CONVEX HYPERSURFACE THEORY IN CONTACT TOPOLOGY

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ABSTRACT. We lay the foundations of convex hypersurface theory (CHT) in contact topology, extending the work of Giroux in dimension three. Specifically, we prove that any closed hypersurface in a contact manifold can be C^0-approximated by a convex one. We also prove that a C^0-generic family of mutually disjoint closed hypersurfaces parametrized by $t \in [0,1]$ is convex except at finitely many times $t_1, \dots, t_N$, and that crossing each t_i corresponds to a bypass attachment. As applications of CHT, we prove the existence of compatible (relative) open book decompositions for contact manifolds and an existence h-principle for codimension 2 contact submanifolds.

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1. Introduction

1.1. **Convex contact structures.** Morse theory is a topologist's favorite tool for exploring the structure of manifolds. The significance of Morse theory — here we mean the traditional finite-dimensional version, not Floer theory — in contact and symplectic topology was advocated by Eliashberg and Gromov in [EG91]. In particular, according to [EG91, Definition 3.5.A], a contact manifold (M, ξ) is *convex* if there exists a Morse function, called a *contact Morse function*, which admits a gradient-like vector field whose flow preserves ξ. Just as a manifold can be reconstructed from its Morse function by a sequence of handle attachments in traditional Morse theory, a contact manifold can be reconstructed from a contact Morse function by a sequence of contact handle attachments. The analogous theory in symplectic topology is known as the theory of Weinstein manifolds.

Eliashberg and Gromov asked in [EG91] whether there exist non-convex contact manifolds. Around 2000 Giroux gave a negative answer to the question by showing that every closed contact manifold is convex; see [Gir02]. This can also be formulated as his celebrated correspondence between contact structures and open book decompositions. This is in sharp contrast to the theory of Weinstein manifolds, where it is relatively easy to see that any compatible Morse function cannot have critical points of index greater than half of the dimension of the manifold.

Remark 1.1.1. It might be the case that "convexity" is one of the most abused terminologies in mathematics. We will not use the term "convex contact manifold" in the sense of Eliashberg and Gromov for the rest of the paper.

At this point, the question bifurcates into two:

Question 1.1.2. *How do we establish Morse theory on contact manifolds?*

Question 1.1.3. *How do we use Morse theory to better understand contact manifolds?*

Let us first address Question 1.1.2, which was first answered by Giroux in both dimension 3 and in higher dimensions. Giroux used two completely different sets of techniques to treat the 3-dimensional and higher-dimensional cases.

We first discuss the 3-dimensional case. In his thesis [Gir91], Giroux introduced what is now known as *convex surface theory* into 3-dimensional contact topology. It is an extremely powerful and efficient way of studying embedded surfaces in contact 3-manifolds, and can recover most of the pioneering results of Bennequin [Ben83] and Eliashberg [Eli92]. Using convex surface theory, Giroux showed that for closed contact 3-manifolds, there is a one-to-one correspondence between isotopy classes of contact structures and compatible open book decompositions up to positive stabilization.

Before moving onto higher dimensions, let us recall the definition of a convex hypersurface following [Gir91]:

Definition 1.1.4. *A hypersurface $\Sigma \subset (M, \xi)$ is* convex *if there exists a contact vector field v, i.e., a vector field whose flow preserves ξ, which is transverse to Σ everywhere.*

Observe that regular level sets of a contact Morse function are convex hypersurfaces.

The situation in dimensions > 3 is quite different. Besides the fact that convex hypersurfaces can be defined in any dimension, until now there has been no systematic *convex hypersurface theory*. Giroux's proof [Gir02, Gir17] that every closed contact manifold is convex involves a completely different technology, i.e., Donaldson's [Don96] technique of *approximately holomorphic sections*, transplanted into contact topology independently by Ibort, Martínez-Torres, and Presas

[IMTP00] and by Mohsen [Moh, Moh19]. Donaldson used the approximate holomorphic technology to construct real codimension 2 symplectic hypersurfaces of a closed symplectic manifold as the zero locus of an approximately holomorphic section of a complex line bundle, while [IMTP00] and [Moh, Moh19] constructed certain codimension 2 contact submanifolds of a closed contact manifold. What Giroux realized is that [Don96], [IMTP00], and [Moh, Moh19] could be used to produce compatible open book decompositions. Roughly speaking, given a closed contact manifold $(M, \xi = \ker\alpha)$, one considers the trivial line bundle $\underline{\mathbb{C}}$ on M equipped with a suitable Hermitian connection determined by α. Then there exists a section $s : M \to \underline{\mathbb{C}}$ whose zero locus $B := s^{-1}(0)$ is a closed codimension 2 contact submanifold called the *binding*, and

$$\frac{s}{|s|} : M \setminus B \to S^1$$

is a smooth fibration defining the compatible open book decomposition of (M, ξ). As a consequence of using the approximate holomorphic technology, the higher-dimensional Giroux correspondence (see Corollary 1.3.1) is a much weaker statement compared to its 3-dimensional counterpart.

1.2. **Main results.** The main goal of this paper is to systematically generalize Giroux's convex surface theory to all dimensions. The main results of *convex hypersurface theory* (CHT) are Theorems 1.2.3 and 1.2.5. In fact, even in dimension 3, our method (see Section 4) differs somewhat from Giroux's original approach, is simpler, and is consistent with our more general approach in higher dimensions.

We first introduce some more terminology describing the anatomy of a convex hypersurface.

Definition 1.2.1. *Let $\Sigma \subset (M, \xi = \ker\alpha)$ be a convex hypersurface with respect to a transverse contact vector field v. Define the* dividing set $\Gamma(\Sigma) := \{\alpha(v) = 0\}$ *and $R_\pm(\Sigma) := \{\pm\alpha(v) > 0\}$ as subsets of Σ.*

It turns out that $\Gamma(\Sigma) \subset (M, \xi)$ is a codimension 2 contact submanifold, and $R_\pm(\Sigma)$ are (complete) Liouville manifolds with Liouville form given by a suitable rescaling of $\alpha|_{R_\pm(\Sigma)}$, respectively. Moreover, the isotopy classes of $\Gamma(\Sigma), R_\pm(\Sigma)$ are independent of the choices of v and α.

In dimensions ≥ 4, there exist Liouville manifolds that are not Weinstein by McDuff [McD91], Geiges [Gei94, Gei95], Mitsumatsu [Mit95], and Massot, Niederkrüger, and Wendl [MNW13]. While these "exotic" Liouville manifolds are great for constructing (counter-)examples, there currently is no systematic understanding of such non-Weinstein Liouville manifolds, partially because of the lack of an appropriate Morse theory on such manifolds. This is slowly starting to change: For example it was recently shown in [EOY] that a stabilized Liouville manifold with the homotopy type of a half-dimensional CW-complex is symplectomorphic to a flexible Weinstein manifold and in [BC] that the stabilizations of Mitsumatsu's Liouville domains are Weinstein domains.[1] This motivates the following definition:

[1] Note the latter involves Liouville *domains*, which is a much harder problem.

Definition 1.2.2. *A convex hypersurface Σ is* Weinstein (resp. 1-Weinstein) convex *if $R_+(\Sigma)$ and $R_-(\Sigma)$ are Weinstein (resp. 1-Weinstein).*

Here a Liouville domain that admits a Liouville vector field which is gradient-like with respect to a "1-Morse" function which also admits critical points of birth-death type) will be called a 1-*Weinstein domain* (instead of a *generalized Weinstein domain*) in this paper.

Weinstein convex hypersurfaces admit a Morse-theoretic interpretation, given in Proposition 2.3.3.

Now we are ready to state the foundational theorems of CHT.

Theorem 1.2.3. *Any closed oriented hypersurface in a contact manifold can be C^0-approximated by a Weinstein convex one.*

Remark 1.2.4. The C^∞-version of Theorem 1.2.3 remains open. Mori's candidate counterexample in [Mor] was shown by Breen [Bre21] to actually admit a C^∞-small approximation by a convex one.

Theorem 1.2.5. *Let ξ be a contact structure on $\Sigma \times [0,1]$ such that the hypersurfaces $\Sigma \times \{0,1\}$ are Weinstein convex. Then, up to a boundary-relative contact isotopy, there exists a finite sequence $0 < t_1 < \cdots < t_N < 1$ such that the following hold:*

- *$\Sigma \times \{t\}$ is 1-Weinstein convex if $t \neq t_i$ for any $1 \leq i \leq N$.*
- *For each i, there exists a small $\epsilon > 0$ such that ξ restricted to $\Sigma \times [t_i - \epsilon, t_i + \epsilon]$ is contactomorphic to a bypass attachment.*

For an initial study of bypass attachments in higher dimensions the reader is referred to [HH].

Remark 1.2.6. Theorem 1.2.5 was conjectured by Paolo Ghiggini in the afternoon of April 10, 2015 in Paris.

1.3. **Applications.** As an immediate application of Theorems 1.2.3 and 1.2.5, we can extend Giroux's 3-dimensional approach to constructing compatible open book decompositions to higher dimensions. This is the content of the following three corollaries. Note, however, that we do not address the stabilization equivalence of the compatible open book decompositions in this paper. We plan to investigate this in future work.

Corollary 1.3.1 ([Gir02])**.** *Any closed contact manifold admits a compatible open book decomposition, all of whose pages are 1-Weinstein.*

Corollary 1.3.2. *Any compact contact manifold with convex boundary admits a compatible partial open book decomposition, all of whose pages are 1-Weinstein domains or 1-Weinstein cobordisms.*

Corollary 1.3.3. *Given a possibly disconnected closed Legendrian submanifold Λ in a closed contact manifold, there exists a compatible open book decomposition such that all the pages are 1-Weinstein and Λ is contained in a page.*

Remark 1.3.4. Although Corollary 1.3.3 does not appear in the literature, asymptotically holomorphic techniques in contact geometry from [IMTP00] and [Moh, Moh19] — more specifically the combination of Giroux's existence theorem of compatible open book decompositions (see Presas [Pre14] for a published proof) and asymptotically holomorphic symplectic special position theorems for Lagrangians in symplectic manifolds [AMnP05] applied to the Lagrangianization of a Legendrian in the symplectization gives the result.

This completes our exploration of Question 1.1.2 for the time being.

Next we turn to Question 1.1.3, which is a much harder question. For example, we would like to obtain classification results for contact structures on higher-dimensional manifolds (e.g., the spheres) besides the "flexible" ones due to Borman, Eliashberg, and Murphy [BEM15]. Unfortunately, our current understanding of contact Morse theory is not good enough for us to classify anything in higher dimensions. Instead, we will use the (mostly dynamical) techniques developed in this paper to address the existence problems of contact manifolds and submanifolds.

The existence problems of contact manifolds and submanifolds were first addressed by Gromov [Gro86] using his magnificent zoo of h-principles. In particular, he proved a full h-principle for contact structures on open manifolds (see [EM02, 10.3.2]) and an existence h-principle for isocontact embeddings $Y \subset (M, \xi)$ under the assumptions that either Y has $\operatorname{codim} Y \geq 4$ or is open with $\operatorname{codim} Y = 2$.

The existence problem turned out to be much harder for closed manifolds. In dimension 3, an existence h-principle for contact structures was proved by Martinet [Mar71] and Lutz [Lut77]. For overtwisted contact 3-manifolds, a full h-principle was proved by Eliashberg [Eli89]. In dimension 5, there is a rich literature of partial results: the existence of contact structures on certain classes of 5-manifolds was established by Geiges [Gei91, Gei97], Geiges-Thomas [GT98, GT01], and Bourgeois [Bou02]. Afterwards, a complete existence h-principle for contact 5-manifolds was established by Casals, Pancholi, and Presas [CPP15] (there is also the approach of of Etnyre [Etn], which currently has a gap). Finally, the existence h-principle for contact manifolds of any dimension, as well as the full h-principle for overtwisted contact manifolds of any dimension, was established by Borman, Eliashberg, and Murphy [BEM15].

So far the story has mostly been about the contact manifolds themselves. Now we turn to the existence problem of contact submanifolds or (iso-)contact embeddings. Results in this direction were surprisingly rare until recently. Besides the aforementioned h-principle of Gromov, there exist constructions of contact submanifolds by Ibort, Martínez-Torres, and Presas [IMTP00], mentioned earlier. In low dimensions, there also exist works by Kasuya [Kas16], Etnyre-Furukawa [EF17], and Etnyre-Lekili [EL] on embedding contact 3-manifolds into certain contact 5-manifolds.

In the rest of the introduction we will explain the existence h-principle for codimension 2 contact submanifolds. Since the case of open submanifolds has already been settled by Gromov, we may assume that all the submanifolds involved are closed.

Definition 1.3.5. *Let (M, ξ) be a contact manifold. A submanifold $Y \subset M$ is an* almost contact submanifold *if there exists a homotopy (η_t, ω_t), $t \in [0,1]$, where $\eta_t \subset TM|_Y$ is a codimension 1 distribution of TM along Y and ω_t is a conformal symplectic structure on η_t, such that:*

(1) $\eta_0 = \xi|_Y$ and ω_0 is induced from $\xi|_Y$; and
(2) $TY \pitchfork \eta_1$ and the normal bundle $T_Y M \subset \eta_1$ is ω_1-symplectic.

A straightforward calculation (see [BCS14, Lemma 2.17]) shows that any even-codimensional submanifold with trivial normal bundle is almost contact.

Corollary 1.3.6. *Any almost contact submanifold can be C^0-approximated by a genuine contact submanifold.*

Corollary 1.3.7. *Any (coorientable) contact submanifold can be C^0-approximated by another contact submanifold with the opposite orientation.*

Remark 1.3.8. Around the same time as our paper, Casals-Pancholi-Presas [CPP] proved the existence of iso-contact embeddings in codimension 2. Their work and ours are equivalent via the work of Pancholi-Pandit [PP] on iso-contact embeddings.

Wrapping up the introduction, by combining Corollary 1.3.6 with Gromov's h-principle for contact structures on open manifolds, one can deduce the following one-half of the groundbreaking(!) theorem of Borman-Eliashberg-Murphy [BEM15] without too much difficulty:

Corollary 1.3.9 (Borman-Eliashberg-Murphy)**.** *The existence h-principle holds for contact structures on closed odd-dimensional manifolds of dimension $4k+1$, $k \in \mathbb{Z}^+$, i.e., there exists a genuine contact structure in the homotopy class of any almost contact structure on M^{4k+1}.*

Our proof of Corollary 1.3.9 follows from producing genuine contact structures representing each homotopy class of almost contact structures on S^{2n-1}, i.e., proving the existence h-principle for S^{2n-1}. This is easy for S^{4k+1} (via connected sums of Brieskorn manifolds) and is much more involved for S^{4k+3}.

The proofs of Corollaries 1.3.6, 1.3.7, and 1.3.9 will be given in Section 11. We note that the proof of [CPP] relies on [BEM15] and cannot be used to prove it. Finally, note that in contrast to the contact structures constructed in [BEM15], the contact submanifolds constructed by Corollary 1.3.6 are not *a priori* overtwisted.

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2. A CONVEXITY CRITERION

Let $\Sigma \subset (M^{2n+1}, \xi)$ be a closed cooriented hypersurface. The goal of this section is to give a sufficient condition for the characteristic foliation Σ_ξ on Σ (see Definition 2.1.1) which guarantees the Weinstein convexity of Σ.

2.1. Characteristic foliations.

Let α be a contact form for ξ. Let $(-\epsilon, \epsilon) \times \Sigma$ be a tubular neighborhood of $\Sigma = \{0\} \times \Sigma \subset M$. Fix an orientation on Σ such that the induced orientation on $(-\epsilon, \epsilon) \times \Sigma$ agrees with the orientation determined by $\alpha \wedge (d\alpha)^n$. We now introduce the characteristic foliation Σ_ξ on Σ.

Definition 2.1.1. *The* characteristic foliation Σ_ξ *is an oriented singular line field on* Σ *defined by*

$$\Sigma_\xi = \ker d\beta|_{\ker \beta},$$

where $\beta := \alpha|_\Sigma \in \Omega^1(\Sigma)$. *The orientation of* Σ_ξ *is determined by requiring that the decomposition* $T\Sigma = \Sigma_\xi \oplus \Sigma_\xi^\perp$ *respect orientations, where the orthogonal complement* $\Sigma_\xi^\perp$, *taken with respect to an auxiliary Riemannian metric on* Σ, *is oriented by* $\beta \wedge (d\beta)^{n-1}|_{\Sigma_\xi^\perp}$.

Remark 2.1.2. The characteristic foliation depends only on the contact structure and the orientation of Σ, and *not* on the choice of the contact form.

