\restriction\alpha$, and fails to have a cap in $T\restriction\alpha$, then abandon the construction and leave $b(\alpha, u)$ undefined. If $\{v_\xi \mid \xi < \gamma\}$ is cofinal in $T\restriction\alpha$, then set $b(\alpha, u)$ to be the cofinal branch of $T\restriction\alpha$ generated by this set.

Most importantly, at successor stages, work as follows to define $v_{\xi+1}$: Let δ be the least element of C_α strictly above the level of v_ξ in T if there is one; otherwise abandon the construction and leave $b(\alpha, u)$ undefined. If $(A_\alpha)_0$ forces in $\mathbb{P}$ that v_ξ extend an element of $h''(A_\alpha)_1$ in T, then let $v_{\xi+1}$ be the least node of T on level δ which extends v_ξ. If there is a condition in $\mathbb{P}$ below $(A_\alpha)_0$ which forces that v_ξ does not extend an element of $h''(A_\alpha)_1$, and forces some node $v \geq_T v_\xi$ on level δ to extend an element of $h''(A_\alpha)_1$, then let $v_{\xi+1}$ be the least node v for which this happens. If neither of these two options holds, then abandon the construction and leave $b(\alpha, u)$ undefined.

If $\alpha \notin E$, or the construction of $b(\alpha, u)$ failed, then by recursion construct nodes $w_\xi^{\alpha,u}$ (w_ξ for short when α, u are clear from the context) forming a cofinal chain in $T\restriction\alpha$ extending u, and let $e(\alpha, u)$ be the cofinal branch of $T\restriction\alpha$ generated by these nodes. Set $w_0 = u$. At limit γ where $\{w_\xi \mid \xi < \gamma\}$ is not yet cofinal in $T\restriction\alpha$, let w_γ be the least cap for this set in $T\restriction\alpha$. We will check below that such a cap exists. At successors stages, let δ be the least element of G_α strictly above the level of w_ξ in T, and let $w_{\xi+1}$ be the least node of T on level δ which extends w_ξ.

This completes the construction of $b(\alpha, u)$ and $e(\alpha, u)$, and with it the construction of the tree order $\leq_T$ and the tree T.

Claim 3.3. *The caps needed in the construction of $e(\alpha, u)$ exist.*

Proof. Fix α and u. Inductively we may assume that $e(\delta, u)$ are constructed and capped in T for all $\delta \in E$ strictly below α with $u \in T\restriction\delta$.

Let $w_\xi^{\alpha,u}$ be constructed as above, for $\xi < \gamma$, with γ a limit ordinal, and with the nodes $w_\xi^{\alpha,u}$ for $\xi < \gamma$ not cofinal in $T\restriction\alpha$. Let δ_ξ be the level of $w_\xi^{\alpha,u}$ in T. Let $\delta_\gamma = \sup\{\delta_\xi \mid \xi < \gamma\}$. We have to show that there is a node on level δ_γ of T which extends all the nodes $w_\xi^{\alpha,u}$, $\xi < \gamma$. Our argument for this is the standard one, using $\square(\kappa, E)$ in a manner similar to its use in Jensen [1].

By construction the levels δ_ξ, $\xi < \gamma$, form an increasing sequence in G_α. Since G_α is club in α, it follows that $\delta_\gamma \in G_\alpha$, and in fact $\delta_\gamma \in \lim(G_\alpha)$. This implies that $\delta_\gamma \notin E$. Hence $e(\delta_\gamma, u)$ was constructed, and (by induction) capped in T. By the coherence of the $\square(\kappa, E)$ sequence, $G_{\delta_\gamma} = G_\alpha \cap \delta_\gamma$. It follows from this that the nodes $w_\xi^{\delta_\gamma,u}$, used in stage δ_γ of the construction of T, are exactly equal to the nodes $w_\xi^{\alpha,u}$, for $\xi < \gamma$. In particular $e(\delta_\gamma, u)$ contains these nodes. Hence the cap for $e(\delta_\gamma, u)$ on level δ_γ of T is a cap for the nodes $w_\xi^{\alpha,u}$, $\xi < \gamma$. □

The tree $(T; \leq_T)$ was constructed with reference to the sequences $\vec{A}$ and $\vec{C}$. When we wish to emphasize this dependence we write $T(\vec{A}, \vec{C})$ and $\leq_T(\vec{A}, \vec{C})$.

