Sample problems for midterm exam

For all of the problems below, you need to fully justify your answer.

(1) True or false. Justify your answer. If something is false, you need to give a counterexample. You will get no credit for simply writing "true" or "false". Let V, W, Z be vector spaces of dimension n, m, ℓ, respectively, and let $T : V \to W$ and $U : W \to Z$ be linear maps.
   (a) If $U \circ T$ is onto, then T is onto.
   (b) If $U \circ T$ is one-to-one, then T is one-to-one.
   (c) $\mathrm{Hom}(V, W) = \mathrm{Hom}(W, V)$.
   (d) There exists a linear map $\phi : \mathrm{Hom}(V, W) \to \mathrm{Hom}(W, V)$ that is one-to-one and onto.
   (e) $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim W$.
   (f) If T is linear, then T takes linearly independent subsets of V to linearly independent subsets of W.
   (g) There exists a linear map $T : \mathbf{R}^3 \to \mathbf{R}^3$ that takes $(1, 1, 1) \mapsto (1, 2, 3)$, $(1, 0, 1) \mapsto (1, 2, 3)$, and $(0, 1, 0) \mapsto (1, 2, 3)$.
   (h) Every subspace of a finite-dimensional vector space is finite-dimensional.
   (i) If S is a subset of a vector space V, then $\mathrm{Span}(S)$ is the intersection of all subspaces of V that contain S.
   (j) If $T : V \to V$ satisfies $T^2 = \mathrm{id}_V$, then $T = \mathrm{id}_V$.
   (k) If α is an ordered basis for V and β is an ordered basis for W, then $[T(v)]_\alpha = [T]_\alpha^\beta [v]_\beta$.

(2) Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be given by $f(x) \mapsto f(x) + f'(x)$. Let $\alpha = [1, x, x^2]$ and $\beta = [1 + x, x, x + x^2]$ be ordered bases of $P_2(\mathbf{R})$.
   (a) Prove that T is a linear map.
   (b) Verify that β is an ordered basis of $P_2(\mathbf{R})$.
   (c) Compute $[T]_\alpha^\alpha$ and $[T]_\alpha^\beta$.
   (d) Is T one-to-one? onto?
   (e) If $f(x) = 1 + x + x^2$, compute $[Tf]_\beta$.

(3) Which of the following are subspaces of the corresponding vector space? If it is a subspace, compute its dimension.
   (a) $\{(x, y, z) \in \mathbf{R}^3 \mid x + y = 5z\}$.
   (b) $\{(x, y, z, w) \in \mathbf{R}^4 \mid x + y = 0, w = 1\}$.
   (c) $\{(x, y, z, w) \in \mathbf{R}^4 \mid x^3 = y^3\}$.
   (d) $\{(x, y, z, w) \in \mathbf{R}^4 \mid x^3 + y = z^3 + w\}$.

(4) Let V be a vector space with basis $\{e_1, \ldots, e_n\}$. Then show that

$$\{e_1, e_1 + e_2, e_1 + e_2 + e_3, \ldots, e_1 + \cdots + e_n\}$$

is also a basis for V.

(5) Consider $W = \{A \in M_{2\times 2}(F) \mid A_{11} + A_{12} = 0\}$.
   (a) Show that W is a subspace of $M_{2\times 2}(F)$.
   (b) Give a basis for W.
   (c) What is $\dim W$?

   (d) Find a basis of $M_{2\times 2}(F)$ extending the one in part (b).

(6) Consider the subset W of all upper triangular matrices in $M_{n\times n}(F)$.
   (a) Show that W is a subspace of $M_{n\times n}(F)$.
   (b) Find a basis for W.
   (c) What is $\dim W$?

(7) Let V be a vector space of dimension n and let W_1, W_2 be subspaces of dimension m and $n-m$, respectively. Show that $V = \mathrm{Span}(W_1 \cup W_2)$ if and only if $W_1 \cap W_2 = \{0\}$.

(8) Let V be a vector space and let S, T be two subsets of V. Show that $\mathrm{Span}(S \cap T) \subset \mathrm{Span}(S) \cap \mathrm{Span}(T)$. Give an example when they are equal and when they are different.

(9) Let V be an n-dimensional F-vector space with an ordered basis α. Show that the map $T : V \to F^n$, $x \mapsto [x]_\alpha$, is linear, one-to-one, and onto. Compute $[T]_\alpha^\beta$ and $[T]_\alpha^\gamma$, where $\beta = \{e_1, \ldots, e_n\}$ and $\gamma = \{e_n, \ldots, e_1\}$.