Note that $x \in \Sigma$ is a singular point of Σ_ξ if $T_x\Sigma = \xi_x$ as unoriented spaces. We say x is *positive* (resp. *negative*) if $T_x\Sigma = \pm\xi_x$ as oriented spaces, respectively.

The significance of the characteristic foliation in 3-dimensional contact topology is that it uniquely determines the germ of contact structures on any embedded surface. The corresponding statement for hypersurfaces in contact manifolds of dimension > 3 is unlikely to hold, i.e., the characteristic foliation by itself is *not* enough to determine the contact germ. Instead we have the following characterization of contact germs on hypersurfaces in any dimension. The proof is a standard application of the Moser technique and is omitted here.

Lemma 2.1.3. *Suppose* $\xi_i = \ker \alpha_i, i = 0, 1$, *are contact structures on* M *such that* $\beta_0 = g\beta_1 \in \Omega^1(\Sigma)$ *for some* $g : \Sigma \to \mathbb{R}_+$, *where* $\beta_i = \alpha_i|_\Sigma$. *Then there exists an isotopy* $\phi_s : M \xrightarrow{\sim} M, s \in [0, 1]$, *such that* $\phi_0 = \mathrm{id}_M, \phi_s(\Sigma) = \Sigma$ *and* $(\phi_1)_*(\xi_0) = \xi_1$ *on a neighborhood of* Σ.

Generally speaking, Σ_ξ can be rather complicated, even when Σ is convex with Liouville $R_\pm(\Sigma)$. For our purposes of this paper, it is more convenient to regard Σ_ξ as a vector field rather than an oriented line field. Of course there is no natural way to specify the magnitude of Σ_ξ as a vector field, which motivates the following definition: Two vector fields v_1, v_2 on Σ are *conformally equivalent* if there exists a positive function $h : \Sigma \to \mathbb{R}_+$ such that $v_1 = hv_2$. This is clearly an equivalence relation among all vector fields, and we will *not* distinguish conformally equivalent vector fields in the rest of the paper unless otherwise stated.

In order to state the convexity criterion, we need to prepare some generalities on gradient-like vector fields in the following subsection. Our treatment on this subject will be kept to a minimum. The reader is referred to the classical works of Cerf [Cer70] and Hatcher-Wagoner [HW73] for more thorough discussions. Indeed the adaptation of the techniques of Cerf and Hatcher-Wagoner to CHT is carried out in [BHH]. Note that similar techniques in symplectic topology have been developed by Cieliebak-Eliashberg in [CE12].

2.2. **Morse and 1-Morse vector fields.** Let Y be a closed manifold of dimension n. A smooth function $f : Y \to \mathbb{R}$ is *Morse* if all the critical points of f (i.e., points $p \in Y$ such that $df(p) = 0$) are *nondegenerate*, i.e., there exist local coordinates $x_1, \dots, x_n$ about p such that locally f takes the form

$$-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2. \tag{2.2.1}$$

Here k is called the *Morse index*, or just the *index*, of the critical point p. We write $\operatorname{ind}(p) = k$.

A smooth function $f : Y \to \mathbb{R}$ is 1-*Morse* if the critical points of f are either nondegenerate or of *birth-death type*. Here a critical point $p \in Y$ of f is of *birth-death type* (also called *embryonic*) if there exist local coordinates $x_1, \dots, x_n$ about p such that f takes the form

$$-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n-1}^2 + x_n^3. \tag{2.2.2}$$

Similarly, k is defined to be the *(Morse) index* of p. The birth-death type critical point fits into a 1-parameter family of 1-Morse functions

$$-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n-1}^2 + tx_n + x_n^3,$$

such that for $t < 0$, there exist two nondegenerate critical points of indices k and $k+1$; for $t = 0$, there exists a birth-death type critical point; and for $t > 0$, there are no critical points.

It is a well-known fact due to Morse that any smooth function can be C^∞-approximated by a Morse function. Moreover, Cerf proved that any 1-parameter family of smooth functions can be C^∞-approximated by a family of 1-Morse functions, where the birth-death type critical points as above occur only at isolated moments.

Given a 1-Morse function $f : Y \to \mathbb{R}$, we say a vector field v on Y is *gradient-like* for f if the following two conditions are satisfied:

(GL1) Near each critical point of f, $v = \nabla f$ with respect to some Riemannian metric; and

(GL2) f is strictly increasing along (non-constant) flow lines of v.

Definition 2.2.1.

(1) A vector field v on Y is Morse *(resp.* 1-Morse*) if there exists a Morse (resp. 1-Morse) function $f : Y \to \mathbb{R}$ such that v is gradient-like for f.*

(2) A 1-parameter family of vector fields $(v_t)_{t\in[0,1]}$ is a 1-Morse family *if each v_t is 1-Morse and there exist $t_1 < \cdots < t_k \in (0,1)$ such that the birth-death type singularities occur only at t_i and there is a single birth-death type singularity at each t_i.*

We also make the slightly nonstandard definition:

Definition 2.2.2. *A Liouville domain is* 1-Weinstein *if its Liouville vector field is gradient-like with respect to a 1-Morse function.*

Remark 2.2.3. 1-Morse functions will be sufficient for the purposes of this paper since we will only encounter 1-parameter families of functions. In [BHH], we will need to deal with generic 2-parameter families of functions (called 2-Morse functions) where new singularities, i.e., the swallowtails, appear.

Convention 2.2.4. *A* flow line *$\ell : (a,b) \to Y$ of a vector field v on Y is assumed to be a* maximal *oriented smooth trajectory $\mathbb{R} \to Y$ that has been precomposed with an orientation-preserving reparametrization $(a,b) \xrightarrow{\sim} \mathbb{R}$. A* partial flow line *is the restriction of a flow line to a subinterval.*

Definition 2.2.5. *A* broken flow line *(resp.* possibly broken flow line*) of a vector field v on Y is a continuous map $\ell : [a,b] \to Y$ such that there exists an increasing sequence $a = a_0 < a_1 < \cdots < a_m = b$ with $m > 1$ (resp. $m \geq 1$) such that $\ell(a_j)$, $j = 0,\dots,m$, are zeros of v and $\ell|_{(a_j,a_{j+1})}$, $j = 0,\dots,m-1$, are flow lines of v. We may also replace $[a,b]$ by half-open or open intervals.*

In the rest of this subsection, we present a simple criterion for a vector field to be Morse which will be useful for our later applications. The corresponding version for 1-Morse vector fields is left to the reader as an exercise.

Proposition 2.2.6. *A vector field v on a closed manifold Y is Morse (resp. 1-Morse) if and only if the following conditions are satisfied:*

(M1) For any point $x \in Y$ with $v(x) = 0$, there exists a neighborhood of x and a locally defined function f of the form given by Eq. (2.2.1) (resp. Eq. (2.2.1) or Eq. (2.2.2)) such that $v = \nabla f$.

(M2) For any point $x \in Y$ with $v(x) \neq 0$, the flow line of v passing through x converges to zeros of v in both forward and backward time.

(M3) There exist no possibly broken loops*, i.e., a possibly broken flow line $\ell : [0,1] \to Y$ such that $\ell(0) = \ell(1)$.*

Proof. The "only if" direction is obvious. To prove the "if" direction, let $Z(v) = \{x_1,\dots,x_k\}$ be the finite set of zeros of v, where the finiteness is guaranteed by (M1) and the compactness of Y. Then we define a partial order on $Z(v)$ such that $x_i \prec x_j$ if there exists a flow line of v from x_i to x_j. The fact that $\prec$ is a partial order follows from (M3).

We then construct a handle decomposition of Y starting from the minimal elements Z_0 of $Z(v)$ (note that a minimal element of $Z(v)$ has index 0 by (M2)) and inductively attaching handles as follows: Starting with a standard neighborhood of Z_0, suppose we have already attached the handles corresponding to Z_j. Then we attach the handles corresponding to the minimal elements of $Z(v) - Z_j$, and then let Z_{j+1} be the union of Z_j and the minimal elements of $Z(v) - Z_j$. □

2.3. **A convexity criterion.** The goal of this subsection is to give a sufficient condition for a hypersurface to be Weinstein convex. To this end, we introduce the notions of *Morse* and *Morse*$^+$ hypersurfaces whose characteristic foliations have particularly simple dynamics.

Definition 2.3.1.

(1) A hypersurface $\Sigma \subset (M, \xi)$ is Morse *(resp.* 1-Morse*) if there exists a representative v in the conformal equivalence class of Σ_ξ which is a Morse (resp.* 1-Morse*) vector field on Σ. We say Σ is* Morse$^+$ *(resp.* 1-Morse$^+$*) if, in addition, there exist no flow trajectories from a negative singular point of v to a positive one.*

(2) A 1-parameter family of hypersurfaces $(\Sigma_t)_{t\in[0,1]}$ is a 1-Morse family*, if $((\Sigma_t)_\xi)_{t\in[0,1]}$ is represented by a 1-Morse family.*

Lemma 2.3.2. *If Σ is a Morse hypersurface, then a C^∞-small perturbation of Σ is Morse$^+$.*

Proof. Choose a contact form $\xi = \ker \alpha$. It suffices to observe that $d\alpha|_\Sigma$ is nondegenerate on a neighborhood of the singular points of Σ_ξ. It is a standard fact (see e.g. [CE12, Proposition 11.9]) that the Morse index $\mathrm{ind}(x) \leq n$ if x is a positive singular point of Σ_ξ, and $\mathrm{ind}(x) \geq n$ if x is negative. The claim therefore follows from the usual transversality argument. □

The following proposition gives a sufficient condition for convexity:

Proposition 2.3.3.

(1) A 1-Morse$^+$ hypersurface Σ is convex.

(2) A hypersurface Σ is Weinstein convex if and only if it is Morse$^+$.

Proof. (1) is a straightforward generalization of the usual proof for surfaces due to Giroux that Σ is convex if it has a Morse$^+$ characteristic foliation.

Let $\mathbf{x} = \{x_1, \dots, x_m\}$ (resp. $\mathbf{y} = \{y_1, \dots, y_\ell\}$) be the positive (resp. negative) singular points of Σ_ξ. Then $d\beta$ is nondegenerate on a small open neighborhood $U(\mathbf{x})$ of $\mathbf{x}$, where $\beta := \alpha|_\Sigma$. Let W_{x_i} be the stable manifold of x_i with respect to the gradient of the 1-Morse function and let the ith skeleton Sk_i be the closure of $W_{x_1} \cup \cdots \cup W_{x_i}$. We order the points of $\mathbf{x}$ so that $W_{x_{i+1}}$ intersects the boundary of a small open neighborhood of Sk_i along a sphere if x_{i+1} is a Morse critical point and along a disk if x_{i+1} is an embryonic point. In particular we necessarily have $\mathrm{ind}(x_1) = 0$, but we do *not* require $\mathrm{ind}(x_i) \geq \mathrm{ind}(x_j)$ for $i > j$. Such an arrangement is possible thanks to the assumption that there is no flow line of Σ_ξ going from $\mathbf{y}$ to $\mathbf{x}$.

There exists a conformal modification $\beta \rightsquigarrow g\beta$, where g is a positive function, so that it becomes Liouville on a tubular neighborhood $U(\mathrm{Sk}_m)$ of Sk_m and $\partial U(\mathrm{Sk}_m)$ is contact: Arguing by induction, suppose that β is Liouville on $U(\mathrm{Sk}_i)$ such that $\partial U(\mathrm{Sk}_i)$ is contact. During the induction we will often reset notation, i.e., modify $\beta \rightsquigarrow g\beta$ and call the result the new β. We will explain the case where x_{i+1} is Morse and $W_{x_{i+1}} \cap \partial U(\mathrm{Sk}_i)$ is a Legendrian sphere $\Lambda \subset \partial U(\mathrm{Sk}_i)$; the other cases are similar. Using the flow of Σ_ξ, we may identify a tubular neighborhood of $W_{x_{i+1}} \setminus (U(\mathrm{Sk}_i) \cup U(x_{i+1}))$ with $[0,1]_r \times Y$, where Y is an open neighborhood of the 0-section in $J^1(\Lambda)$ such that:

- $\{0\} \times Y \subset \partial U(\mathrm{Sk}_i)$;
- $\{1\} \times Y \subset \partial U(x_{i+1})$; and
- ∂_r is identified with Σ_ξ.

It follows that one can write $\beta = g\lambda$ on $[0,1] \times Y$, where λ is a contact form on Y and g is a positive function on $[0,1] \times Y$. Note that

$$d\beta = \partial_r g dr \wedge \lambda + d_Y g \wedge \lambda + g d\lambda$$

is symplectic if $\partial_r g > 0$. By assumption we have $\partial_r g > 0$ when r is close to 0 or 1. Rescaling $\beta|_{U(x_{i+1})}$ by a large constant $K \gg 0$, we can extend $\beta|_{U(\mathrm{Sk}_i) \cup U(x_{i+1})}$ to a Liouville form on $U(\mathrm{Sk}_{i+1})$. Moreover, we can assume $\partial U(\mathrm{Sk}_{i+1})$ is transverse to Σ_ξ by slightly shrinking $U(\mathrm{Sk}_{i+1})$. Hence by induction we can arrange so that β is a Liouville form on $U(\mathrm{Sk}_m)$.

The treatment of the negative singular points of Σ_ξ is similar. Let Sk'_ℓ be the closure of the union of the unstable manifolds of $\mathbf{y}$. Then by the same argument we can assume that β is a Liouville form on $-U(\mathrm{Sk}'_\ell)$, where the minus sign indicates the opposite orientation.

Using the flow of Σ_ξ, we can identify $\Sigma \setminus (U(\mathrm{Sk}_m) \cup U(\mathrm{Sk}'_\ell))$ with $\Gamma \times [-1,1]_s$ such that:

- $\Gamma \times \{-1\}$ is identified with $\partial U(\mathrm{Sk}_m)$;
- $\Gamma \times \{1\}$ is identified with $\partial U(\mathrm{Sk}'_\ell)$; and
- $\mathbb{R}\langle \partial_s \rangle = \Sigma_\xi$.

We can write $\beta = h\eta$ near $\Gamma \times \{-1,1\}$, where η is a contact form on Γ and $h = h(s)$ is a positive function such that $h'(s) > 0$ near $\Gamma \times \{-1\}$ and $h'(s) < 0$ near $\Gamma \times \{1\}$. Extend h to a positive function $\Gamma \times [-1,1] \to \mathbb{R}$ such that $h'(s) > 0$ for $s < 0$, $h'(0) = 0$, and $h'(s) < 0$ for $s > 0$. Let $f = f(s) : \Gamma \times [-1,1] \to \mathbb{R}$ be a nonincreasing function of s such that $f(-1) = 1$, $f(0) = 0$, $f(1) = -1$, $f'(0) < 0$, and $f^{(n)}(-1) = 0 = f^{(n)}(1)$ for all $n \geq 1$. Then define $\rho = f dt + h\eta$ on $\mathbb{R}_t \times \Gamma \times [-1,1]$, $\rho = dt + \beta$ on $\mathbb{R} \times U(\mathrm{Sk}_m)$, and $\rho = -dt + \beta$ on $\mathbb{R} \times U(\mathrm{Sk}'_{\mathbf{y}_\ell})$. We leave it to the reader to check that ρ is contact and that $\rho|_{\{0\} \times \Sigma}$ agrees with $\alpha|_\Sigma$ up to an overall positive function. (1) now follows from Lemma 2.1.3.

(2) The "if" direction follows from the proof of (1) and the "only if" direction is clear. $\square$

3. Construction of mushrooms in dimension 3

In order to make a hypersurface $\Sigma \subset (M, \xi)$ Weinstein convex, we would like to modify the characteristic foliation Σ_ξ so it is directed by a Morse vector field and then apply Proposition 2.3.3. (Note that going from Morse to Morse$^+$ is a C^∞-generic operation which can always be done by Lemma 2.3.2.) This will be achieved by certain C^0-small perturbations of Σ which we call *mushrooms*. The mushrooms are most easily described in dimension 3 and the general case will be constructed in Section 5 using 3-dimensional mushrooms. It turns out that mushrooms alone are enough to make any Σ_ξ Morse if $\dim \Sigma = 2$. If $\dim \Sigma > 2$, then mushrooms are not quite sufficient and we will need an additional technical construction in Section 7.

The standard model of a mushroom will be constructed in a Darboux chart

$$(\mathbb{R}^3_{z,s,t}, \xi = \ker \alpha), \quad \alpha = dz + e^s dt.$$

Let $\Sigma = \{z = 0\}$ be the surface under consideration with normal orientation ∂_z and characteristic foliation Σ_ξ directed by ∂_s. The goal of this section is to "fold" Σ to obtain another surface Z which coincides with Σ outside of a compact set, and analyze the change in the dynamics of the characteristic foliations.