Let Δ_{mult}, Δ_{rem}, Δ_f, Δ_g, and Δ_h be predicates on powers of κ that code the multiplication function on κ, the remainder function on ω, the sequences of functions f_α and g_α, and the function h used above. Let $\vec{\Delta}$ denote the sequence of these predicates.

Claim 3.4. *There is a sentence ψ_{tree} in the language of the structure $(\kappa; \in, \leq_{\mathbb{P}}, \vec{\Delta}, \vec{G}, \vec{A}, \vec{C}, E_0, E_1, T, \leq_T)$, where E_0 and E_1 are ternary predicates, so that:*

1. *If $T = T(\vec{A}, \vec{C})$ and $\leq_T = \leq_T(\vec{A}, \vec{C})$ then there are unique E_0 and E_1 so that $(\kappa; \in, \leq_{\mathbb{P}}, \vec{\Delta}, \vec{G}, \vec{A}, \vec{C}, E_0, E_1, T, \leq_T) \models \psi_{\text{tree}}$.*
2. *If there are E_0, E_1 so that $(\kappa; \in, \leq_{\mathbb{P}}, \vec{\Delta}, \vec{G}, \vec{A}, \vec{C}, E_0, E_1, T, \leq_T) \models \psi_{\text{tree}}$, then $T = T(\vec{A}, \vec{C})$ and $\leq_T = \leq_T (\vec{A}, \vec{C})$.*

Proof. Take ψ_{tree} to express the statement that E_0, E_1, T, and $\leq_T$ are constructed as above, where E_0 and E_1 consist of all the tuples $\langle \alpha, u, v_\xi^{\alpha,u} \rangle$ and $\langle \alpha, u, w_\xi^{\alpha,u} \rangle$ respectively. It is clear that the construction rules are first order over the resulting structure. □

Claim 3.5. *There is a sentence ψ_{mac} so that, letting $T = T(\vec{A}, \vec{C})$ and $\leq_T = \leq_T(\vec{A}, \vec{C})$:*

(1) *If* $(\kappa; \in, \leq_{\mathbb{P}}, \vec{\Delta}, \vec{G}, \vec{A}, \vec{C}, A) \models \psi_{\rm mac}$ *then* $(A)_0$ *forces in* $\mathbb{P}$ *that* $h''(A)_1$ *is a maximal antichain of size* κ *in* $(T; \leq_T)$*, and* $(A)_2$ *is a club of* $\alpha < \kappa$ *which are elementary in* $(\kappa; \Delta_h, \leq_{\mathbb{P}}, T, \leq_T, (A)_0, h''(A)_1)$*.*

(2) *If* p *forces in* $\mathbb{P}$ *that* $h''Z$ *is a maximal antichain of size* κ *in* $(T; \leq_T)$*, and* U *is a club of* $\alpha < \kappa$ *which are elementary in* $(\kappa; \Delta_h, \leq_{\mathbb{P}}, T, \leq_T, \{p\}, h''Z)$*, then there exists* A *so that* $(\kappa; \in, \leq_{\mathbb{P}}, \vec{\Delta}, \vec{G}, \vec{A}, \vec{C}, A) \models \psi_{\rm mac}$*, with* $(A)_0 = \{p\}$*,* $(A)_1 = Z$*, and* $(A)_2 = U$*.*

Proof. Immediate from Claim 3.4, using $(A)_3$ to code E_0, E_1, T, and $\leq_T$, and rephrasing $\psi_{\rm tree}$ to refer to the coded objects. The coding can be done definably over $(\kappa; \in, \leq_{\mathbb{P}}, \vec{\Delta}, \vec{G}, \vec{A}, \vec{C}, A)$ using the function h. The properties of $(A)_0$, $(A)_1$, $\leq_{\mathbb{P}}$, T, $\leq_T$, and $(A)_2$ in the current claim are clearly first order. □

We are now ready to pick the tree that witnesses Lemma 3.2. Let $\vec{A}$ and $\vec{C}$ witness coherent diamond on κ for the predicates $\in$, $\leq_{\mathbb{P}}$, $\vec{\Delta}$, and $\vec{G}$, with the sentence $\psi_{\rm mac}$, strengthened to incorporate an explicit statement that $(A)_2$ is club. Let $T = T(\vec{A}, \vec{C})$, and let $\leq_T \; = \; \leq_T(\vec{A}, \vec{C})$. We will prove that $(T; \leq_T)$ is $\mathbb{P}$-indestructibly Suslin.

