A CARDINAL PRESERVING EXTENSION MAKING THE SET OF POINTS OF COUNTABLE V COFINALITY NONSTATIONARY

MOTI GITIK, ITAY NEEMAN, AND DIMA SINAPOVA

Abstract. Assuming large cardinals we produce a forcing extension of V which preserves cardinals, does not add reals, and makes the set of points of countable V cofinality in κ^+ nonstationary. Continuing to force further, we obtain an extension in which the set of points of countable V cofinality in ν is nonstationary for every regular $\nu \geq \kappa^+$. Finally we show that our large cardinal assumption is optimal.

The results in this paper were inspired by the following question, posed in a preprint (http://arxiv.org/abs/math/0509633v1, 27 September 2005) to the paper Viale [9]: Suppose $V \subset W$ and V and W have the same cardinals and the same reals. Can it be shown, in ZFC alone, that for every cardinal κ, there is in V a partition $\{A_s \mid s \in \kappa^{<\omega}\}$ of the points of κ^+ of countable V cofinality, into disjoint sets which are stationary in W?

In this paper we show that under some assumptions on κ there is a reals and cardinal preserving generic extension W which satisfies that the set of points of κ^+ of countable V cofinality is nonstationary. In particular, a partition as above cannot be found for each κ.

Continuing to force further, we produce a reals and cardinal preserving extension in which the set of points of λ of countable V cofinality is nonstationary for every regular $\lambda \geq \kappa^+$. All this is done under the large cardinal assumption that for each $\alpha < \kappa$ there exists $\theta < \kappa$ with Mitchell order at least α. We prove that this assumption is optimal.

It should be noted that our counterexample (Theorem 1) leaves open the possibility that a partition as above, but of the points of κ^+ of countable W (rather than V) cofinality, can be found provably in ZFC. This is enough for Viale's argument, and this weaker question is posed in the published paper.

There has been work in the past leading to forcing extensions making the set of points of κ^+ of countable V cofinality nonstationary in the extension, specifically in the context of making the nonstationary ideal on κ precipitous, see Gitik [1]. But preservation of cardinals was not an issue in that context, and the extensions involved did not in fact preserve cardinals. There has also been work on forcing to add clubs consisting of regulars in V, see Gitik [2].

Theorem 1. *Suppose that* $\mathrm{cf}(\kappa) = \omega$, $(\forall\alpha < \kappa)(\exists\theta < \kappa)(o(\theta) \geq \alpha)$, *and* $2^\kappa = \kappa^+$. *Then there is a generic extension W of V such that V and W have the same cardinals and same reals and $W \models A$ is nonstationary, where* $A = \{\alpha < \kappa^+ \mid \mathrm{cf}^V(\alpha) = \omega\}$.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-0094174.

Proof. First we construct a model M by forcing over V with a restricted product of the posets defined in Magidor [4]. We get a generic function $g^*\colon \kappa \to \kappa$ in M, such that the range of g^* is a club in κ of measurable cardinals of V. Then we construct W by forcing over M to shoot a club C through the complement of A.

Let θ_i, $i < \omega$, be a sequence of cardinals, cofinal in κ, so that $o(\theta_{i+1}) > \theta_i$ for each i. Let $\mathbb{P}_i$ be the forcing of Magidor [4], to change the cofinality of θ_{i+1} to θ_i. Let $\mathbb{P}^*$ be the full support product of the posets $\mathbb{P}_i$. Let $\mathbb{P}$ consist of all conditions $\langle\langle g_i, H_i\rangle \mid i < \omega\rangle$ in $\mathbb{P}^*$ so that $g_i = \emptyset$ for all but finitely many $i < \omega$.

Let M be obtained by forcing with $\mathbb{P}$ over V. Let $g_i^*\colon \theta_i \to \theta_{i+1}$ be the generic function added by the part of the forcing corresponding to $\mathbb{P}_i$. The range of g_i^* consists of measurable cardinals of V, and is (see for example Jech [3, §36]) closed and unbounded in θ_{i+1}. Let $X = \bigcup_{i<\omega}((\operatorname{ran}(g_i^*) - \theta_i) \cup \{\theta_{i+1}\})$ and let $g^*\colon \kappa \to \kappa$ enumerate X in increasing order. Then $X = \operatorname{ran}(g^*)$ is closed unbounded in κ, and consists entirely of measurables of V. This fact will be used in the proof of Claim 3 below.