In §3.1 we construct a piecewise linear (PL) model Z_{PL} and then in §3.2 we round the corners of Z_{PL} to obtain a suitably generic smooth surface Z such that the characteristic foliation Z_ξ has the desired properties.

Remark 3.0.1. In an earlier version of the paper we constructed a mushroom whose base was smaller than the cap and discussed "mushroom packing ratios". In the current version they approximately have the same size and can be packed tightly.

3.1. Construction of Z_{PL}.

Choose a rectangle $\square = [0, s_0] \times [0, t_0] \subset \Sigma$, where $s_0, t_0 > 0$. We define Z_{PL} to coincide with Σ outside of $\square$.

Remark 3.1.1. The more general case $[s_{-1}, s_0] \times [t_{-1}, t_0]$ can be computed analogously. In what follows we can replace $1 - e^{-s_0}$ by $e^{-s_{-1}} - e^{-s_0}$ and the s-width $\mathscr{S}(Z_{PL})$ becomes $s_0 - s_{-1}$.

Choose $z_0 > 0$. We construct three rectangles P_0, P_2, P_4 and two parallelograms P_1, P_3 in $\mathbb{R}^3$, which, together with $\mathbb{R}^2_{s,t} \setminus \square$, glue to give Z_{PL}, i.e., we define

- $P_0 := [0, s_0] \times [-e^{-s_0/2} z_0, -e^{-s_0/2} z_0 + t_0] \subset \{z = z_0\}$;
- $P_i, i = 1, \dots, 4$, are the faces ($\neq P_0, \square$) of the convex hull of $P_0 \cup \square$ (which is a parallelepiped), ordered counterclockwise so that $P_1 \subset \{s = 0\}$.

Definition 3.1.2. *We define the PL surface*

$$Z_{\mathrm{PL}} := (\Sigma \setminus \square) \cup (\cup_{0 \le i \le 4} P_i).$$

The rectangle $\square \subset \Sigma$ (resp. P_0) is called the base (resp. cap) of the PL mushroom *and $\cup_{0 \le i \le 4} P_i$ is called the* PL mushroom. *We refer to the modification $\Sigma \rightsquigarrow Z_{\mathrm{PL}}$ as "growing a PL mushroom".*

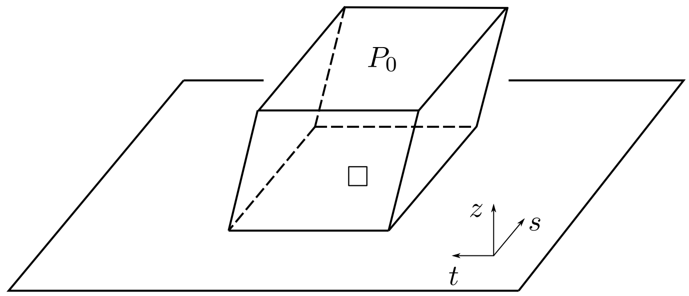

FIGURE 3.1.1. The PL model Z_{PL}. Here P_0 is the top face; P_1 and P_3 are the front and back faces, respectively; P_2 and P_4 are the right and left faces, respectively; and $\square$ is the bottom face which is *not* part of Z_{PL}.

Observe that, away from the corners, the characteristic foliation $(Z_{\mathrm{PL}})_\xi$ on Z_{PL} satisfies

- $(Z_{\mathrm{PL}})_\xi = \mathbb{R}\langle\partial_s\rangle$ on $\Sigma \setminus \square$ and P_0;
- on P_2 (resp. P_4), $(Z_{\mathrm{PL}})_\xi$ is directed by ∂_s (resp. $-\partial_s$) for $s \in [0, \frac{s_0}{2})$, is singular along $s = \frac{s_0}{2}$, and is directed by $-\partial_s$ (resp. ∂_s) for $s \in (\frac{s_0}{2}, s_0]$;
- $(Z_{\mathrm{PL}})_\xi$ is the linear foliation on P_1 and P_3 with "slopes" -1 and $-e^{-s_0}$, respectively, where "slope" refers to the value of $dt/dz = -e^{-s}$. See Figure 3.1.2.

We refer to the singular line segments on P_2 and P_4 by $\mathscr{S}_-$ and $\mathscr{S}_+$ indicating their signs.

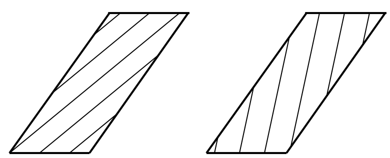

FIGURE 3.1.2. The linear characteristic foliations on P_1 (left) and P_3 (right) where Z_{PL} is sufficiently thin.

We now analyze the dynamics of the PL flow on Z_{PL}. Note that the flow lines are not necessarily uniquely determined by the initial conditions due to the presence of corners.

We begin by introducing a few quantities which characterize the various sizes of the mushrooms.

Definition 3.1.3. *Given* Z_{PL} *as above, its* z-height, s-width, *and* t-width *are given by:*

$$\mathscr{Z}(Z_{\mathrm{PL}}) := z_0, \quad \mathscr{S}(Z_{\mathrm{PL}}) := s_0, \quad \mathscr{T}(Z_{\mathrm{PL}}) := t_0.$$

The following lemma characterizes a key feature of $(Z_{\mathrm{PL}})_\xi$ when the parameters of the mushrooms are appropriately adjusted.

Lemma 3.1.4. *Fix* $s_0, z_0 > 0$. *If* $t_0 < (1 - e^{-s_0})z_0$, *then*

(1) the unique flow line of $(Z_{\mathrm{PL}})_\xi$ passing through $(-1,a) \in \mathbb{R}^2_{s,t}$, where $a \in (0,t_0)$, either hits $P_0 \cap P_2$ or converges to $\mathscr{S}_-$ in forward time;
(2) the unique flow line of $(Z_{\mathrm{PL}})_\xi$ passing through $(s_0+1,a), a \in (0,t_0)$, either hits $P_0 \cap P_4$ or converges to $\mathscr{S}_+$ in backward time; and
(3) all the flow lines of $(Z_{\mathrm{PL}})_\xi$ passing through $(-1,a)$, $a \notin [0,t_0]$, or (s_0+1,a), $a \notin [0,t_0]$, are unaffected.

Proof. (1) Since $t_0 - z_0(1-e^{-s_0}) < 0$, it follows that the unique flow line passing through the point $(-1,a) \in \Sigma$ with $a \in (0,t_0)$ does one of three things in forward time:

- travels over P_0, enters P_1, and ends at $\mathscr{S}_-$;
- travels over P_0 into $P_0 \cap P_2$ (this happens with only one flow line); or
- travels over P_0, P_1, P_2 in that order, enters P_1, and ends at $\mathscr{S}_-$.

See Figure 3.1.3. (2) is similar and (3) is clear. □

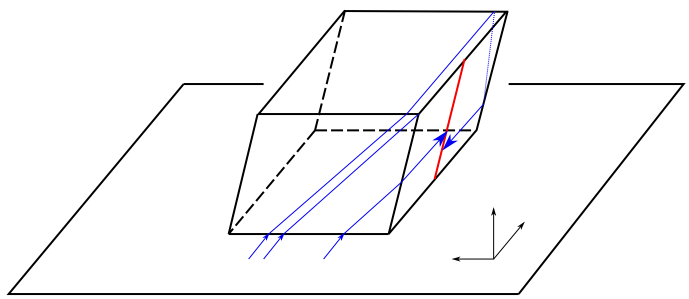

FIGURE 3.1.3. Flow lines of $(Z_{\mathrm{PL}})_\xi$ limiting to $\mathscr{S}_-$ in red (the generic case) and one flow line limiting to $P_0 \cap P_2$.

3.2. **Smoothing of Z_{PL}.** In this subsection we construct a smoothing of Z_{PL}.

Construction of the smoothing Z. Choose a *small smoothing parameter* $\delta > 0$ and a smooth "profile function" $\phi : [0,z_0] \to [-\delta,\delta]$ such that $\phi(0)=\delta$, $\phi(z_0) = -\delta$ and has "derivative $-\infty$" at $z = 0, z_0$.

For each $z' \in (0,z_0)$, the slices $R_{z'} := Z_{\mathrm{PL}} \cap \{z = z'\}$ are rectangles. When $z' = 0$ or z_0, we take $R_{z'} = \partial(Z_{\mathrm{PL}} \cap \{z = z'\})$. For each $\delta' \in \mathbb{R}$ with $|\delta'|$ small, let $R^{\delta'}_{z'} \subset \{z = z'\}$ be the rectangle concentric to $R_{z'}$ and whose side lengths are δ' larger. Then let $\widetilde{R}^{\delta'}_{z'} \subset \{z = z'\}$ be a smoothing of $R^{\delta'}_{z'}$ contained in the closure of the region between $R^{\delta'}_{z'}$ and $R^{\delta'-\delta}_{z'}$, which:

(R1) agrees with $R^{\delta'}_{z'}$ on the complement of the open δ-neighborhoods of the edges of $R^{\delta'}_{z'}$ parallel to $t = const$;
(R2) has nonzero curvature on these δ-neighborhoods and is tangent to $R^{\delta'}_{z'}$ precisely at the midpoints of the edges of $R^{\delta'}_{z'}$ (i.e., when $s = s_0/2$); and
(R3) is smoothly varying with z' and δ'.

The *ϕ-smoothing Z of Z_{PL} with smoothing parameter $\delta > 0$ and profile function* ϕ is obtained by modifying Z_{PL} as follows:

(1) replace $R_{z'}$ by $\widetilde{R}_{z'}^{\phi(z')}$ for $z \in (0, z_0)$;
(2) remove the bounded component of $\{z = 0\} \setminus \widetilde{R}_0^{\phi(0)}$; and
(3) adjoin the closure of the bounded component of $\{z = z_0\} \setminus \widetilde{R}_{z_0}^{\phi(z_0)}$.

The *base of the mushroom* $\widetilde{\square} \subset \Sigma$ of Z is the closure of $\Sigma \setminus Z$ and the *mushroom* is the closure of $Z \setminus \Sigma$. By construction $\widetilde{\square}$ converges to $\square$ when all the parameters tend to zero.

The following proposition describes the key dynamical properties of Z_ξ.

Proposition 3.2.1. *Given Z_{PL} with parameters s_0, z_0, t_0, ϵ satisfying $t_0 < (1 - e^{-s_0})z_0$, there exists a smoothing Z of Z_{PL} with small smoothing parameter $\delta > 0$ and profile function $\phi : [0, z_0] \to [-\delta, \delta]$ whose vector field Z_ξ satisfies the following properties:*

(TZ0) Z_ξ is gradient-like with respect to a Morse function $f_a : \widetilde{Z} \to \mathbb{R}$ which agrees with s outside of $\widetilde{\square}$.
(TZ1) Z_ξ has four nondegenerate singularities: a positive source e_+ and a positive saddle h_+ near the midpoint of $\mathbb{R}^2_{s,t} \cap P_4$, and a negative sink e_- and a negative saddle h_- near the midpoint of $\mathbb{R}^2_{s,t} \cap P_2$.
(TZ2) There is a unique flow line each from e_+ to h_+, from e_+ to h_-, from h_+ to e_-, and from h_- to e_-, and the four flow lines bound a quadrilateral whose interior consists of flow lines from e_+ to e_-.
(TZ3) There exist $\kappa_1 > \kappa_2 > \kappa_3 > 0 > \kappa_4 > \kappa_5 > \kappa_6$ such that all $\kappa_i \to 0$ as $\delta \to 0$ and the following hold:
  (1) the stable manifold of h_+ (resp. h_-) intersects the line $\{s = -1\}$ at $(-1, t_0 + \kappa_2)$ (resp. $(-1, \kappa_4)$),
  (2) the unstable manifold of h_+ (resp. h_-) intersects the line $\{s = s_0+1\}$ at $(s_0 + 1, t_0 + \kappa_3)$ (resp. $(s_0 + 1, \kappa_5)$),
  (3) any flow line passing through $(-1, a)$, $a \in (\kappa_4, t_0 + \kappa_2)$, converges to e_- in forward time,
  (4) any flow line passing through $(s_0 + 1, a)$, $a \in (\kappa_5, t_0 + \kappa_3)$, converges to e_+ in backward time,
  (5) a flow line passes through $(-1, a)$, $a \notin [\kappa_6, t_0 + \kappa_1]$, if and only if it passes through $(s_0 + 1, a)$, $a \notin [\kappa_6, t_0 + \kappa_1]$,
  (6) a flow line passes through $(-1, a)$, $a \in (t_0 + \kappa_2, t_0 + \kappa_1)$, if and only if it passes through $(s_0 + 1, a')$, $a' \in (t_0 + \kappa_3, t_0 + \kappa_1)$,
  (7) a flow line passes through $(-1, a)$, $a \in (\kappa_6, \kappa_4)$ if and only if it passes through $(s_0 + 1, a')$, $a' \in (\kappa_6, \kappa_5)$.
(TZ4) The flow lines described in (TZ2) and (TZ3) are all the flow lines that nontrivially intersect the mushroom.

In words, Z_ξ blocks all flow lines that pass through an open interval that is close to $\{-1\} \times (0, t_0)$, slightly bends unblocked flow lines that pass through points close to $(-1, 0)$ and $(-1, t_0)$ (with bending $\to 0$ as the smoothing parameter $\delta \to 0$), and leaves all other flow lines passing through $s = -1$ untouched.

See Figure 3.2.1 for an illustration of the effect of a mushroom on the characteristic foliation and also the values of t where certain flow lines in (TZ3) intersect $s = -1$ or $s = s_0 + 1$.

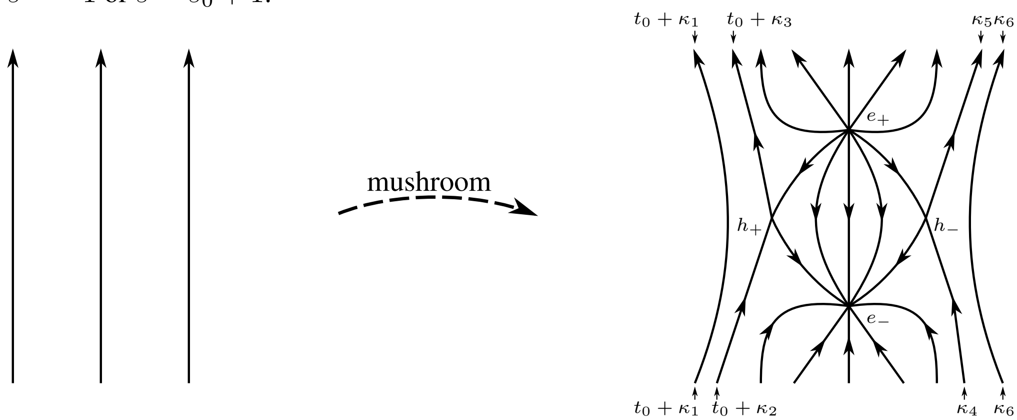

FIGURE 3.2.1. The characteristic foliations before and after a mushroom. The numbers $t_0 + \kappa_1$ etc. are the t-coordinates where the indicated flow lines intersect $s = -1$ or $s = s_0 + 1$; see (TZ3).

Proof. Let Z be the smoothing of Z_{PL} from above.

(TZ1) We determine the singular points of Z_ξ as follows: The first requirement for ξ to be tangent to Z is for Z to be tangent to ∂_s. Since $\widetilde{R}^{\delta'}_{z'}$ is tangent to ∂_s exactly at two points by (R2), i.e., when $s = s_0/2$, the singular points lie on the restriction of Z to the slice $s = s_0/2$.

We now choose ϕ such that $\phi' < 0$ on $(0, c_0)$ and (c_1, z_0) and $\phi' > 0$ on (c_0, c_1), where $0 < c_0 < c_1 \ll z_0$. By the choice of ϕ, there are four points where $Z \cap \{s = s_0/2\}$ is tangent to $\xi \cap \{s = s_0/2\}$. They occur as described in (TZ1) since $\phi' = 0$ at $z = c_0, c_1$, which are both close to $z = 0$.

(TZ0), (TZ2)–(TZ5) then follow from Lemma 3.1.4. The conditions $\kappa_2 > \kappa_3$ and $\kappa_4 > \kappa_5$ are required since the surface Z was obtained from Σ by pushing in the positive z-direction. □

The following remark will also be very useful later:

Remark 3.2.2. Proposition 3.2.1 also holds with (TZ1) replaced by:

(TZ1') Z_ξ has two singularities: a positive birth-death singularity near the midpoint of $\mathbb{R}^2_{s,t} \cap P_4$ and a negative birth-death singularity near the midpoint of $\mathbb{R}^2_{s,t} \cap P_2$.