Suppose otherwise. Then there is a condition in $\mathbb{P}$ forcing the existence of a maximal antichain in $(T; \leq_T)$ of size κ. Using condition (2) of Claim 3.5 it follows that there exists $A \subseteq \kappa$ so that $(\kappa; \in, \leq_{\mathbb{P}}, \vec{\Delta}, \vec{G}, \vec{A}, \vec{C}, A) \models \psi_{\rm mac}$. Using coherent diamond, fix such an A which is guessed coherently by $\vec{A}$ and $\vec{C}$ on a club $C \subseteq \kappa$. Moreover, using the final clause in Definition 2.1, do this in such a way that $(\alpha; \in, \leq_{\mathbb{P}}\restriction\alpha, \vec{\Delta}\restriction\alpha, \vec{G}\restriction\alpha, \vec{A}\restriction\alpha, \vec{C}\restriction\alpha, A \cap \alpha) \models \psi_{\rm mac}$ for every $\alpha \in C$. Since we incorporated the statement that $(A)_2$ is club into $\psi_{\rm mac}$, this implies in particular that $(A)_2$ is unbounded in α for each $\alpha \in C$, and hence, using the closure of $(A)_2$, $C \subseteq (A)_2$.

Using condition (1) of Claim 3.5, and letting $p \in \mathbb{P}$ be such that $(A)_0 = \{p\}$, $Z = (A)_1$, $Y = h''Z$, and $U = (A)_2$, we have that p forces Y to be a maximal antichain of $(T; \leq_T)$ of size κ, and U is a club of $\alpha < \kappa$ which are elementary in $(\kappa; \Delta_h, \leq_{\mathbb{P}}, T, \leq_T, \{p\}, Y)$. We also saw above that $C \subseteq U$.

Recall that Y is a subset of $T \times \mathbb{P} \subseteq \kappa^3$. We use $Y\restriction\alpha$ to denote $Y \cap \alpha^3$. Using the elementarity of α relative to Δ_h, and the fact that α is a limit ordinal, it is easy to see that $Y\restriction\alpha = h''(A \cap \alpha)_1$.

Claim 3.6. *Let* $\alpha \in U$*, let* $q \in \mathbb{P} \cap \alpha$*, and let* $w \in T\restriction\alpha$*. Then* q *forces in* $\mathbb{P}$ *that* w *extends an element of* Y *iff* q *forces in* $\mathbb{P}$ *that* w *extends an element of* $Y\restriction\alpha$*.*

Proof. Clear using the elementarity of α in $(\kappa; \Delta_h, \leq_{\mathbb{P}}, T, \leq_T, \{p\}, Y)$, and the fact that $\mathbb{P}$ is κ-c.c. □

Claim 3.7. (1) *For every* $u \in T$*, there is* $w \in T$ *extending* u*, so that* w *is forced by* p *to extend an element of* Y*.*

(2) *For every* $\alpha \in C$*, and for every* $u \in T\restriction\alpha$*, there is* w' *on level* α *of* T *extending* u*, so that* w' *is forced by* p *to extend an element of* Y*.*

Proof. To prove the second condition from the first, note that $C \subseteq U$, so that α is elementary in $(\kappa; \Delta_h, \leq_{\mathbb{P}}, T, \leq_T, \{p\}, Y)$. Assuming the first condition of the current claim, this implies that every $u \in T\restriction\alpha$ has an extension w in $T\restriction\alpha$ which is forced by p to extend an element of Y. Any extension w' of w to level α witnesses the second condition.

We prove the first condition. Fix $u \in T$. Suppose for contradiction that no extension of u is forced by p to extend an element of Y. Construct a chain in T extending u as follows. Set $v_0 = u$, and let δ_0 be the level of u. At limit γ, if $\{v_\xi \mid \xi < \gamma\}$ does not have a cap on level $\delta_\gamma = \sup\{\delta_\xi \mid \xi < \gamma\}$, end the construction. If $\{v_\xi \mid \xi < \gamma\}$ does have a cap on level δ_γ, let v_γ be the least one. For the successor stage, having defined v_ξ on level δ_ξ of T, and letting $\delta_{\xi+1}$ be the first element of C above δ_ξ, if there is a condition in $\mathbb{P}$ below p which forces that v_ξ does not extend an element of Y, and forces some node $v \geq_T v_\xi$ on level $\delta_{\xi+1}$ to extend an element of Y, then let $v_{\xi+1}$ be the least node v for which this happens. If no such node exists, end the construction.