Magidor's posets do not collapse cardinals, and it is easy to see that they do not add reals. (Magidor shows that if H_i is $\mathbb{P}_i$-generic, and g_i^* is the generic function, $\beta \in \operatorname{dom}(g_i^*)$, $a \subset \delta$ for some $\delta \le g_i^*(\beta)$ and $a \in V[H_i]$, then $a \in V[H_\beta]$, where $H_\beta = \{\langle g \restriction \beta+1, G \restriction \beta+1\rangle \mid \langle g, G\rangle \in H_i\}$. Taking $\delta = \omega$, and $\beta = 0$, we get that no new reals are added in M.) An argument similar to the one in Magidor [4] establishes that the combined poset $\mathbb{P}$ does not add reals and does not collapse cardinals. Let us just comment that the restriction in the definition of $\mathbb{P}$ to conditions $\langle\langle g_i, H_i\rangle \mid i < \omega\rangle$ with $g_i = \emptyset$ for all but finitely many $i < \omega$ is used to establish that $\mathbb{P}$ does not collapse cardinals above κ.

Given the model M, force with $\mathbb{Q} = \{p \subset A^c \mid p \text{ is closed and bounded in } \kappa^+\}$. If G is $\mathbb{Q}$-generic over M, then $C = \bigcup G$ is a club in $W = M[G]$ contained in the complement of A.

It remains to show that the forcing preserves cardinals and reals.

Lemma 2. *In M, $\mathbb{Q}$ is λ-distributive, for all $\lambda \le \kappa$.*

Proof. Fix $\lambda \le \kappa$, a regular cardinal of M. Jech [3, §23] gives a proof that $\mathbb{Q}$ is ω-distributive. So, we may assume that $\lambda > \omega$.

Suppose p is a condition in $\mathbb{Q}$ that forces $\dot{f}\colon \lambda \to Ord$. Let Ω be a regular cardinal with $\kappa, \mathbb{Q}, \dot{f} \in V_\Omega^M$.

Claim 3. *There is an elementary substructure H of V_Ω^M so that:*

1. *H has cardinality κ. $\mathbb{Q}$, p, $\dot{f}$, κ, and all bounded subsets of κ belong to H.*
2. *$\gamma = \sup(H \cap \kappa^+)$ has cofinality λ in M.*
3. *There is a normal sequence $\langle\gamma_\alpha \mid \alpha < \lambda\rangle$ in M so that $\lim(\gamma_\alpha) = \gamma$, $\langle\gamma_\alpha \mid \alpha < \beta\rangle$ belongs to H for each $\beta < \lambda$, and $(\forall\alpha < \lambda)\operatorname{cf}^V(\gamma_\alpha) \neq \omega$.*

Proof. Let $g = g^* \restriction \lambda$. We have $g\colon \lambda \to \tau$, where $\tau = \sup(\operatorname{ran}(g))$. The range of g is a club in τ, and $\operatorname{cf}^M(\tau) = \lambda$.

A standard elementary chain argument in V produces an elementary substructure $\bar{H}$ of V_Ω, and a normal function $R\colon \tau \to Ord$, so that: $\bar{H}$ has κ, $\mathbb{P}$, all bounded subsets of κ, and $\mathbb{P}$ names for $\mathbb{Q}$, p, and $\dot{f}$ as elements; $\operatorname{card}(\bar{H}) = \kappa$; R is cofinal in $\sup(\bar{H} \cap \kappa^+)$; and $R \restriction \beta \in \bar{H}$ for each $\beta < \tau$.

Let $H = \bar{H}[g^*]$. Notice that $\sup(H \cap \kappa^+) = \sup(\bar{H} \cap \kappa^+)$, since the poset $\mathbb{P}$, which adds g^*, has the κ^+ chain condition. H is therefore an elementary substructure of V^M_Ω satisfying conditions (1) and (2) of the claim. As for condition (3): Set $\gamma_\alpha = R(g(\alpha))$. Then $\langle \gamma_\alpha \mid \alpha < \lambda\rangle$ is a normal sequence, $\lim_\alpha \gamma_\alpha = \gamma$, $\langle \gamma_\alpha \mid \alpha < \beta\rangle$ belongs to H for each $\beta < \lambda$ since $R \restriction g(\beta)$ belongs to $\bar{H}$, and

$$\begin{aligned} \mathrm{cf}^V(\gamma_\alpha) &= \mathrm{cf}^V(R(g(\alpha)) \\ &=^{(1)} \mathrm{cf}^V(g(\alpha)) \\ &\neq^{(2)} \omega. \end{aligned}$$