Moreover, there is a foliated 1-parameter family of surfaces from $\{z = 0\}$ to a slight upward translate of Z (i.e., in the z-direction) whose characteristic foliations have no singularities except for Z. For this Z we take ϕ such that $\phi' < 0$ for $z \neq c_0$ and $\phi'(c_0) = 0$. Note that the singularities vanish if we take ϕ such that $\phi' < 0$ for all z.

Convention 3.2.3. *In view of Proposition 3.2.1, from now on we assume that all mushrooms satisfy* $t_0 < (1 - e^{-s_0})z_0$.

4. Convex surface theory revisited

The goal of this section is to give elementary, Morse-theoretic proofs of Theorems 1.2.3 and 1.2.5 in dimension 3 using the folding techniques developed in Section 3. In dimension 3, Theorem 1.2.3 was proved by Giroux in [Gir91] in a stronger form where C^0 is replaced by C^∞. Theorem 1.2.5 can be inferred from Giroux's work on bifurcations [Gir00] and the bypass-bifurcation correspondence. The technical heart of Giroux's work is based on the study of dynamical systems of vector fields on surfaces, a.k.a., Poincaré-Bendixson theory. In particular, one invokes a deep theorem of Peixoto [Pei62] to prove the C^∞-version of Theorem 1.2.3 and much more work to establish Theorem 1.2.5.

Our proof strategy is the following: First apply a C^∞-small perturbation of $\Sigma \subset (M^3, \xi)$ such that the singularities of Σ_ξ become Morse. There exists a finite collection of pairwise disjoint transverse arcs $\gamma_i, i \in I$, in Σ such that any flow line of Σ_ξ passes through some γ_i. In §4.1 we will construct a 3-dimensional plug supported on a small flow box $B_i = [0, s_i] \times [0, t_i]$, where $\gamma_i = \{\frac{s_i}{2}\} \times (\epsilon, t_i - \epsilon)$ and $\epsilon > 0$ is small, such that no flow line that enters through $\{0\} \times (\epsilon, t_i - \epsilon)$ can leave through $\{s_0\} \times [0, t_i]$, i.e., they all necessarily converge to singularities in the plug. Each plug consists of a large number of mushrooms constructed in Section 3. This proves Theorem 1.2.3. To prove Theorem 1.2.5, we slice $\Sigma \times [0, 1]$ into thin layers using $\Sigma_i := \Sigma \times \{\frac{i}{N}\}, 0 \le i \le N$, for large N such that the difference between $(\Sigma_i)_\xi$ and $(\Sigma_{i+1})_\xi$ is small. (By "small" we mean the vector fields in question are C^0-close to each other. The global dynamics of $(\Sigma_i)_\xi$ may still drastically differ from that of $(\Sigma_{i+1})_\xi$.) Within each layer we insert plugs on Σ_i as in the case of a single surface so that the isotopy from Σ_i to Σ_{i+1} is through a 1-Morse family of surfaces, i.e., $(\Sigma_t)_\xi$ is a 1-Morse family for all $\frac{i}{N} \le t \le \frac{i+1}{N}$. For technical reasons, it is desirable to eliminate the plugs created on Σ_i when we reach Σ_{i+1}, replacing them by new plugs on Σ_{i+1}, so that one can inductively run from $i = 0$ to $i = N$ and make all intermediate surfaces 1-Morse. Then the only obstructions to convexity occur at finitely many instances where the surface is 1-Morse but not 1-Morse$^+$, corresponding to bypass attachments.

This section is organized as follows: In §4.1 we describe 3-dimensional plugs and in §4.4 we explain how to "install" and "uninstall" plugs. The higher-dimensional plugs will be described in Section 7. We then use this technology to prove Theorem 1.2.3 in §4.3 and Theorem 1.2.5 in §4.5.

4.1. **3-dimensional plugs.** The construction of a plug is local. Consider $M = [-z_0, z_0] \times [0, s_0] \times [0, t_0]$ with coordinates (z, s, t), contact form $\alpha = dz + e^s dt$, and contact structure $\xi = \ker \alpha$. Here $z_0, s_0, t_0 > 0$ are arbitrary, but for most of our applications, we should think of z_0, s_0 as being much smaller than t_0. In other words, the condition $t_0 < (1 - e^{-s_0}) z_0$ in Proposition 3.2.1 will not be satisfied.

Consider the surface $B = \{0\} \times [0, s_0] \times [0, t_0]$ with $B_\xi = \mathbb{R}\langle \partial_s \rangle$. We will refer to (M, α) as a *standard contact neighborhood of B with parameters s_0, t_0, z_0*. Let $\partial_- B = \{0\} \times \{0\} \times [0, t_0]$ and $\partial_+ B = \{0\} \times \{s_0\} \times [0, t_0]$. Pick a large integer

N such that $N \equiv 3 \bmod 4$. Let $\square_{k,l} \subset B$ be boxes defined by

$$\square_{k,l} := \left[\frac{2l-1}{5}s_0, \frac{2l}{5}s_0\right] \times \left[\frac{4k+2l-1}{N}t_0, \frac{4k+2l+2}{N}t_0\right],$$

where $0 \leq k < \lfloor N/4 \rfloor, l = 1, 2$. See Figure 4.1.1. In words, since the mushrooms can be packed "tightly", it suffices to arrange two rows of mushrooms.

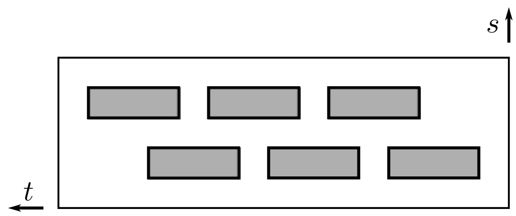

FIGURE 4.1.1. Bases of the mushrooms on B. Here $N = 15$.

Applying Proposition 3.2.1, we install pairwise disjoint mushrooms $Z_{k,l}$ on B such that the base of each $Z_{k,l}$ approximately equals $\square_{k,l}$. The key property of the dynamics of the resulting $B_\xi^\vee$ is as follows: Suppose $\epsilon > 0$ is small and we choose $N \gg 1/\epsilon$. Let $x = (z(x), s(x), t(x)) \in \partial_- B$.

(1) If $t(x) \in (\epsilon, t_0 - \epsilon)$, then the maximal possibly broken flow line of $B_\xi^\vee$ passing through x converges to a sink in forward time.
(2) If a flow line passes through x and does not converge to sink, then it exits along $y \in \partial_+ B$ and $|t(y) - t(x)| < \epsilon$.

4.2. **Barricades.** *In this subsection only, let Σ be a manifold of dimension m and v a vector field on Σ with only Morse singularities.* We also fix a Riemannian metric on Σ.

Definition 4.2.1. *A* flow box *is an embedded cylinder $B = [0, s_0] \times D^{m-1} \subset \Sigma$ with coordinates (s, x) over the disk D^{m-1} such that $v|_B = \partial_s$.*

Given $\epsilon > 0$ small, let $B^\epsilon = [\epsilon, s_0 - \epsilon] \times D_\epsilon^{m-1}$ be a slightly smaller flow box (called a *shrinkage of* B), where $D_\epsilon^{m-1} \subset D^{m-1}$ is a disk such that every point $x \in D^{m-1} \setminus D_\epsilon^{m-1}$ has metric distance $< \epsilon$ from $\partial D_\epsilon^{m-1}$ and ∂D^{m-1}.

Definition 4.2.2 (Barricade)**.** *A collection $B_I = \{B_i = [0, s_i] \times D^{m-1}\}_{i \in I}$ of pairwise disjoint flow boxes for Σ is a* barricade for v *if for each B_i there exist a small constant $\epsilon_i > 0$ and a shrinkage $B_i^{\epsilon_i}$ and the following hold:*

(*) each flow line of v intersects some B_i^ϵ and for any $x \in \Sigma$ which neither is a singularity of v nor is contained in any B_i^ϵ, the flow line of v passing through x enters some B_i^ϵ or limits to some Morse singularity in forward time (resp. in backward time).
(**) B_I is locally finite, i.e., each compact subset of Σ intersects only a finite number of B_i.

The following theorem of Wilson [Wil66, Theorem A], slightly adapted to our situation, guarantees the existence of barricades:

Theorem 4.2.3 (Wilson)**.** *A vector field v on a manifold Σ with only Morse singularities has a barricade B_I.*

The following lemma on taking refinements is immediate:

Lemma 4.2.4 (Refinement)**.** *Given a flow box $B = [0, s_0] \times D^{m-1}$, a properly embedded submanifold Z of D^{m-1}, and $\epsilon, \delta > 0$ small, there exists a finite disjoint collection $\mathscr{B}$ of flow boxes B_{1j}, $j = 1, \dots, k_1$, and B_{2j}, $j = 1, \dots, k_2$, such that:*

(1) $B_{1j} \subset (0, s_0/2) \times D^{m-1}$ and $B_{2j} \subset (s_0/2, s_0) \times D^{m-1}$;
(2) $\pi_{D^{m-1}}(\cup_{j=1}^{k_1} B_{1j})$ does not intersect Z and $\pi_{D^{m-1}}(\cup_{j=1}^{k_2} B_{2j})$ is contained in a δ-neighborhood of Z; and
(3) $\mathscr{B}$ is a barricade for the shrinkage B^ϵ.

Here $\pi_{D^{m-1}} : B \to D^{m-1}$ is the projection onto the second factor.

4.3. **Proof of Theorem 1.2.3 in dimension 3.** Given any closed surface $\Sigma \subset (M, \xi)$, it is well-known (see e.g. [Gei08, Section 4.6]) that, after a C^∞-small perturbation, we can assume that Σ_ξ has only Morse type singularities. By Theorem 4.2.3, a barricade $B_I = \{B_i = [0, s_i] \times [0, t_i]\}_{i \in I}$ exists for Σ_ξ; moreover I can be taken to be finite since Σ is closed. Each B_i has a standard contact neighborhood with parameters s_i, t_i, z_i, where $z_i > 0$ is small and we construct a C^0-small modification $\Sigma^\vee$ of Σ by replacing every B_i by the plug $B_i^\vee$. The characteristic foliation $\Sigma_\xi^\vee$ satisfies Conditions (M1)–(M3) of Proposition 2.2.6 and is Morse.

After a further C^∞-small perturbation if necessary, $\Sigma^\vee$ can be made Morse$^+$: If there exists a "retrogradient" flow line ℓ from a negative index 1 singularity to a positive index 1 singularity, we take a flow box $B = \{0\} \times [0, s_0] \times [0, t_0] \subset \Sigma^\vee$ such that ℓ intersects B^ϵ exactly once and B has a standard contact neighborhood with parameters s_0, t_0, z_0. The retrogradient flow line can be eliminated by taking a small *nonnegative* function $h : B \to \mathbb{R}_{\geq 0}$ with support on $B^{\epsilon/2}$ and replacing B by $z = h(s, t)$; the modification has the effect of pushing the holonomy from $s = 0$ to $s = s_0$ in the negative t-direction.

Since $\Sigma^\vee$ is now Morse$^+$, it is Weinstein convex by Proposition 2.3.3.

4.4. **Installing and uninstalling plugs.** The construction of a plug $B^\vee$ was sufficient to prove Theorem 1.2.3 in dimension 3. In order to prove Theorem 1.2.5, we also need to interpolate between B and $B^\vee$ with some control of the intermediate dynamics. We now explain this procedure.

Let $(M = [-z_0, z_0] \times [0, s_0] \times [0, t_0], \xi = \ker(dz + e^s dt))$ be as before and let $B_z := \{z\} \times [0, s_0] \times [0, t_0]$. Replace $B_{z_0/2}$ by a plug $B^\vee_{z_0/2}$ such that its analogously defined z-height satisfies $\mathscr{Z}(B^\vee_{z_0/2}) \ll \frac{z_0}{2}$ and in particular $B^\vee_{z_0/2}$ is still contained in M.

For the moment consider the PL model of the plug $B^\vee_{z_0/2}$, i.e., each mushroom Z involved in the construction is replaced by the corresponding Z_{PL}. It is fairly straightforward to foliate the regions bounded between B_0 and $B^\vee_{z_0/2}$ and between $B^\vee_{z_0/2}$ and B_{z_0} by a family of PL surfaces; see Figure 4.4.1 for a schematic picture. Then one can apply the smoothing scheme from §3.2 to smooth the corners of the

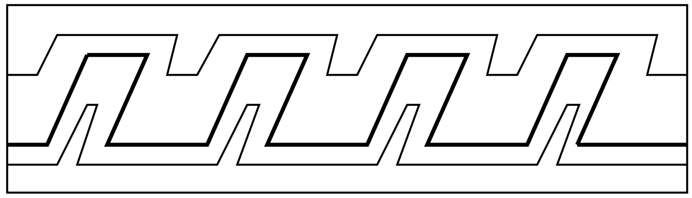

FIGURE 4.4.1. The interpolation between $B_0, B^\vee_{z_0/2}$ and B_{z_0}; the z-height of $B^\vee_{z_0/2}$ not drawn to scale.

leaves simultaneously and obtain the desired foliation $M = \cup_{0\le a\le z_0}\widetilde{B}_a$, where $\widetilde{B}_0 = B_0$, $\widetilde{B}_{z_0} = B_{z_0}$, and $\widetilde{B}_{z_0/2}$ is the smoothed version of $B^\vee_{z_0/2}$.

To analyze the dynamics of $(\widetilde{B}_a)_\xi$ for each $a \in [0, z_0]$, we introduce the partially defined and possibly multiple-valued holonomy map $\rho_a : \partial_-\widetilde{B}_a \dashrightarrow \partial_+\widetilde{B}_a$, where $\partial_+\widetilde{B}_a$ (resp. $\partial_-\widetilde{B}_a$) is the side $s = s_0$ (resp. $s = 0$) of $\widetilde{B}_a$ as before: Given $x \in \partial_-\widetilde{B}_a$, if there exists a possibly broken partial flow line of $(\widetilde{B}_a)_\xi$ starting from x and ending at $y \in \partial_+\widetilde{B}_a$, then $y \in \rho_a(x)$. Note that such y may not be unique. If there is no such flow line, then $\rho_a(x)$ is not defined.

Define the *internal discrepancy*

$$\|\rho_a\| := \sup_{x\in\partial_-\widetilde{B}_a} |t(x) - t(\rho_a(x))|,$$

where $t(x)$ refers to the t-coordinate of the point x; $|t(x) - t(\rho_a(x))| = 0$ if $\rho_a(x)$ is not defined; and the supremum is taken over all possible $\rho_a(x)$ if ρ_a is not single-valued at x.

The following lemma will be important for our applications.

Lemma 4.4.1. *The internal discrepancies* $\sup_{0\le a\le z_0}\|\rho_a\| \to 0$ *as* $N \to \infty$.

Proof. The lemma is not a statement about blocking and is rather a statement about the t-widths of the mushrooms $Z_{k,l}$: We will treat the case where $a \in [0, \frac{z_0}{2}]$. If a flow line enters a box $\square_{k,l}$ along the bottom and exits from the top, the maximum it is moved in the t-direction is the width $\frac{3t_0}{N}$ of the box. Since a flow line or broken flow line passes through at most 2 boxes, $\|\rho_a\| \le \frac{6t_0}{N}$. □

We call the foliation from $\widetilde{B}_0 = B_0$ to $\widetilde{B}_{z_0/2}$ *installing a plug* and the foliation from $\widetilde{B}_{z_0/2}$ to $\widetilde{B}_{z_0} = B_{z_0}$ *uninstalling a plug*. Then Lemma 4.4.1 basically says that neither installing nor uninstalling a plug affects the local holonomy by much. For the rest of Section 4, we assume that $N \gg 0$ without further mention.

The following is based on the construction of mushrooms in §3.2 and its slight generalization to 1-parameter families:

Lemma 4.4.2. $(\widetilde{B}_a)_\xi$, $a \in [0, z_0]$, *is gradient-like with respect to a* 1*-Morse function* $f_a : \widetilde{B}_a \to \mathbb{R}$ *which agrees with* s *on* $\partial\widetilde{B}_a$.

4.5. **Proof of Theorem 1.2.5 in dimension 3.** Consider a contact structure ξ on $\Sigma \times [0,1]$ such that $\Sigma \times \{0,1\}$ is Morse$^+$ in the sense of Definition 2.3.1. The goal is to show that up to an isotopy relative to the boundary, $((\Sigma_t)_\xi)_{t\in[0,1]}$ is a 1-Morse family, where $\Sigma_t := \Sigma \times \{t\}$.

