Sample problems for final exam

Please also review the midterm sample problems. For all of the problems below, you need to fully justify your answer.

(1) True or false. Justify your answer. If something is true, you need to give a proof or cite the relevant theorem. If something is false, you need to give a counterexample. You will get no credit for simply writing "true" or "false".
  (a) A matrix A is invertible if and only if $\det A = 0$.
  (b) If v is an eigenvector of a linear map $T : V \to V$ with eigenvalue 100, then $3v$ is an eigenvector of T with eigenvalue 300.
  (c) If a matrix $A \in M_{n\times n}(\mathbb{C})$ satisfies $A^2 = 0$, then $A = 0$.
  (d) If $A, B \in M_{n\times n}(F)$ satisfy $AB = I$, then A and B are invertible.
  (e) If $A \in M_{n\times n}(F)$ is diagonalizable, then A is invertible.

(2) Let $S = \{v_1, \dots, v_k\}$ be a finite subset of a vector space V. Show that if S' is obtained from S by an "elementary row operation":
  (a) replacing v_i by av_i, $a \in F$ nonzero, or
  (b) replacing v_i by $v_i + av_j$, where $a \in F$ and $j \neq i$,
  then $\operatorname{Span} S = \operatorname{Span} S'$.

(3) Let $T : P_{30}(\mathbb{R}) \to P_{30}(\mathbb{R})$ be given by $T(f) = (x + 3)f'(x)$.
  (a) Prove that T is linear.
  (b) Is T one-to-one? onto?

(4) If $T : V \to W$ is a linear map, then define the *cokernel* $\operatorname{coker}(T)$ *of* T as $W/\operatorname{Im} T$. Prove that if T is a linear map from a finite-dimensional vector space V to itself, then $\operatorname{coker} T$ is isomorphic to $\ker T$.

(5) Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 2a+b & c \\ b & 2d+c \end{pmatrix}$. Consider the ordered bases

$$\alpha = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

$$\beta = \left[\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

  for $M_{2\times 2}(\mathbb{R})$.
  (a) Show that β is a basis for $M_{2\times 2}(\mathbb{R})$.
  (b) Compute $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$.
  (c) If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then compute $[T(A)]_\beta$.

(6) Let $V = P_2(\mathbb{R})$ with ordered basis $\alpha = [1, x, x^2]$ and let $S, T : V \to V$ be given by

$$T(f) = f(1) + f(-1)x + f(0)x^2$$
$$S(ax^2 + bx + c) = cx^2 + bx + a.$$

Compute $[S]_\alpha^\alpha$, $[T]_\alpha^\alpha$, $[TS]_\alpha^\alpha$, $[(TS)^{-1}]_\alpha^\alpha$.

(7) Let $A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$.

(a) Compute the characteristic polynomial $p(t)$ of A and the eigenvalues. Does $p(t)$ split?

(b) Is A diagonalizable? Is yes, determine a basis of eigenvectors of A.

(8) Determine precisely when an upper triangular matrix $A \in M_{n\times n}(F)$ with only one eigenvalue λ is diagonalizable.

(9) Let A and B be $n \times n$ matrices such that $AB = BA$, i.e., they commute. Let λ be an eigenvalue of A with eigenvector v.

(a) Show that Bv is an eigenvector of A with eigenvalue λ (assuming $Bv \neq 0$).

(b) Show that if v is not an eigenvector of B, then $\dim E_\lambda > 1$. (Here E_λ is the λ-eigenspace for A.)

(c) Suppose that A has n distinct eigenvalues. Then show that B is diagonalizable.

(10) Let $T : V \to V$ be a linear map. If $W \subset V$ is a subspace of and $T(W) \subset W$ (we say that T is W-*invariant*), then we can restrict T to W, i.e., we have a map $T|_W : W \to W$ given by $w \mapsto T(w)$. Let $\{\lambda_1, \dots, \lambda_k\}$ be the set of eigenvalues of T and let $W = E_{\lambda_1} + \cdots + E_{\lambda_k}$.

(a) Show that $W = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$.

(b) Show that $T(W) \subset W$.

(c) Show that $T|_W : W \to W$ is diagonalizable.

(d) If $Z \subset V$ is another T-invariant subspace such that $T|_Z$ is diagonalizable, then $\dim Z \leq \sum_{i=1}^k \dim E_{\lambda_i}$.

(11) Consider $P_1(\mathbb{R})$ with inner product $\langle f, g\rangle = \int_0^2 f(t)g(t)dt$.

(a) Compute $\|x\|$.

(b) Compute the orthogonal projection of x onto the subspace generated by 1.

(c) Apply the Gram-Schmidt process to the ordered basis $[1, x]$ to obtain an orthonormal basis for $P_1(\mathbb{R})$.

(12) Let $V = \mathbb{R}^4$ with the standard inner product $\langle , \rangle$. Consider the linearly independent subset $S = \{w_1 = (1, 0, 1, 0), w_2 = (1, 1, 1, 1), w_3 = (2, 2, 0, 2)\}$.

(a) Apply the Gram-Schmidt process to S to compute an orthonormal basis α for $\operatorname{Span} S$.

(b) Determine the Fourier coefficients of $v = (1, 2, 3, 2) \in \operatorname{Span} S$ with respect to α.

(13) Let $(V, \langle , \rangle)$ be a finite-dimensional inner product space.

(a) If $T : V \to V$ is a linear operator such that $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$, then show that T is one-to-one.

(b) Show that if V is an $\mathbb{R}$-vector space, then $\langle x, y\rangle = \frac{1}{2}(\|x + y\|^2) - \|x\|^2 - \|y\|^2)$ for all $x, y \in V$. Hence an inner product can be recovered from the norm.

(14) Let $(V, \langle , \rangle)$ be a finite-dimensional inner product space. Let λ be an eigenvalue of T with eigenvector v.

(a) Show that $v \in (\operatorname{Im}(T^* - \overline{\lambda} I))^\perp$.

(b) Show that $T^* - \overline{\lambda} I$ is not onto.

(c) Show that $\overline{\lambda}$ is an eigenvalue for T^*.

(d) Show that if z is an eigenvector of T^* and $W = \mathrm{Span}\{z\}$, then $W^\perp$ is T-invariant, i.e., $T(W^\perp) \subset W^\perp$.

Now suppose that the characteristic polynomial $p(t)$ of T splits.

(e) Show that the characteristic polynomial $q(t)$ of $T|_{W^\perp} : W^\perp \to W^\perp$ divides $p(t)$, i.e., $p(t) = q(t)r(t)$ where $r(t)$ is a polynomial.

(f) Show by induction on $\dim(V)$ that V has an ordered orthonormal basis α such that $[T]_\alpha^\alpha$ is an upper triangular matrix.

(15) Determine whether the following linear maps are diagonalizable:

(a) $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $(a_1, \ldots, a_n, b_1, \ldots, b_n) \mapsto (b_1, \ldots, b_n, a_1, \ldots, a_n)$.

(b) $T : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$, $(a_1, \ldots, a_n, b_1, \ldots, b_n) \mapsto (ib_1, \ldots, ib_n, -ia_1, \ldots, -ia_n)$.

(c) $T : \mathbb{R}^n \to \mathbb{R}^n$, $(x_1, \ldots, x_n) \mapsto (x_1 + x_2, x_2 + x_3, \ldots, x_{n-1} + x_n, x_n)$.

(d) $T : \mathbb{R}^n \to \mathbb{R}^n$, $(x_1, \ldots, x_n) \mapsto (x_1+x_2, x_1+x_2+x_3, \ldots, x_{n-2}+x_{n-1}+x_n, x_{n-1}+x_n)$.