Note that our construction exactly matches the construction of the nodes $v_\xi^{\alpha,u}$, for $\alpha \in C$. This is because $C_\alpha = C \cap \alpha$, $A_\alpha = A \cap \alpha$, because for every $\alpha \in C$, our assumption that no extension of u is forced by p to extend an element of Y eliminates the corresponding case in the definition of $v_{\xi+1}^{\alpha,u}$, and because, using the elementarity of α, Claim 3.6 equates forcing a node in $T\restriction\alpha$ to extend an element of Y with forcing it to extend an element of $h''(A_\alpha)_1$.

Thus, for every $\alpha \in C$ above the level of u, and every ξ so that $\delta_\xi < \alpha$ (if defined), our v_ξ is defined iff $v_\xi^{\alpha,u}$ is defined, and the two are equal when defined.

This implies in particular that the construction of v_ξ cannot fail first at a limit γ. This is because, by construction of T, there is a node on level δ_γ which caps $b(\delta_\gamma, u) = \{v_\xi \mid \xi < \gamma\}$.

The construction of v_ξ cannot fail at a successor stage either. To see this, note that by assumption there is $q \leq p$ forcing that v_ξ does not extend an element of Y. Since p forces Y to be a maximal antichain in $(T; \leq_T)$, we can find a $w \in T$ extending v_ξ, and a condition $q' \leq q$, forcing that w extends an element of Y. Since $\delta_{\xi+1} \in C \subseteq U$, and by the elementarity of all elements of U in $(\kappa; \Delta_h, \leq_\mathbb{P}, T, \leq_T, \{p\}, Y)$, such w can be found on a level below $\delta_{\xi+1}$. Then any extension v of w to level $\delta_{\xi+1}$ witnesses the condition in the definition of $v_{\xi+1}$.

So the construction of v_ξ proceeds without fail at all $\xi < \kappa$.

For each ξ, by construction there is a condition below p forcing that v_ξ does not extend an element of Y, but $v_{\xi+1}$ does. Let q_ξ be such a condition. If $\xi < \eta$ then $v_{\xi+1} \leq_T v_\eta$ so q_η forces that $v_{\xi+1}$ does not extend an element of Y, and hence q_η is incompatible with q_ξ. This gives an antichain in $\mathbb{P}$ of size κ, contradicting the κ-chain condition. □

Claim 3.8. *At every $\alpha \in C$, the construction of $b(\alpha, u)$ succeeds, and all nodes in $b(\alpha, u)$ except possibly u are forced by p to extend an element of Y.*

Proof. Suppose not, and fix the least $\alpha \in C$ for which the construction fails, or includes nodes other than u which are not forced by p to extend a node in Y.

Note that the construction cannot fail at a successor stage. This is a consequence of Claim 3.7: since $\alpha \in C$ we have that $C_\alpha = C \cap \alpha$, hence the least $\delta \in C_\alpha$ above the level of $v_\xi^{\alpha,u}$ is an element of C, and hence by Claims 3.7 and 3.6, there is $v \geq_T v_\xi^{\alpha,u}$ on level δ which is forced by p to extend an element of $Y\restriction\alpha = h''(A_\alpha)_1$.

This also shows that the very first node above u, $v_1^{\alpha,u}$, is forced by p to extend an element of Y, and hence so are all subsequent nodes.

Finally, the construction cannot fail at a limit stage γ either. To see this, let δ_ξ for $\xi < \gamma$ be the level of $v_\xi^{\alpha,u}$, and let $\delta = \sup\{\delta_\xi \mid \xi < \gamma\}$. Since $\delta \in C_\alpha = C \cap \alpha$, we have $A_\delta = A \cap \delta = A_\alpha \cap \delta$ and $C_\delta = C \cap \delta = C_\alpha \cap \delta$. From this agreement, and from Claim 3.7, it follows that the constructions of $v^{\delta,u}_{\xi}$ and $v^{\alpha,u}_{\xi}$ are identical for $\xi < \gamma$. By the minimality of α, $v^{\delta,u}_{\xi}$ for $\xi < \gamma$ are all defined, and capped at level δ. So $v^{\alpha,u}_{\gamma}$ is defined, and equal to the least such cap. $\square$

Having produced enough nodes which are forced by p to extend elements of Y, we can now conclude the proof of Lemma 3.2 following the usual lines in a Suslin tree construction.