Define $L := \cup_{t\in[0,1]}\{x \in \Sigma_t \mid \xi_x = T_x\Sigma_t\}$. Up to a C^∞-small perturbation of ξ, we can assume that L satisfies the following:

- (S1) L is a properly embedded 1-submanifold;
- (S2) the singularities of $(\Sigma_t)_\xi$ are 1-Morse for all t; and
- (S3) the restricted coordinate function $t|_L : L \to [0,1]$ is Morse and all its critical points have distinct critical values.

Suppose $0 < a_1 < \cdots < a_m < 1$ are the critical values of $t|_L$, which we assume to be irrational. For each $t \in [0,1]$ there exists a barricade B_{I^t} for Σ_t; moreover B_{I^t} is a barricade for any vector field that is sufficiently close to $(\Sigma_t)_\xi$. By the compactness of $[0,1]$, there exists an integer $K \gg 0$ such that, for $i = 0,1,\dots,K$, B_{I_i}, $I_i = I^{i/K}$, is a barricade for all Σ_t, $t \in [\frac{i-1}{K},\frac{i+1}{K}] \cap [0,1]$. Note that, for each a_j, there exist unique j_+, j_- such that $j_+ = j_- + 1$ and $\frac{j_-}{K} < a_j < \frac{j_+}{K}$.

Let $\pi : \Sigma \times [0,1] \to \Sigma$ be the projection onto the first factor.

We claim that for each $i = 0,\dots,K-1$ there are refinements of B_{I_i} and $B_{I_{i+1}}$ (by abuse of notation we keep the same notation for the refinements) such that

$$\pi(B_{I_{i+1}}) \cap \pi(B_{I_i}) = \varnothing, \tag{4.5.1}$$

and moreover we may choose the refinement so that the new $B_{I_{i+1}}$ remains a barricade for all Σ_t, $t \in [\frac{i}{K},\frac{i+2}{K}]\cap[0,1]$. The claim follows from viewing B_{I_i} and $B_{I_{i+1}}$ as thin neighborhoods of collections γ_i and γ_{i+1} of arcs, taking their intersection $Z = \gamma_i \cap \gamma_{i+1}$ which we may take to be transverse, and applying Lemma 4.2.4.

We divide the proof into several steps.

STEP 1. *From Σ_0 to $\Sigma^\vee_{1/(N'K)}$ where $N' > 0$ is a large integer.*

Consider a flow box $B_i = [0,s_0] \times [0,t_0]$ of B_{I_0}. Let $\partial_+ B_i = \{s_0\} \times [0,t_0]$ and $\partial_- B_i = \{0\} \times [0,t_0]$.

For each positive integer r, define the *external holonomy* $\widehat{\rho}_{i,r} : \partial_+ B_i \dashrightarrow \partial_- B_i$ — a multiple-valued, partially defined, rth return map from $\partial_+ B_i$ to $\partial_- B_i$ of $(\Sigma_0)_\xi$ — as follows: For any $x \in \partial_+ B_i$, a point $y \in \partial_- B_i$ is in the image $\widehat{\rho}_{i,r}(x)$ if there exists a possibly broken flow line $c : [0,1] \to \Sigma_0$ such that $c(0) = x$, $c(1) = y$, and c passes through $\operatorname{int}(B_i)$ $(r-1)$ times. Of course $\widehat{\rho}_{i,r}$ is not necessarily defined on all of $\partial_+ B_i$ and when it is defined, it is not necessarily single-valued.

Since $(\Sigma_0)_\xi$ is Morse by assumption, (A) there exists $\delta > 0$ such that

$$\|\widehat{\rho}_{i,r}\| := \inf_{x\in\partial_+ B_i} |t(x) - t(\widehat{\rho}_{i,r}(x))| > \delta,$$

where we are taking $t(\widehat{\rho}_{i,r}(x)) = \infty$ if $\widehat{\rho}_{i,r}(x)$ does not exist. Otherwise, there is a sequence of points $x_j \in \partial_+ B_i$ such that $|t(x_j) - t(\widehat{\rho}_{i,r}(x_j))| \to 0$ and the compactness of the sequence of broken flow lines gives us $x_\infty \in \partial_+ B_i$ such that

$|t(x_\infty) - t(\widehat{\rho}_{i,r}(x_\infty))| = 0$, which contradicts (M3) from Proposition 2.2.6. Moreover, the Morse condition implies that (B) there exists $r_0 > 0$ finite such that $\|\widehat{\rho}_{i,r_0}\| = \infty$ for all $i \in I$.

We then install plugs $B_{I_0}^\vee$ on B_{I_0} as described in §4.1 and §4.4 and obtain a foliation between Σ_0 and $\Sigma_0^\vee$. As long as we take $N \gg 0$, i.e., the individual mushrooms are very small, the internal discrepancies are $\ll \delta$ by Lemma 4.4.1. Together with (A) and (B), it follows that all the leaves of the foliation are Morse.

For convenience we assume that $\Sigma_0^\vee$ agrees with Σ_0 on the complement of B_{I_0}, and that the difference is contained in a small invariant neighborhood of B_{I_0}. Also, below we construct 1-parameter families of embedded surfaces that are disjoint away from a subset on which they all agree; the perturbation into a family of disjoint embedded surfaces is done by flowing in the transverse direction for a short time and will not be done explicitly.

In order to interpolate between $\Sigma_0^\vee$ and $\Sigma_{1/(N'K)}^\vee$ for a large integer $N' > 0$ which we take to be odd, we use B_{I_1} satisfying Eq. (4.5.1). If $N' \gg 0$, then there is a 1-parameter family of embedded surfaces $F_s \subset \Sigma \times [0, \frac{1}{N'K}]$, $s \in [0,1]$, such that $F_0 = \Sigma_0 \setminus N(B_{I_0})$, $F_1 \cap \Sigma_{1/(N'K)} \supset N(B_{I_1}) \times \{\frac{1}{N'K}\}$, $\partial F_s = \partial N(B_{I_0}) \times \{0\}$ for all $s \in [0,1]$, the interiors of F_s are disjoint, and the $(F_s)_\xi$, $s \in [0,1]$, are ϵ-close to one other so that B_{I_0} is a barricade for all $(N(B_{I_0}) \times \{0\}) \cup F_s$; in particular, no new singularities are introduced in this process. The barricading condition can be guaranteed by having chosen $N' \gg 0$. See the upper-left corner of Figure 4.5.1 for an illustration of this procedure. By the barricading condition the surfaces $(N(B_{I_0}) \times \{0\})^\vee \cup F_s$ are Morse for all $s \in [0,1]$, where $(N(B_{I_0}) \times \{0\})^\vee$ is $N(B_{I_0}) \times \{0\}$ with $B_{I_0}^\vee$ installed.

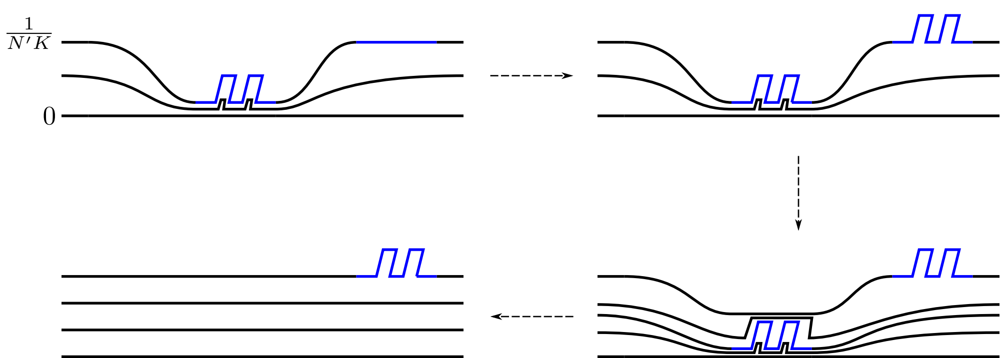

FIGURE 4.5.1. Interpolation between Σ_0 and $\Sigma_{1/(N'K)}^\vee$ by Morse surfaces. The blue parts represent B_{I_i}, $i = 0, 1$.

Next we install a plug on $B_{I_1} \times \{\frac{1}{N'K}\} \subset (N(B_{I_0}) \times \{0\})^\vee \cup F_1$, uninstall the plug on $B_{I_0}^\vee$, and lift the resulting surface up to $\Sigma_{1/(N'K)}$, as shown in the upper-right, lower-right, and lower-left corners of Figure 4.5.1, respectively. Moreover all the intermediate surfaces are Morse by analogous reasons. This finishes our construction of the foliation from Σ_0 to $\Sigma_{1/(N'K)}^\vee$.

STEP 2. *From* $\Sigma^\vee_{1/(N'K)}$ *to* $\Sigma^\vee_{1_-/K}$, *where* $1_-/K < a_1 < 1_+/K$.

Switching back and forth between B_{I_0} and B_{I_1}, we similarly construct the Morse foliation from $\Sigma^\vee_{1/(N'K)}$ to $\Sigma^\vee_{2/(N'K)}$, from $\Sigma^\vee_{2/(N'K)}$ to $\Sigma^\vee_{3/(N'K)}$, and so on as in Step 1, until we get to $\Sigma^\vee_{1/K}$. Between $\Sigma^\vee_{1/K}$ and $\Sigma^\vee_{2/K}$ we use B_{I_1} and B_{I_2}. Continuing in this manner we get to $\Sigma^\vee_{1_-/K}$.

STEP 3. *From* $\Sigma^\vee_{1_-/K}$ *to* $\Sigma^\vee_{1_+/K}$.

The only modification needed in this step is due to the fact that the vector fields $(\Sigma_{1_-/K})_\xi$ and $(\Sigma_{1_+/K})_\xi$ are not C^∞-close to each other in the usual sense. Rather, one observes either the birth or the death of a pair of nearby Morse singularities as we go from $(\Sigma_{1_-/K})_\xi$ to $(\Sigma_{1_+/K})_\xi$. In either case, we slightly modify the notion of barricades $B_{I_{1_\pm}}$ so that the unique (short) flow line connecting the pair of Morse singularities is the only flow line that does not pass through $B_{I_{1_\pm}}$. Similar remarks apply to all $a_i, 1 \leq i \leq m$.

STEP 4. *From* $\Sigma^\vee_{(K-1)/K}$ *to* Σ_1.

In this final step, the only new ingredient is to uninstall the plugs as we go from $\Sigma^\vee_1$ to Σ_1. By assumption Σ_1 is Morse and in fact convex. Hence by the same holonomy bound as in Step 1, all the intermediate surfaces are Morse.

Finally we have foliated $\Sigma \times [0,1]$ by surfaces of the form Σ_t which are all Morse. The only obstruction to convexity occurs when $(\Sigma_t)_\xi$ is Morse but not Morse$^+$ and this corresponds to a bypass attachment (see Proposition 8.3.2). This concludes the proof of Theorem 1.2.5 in dimension 3.

4.6. **Further remarks.** Compared to earlier groundbreaking works of Bennequin [Ben83] and Eliashberg [Eli92], convex surface theory is a more systematic framework for studying embedded surfaces in contact 3-manifolds. It is sufficiently powerful that basically all known classification results of contact structures or Legendrian knots in this dimension follow from this theory.

The only "drawback" of convex surface theory, at least in its original form [Gir91, Gir00], is that the monster of dynamical systems on surfaces is always lurking behind the story. More precisely, if one just wants to classify contact structures or Legendrian knots up to isotopy, then the problem often reduces to a combinatorial one by combining Giroux's theory with, say, the bypass approach of [Hon00]. However, if one wants to obtain higher homotopical information of the space of contact structures (say π_n for $n \geq 1$), then some serious work on higher codimensional degenerations of Morse-Smale flows seems inevitable.

As an example, in [Eli92] Eliashberg outlined the proof that the space of tight contact structures on S^3 is homotopy equivalent to S^2. This particular result is based on the study of characteristic foliations on $S^2 \subset S^3$, which is particularly simple since we never have periodic orbits. In more general contact manifolds such as T^3, one cannot necessarily rule out periodic orbits from characteristic foliations, and hence the bifurcation theory quickly becomes unwieldy (the work

[Ngo] probably comes close to the limit of what one can do). However, in light of our reinterpretation/simplification of Giroux's theory, it suffices to understand the space of Morse gradient vector fields, instead of general Morse-Smale vector fields.

We hope our techniques can be applied to future studies of homotopy types of the space of contact structures. This topic however will not be pursued any further in this paper.

5. Construction of mushrooms in dimension > 3

The goal of this section is to generalize the construction of mushrooms in dimension 3 in Section 3 to higher dimensions.

Notation. Throughout this section, we will write $Z^3 \subset \mathbb{R}^3$ for the mushroom constructed in Section 3 and write Z for the higher-dimensional mushroom to be constructed.

5.1. Introduction.

We first introduce some notation which will be used throughout this paper.

Definition 5.1.1 (Contact handlebodies and generalized contact handlebodies)**.**

(1) A contact handlebody over a Weinstein domain (X, μ) *is a contact manifold contactomorphic to*

$$([0, C]_t \times X, \ker(dt + \mu)),$$

where $C > 0$ is the thickness *of the handlebody.*

(2) A generalized contact handlebody over a Weinstein domain (X, μ) *is a contact manifold contactomorphic to*

$$\{(t, x) \mid f_0(x) < t < f_1(x)\} \subset (\mathbb{R}_t \times X, \ker(dt + \mu)),$$

where there exists a 1-parameter family $f_t : X \to \mathbb{R}$, $t \in [0, 1]$, of smooth functions such that $f_t(x) < f_{t'}(x)$ for all $t < t'$, $x \in X$ and the graphs $\{t = f_{t_0}(x)\}$ are Weinstein for all $t_0 \in [0, 1]$.

A contact handlebody is a compact contact manifold with a contact form such that all its Reeb orbits are chords of the same length and a generalized contact handlebody is one such that that all the Reeb orbits are chords but they need not have the same length.

Let (W, λ) be a complete Weinstein manifold of dimension $2n - 2 > 0$ and $\mathbb{R}_t \times W$ be the contactization of W with contact form $\beta = dt + \lambda$. Let $W^c \subset W$ be a compact subdomain such that $W = W^c \cup ([0, \infty)_\tau \times \Gamma)$, $\Gamma := \partial W^c$ is the contact boundary, and $[0, \infty)_\tau \times \Gamma$ is the positive half-symplectization of Γ. Let $\eta := \lambda|_\Gamma$ be the contact form on Γ; then $\lambda|_{[0,\infty)\times\Gamma} = e^\tau \eta$. For $\tau' > 0$ we also define

$$W^c_{\tau'} := W^c \cup ([0, \tau'] \times \Gamma). \tag{5.1.1}$$

The ambient contact manifold of a mushroom is

$$(M = \mathbb{R}^3_{z,s,t} \times W, \xi = \ker \alpha), \quad \alpha = dz + e^s \beta.$$

The hypersurface on which we construct the mushroom is $\Sigma = \{z = 0\} \subset M$ with characteristic foliation $\Sigma_\xi = \partial_s$.

Remark 5.1.2. For ease of notation, we will not distinguish between the characteristic foliation, which is an oriented singular line field, and a trivializing vector field.

A *mushroom Z with contact handlebody profile $H = ([0, t_0] \times W^c_{\tau_0}, dt + \lambda)$, $\tau_0 > 0$,* is constructed by first taking the product hypersurface $Z^3 \times W^c$, where Z^3_{PL} has base $[0, s_0] \times [0, t_0]$, and then damping out the Z^3-factor on $W^c_{\tau_0} - W^c$. Roughly speaking, the goal is to fold Σ using Z, so that the resulting characteristic foliation cannot pass through a region which approximates H.

Remark 5.1.3. One can think of the constructions in Section 3 as a special case where W is a point and $H = [0, t_0]$ is equipped with the contact form dt.

5.2. **Product hypersurface.** Recall that in Section 3 we constructed the mushroom

$$Z^3 \subset (\mathbb{R}^3_{z,s,t}, \ker(dz + e^s dt))$$

which agrees with $\mathbb{R}^2_{s,t}$ outside of a rectangle $\square = [0, s_0] \times [0, t_0]$. Let Z^3_ξ be the characteristic foliation on Z^3.

We will compute the characteristic foliation Z'_ξ on the product hypersurface $Z' := Z^3 \times W^c \subset M$.

Choose vector fields v on Z^3, defined away from the singularities of Z^3_ξ, such that $\alpha|_{Z^3}(v) = 1$ and w on W^c, defined away from the zero set of λ, such that $\lambda(w) = 1$.

Lemma 5.2.1. *Away from the zeros of $\alpha|_{Z^3}$ and λ, the characteristic foliation Z'_ξ is given by*

$$Z'_\xi = \mathbb{R}\langle Z^3_\xi + dz \wedge ds(Z^3_\xi, v) X_\lambda \rangle, \tag{5.2.1}$$

where X_λ is the Liouville vector field of λ.

