Sample problems for midterm exam

Solutions will not be provided.

(1) (Warm-up) Express the following numbers in the form $a + ib$, where $a, b \in \mathbf{R}$:

$$(1+5i)/\overline{(3-2i)}, \quad (-3+i)\overline{(4+2i)}, \quad \frac{e^{1+3\pi i}}{e^{-1+i\pi/2}}, \quad \mathrm{Log}(1+i), \quad \sqrt{-1-i},$$

where $\sqrt{-}$ refers to the principal branch.

(2) (Warm-up) Find all the values of $\log(1-i)$ and $i^{1/5}$.

(3) (Next step) Make sure you can do all the quiz problems *100% correctly!*

(4) Find all the solutions of $z^7 = -7$. Same for $z^7 - 7z = 0$.

(5) Find and sketch the set described by $|z-1| = |z+i|$.

(6) Find and prove a formula for $\sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$, where $\theta \in \mathbf{R}$.

(7) Sketch:
  (a) $\{z = re^{i\theta} \mid r > 0, \frac{\pi}{4} < \theta < \frac{3\pi}{4}\}$ and its image under $f(z) = iz^2$.
  (b) $\{1 \leq \mathrm{Re}(z) \leq 2\}$ and $\{1 \leq \mathrm{Im}(z) \leq 2\}$ and their images under $f(z) = e^z$.
  (c) $\{|z| \leq 1, \mathrm{Re}(z) \geq 0\}$ and its image under $\mathrm{Log}((1-i)z)$.
  (d) $\{1 \leq \mathrm{Im}(z) \leq 2\}$ and its image under $f(z) = 1/z$.

(8) Determine whether the following functions $f(z) = u(z) + iv(y)$ satisfy the Cauchy-Riemann equations:

$$z^2, \quad \mathrm{Log}\, z, \quad \mathrm{Re}(z), \quad |z|^2, \quad \mathrm{Re}(z) + 2i\,\mathrm{Im}(z), \quad 1/z.$$

(9) Let $u(x,y) = \frac{y}{x^2+y^2}$, defined on $\mathbf{C} \setminus \{0\}$.
  (a) Show that u is a harmonic function.
  (b) Find its harmonic conjugate.

(10) Find a conformal map of the horizontal strip $\{-A \leq \mathrm{Im}(z) \leq A\}$ onto the right half-plane $\{\mathrm{Re}(w) \geq 0\}$, where A is a positive number. Do the same for the vertical strip $\{-A \leq \mathrm{Re}(z) \leq A\}$.

(11) Write down fractional linear transformations taking
  (a) $(0, 1, \infty) \mapsto (0, \infty, i)$;
  (b) $(1, i, -1) \mapsto (1, 0, -1)$.

(12) Problems II.7: 2,3,4.

(13) Problem II.7: 9.

(14) Suppose $f : U \to \mathbf{C}$ is holomorphic and satisfies $\mathrm{Re}\, f(z) = \mathrm{Im}\, f(z)$ for all $z \in U$. Then show that f is constant. Similarly, suppose $f : U \to \mathbf{C}$ and $\overline{f} : U \to \mathbf{C}$ are both holomorphic. Then show that f is constant.

(15) State Green's Theorem.

(16) Problems III.1: 2,3,4.

(17) Evaluate $\int_\gamma \frac{1}{z} dz$, where:
  (a) γ is the circle of radius $R > 0$ centered at 0, oriented counterclockwise. (Do the same for the clockwise orientation.)

(b) γ is the quarter circle centered at 0, from $-1-i$ to $-1+i$.
(c) γ is the line from $-1-i$ to $-1+i$.

(18) Evaluate $\int_{\partial D} z dz$, where D is the square $\{-1 \leq x \leq 1, -1 \leq y \leq 1\}$.