Fix $\alpha \in C \cap E$. Since, by Claim 3.8, $b(\alpha, u)$ is defined for all $u \in T{\restriction}\alpha$, and since $\alpha \in E$, we do not construct the branches $e(\alpha, u)$ for any $u \in T{\restriction}\alpha$. Thus, level α of T consists only of caps for the branches $b(\alpha, u)$. By Claim 3.8, these caps are all forced by p to extend elements of Y. This implies that all nodes of T on levels α and higher are forced by p to extends nodes in Y. Since Y is forced by p to be an antichain, it follows that it is forced by p to not have any nodes on levels α or greater. This contradicts the fact that p forces Y to have size κ. $\square$

Remark 3.9. In the case of $\kappa = \omega_1$, the assumption in Lemma 3.2 that there exists a stationary E so that $\square(\kappa, E)$ becomes vacuous, since it holds trivially with E consisting of all limit ordinals below ω_1, as witnessed by taking G_α for limit $\alpha < \omega_1$ to be a cofinal subset of α of ordertype ω containing only successor ordinals. With these G_αs, the proof of Lemma 3.2 simplifies slightly: $e(\alpha, u)$ is constructed only if the construction of $b(\alpha, u)$ fails, and Claim 3.3 is not needed, since there are no limit cases in the construction of $e(\alpha, u)$.

Corollary 3.10. *In L, and in $L(x)$ for every real x, the indestructible Suslin tree property holds at every regular uncountable κ which is not weakly compact.*

Proof. Immediate from Lemmas 2.3, 3.1, and 3.2. $\square$

Remark 3.11. If $\kappa = \mu^+$ for a strongly inaccessible cardinal μ, then the assumption in Lemma 3.2 that there exists a stationary E so that $\square(\kappa, E)$ can be dropped. This fact, and the necessary modification to the proof of Lemma 3.2, exactly parallel the situation with Jensen's construction of a κ-Suslin tree. The modifications are as follows: Take E to be the set of $\alpha < \kappa$ of cofinality μ. Set G_α for each limit $\alpha < \kappa$ to be a cofinal club in α of ordertype $\leq \mu$ consisting entirely of ordinals of cofinality $< \mu$, so that G_α is disjoint from E. Modify the tree construction in the proof of Lemma 3.2 to cap *all* cofinal branches through $T{\restriction}\alpha$ when α has cofinality $< \mu$. This modification is compatible with the narrowness requirements of the tree since μ is strongly inaccessible. It removes the need for Claim 3.3 and with it the need for any coherence assumptions on the clubs G_α.

Corollary 3.12. *In the standard Jensen-indexed fine structural inner models, the indestructible Suslin tree property holds at every regular uncountable cardinal which is not weakly compact.*

Proof. The proof parallels the proof of Corollary 0.3 in Zeman [5]. By Lemma 3.2, Lemma 2.3, and Remark 2.4, it is enough to either establish that there is a stationary E so that $\square(\kappa, E)$, or, using Remark 3.11, argue that κ is the successor of a strongly inaccessible cardinal. This is done in cases. If κ itself is inaccessible (but not weakly compact), then by Theorem 0.1 of [5] there is a stationary E so that $\square(\kappa, E)$ holds. If $\kappa = \mu^+$ where μ is not subcompact, then the results of Schimmerling-Zeman [2] give the existence of a $\square_\mu$ sequence, and any such sequence is a $\square(\kappa, E)$ sequence where E consists of $\alpha < \kappa$ of cofinality μ. Finally, if $\kappa = \mu^+$

where μ is subcompact, then κ is the successor of a strongly inaccessible cardinal. $\square$

Since weakly compact cardinals cannot carry Suslin trees, it follows from Corollary 3.12 that, in Jensen-indexed fine structural models, for every regular uncountable κ, the indestructible Suslin tree property at κ is equivalent to the existence of a κ-Suslin tree.

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Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90095-1555, USA

Email address: `ineeman@math.ucla.edu`