Proof. One can easily check that

$$T(Z^3 \times W^c) \cap \xi = \mathbb{R}\langle Z^3_\xi, w - e^s v, \ker \lambda \rangle.$$

Basically the calculation of Z'_ξ is reduced to computing the kernel $K = aX + bY + cZ$ of the 3-dimensional vector space $\mathbb{R}\langle X, Y, Z \rangle$ with a maximally nondegenerate alternating 2-form $\langle \cdot, \cdot \rangle$. One can easily verify that

$$K = \langle Y, Z \rangle X + \langle Z, X \rangle Y + \langle X, Y \rangle Z$$

works. We have $e^{-s} d\alpha = ds \wedge dt + ds \wedge \lambda + d\lambda$, and if we write $\langle \cdot, \cdot \rangle := e^{-s} d\alpha(\cdot, \cdot)$ and $Z'_\xi = aX + bY + cZ + dA$, where $X = Z^3_\xi$, $Y = X_\lambda$, $Z = w - e^s v$, and $A \in \ker \lambda$ and is not parallel to X_λ, then $d = 0$ since otherwise there exists

$B \in \ker \lambda$ such that $\langle A, B\rangle \neq 0$ and $\langle w, B\rangle = 0$. The remaining/relevant part of the pairing is given as follows:

$$\begin{aligned}\langle Z^3_\xi, X_\lambda\rangle &= 0,\\ \langle Z^3_\xi, w - e^s v\rangle &= ds(Z^3_\xi) - e^s ds \wedge dt(Z^3_\xi, v),\\ \langle X_\lambda, w - e^s v\rangle &= 1.\end{aligned}$$

Hence $Z'_\xi = K = Z^3_\xi - (ds(Z^3_\xi) - e^s ds \wedge dt(Z^3_\xi, v))X_\lambda$.

Finally, since $\alpha - (dz + e^s dt) = 0$ when evaluated on vectors on Z^3 and hence

$$(ds \wedge \alpha + dz \wedge ds - e^s ds \wedge dt)(Z^3_\xi, v) = 0,$$

it follows that $Z'_\xi = Z^3_\xi + dz \wedge ds(Z^3_\xi, v)X_\lambda$. □

At the zeros of $\alpha|_{Z^3}$ and λ, Eq. (5.2.1) can be interpreted as saying that Z'_ξ contains the limit of the right-hand side as the points on $Z^3 \times W^c$ approach the zero.

Remark 5.2.2. Lemma 5.2.1 is rather general and holds for Z^3 replaced by any surface in $\mathbb{R}^3_{z,s,t}$.

5.3. **Dynamics of Z'_ξ.** We now investigate the dynamics of Z'_ξ.

Let us first consider the PL case $Z'_{PL} = Z^3_{PL} \times W^c$.

Lemma 5.3.1. *The flow lines of $(Z'_{PL})_\xi$ passing through $\{-1\}_s \times (0, t_0)_t \times W^c$ eventually limit to a negative singularity of $(Z'_{PL})_\xi$ and in particular do not leave Z'_{PL}.*

Proof. The lemma follows from two observations: (i) Since $dz \wedge ds((Z^3_{PL})_\xi, v)$ is positive on P_4, negative on P_2, and vanishes on $P_0 \cup P_1 \cup P_3$, the term $dz \wedge ds((Z^3_{PL})_\xi, v)X_\lambda$ in Eq. (5.2.1) is a positive multiple of X_λ on P_4, a negative multiple of X_λ on P_2, and zero on $P_0 \cup P_1 \cup P_3$. [Sample sign calculation on P_4 (it is useful to refer to Figure 3.1.2): $(Z^3_{PL})_\xi = -\partial_s$ and v, which we take to be parallel to the zt-plane, has positive ∂_z-component. Hence $dz \wedge ds((Z^3_{PL})_\xi, v) > 0$ on P_4.] (ii) By Lemma 3.1.4, if a flow line of $(Z'_{PL})_\xi$ passes through $\{-1\}_s \times (0, t_0)_t \times W^c$, then its projection to Z^3_{PL} only passes through P_0, P_1, P_2, and P_3. □

Next we describe the smoothed version Z'_ξ. We identify the singular points of Z'_ξ: Recall from Lemma 3.1.4 that Z^3_ξ has four singular points $e_\pm, h_\pm$. By the sign calculations of $dz \wedge ds((Z^3_{PL})_\xi, v)$ from the proof of Lemma 5.3.1 and continuity, $dz \wedge ds(Z^3_\xi, v) > 0$ on neighborhoods of e_+, h_+ and < 0 on neighborhoods of e_-, h_-. Hence for each singular point $x \in W^c$ of the Liouville vector field X_λ, there exist four singular points $e^x_\pm, h^x_\pm$ of Z'_ξ whose Morse indices are given by:

$$\begin{aligned}\mathrm{ind}(e^x_+) &= \mathrm{ind}_W(x), & \mathrm{ind}(h^x_+) &= \mathrm{ind}_W(x) + 1,\\ \mathrm{ind}(e^x_-) &= 2n - \mathrm{ind}_W(x), & \mathrm{ind}(h^x_-) &= 2n - 1 - \mathrm{ind}_W(x),\end{aligned}$$

where $\mathrm{ind}_W(x)$ is the Morse index of $x \in W^c \subset W$ and we recall that $\dim W = 2n - 2$.

See the top left figure in Figure 5.3.1 for Z^3_ξ and the regions indicating the signs of $dz \wedge ds(Z^3_\xi, v)$. The red (resp. blue, white) region indicates where $dz \wedge ds(Z^3_\xi, v)$ or $dz \wedge ds(S_{\tau,\xi}, v)$ is positive (resp. negative, zero).

Remark 5.3.2. In view of Remark 3.2.2, we may replace the nondegenerate singular points by birth-death singularities as in the top right of Figure 5.3.1. The advantage of the birth-death singularities is that Lemma 5.3.3 still holds but the singular points can be immediately eliminated; this will be useful for example when damping out in Section 5.4.

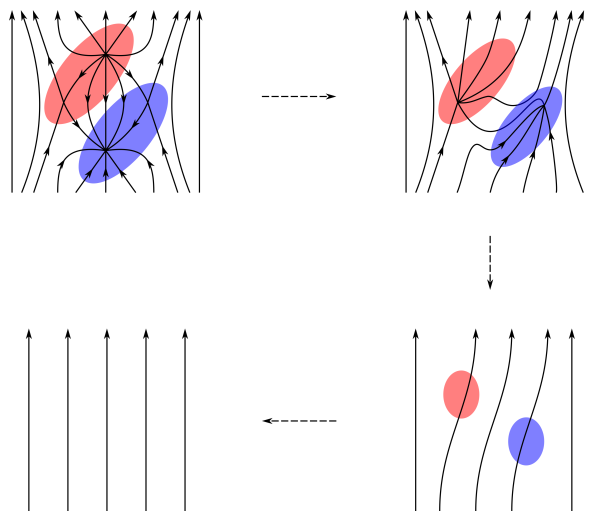

FIGURE 5.3.1. The top left is Z^3_ξ and the top right is an alternate perturbation of $Z^3_{PL,\xi}$ corresponding to $\tau = 0$. The top right, bottom right, and bottom left are $S_{\tau,\xi}$ for some as τ goes from 0 to τ_0.

Let $\mathrm{Sk}(W)$ be the isotropic skeleton of W^c with respect to X_λ. Let $\kappa_1 > \kappa_2 > \kappa_3 > 0 > \kappa_4 > \kappa_5 > \kappa_6$ with all κ_i small as in Proposition 3.2.1, and let $a \geq 0$ be small. We define

$$I^-_a := \{-1\} \times [\kappa_4, t_0 + \kappa_2 - a] \subset \mathbb{R}^2_{s,t},$$
$$I^+_a := \{s_0 + 1\} \times [\kappa_5 - a, t_0 + \kappa_3] \subset \mathbb{R}^2_{s,t},$$

so that I^-_0 (resp. I^+_0) is the maximal interval with the property that any flow line of Z^3_ξ passing through the interval converges to a singularity of Z^3_ξ in forward (resp. backward) time.

We now give a description of all the flow lines passing through $Z^3 \times W^c$:

Lemma 5.3.3 (Description of all flow lines passing through $Z^3 \times W^c$)**.** *There exist functions $\sigma^-_0, \sigma^+_0 : W^c \to \mathbb{R}_{\geq 0}$, which vanish exactly on* $\mathrm{Sk}(W)$ *such that:*

(1) *each flow line of Z'_ξ passing through $I_0^- \times \mathrm{Sk}(W)$ (resp. $I_0^+ \times \mathrm{Sk}(W)$) converges to a singularity of Z'_ξ in forward (resp. backward) time;*
(2) *for $x \in W^c \setminus \mathrm{Sk}(W)$, each flow line passing through $I^-_{\sigma_0^-(x)} \times \{x\}$ (resp. $I^+_{\sigma_0^+(x)} \times \{x\}$) converges to a singularity of Z'_ξ in forward (resp. backward) time;*
(3) *for $x \in W^c \setminus \mathrm{Sk}(W)$, each flow line passing through $(I_0^- \setminus I^-_{\sigma_0^-(x)}) \times \{x\}$ (resp. $(I_0^+ \setminus I^+_{\sigma_0^+(x)}) \times \{x\}$) exits $Z^3 \times W^c$ along $Z^3 \times \partial W^c$ in finite forward (resp. backward) time;*
(4) *for $x \in W^c$, each flow line passing through $\{-1\} \times (t_0+\kappa_2, t_0+\kappa_1] \times \{x\}$ (resp. $\{s_0+1\} \times [\kappa_6, \kappa_5) \times \{x\}$) exits from either $\{s_0+1\} \times (t_0+\kappa_3, t_0+\kappa_1] \times W^c$ (resp. $\{-1\} \times [\kappa_6, \kappa_4) \times W^c$) or $Z^3 \times \partial W^c$ in finite forward (resp. backward) time; in the former case, the W^c-coordinate x' of the exit point of the flow line is on the time ≥ 0 flow line of X_λ starting at x;*
(5) *for $x \in W^c$, each flow line passing through $\{-1\} \times [\kappa_6, \kappa_4) \times \{x\}$ (resp. $\{s_0+1\} \times (t_0+\kappa_3, t_0+\kappa_1] \times \{x\}$) exits from $\{s_0+1\} \times [\kappa_6, \kappa_5) \times W^c$ (resp. $\{-1\} \times (t_0+\kappa_2, t_0+\kappa_1] \times W^c$) in finite forward (resp. backward) time; the W^c-coordinate x' of the exit point of the flow line is on the time ≤ 0 flow line of X_λ starting at x;*
(6) *each flow line outside of $\mathbb{R}_s \times [\kappa_6, t_0+\kappa_1] \times W^c$ has trivial holonomy;*
(7) *all other flow lines are (i) flow lines between singularities, (ii) flow lines from a singularity to $Z^3 \times \partial W^c$, or (iii) flow lines from $Z^3 \times \partial W^c$ to a singularity.*

Moreover, as $Z^3 \to Z^3_{PL}$, all $\kappa_i \to 0$ and $|\sigma_0^\pm|_{C^0} \to 0$.

Proof. This is an immediate consequence of Lemmas 5.2.1 and 5.3.1, taking the limit $Z^3 \to Z^3_{PL}$, and a case-by-case analysis of the various regions of the top left figure of Figure 5.3.1.

Suppose the flow line passes through the red region times W^c. Then either the flow line exits from $Z^3 \times \partial W^c$ or escapes to the white region times W^c. Once in the white region, the flow line either reaches $s = s_0 + 1$ or enters the blue region times W^c and reaches a negative singularity.

Suppose the flow line passes through the white region (e.g., passes through $s = -1$). Then the flow line reaches $s = s_0 + 1$, enters the blue region times W^c (and hence reaches a negative singularity), or enters the red region times W^c (already considered).

All $\kappa_i \to 0$ and $|\sigma_0^\pm|_{C^0} \to 0$ as $Z^3 \to Z^3_{PL}$ by construction. □

Technically, the functions $\sigma_0^\pm$ account for the speed of convergence of flow lines of $(Z^3)_\xi$ towards its singularities and those of X_λ in W^c towards $\mathrm{Sk}(W)$.

5.4. **Damping.** In order for the mushroom to be the image of a continuous map $\Sigma \to M$, one must damp out the Z^3-fiber over $W^c_{\tau_0} \setminus \mathrm{int}(W^c) = [0, \tau_0] \times \Gamma$ as τ grows. Recall the notation from Section 5.1.

The damping procedure amounts to choosing an isotopy of surfaces S_τ, $\tau \in [0, \tau_0]$, in $\mathbb{R}^3_{z,s,t}$ from $S_0 = Z^3$ to the flat $S_{\tau_0} = \mathbb{R}^2_{s,t}$. We take $S_0 = Z^3$ to have birth-death type singular points as in Figure 5.3.1; see Remark 5.3.2. In practice we also take $\tau_0 > 0$ to be arbitrarily small. We then set

$$\mathscr{I}_0 := \cup_{0\leq\tau\leq\tau_0}(S_\tau \times \{\tau\}) \subset \mathbb{R}^4_{z,s,t,\tau}$$

and the actual hypersurface in M will be $\mathscr{I}_0 \times \Gamma$.

The PL model of S_τ is obtained by replacing P_0 by the rectangle $[0, s_0] \times [-e^{-s_0/2}z, -e^{-s_0/2}z + t_0]$ as in Section 3.1, where the parameter z ranges from z_0 to 0 as τ goes from 0 to τ_0 (in other words, we are pushing the top face of the parallelepiped into the parallelepiped); its smoothing for $\tau > 0$ will use a profile function ϕ such that $\phi' < 0$ everywhere so that there are no singularities of the characteristic foliation. We have

$$T\mathscr{I}_0 = \mathbb{R}\langle TS_\tau, \partial_\tau + fw_0\rangle,$$

where $w_0 = \partial_z - K_0\partial_t$ is parallel to P_2 and P_4 and $f \leq 0$ is a τ-dependent smooth function on $\mathbb{R}^3_{z,s,t}$ which vanishes when τ is close to $\{0, \tau_0\}$ or $z = 0$.

We are now ready to compute the characteristic foliation $(\mathscr{I}_0 \times \Gamma)_\xi$. Let $S_{\tau,\xi}$ be the characteristic foliation on S_τ, i.e., $\alpha|_{\mathbb{R}^3}(S_{\tau,\xi}) = 0$, and let v be a vector field on S_τ, defined away from the singularities of $S_{\tau,\xi}$, such that $\alpha|_{\mathbb{R}^3}(v) = 1$.

Lemma 5.4.1. *The characteristic foliation $(\mathscr{I}_0 \times \Gamma)_\xi$ is given by*

(5.4.1)

$$\begin{aligned}(\mathscr{I}_0 \times \Gamma)_\xi = S_{\tau,\xi} &+ dz \wedge ds(S_{\tau,\xi}, v)(\partial_\tau + fw_0) + f\,(ds(S_{\tau,\xi})v - ds(v)S_{\tau,\xi}) \\ &+ e^{-\tau}f(-ds \wedge dt(S_{\tau,\xi}, v) + K_0 dz \wedge ds(S_{\tau,\xi}, v))R_\eta,\end{aligned}$$

on the subset of $\mathscr{I}_0 \times \Gamma$ where v is defined. Here R_η is the Reeb vector field of η.