Equality (1) follows from the fact that R is normal and belongs to V, and equality (2) follows since the points in $\mathrm{ran}(g)$ are inaccessible (and in fact measurable) cardinals of V. $\square$

Let H be the structure given by the last claim, and let $\langle \gamma_\alpha \mid \alpha < \lambda\rangle$ witness condition (3) of the claim. Let $C = \{\gamma_\alpha \mid \alpha < \lambda\}$. Let $\sqsubseteq$ in H be a wellordering of the conditions in $\mathbb{Q}$. Define now an increasing sequence of conditions in $\mathbb{Q}$, $\langle p_\eta \mid \eta \leq \lambda\rangle$ as follows: Set $p_0 = p$. Given p_η and assuming that p_η belongs to H, let $p_{\eta+1}$ be the $\sqsubseteq$-least condition so that $p_{\eta+1} \leq p_\eta$, $p_{\eta+1}$ decides $\dot{f}(\eta)$, and there is at least one element of C between $\max(p_\eta)$ and $\max(p_{\eta+1})$. Notice that if p_η belongs to H then by elementarity so does $p_{\eta+1}$. For η limit, set $p_\eta = \bigcup_{\rho<\eta} p_\rho \cup \{\delta_\eta\}$ where $\delta_\eta = \sup\{\max(p_\rho) \mid \rho < \eta\}$. Notice that δ_η belongs to $C \cup \{\gamma\}$ by construction, and hence in particular δ_η has uncountable cofinality in V, namely $\delta_\eta \in A^c$. p_η as defined here is therefore a condition in $\mathbb{Q}$.

Using the fact that all strict initial segments of C belong to H, it is easy to prove by induction on η that p_η belongs to H for each $\eta < \lambda$. The construction described above therefore continues to stage λ, producing a condition p_λ that decides all values of $\dot{f}$. $\square$

Remark. The authors thank James Cummings for noticing and correcting an error in a previous version of the proof of Lemma 2.

Let G be $\mathbb{Q}$-generic over M. Let $W = M[G]$. Recall that M is obtained by forcing over V using Magidor's product $\mathbb{P}$. V and M have the same cardinals and the same reals. $\mathbb{Q}$ has size κ^+, and is λ-distributive for all $\lambda \leq \kappa$ by the last lemma. M and W therefore have the same cardinals and the same reals. W has a club disjoint from A. This completes the proof of Theorem 1. $\square$

Lemma 2 rests on Claim 3, and the claim in turn uses just the following two properties of V, M, and κ^+: (1) M is an extension of V by a κ^+-c.c. poset, and (2) for every $\lambda < \kappa^+$ which is regular in M, there is a $\tau < \kappa^+$ so that $\mathrm{cf}^M(\tau) = \lambda$ and τ has a club subset in M which completely avoids points of countable V cofinality. Abstracting from Lemma 2 we therefore obtain the following claim:

Claim 4. *Let $V \subset N$, let ν be regular in N, and suppose that:*

1. *N is an extension of V by a ν-c.c. poset.*
2. *For every $\lambda < \nu$ which is regular in N, there is $\tau < \nu$ so that $\mathrm{cf}^N(\tau) = \lambda$ and τ has a club subset in N which avoids points of countable V cofinality.*

In N let $\mathbb{Q} = \{p \subset \nu \mid p$ is closed and bounded in ν and avoids points of countable V cofinality$\}$. Then, in N, $\mathbb{Q}$ is λ-distributive for every regular $\lambda < \nu$.

Notice that the assumptions of Claim 4 hold for the model W produced by Theorem 1 with $\nu = \kappa^{++}$. (The main condition is (2), which holds for $\lambda \leq \kappa$ using the generic for Magidor's forcing, and holds for $\lambda = \kappa^+$ directly by the conclusion of Theorem 1, taking $\tau = \lambda = \kappa^+$.) This suggests forcing over W with the poset $\{p \subset \nu \mid p$ is closed and bounded in ν and avoids points of countable V cofinality$\}$. More generally it suggests the iteration leading to the following theorem:

Theorem 5. *Suppose that* $\mathrm{cf}(\kappa) = \omega$, $(\forall \alpha < \kappa)(\exists \theta < \kappa)(o(\theta) \geq \alpha)$, *and the* GCH *holds from* κ *upward. Then there is a generic extension* W^* *of* V *so that* V *and* W^* *have the same cardinals and same reals, and so that for each regular* $\nu \geq \kappa^+$, $W^* \models A$ *is nonstationary, where* $A = \{\alpha < \nu \mid \mathrm{cf}^V(\alpha) = \omega\}$.

