Sample problems for midterm exam

(1) True or false. Justify your answer. If something is false, you need to give a counterexample. You will get no credit for simply writing "true" or "false".

(a) If a topological space X is second countable, then every open cover of X has a finite subcover.

(b) A normal topological space is Hausdorff.

(c) Every open subset of $\mathbf{R}$ (with the usual metric topology) is a union of disjoint open intervals (finite, semi-infinite, or infinite).

(d) Every closed subset of $\mathbf{R}$ (with the usual metric topology) is the union of a sequence of points and a (possibly uncountable) union of disjoint closed intervals (finite, semi-infinite, or infinite).

(e) A totally bounded metric space is bounded.

(f) A bounded metric space is totally bounded.

(g) The closure of a subset S of a topological space X is closed. (Here the closure of S is the set of points in X which are adherent to S; and a point of $x \in X$ is adherent to S if S meets every neighborhood of x.)

(2) Let (X, d) be a metric space and E be a subset of X. Show that the boundary ∂E of E is closed in X.

(3) Let (X, d) be a metric space and E be a subset of X. Show that if E is compact, then it must be closed in X.

(4) Let $f : X \to Y$ and $g : X \to Z$ be continuous maps between topological spaces. Show that $h : X \to Y \times Z$ given by $h(x) = (f(x), g(x))$ is continuous where $Y \times Z$ has the product topology.

(5) Show that if X and Y are regular, then so is $X \times Y$.

(6) Let $\mathcal{B} = \{[a, b) \subset \mathbf{R} \mid -\infty < a < b < \infty\}$.

(a) Prove that $\mathcal{B}$ is a basis for a topology $\mathcal{T}_{\mathcal{B}}$ of $\mathbf{R}$.

(b) Show that $(\mathbf{R}, \mathcal{T}_{\mathcal{B}})$ is a T_3-space. (You actually showed T_4 in your HW, which is a bit hard. T_3 is much easier and could be a reasonable exam question.)

(7) Let X be a metric space and let $Y \subset X$ be a subset.

(a) Define the closure $\overline{Y}$ of Y.

(b) Show that the closure of $\overline{Y}$ is equal to $\overline{Y}$.

(8) Let $B([0, 1])$ be the space of bounded functions $f : [0, 1] \to \mathbf{R}$. Show that $d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|$ is a metric on $B([0, 1])$. Is it separable?

(9) Prove that $[0, 1]/(0 \sim 1)$ and the unit circle $S^1 = \{x^2 + y^2 = 1\} \subset \mathbf{R}^2$ are homeomorphic.

(10) Let Y, Z be disjoint closed subsets of a normal topological space X. Let $f : Y \to \mathbf{R}$ and $g : Z \to \mathbf{R}$ be bounded continuous functions. Show that there exists a bounded continuous function $h : X \to \mathbf{R}$ such that $h|_Y = f$ and $h|_Z = g$.