Proof. This is similar to the calculation of Lemma 5.2.1. We compute

$$T(\mathscr{I}_0 \times \Gamma) \cap \xi = \mathbb{R}\langle S_{\tau,\xi}, e^{\tau+s}v - R_\eta, \partial_\tau + fw_0 + (-e^{-\tau-s} + K_0e^{-\tau})fR_\eta, \ker\eta\rangle.$$

Next we have

$$\begin{aligned}\alpha &= dz + e^s(dt + e^\tau\eta),\\ e^{-s}d\alpha &= ds \wedge dt + e^\tau ds \wedge \eta + e^\tau(d\tau \wedge \eta + d\eta).\end{aligned}$$

Setting $X = S_{\tau,\xi}$, $Y = e^{\tau+s}v - R_\eta$, $Z = \partial_\tau + fw_0 + (-e^{-\tau-s} + K_0e^{-\tau})fR_\eta$,

$$\begin{aligned}\langle X, Y\rangle &= e^{\tau+s}ds \wedge dt(S_{\tau,\xi}, v) - e^\tau ds(S_{\tau,\xi}) = e^\tau(dz \wedge ds(S_{\tau,\xi}, v)),\\ \langle X, Z\rangle &= e^\tau ds(S_{\tau,\xi})(-e^{-\tau-s}f) = -e^{-s}f ds(S_{\tau,\xi}),\\ \langle Y, Z\rangle &= e^{\tau+s}(e^\tau ds(v)(-e^{-\tau-s}f)) + e^\tau = e^\tau(1 - f ds(v)).\\ (\mathscr{I}_1 \times \Gamma)_\xi &= (1 - f ds(v))S_{\tau,\xi} + e^{-\tau-s}f ds(S_{\tau,\xi})(e^{\tau+s}v - R_\eta)\\ &\quad + dz \wedge ds(S_{\tau,\xi}, v)(\partial_\tau + fw_0 + (-e^{-\tau-s} + K_0e^{-\tau})fR_\eta).\end{aligned}$$

A rearrangement of the terms gives the lemma. □

Note that Eq. (5.4.1) agrees with Eq. (5.2.1) at $\tau = 0$. The first two terms of $(\mathscr{I}_0 \times \Gamma)_\xi$ are analogous to those of Z'_ξ; see Lemma 5.2.1. The third term $f(ds(S_{\tau,\xi})v - ds(v)S_{\tau,\xi})$ lies in $\ker ds$ and, away from the corners,

- vanishes on $P_1 \cup P_3$,
- has negative ∂_t-component on $P_0 \cup P_4$, and
- has positive ∂_t-component on P_2.

See Figure 5.4.1. In other words, the third term, when we project out the s- and W-directions, is a flow in the clockwise direction around ∂P_1 as seen in the picture. The last term of Eq. (5.4.1) has a substantial contribution in the R_η-direction when the damping happens quickly, i.e., when τ_0 is small and f is large. This is something we need to be careful about, but ultimately can be finessed away by stacking the mushrooms in a particular way in Section 7.

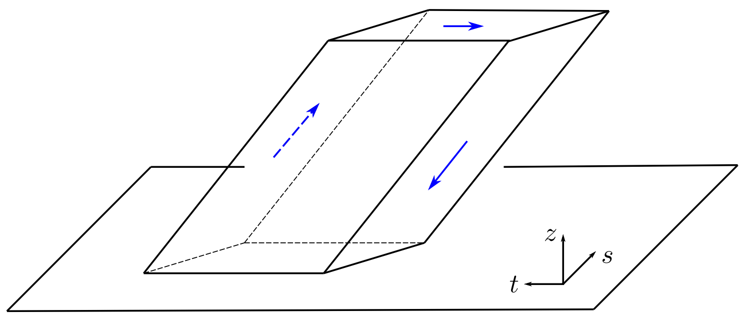

FIGURE 5.4.1. The vector field $f(ds(S_{\tau,\xi})v - ds(v)S_{\tau,\xi})$ is depicted in blue.

5.5. **Description of the characteristic foliation of the mushroom.** In this subsection we summarize the dynamics of the characteristic foliation of the mushroom Z_H of $\Sigma = \{z = 0\} \subset M = \mathbb{R}^3_{z,s,t} \times W$ with profile $H = [0, t_0] \times W^c_{\tau_0}$.

Definition 5.5.1. *The* mushroom of Σ with profile H *is the hypersurface*

$$Z_H := \big(\Sigma \setminus (\square \times W^c_{\tau_0})\big) \cup Z'_{PL} \cup (\mathscr{I}_0 \times \Gamma)_{PL}, \tag{5.5.1}$$

modulo smoothing. (The smoothed versions do not have the subscripts PL.) The region $\square \times W^c_{\tau_0} \subset \Sigma$, where $\square = [0, s_0] \times [0, t_0]$, is the base *of Z_H, and the region $\mathscr{I}_0 \times \Gamma$ is the* damping region.

Let $\tau'_0 \in (0, \tau_0)$, let $\kappa_1 > \kappa_2 > \kappa_3 > 0 > \kappa_4 > \kappa_5 > \kappa_6$ with all κ_i small as in Proposition 3.2.1, and let $\sigma_1^\pm, \sigma_2^\pm : W^c_{\tau'_0} \to \mathbb{R}_{\geq 0}$ be functions such that:

- $\sigma_1^\pm$ vanishes exactly on $\mathrm{Sk}(W)$ and $\sigma_2^\pm$ vanishes on W^c;
- on $\{0 \leq \tau \leq \tau'_0\}$, both $\sigma_1^\pm = \sigma_1^\pm(\tau)$ and $\sigma_2^\pm = \sigma_2^\pm(\tau)$ are strictly increasing and reach their maximum at $\tau = \tau'_0$;
- $\sigma_1^-(\tau'_0) + \sigma_2^-(\tau'_0) = t_0 + \kappa_2 - \kappa_4$ and $\sigma_1^+(\tau'_0) + \sigma_2^+(\tau'_0) = t_0 + \kappa_3 - \kappa_5$.

As the smooth version of Z_H limits to the PL version, all $\kappa_i \to 0$ and $|\sigma_i^\pm|_{C^0} \to 0$ on $W^c_{\tau_0}$.

We then define the compact submanifolds

$$H_{\text{in}} := \{(t, x) \mid x \in W^c_{\tau'_0}, \kappa_4 + \sigma_2^-(x) \leq t \leq t_0 + \kappa_2 - \sigma_1^-(x)\}, \tag{5.5.2}$$

$$H_{\text{out}} := \{(t, x) \mid x \in W^c_{\tau'_0}, \kappa_5 + \sigma_1^+(x) \leq t \leq t_0 + \kappa_3 - \sigma_2^+(x)\}, \tag{5.5.3}$$

which approximate H when all the smoothing parameters involved in the construction tend to 0. See Figure 5.5.1. We use the notation X° (and also $\mathrm{int}(X)$) as in H_{in}° to denote the interior of a space X.

Note that ∂H (which we assume has rounded corners) is convex; this follows from observing that $([-1,1]_t \times W, dt+\lambda)$ has contact vector field $t\partial_t + X_\lambda$.

Proposition 5.5.2. *Assuming all the corner rounding parameters are sufficiently small, there exists a tubular neighborhood $[-\epsilon,\epsilon]_\ell \times \partial H$ of $\partial H = \{0\}\times \partial H$ and H_{in} and H_{out} that approximate H such that:*

- *(Z1) $\partial H_{\mathrm{in}}, \partial H_{\mathrm{out}} \subset [-\epsilon,\epsilon]\times \partial H$ are graphical over ∂H.*
- *(Z2) $Z_{H,\xi}$ is "1-Morse" in the following sense: it satisfies (M1) and (M3) of Proposition 2.2.6 and*
  - *(M2') every flow line passing through $x\in Z_H$ with $Z_{H,\xi}(x)\neq 0$ converges to a singularity or goes to $\{s=+\infty\}$ in forward time and converges to a singularity or goes to $\{s=-\infty\}$ in backward time.*
- *(Z3) Any flow line of $Z_{H,\xi}$ that passes through $H_{\mathrm{in}}^\circ \subset \{s=-1\}$ converges to a negative singularity of $Z_{H,\xi}$ in forward time. Similarly, any flow line of $Z_{H,\xi}$ that passes through $H_{\mathrm{out}}^\circ \subset \{s=s_0+1\}$ converges to a positive singularity of $Z_{H,\xi}$ in backward time.*
- *(Z4) Any flow line of $Z_{H,\xi}$ that does not pass through $H\cup([-\epsilon,\epsilon]\times\partial H)\subset \{s=-1\}$ has trivial holonomy.*
- *(Z5) There exists a Morse function F on ∂H such that ∂H_ξ is gradient-like for F (and hence flows "from $R_+(\partial H)$ to $R_-(\partial H)$") and such that any flow line of $Z_{H,\xi}$ that passes through $(\ell,x)\in[-\epsilon,\epsilon]\times\partial H\subset\{s=-1\}$ and does not converge to a singularity of $Z_{H,\xi}$:*
  - *(1) passes through $(\ell',y)\in[-\epsilon,\epsilon]\times\partial H\subset\{s=s_0+1\}$ with $F(y)\geq F(x)$; and*
  - *(2) is parallel to X_λ (resp. $-X_\lambda$) on $[-\epsilon,\epsilon]\times W_+^c$ (resp. $[-\epsilon,\epsilon]\times W_-^c$) when projected to $[-\epsilon,\epsilon]\times\partial H$.*

  Here W_+^c is the portion of ∂H corresponding to $\{t_0\}\times W^c$ and W_-^c is the portion corresponding to $\{0\}\times W^c$.

Proof. (Z1) is by construction. (Z4) is clear. (Z2)–(Z5) follow from Lemmas 5.3.3 and 5.4.1. In (Z5) we take $\{\tau\}\times\Gamma$ to be level sets of F so that the component of $Z_{H,\xi}$ in the direction of the Reeb vector field R_η vanishes on dF. $\square$

6. Quantitative stabilization of an open book decomposition for S^{2n-1}

6.1. **Some definitions.** Let M be a closed manifold. An *open book decomposition* (abbreviated OBD) of M is a pair (B,π), where $B\subset M$ is a closed codimension 2 submanifold and

$$\pi : M\setminus B \to S^1 \subset \mathbb{C}$$

is a fibration which agrees with the angular coordinate θ on a neighborhood $B\times D^2$ of $B = B\times\{0\}$. We call $S_\theta := \pi^{-1}(e^{i\theta})$, $e^{i\theta}\in S^1$, the *pages* of the OBD, and call B the *binding*.

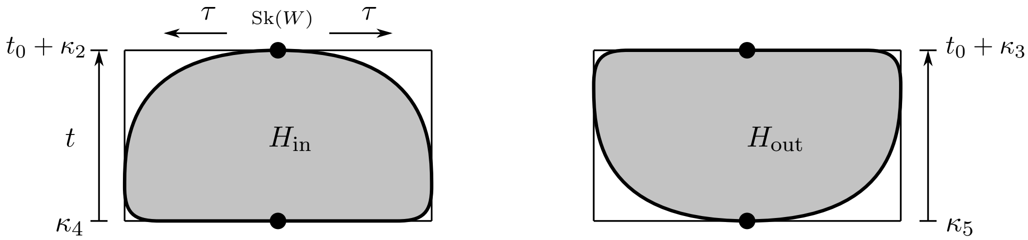

FIGURE 5.5.1. The shaded regions are H_{in} and H_{out}, respectively. The area of the complements of H_{in} and H_{out} in the rectangles tend to 0 as all the parameters involved in the construction tend to 0.

Let ξ be a contact structure on M.

Definition 6.1.1.

(1) An OBD (B, π) is ξ-compatible *if there exists a contact form α for ξ, called an* adapted *contact form, such that the Reeb vector field R_α of α is transverse to all the pages and is tangent to B, and $\alpha|_B$ is a contact form on B. We also say "α-compatible" or simply "compatible" if the contact structure is understood.*

(2) An α-compatible (B, π) is strongly Weinstein *if all its pages $(S_\theta, \alpha|_{S_\theta})$ are Weinstein.*

Let (B, π) be an α-compatible OBD. Let $\arg : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$ be the map $e^{i\theta} \mapsto \theta$. Then define $\rho : M \setminus B \to \mathbb{R}_{>0}$ by

$$\rho(x) = d(\arg \circ \pi)(R_\alpha(x)). \tag{6.1.1}$$

Roughly speaking, $\rho(x)$ measures infinitesimally how fast the orbit of R_α through x traverses the pages.

Definition 6.1.2. *The* infinitesimal variation *on the page S_θ is*

$$V_\theta \coloneqq \sup_{x \in S_\theta} \rho(x) / \inf_{x \in S_\theta} \rho(x) \in [1, \infty), \tag{6.1.2}$$

and the total infinitesimal variation *is $V \coloneqq \sup_{\theta \in [0, 2\pi]} V_\theta$.*

The following is standard:

Lemma 6.1.3. *If $V \equiv 1$, then $M \setminus \overline{S}_0$ is the interior of a contact handlebody.*

Proof. Let t be the coordinate obtained by flowing in the direction of R_α starting from S_0. Then $M \setminus \overline{S}_0 \simeq (0, C)_t \times S_0$ and $\alpha = f_t dt + \beta_t$, where f_t (resp. β_t) is a function (resp. 1-form) on S_0 that depends on t.

We claim that $R_\alpha = \partial_t$ implies that $f_t = 1$ and $\dot{\beta}_t = 0$, where the dot denotes the derivative in the t-direction: Since $\alpha(R_\alpha) = 1$, we have $f_t = 1$. Then $d\alpha$ becomes $dt \wedge \dot{\beta}_t + d_{S_0}\beta_t$, where d_{S_0} is the exterior derivative in the S_0-direction. Finally, $i_{R_\alpha} d\alpha = 0$ forces $\dot{\beta}_t = 0$. $\square$

6.2. **Quantitative stabilization for S^{2n-1}.** Before starting, we warn the reader that *the type of stabilization in this section is different from the notion of stabilization of an OBD in the Giroux correspondence which involves changing the topology of the page by a handle attachment and composing the monodromy with a suitable Dehn twist.*

Let ξ_{std} be the standard contact structure on $S^{2n-1} = \{|z_1|^2 + \cdots + |z_n|^2 = 1\} \subset \mathbb{C}^n$ given by the restriction of $\alpha = \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)$ which we denote by α_{std}. The standard OBD for ξ_{std} can be constructed as follows: Starting with $z_1 : S^{2n-1} \to \mathbb{C}$ which is a submersion away from $|z_1| = 1$, we set $B = z_1^{-1}(0) = S^{2n-3} = \{|z_2|^2 + \cdots + |z_n|^2 = 1\}$ and

$$\pi = \tfrac{z_1}{|z_1|} : S^{2n-1} \setminus B \to S^1 \subset \mathbb{C}.$$

The pages S_θ are Weinstein $(2n-2)$-disks and the Reeb vector field is

$$R_{\alpha_{std}} = \textstyle\sum_{i=1}^n (x_i \partial_{y_i} - y_i \partial_{x_i}) = \sum_{i=1}^n \partial_{\theta_i},$$

where θ_i is the ith angular coordinate. Then $\rho(z_1, \dots, z_n) = d\theta_1(\partial_{\theta_1}) = 1, V \equiv 1$, and $S^{2n-1} \setminus \overline{S}_0$ is the interior of a genuine contact handlebody of thickness 2π by Lemma 6.1.3. Note that B has an analogous OBD derived from $z_2 : S^{2n-3} = \{|z_2|^2 + \cdots + |z_n|^2 = 1\} \to \mathbb{C}$.

Lemma 6.2.1. *For any positive integer $k > 0$ and $\epsilon' > 0$ small, there exists an α_{std}-compatible, strongly Weinstein OBD (B_k, π_k) of S^{2n-1} such that each page is C^∞-close to*

$$S_{\theta_0} \cup S_{\theta_0 + 2\pi/k} \cup \cdots \cup S_{\theta_0 + 2\pi(k-1)/k}$$

(i.e., the union of k evenly spaced pages for some θ_0) outside an ϵ'-small neighborhood of B (with respect to the standard Euclidean metric on $\mathbb{C}^n$) and such that S^{2n-1}, cut open along a new page, is the interior of a contact handlebody of thickness $2\pi/k$.

Proof. We would like to "stabilize" (B, π) by replacing z_1 by z_1^k. However, since 0 is not a regular value of z_1^k, we use

$$f_k := z_1^k + \epsilon z_2^k + \epsilon^2 z_3^k + \cdots + \epsilon^{n-1} z_n^k,$$

where $\epsilon > 0$ is small. We are thinking of f_k as inductively defined as z_1^k plus ϵ times f_k corresponding to the binding $|z_2|^2 + \cdots + |z_n|^2 = 1$. We write $R = R_{\alpha_{std}}$.

Step 1. Verification that 0 is a regular value of f_k. Let F_k be f_k viewed as a map $\mathbb{C}^n \to \mathbb{C}$. Then $dF_k(z_1, \dots, z_k) = k(z_1^{k-1}, \epsilon z_2^{k-1}, \dots, \epsilon^{n-1} z_n^{k-1})$. Next we precompose with the derivative of the inclusion map $i : S^{2n-1} \hookrightarrow \mathbb{C}^n$. Let $z = (z_1, \dots, z_n) \in S^{2n-1}$. Suppose there exists $z_j \neq 0, 1$. Then there exists $v \in T_z S^{2n-1}$ with a nontrivial component in the z_j-direction and df_k is surjective since $\epsilon^{j-1} z_j^{k-1} \neq 0$. Otherwise, some $z_j = 1$ and $z_i = 0$ for all $i \neq j$. Then $f_k(z) = \epsilon^{j-1} z_j^k \neq 0$. Hence 0 is a regular value of f_k.

