Sample problems for final exam

(1) True or false. Justify your answer. If something is false, you need to give a counterexample. You will get no credit for simply writing "true" or "false".
  (a) Let X be a locally compact Hausdorff space that is not compact and let $Y = X \cup \{\infty\}$ be its one-point compactification. If X is connected, then Y is connected.
  (b) Let X be a locally compact Hausdorff space that is not compact and let $Y = X \cup \{\infty\}$ be its one-point compactification. If Y is connected, then X is connected.
  (c) If X is path-connected and $f : X \to Y$ is a continuous map, then $f(X)$ is path-connected.
  (d) If Y is connected and $f : X \to Y$ is a continuous map, then $f(X)$ is connected.
  (e) The (possibly infinite) product of Hausdorff spaces is Hausdorff.

(2) If X is a second-countable space and every countable cover of X has a finite subcover, then show that X is compact.

(3) Let X be a T_3-space and let A be a closed subset of X. If $\pi : X \to X/A$ be the quotient map that collapses A to a point, then show that X/A is Hausdorff.

(4) Show that $X = \prod_{i=1}^{\infty} X_i$ be an infinite product of topological spaces. If each X_i is compact Hausdorff, show that X is T_4.

(5) Prove that the one-point compactification of $\mathbf{R}^n$ is homeomorphic to the n-sphere S^n. Prove that the one-point compactification of any open ball in $\mathbf{R}^n$ is homeomorphic to the n-sphere.

(6) Define what it means for a topological space X to be simply connected.
  (a) Prove that if $X = U \cup V$, where U and V are simply connected open sets and $U \cap V$ is nonempty and path-connected, then X is simply connected.
  (b) Prove that the product of simply connected spaces is simply connected.
  (c) Prove that S^n is simply connected for $n \geq 2$.
  (d) Prove that $\mathbf{R}^n - \{0\}$ is simply connected for $n \geq 3$.
  (e) A space X *deformation retracts to* $x_0 \in X$ if there is a map $F : X \times [0,1] \to X$ such that $F(x,0) = x_0$ and $F(x,1) = x$ for all $x \in X$ and $F(x_0,t) = x_0$ for all $t \in [0,1]$. Show that such an X is simply connected.

(7) Define what it means for two continuous maps $f, g : X \to Y$ between topological spaces to be homotopic.

(8) Given two points x_0, x_1 in a topological space X, explain the relationship between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

(9) What are the covering transformations of $\mathbf{R} \to S^1$?