Proof. Let W be the model given by Theorem 1. Working in W let $\langle \mathbb{P}_\nu, \dot{\mathbb{Q}}_\nu \mid \nu > \kappa^+$ regular$\rangle$ be the Easton support iteration obtained by letting $\dot{\mathbb{Q}}_\nu$ name the poset $\{p \subset \nu \mid p$ is closed and bounded in ν and avoids points of countable V cofinality$\}$ as defined in $W^{\mathbb{P}_\nu}$.

$\mathbb{P}_\nu$ adds a club disjoint from $\{\alpha < \lambda \mid \mathrm{cf}^V(\alpha) = \omega\}$ for each regular $\lambda \in (\kappa^+, \nu)$. By Claim 4, $\dot{\mathbb{Q}}_\nu$ is forced to be λ-distributive in $W^{\mathbb{P}_\nu}$ for each regular $\lambda < \nu$. It follows that the iteration preserves reals and preserves cardinals. The model $V^{\mathbb{P}_\infty}$ satisfies that $\{\alpha < \nu \mid \mathrm{cf}^V(\alpha) = \omega\}$ is nonstationary for each regular $\nu \geq \kappa^+$. □

Remark 6. The conclusion of Theorem 1 can be improved to state that the set $\{\alpha < \kappa^+ \mid \mathrm{cf}^V(\alpha)$ is not measurable in $V\}$ is nonstationary in W. Similarly the conclusion of Theorem 5 can be improved to state that for each regular $\nu \geq \kappa^+$, the set $\{\alpha < \nu \mid \mathrm{cf}^V(\alpha) \leq \kappa$ and $\mathrm{cf}^V(\alpha)$ is not measurable in $V\}$ is nonstationary in W^*.

Remark 7. Similar changes to the conclusions of Theorems 1 and 5 are possible addressing the set $\{\alpha \mid \mathrm{cf}^V(\alpha) = \gamma\}$ for any fixed $\gamma < \kappa$, and the set $\{\alpha \mid \mathrm{cf}^V(\alpha) \in X\}$ for any fixed X bounded in κ, or for that matter any $X \subset \kappa$ which is given measure zero by enough measures to witness $(\forall \alpha < \kappa)(\exists \theta < \kappa)(o(\theta) \geq \alpha)$. Moreover X need not belong to V but may instead belong to a generic extension of V by a forcing of size less than κ which does not add reals and does not collapse cardinals.

Remark 8. In a context where preservation of cardinals is not required, Shelah [8] showed that the following is equiconsistent with a 2-Mahlo cardinal: there is a generic extension W of V such that V and W have the same reals and the set $\{\alpha < \omega_2^W \mid \mathrm{cf}^V(\alpha) = \omega\}$ is nonstationary in W. Using an iteration as in Theorem 5 one can force further and obtain a model W^* with the same reals, such that for each regular $\nu > \omega_1$ the set $\{\alpha < \nu \mid \mathrm{cf}^V(\alpha) = \omega\}$ is nonstationary in W^*.

Theorem 5 produces in particular a set forcing extension of V in which $\{\alpha < \nu \mid \mathrm{cf}^V(\alpha) = \omega\}$ is nonstationary for some regular ν. The following theorem shows that this consequence already requires precisely the large cardinal assumed in Theorem 5.

Theorem 9. *Let* K *be Mitchell's core model for sequences of measures* [7]. *Suppose that there is a forcing extension* W *of* K, *and some* ν *which is regular in* W, *so that* W *and* K *have the same reals and the same cardinals, and so that* $\{\alpha < \nu \mid \mathrm{cf}^K(\alpha) = \omega\}$ *is nonstationary in* W. *Then in* K *there is a cardinal* $\kappa < \nu$ *so that* $(\forall \alpha < \kappa)(\exists \theta < \kappa)(o(\theta) \geq \alpha)$.

Proof. Set $\rho_0 = \omega$ and define by induction ρ_{n+1} equal to the least cardinal of Mitchell order ρ_n in K if there is such a cardinal below ν. Otherwise stop the construction.

Recall the following result, due to Mitchell: Let θ be regular in K and suppose that $\omega < \delta = \mathrm{cf}^W(\theta) < \theta$. Then the Mitchell order of θ in K is at least δ. (Mitchell [5] proves this for countably closed δ, but the need for countable closure is eliminated in [6, Remark 4.20] assuming that there is a successor of a regular cardinal in W between $\delta = \mathrm{cf}^W(\theta)$ and θ. This assumption always holds in our case, as K and W have the same cardinals.)

Assume for contradiction that the construction stops at a finite stage. Suppose for simplicity that only ρ_0, ρ_1, and ρ_2 are defined. By Mitchell's covering quoted above, no regular cardinal of K below ν can change its cofinality to ρ_2^+ in W. Similarly no regular cardinal of K below ρ_2^+ can change its cofinality to ρ_1^+ in W, and no regular cardinal of K below ρ_1^+ can change its cofinality to ρ_0^+ in W. Thus:

(1) $\xi < \nu$ and $\mathrm{cf}^W(\xi) = \rho_2^+$ implies that $\mathrm{cf}^K(\xi) = \rho_2^+$.
(2) $\mathrm{cf}^K(\xi) \leq \rho_2$ and $\mathrm{cf}^W(\xi) = \rho_1^+$ implies that $\mathrm{cf}^K(\xi) = \rho_1^+$.
(3) $\mathrm{cf}^K(\xi) \leq \rho_1$ and $\mathrm{cf}^W(\xi) = \rho_0^+$ implies that $\mathrm{cf}^K(\xi) = \rho_0^+$.

Let $C \subset \nu$ be a club in W avoiding ordinals of cofinality ω in K. Pick a limit point ξ of C of cofinality ρ_2^+ in W. Then $\mathrm{cf}^K(\xi) = \rho_2^+$ by (1).

Let $D \subset \xi$ be a club in K of order type ρ_2^+, consisting of ordinals of cofinality at most ρ_2 in K. In W (which has the original club C), pick a limit point ζ of $D \cap C$ of cofinality ρ_1^+. Then $\mathrm{cf}^K(\zeta) = \rho_1^+$ by (2).

Repeating, fix a club $E \subset \zeta$ in K, of order type ρ_1^+ and consisting of ordinals of cofinality at most ρ_1 in K. In W, pick a limit point μ of $E \cap D \cap C$ of cofinality $\omega_1 = \rho_0^+$. Then $\mathrm{cf}^K(\mu) = \omega_1$ by (3).

Finally, fix a club $F \subset \mu$ in K, of order type ω_1 and consisting of ordinals of cofinality at most ω in K.

Let α belong to $F \cap E \cap D \cap C$. Then $\mathrm{cf}^K(\alpha) = \omega$ since α belongs to F, and $\mathrm{cf}^K(\alpha) \neq \omega$ since α belongs to C. This contradiction completes the proof. $\square$

References

[1] Moti Gitik. Changing cofinalities and the nonstationary ideal. *Israel J. Math.*, 56(3):280–314, 1986.
[2] Moti Gitik. On closed unbounded sets consisting of former regulars. *J. Symbolic Logic*, 64(1):1–12, 1999.
[3] Thomas Jech. *Set theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.
[4] Menachem Magidor. Changing cofinality of cardinals. *Fund. Math.*, 99(1):61–71, 1978.
[5] W. Mitchell. Applications of the covering lemma for sequences of measures. *Trans. Amer. Math. Soc.*, 299(1):41–58, 1987.
[6] William J. Mitchell. The covering lemma. In Foreman, Kanamori, and Magidor, editors, *Handbook of set theory.* To appear.
[7] William J. Mitchell. The core model for sequences of measures. I. *Math. Proc. Cambridge Philos. Soc.*, 95(2):229–260, 1984.
[8] Saharon Shelah. *Proper and improper forcing.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.
[9] Matteo Viale. The proper forcing axiom and the singular cardinal hypothesis. *J. Symbolic Logic*, 71(2):473–479, 2006.

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90095-1555, U.S.A

E-mail address: `gitik@post.tau.ac.il`
E-mail address: `ineeman@math.ucla.edu`
E-mail address: `sinapova@math.ucla.edu`