We set $B_k = f_k^{-1}(0)$ $\pi_k = \frac{f_k}{|f_k|} : S^{2n-1} \setminus B_k \to S^1$, and $S_{k,\theta} = \pi_k^{-1}(\theta)$.

Step 2. Computation of $df_k(R)$. For $(z_1,\dots,z_k)\in S^{2n-1}\setminus B_k$, we use polar coordinates (r_i,θ_i) for z_i and compute:

$$\begin{aligned} df_k(R) &= d(r_1^k e^{ik\theta_1}+\epsilon r_2^k e^{ik\theta_2}+\cdots+\epsilon^{n-1} r_n^k e^{ik\theta_n})(\textstyle\sum_{j=1}^n \partial_{\theta_j}) \\ &= ik(r_1^k e^{ik\theta_1}+\epsilon r_2^k e^{ik\theta_2}+\cdots+\epsilon^{n-1} r_n^k e^{ik\theta_n}) = ikf_k. \end{aligned} \tag{6.2.1}$$

Observe that this equation is the version of the equation in [Gir00, p.411, last line] when F_k is holomorphic.

Step 3. Verification of the properties. Eq. (6.2.1) implies that R is tangent to B_k and transverse to $S_{k,\theta}$ and that π_k is a fibration. Moreover, for (B_k,π_k), $\rho(z)=1$ and $V\equiv 1$, and $S^{2n-1}\setminus \overline{S}_{k,0}$ is the interior of a contact handlebody of thickness $2\pi/k$. Since R is tangent to B_k, for each $z\in B_k$, $df_k(\ker\alpha_{std}(z))=\mathbb{C}$. This, together with the invariance of $\ker\alpha_{std}$ under the standard almost complex structure on $\mathbb{C}^n$, implies that B_k is a codimension two contact submanifold of $\ker\alpha_{std}$.

Next we apply Lemma 6.3.1, proved in §6.3, to show that $S_{k,\theta}$ is Weinstein after perturbing f_k by adding $\sum_{i=1}^n c_i z_i^k$ for c_i small. Using the notation from §6.3, we have $d\theta(R)=1$ and $d\phi(R)=0$; hence $d\phi(X_\beta)=0$ if and only if $d\phi=0$, where ϕ is viewed as a function on $S_{k,\theta=0}$. One can compute that if $z\in S_{k,0}$ is a critical point of ϕ, then all $|z_i|\neq 0$; at such a point $d(z_1^k),\dots,d(z_n^k)$ are linearly independent. This provides enough perturbations to make ϕ Morse.

Finally, the C^∞-closeness property is immediate from taking $\epsilon>0$ small. □

6.3. **Verification of the Weinstein property.** Let $f:\mathbb{C}^n\to\mathbb{C}$ be a holomorphic function such that $f(0)=0$ with an isolated critical point at the origin. For $\kappa>0$ sufficiently small, f defines an OBD (B,θ) of the sphere S_κ of radius κ, where $B=S_\kappa\cap f^{-1}(0)$ and $\theta=\arg f: S_\kappa\setminus B\to\mathbb{R}/2\pi\mathbb{Z}$. Also let $\phi:=-\log|f|$ on $S_\kappa\setminus B$.

Let

- $\alpha=\frac{1}{2}\sum_{i=1}^n(x_i dy_i - y_i dx_i)=\frac{1}{2}\sum_{i=1}^n r_i^2 d\theta_i$ be the standard Liouville form on $\mathbb{C}^n$ with Liouville vector field $X_\alpha=\frac{1}{2}\sum_{i=1}^n r_i\frac{\partial}{\partial r_i}$, where (r_i,θ_i) are polar coordinates corresponding to (x_i,y_i);
- α_{std} be the induced contact form on S_κ for $\kappa>0$ small and $R=\frac{2}{r^2}\sum_i\frac{\partial}{\partial\theta_i}$ be the Reeb vector field on S_κ, where $r^2=\sum_i r_i^2$ (note that R is defined on all of $\mathbb{C}^n$, not just on S_κ);
- β be the 1-form induced by α_{std} on any page of the OBD with Liouville vector field X_β; we view X_β as a vector field on $S_\kappa\setminus B$ that is tangent to the pages;
- X_θ be the Hamiltonian vector field of θ (satisfying $i_{X_\theta}d\alpha=d\theta$), viewed as a function on $\mathbb{C}^n\setminus f^{-1}(0)$; and
- J be the standard complex structure on $\mathbb{C}^n$.

The following is due to Emmanuel Giroux (presented here with his permission):

Lemma 6.3.1 (Giroux)**.** *On $S_\kappa\setminus B$, if $d\phi(R)\equiv 0$, then $d\phi(X_\beta)=\frac{1}{d\theta(R)}|d\phi|^2$.*

Proof. We first claim that, at every point of $S_\kappa \setminus B$, the following identity holds:

$$X_\alpha = X_\beta + aX_\theta + bR, \tag{6.3.1}$$

where $a = \frac{1}{d\theta(R)}$ and $b = \frac{d\theta(X_\alpha)}{d\theta(R)}$. First note that the $d\alpha$-symplectic orthogonal complement of the tangent space TS of a page S is spanned by R and X_θ. [Verification: $d\alpha(R, TS) = 0$ since R, TS are tangent to S_κ; $d\alpha(X_\theta, TS) = d\theta(TS) = 0$; and $d\alpha(X_\theta, R) = d\theta(R) > 0$.] Hence we can write $X_\alpha = Y + aX_\theta + bR$ for some $Y \in TS$. Evaluating on TS gives $Y = X_\beta$. We can then determine a and b by applying $d\theta$ and $d(r^2)$ to Eq. (6.3.1): $d(r^2)(X_\alpha) = 2rdr(\frac{1}{2}\sum_i r_i \frac{\partial}{\partial r_i}) = r^2$ and $d(r^2)(X_\alpha) = ad(r^2)(X_\theta) = -ad\theta(X_{r^2}) = -ad\theta(-r^2 R)$, so $a = \frac{1}{d\theta(R)}$. Similarly, $d\theta(X_\alpha) = bd\theta(R)$, so $b = \frac{d\theta(X_\alpha)}{d\theta(R)}$.

It follows from Eq. (6.3.1) that

$$d\phi(X_\beta) = -\frac{d\phi(X_\theta)}{d\theta(R)} + d\phi(X_\alpha) - d\phi(R)\frac{d\theta(X_\alpha)}{d\theta(R)}.$$

Now, since the function $-\phi + i\theta$ is holomorphic when viewed as a function on $\mathbb{C}^n \setminus \{f = 0\}$, we have $d\phi \circ J = d\theta$. Hence

$$\begin{aligned}
d\phi(X_\alpha) &= -d\phi \circ J(JX_\alpha) = -d\theta(\tfrac{1}{2}\textstyle\sum_i \frac{\partial}{\partial \theta_i}) = -\frac{r^2}{4}d\theta(R),\\
d\phi(X_\theta) &= -d\phi \circ J(JX_\theta) = -d\theta(JX_\theta) = -d\alpha(X_\theta, JX_\theta)\\
&= -|X_\theta|^2 = -|d\theta|^2 = -|d\phi|^2,\\
d\phi(R) &= -d\theta(JR) = d\theta(\tfrac{4}{r^2}X_\alpha),
\end{aligned}$$

and

$$d\phi(X_\beta) = \frac{1}{d\theta(R)}(|d\phi|^2 - (d\phi(\tfrac{r}{2}R))^2 - (d\phi(\tfrac{2}{r}X_\alpha))^2).$$

Observe that $\frac{2}{r}X_\alpha$ and $\frac{r}{2}R$ are orthonormal unit vectors. Finally, since $d\phi(R) = 0$, $d\phi(X_\beta) = \frac{1}{d\theta(R)}|d\phi|^2$ on each page. □

7. Construction of the plug

The goal of this section is to generalize the 3-dimensional plug constructed in §4.1 to higher dimensions. This is the key construction that will allow us to prove Theorems 1.2.3 and 1.2.5 in Section 9 in essentially the same way as in the 3-dimensional case.

7.1. **Definition of a plug.**

Let us rephrase the 3-dimensional case considered in §4.1 in a way that is amenable to higher-dimensional generalization. Consider the standard contact space $(\mathbb{R}^3, \ker(dz + e^s dt))$ and the surface $\Sigma = \{z = 0\} \subset \mathbb{R}^3$. The plug is obtained by growing a mushroom along a box $U = [0, s_0] \times [0, t_0] \subset \Sigma$, where we are viewing U as the truncated symplectization of the 1-dimensional compact contact manifold $\partial_- U = \{0\} \times [0, t_0]$ with contact form dt.

In higher dimensions, let $(Y, \ker \eta)$ be a compact contact manifold of dimension $2n - 1$ with convex boundary. Let

$$(N_{\epsilon_0}(Y) := Y \cup ([0, \epsilon_0] \times \partial Y), \ker \eta)$$

be a small extension of $(Y, \ker\eta)$. Now we consider

$$(M^{2n+1} := \mathbb{R}^2_{z,s} \times N_{\epsilon_0}(Y), \xi = \ker(dz + e^s\eta))$$

and the hypersurface $\Sigma := \{z = 0\}$. Let $U := [0, s_0] \times N_{\epsilon_0}(Y)$ and let

$$\partial_- U := \{-1\} \times N_{\epsilon_0}(Y) \quad \text{and} \quad \partial_+ U := \{s_0 + 1\} \times N_{\epsilon_0}(Y).$$

From now on, we fix a Riemannian metric on M, which induces a metric on any submanifold and such that $[0, \epsilon_0] \times \partial Y$ has thickness ϵ_0 with respect to this metric.

Definition 7.1.1. *A Y-*shaped plug *with parameter $\epsilon > 0$ is a C^0-small perturbation $\widetilde{U}$ of U supported in the interior U° of U such that:*

1. *all the flow lines of $\widetilde{U}_\xi$ that pass through $\{-1\} \times Y^\circ$ flow to a negative singularity;*
2. *all the flow lines of $\widetilde{U}_\xi$ that pass through $\{s_0+1\} \times Y^\circ$ flow from a positive singularity;*
3. *for all possibly broken flow lines of $\widetilde{U}_\xi$ that go from $\partial_- U$ to $\partial_+ U$, the holonomy map is ϵ-close to the identity when defined;*
4. *$\widetilde{U}_\xi$ is gradient-like with respect to a Morse function $f : \widetilde{U} \to \mathbb{R}$ which agrees with s on $\partial\widetilde{U}$; in particular there are no possibly broken loops of $\widetilde{U}_\xi$.*

Definition 7.1.2. *A Y-*shaped pre-plug *$\widetilde{U}$ satisfies Definition 7.1.1 with (3) replaced by:*

(3') *for each possibly broken flow line of $\widetilde{U}_\xi$ that goes from $\partial_- U$ to $\partial_+ U$, the holonomy map is obtained by following a small perturbation of $(\partial Y)_{\ker\eta}$.*

7.2. **A Peter-Paul contactomorphism.** Let (Y, η) be a contact manifold with a fixed choice of contact form η. Let S be a hypersurface of Y transverse to the Reeb vector field R_η. Then S has a neighborhood $S \times [-\epsilon, \epsilon]_\tau \subset Y$ on which $R_\eta = \partial_\tau$.

The following is well-known:

Lemma 7.2.1. *If $R_\eta = \partial_\tau$ on $S \times [a, b]_\tau \subset Y$, $a < b$, then $\eta = d\tau + \beta$, where β is the pullback of a 1-form on S. Moreover, $d\beta$ is symplectic on S.*

In other words, η is the *contactization of* (S, β). In particular, if (S, β) is Weinstein then $S \times [a, b]$ is a contact handlebody.

Proof. We first write $\eta = f d\tau + \beta$, where $f(\tau) \in \Omega^0(S)$ and $\beta(\tau) \in \Omega^1(S)$. Since $\eta(R_\eta) = 1$, we have $f = 1$. Also, since $\mathscr{L}_{R_\eta}\eta = 0$, $\beta(\tau)$ must be τ-independent. Finally, $d\beta$ is symplectic on S due to the contact condition on Y. □

Given (Y, η), let $(M, \alpha) = (\mathbb{R}^2_{z,s} \times Y, dz + e^s\eta)$ and let $\phi_t : Y \xrightarrow{\sim} Y$ be the time-t flow of R_η.

Lemma 7.2.2. *The diffeomorphism*

$$\Psi : M \xrightarrow{\sim} M, \quad (z, s, y) \mapsto \left(e^{(-1+1/C)s} \cdot Cz, s/C, \phi_{(1-C)e^{-s}z}(y)\right), \tag{7.2.1}$$

where $C > 0$, is a contactomorphism.

Proof. We compute

$$\begin{aligned}\Psi^*(\alpha) &= d(e^{(-1+1/C)s}\cdot Cz) + e^{s/C}(\eta + d((1-C)e^{-s}z))\\ &= e^{(-1+1/C)s}(1-C)zds + e^{(-1+1/C)s}Cdz + e^{s/C}\eta\\ &\qquad - e^{s/C}(1-C)e^{-s}zds + e^{s/C}(1-C)e^{-s}dz\\ &= e^{(-1+1/C)s}(dz+e^s\eta) = e^{(-1+1/C)s}\alpha.\end{aligned}$$

We explain the first line: By Lemma 7.2.1, η can locally be written as $d\tau+\beta$, where β is a 1-form on a hypersurface $S\subset Y$ transverse to $R_\eta=\partial_\tau$. (Note that we can use the immersion $i: S\times\mathbb{R}_\tau\to Y$ with $i^*R_\eta=\partial_\tau$ instead in Lemma 7.2.1.) Then $\phi_{(1-C)e^{-s}z}(\tau,x)=(\tau+(1-C)e^{-s}z,x)$, where x is the coordinate on S, and $\phi^*_{(1-C)e^{-s}z}\eta=\eta+d((1-C)e^{-s}z)$. □

Let $M_{(z_0,s_0)}=[-z_0,z_0]\times[0,s_0]\times Y\subset(M,\alpha)$. As an immediate corollary, by taking $C\gg 0$, we have:

Lemma 7.2.3. *For any $0<s_0'\leq s_0$ and $z_0'>0$, there exists $0<z_0\leq z_0'$, such that $M_{(z_0,s_0)}$ contactly embeds into $M_{(z_0',s_0')}$ and takes $\{z=0\}\cap M_{(z_0,s_0)}$ to $\{z=0\}\cap M_{(z_0',s_0')}$.*

We call the contactomorphism Ψ given by (7.2.1) a *Peter-Paul contactomorphism* for the following reason: In Lemma 7.2.3, $\Sigma=\{0\}\times[0,s]\times Y\subset M_{(z_0,s_0)}$ is the hypersurface on which we want to create mushrooms. The length of the interval $[-z_0,z_0]$ can be regarded as the given size of a neighborhood of Σ. The map Ψ then allows us to rob the (already small) size of the neighborhood of Σ to pay for a large size in the s-direction.

Observe that the Peter-Paul contactomorphism was not needed in Section 4 to make any 2-dimensional surface convex.

7.3. **A pre-plug.** Given *a standard Darboux ball (Y^{2n-1},η') with convex boundary,* we explain how to construct a Y-shaped pre-plug $\widetilde{U}$ on

$$\Sigma := \{z=0\}\subset M=\mathbb{R}^2_{z,s}\times N_{\epsilon_0}(Y).$$

The Y-shaped pre-plug will be upgraded to a Y-shaped plug with parameter $\epsilon>0$ in the next two subsections.

Modulo corner rounding, we may assume that $(Y,\ker\eta')$ is contactomorphic to $(Y_0\cup Y_1,\ker\eta)$ such that:

1. (1) $Y_0=(z_1^{-1}(\{|\zeta|\leq\epsilon'\}),\eta=\alpha_{std})$, where $z_1: S^{2n-1}\to\mathbb{C}_\zeta$ and $\alpha_{std}=\frac{1}{2}\sum_i(x_idy_i-y_idx_i)$ are as in §6.2 and $\epsilon'>0$ is small;
2. (2) $Y_1=([0,2\pi]_t\times D^{2n-2},\eta=dt+\beta_{D^{2n-2}})$, where $\beta_{D^{2n-2}}$ is a standard Liouville form on D^{2n-2} with one elliptic singular point; and
3. (3) for each $t\in[0,2\pi]$, $\{t\}\times\partial D^{2n-2}$ is glued to $z_1^{-1}(\epsilon' e^{it})$ so that the contact forms (and the Reeb vector fields) match.

(In particular, $Y_0\cup Y_1$ has a partial open book structure, where Y_0 is a neighborhood of the binding and $\{t\}\times D^{2n-2}$ is a retraction of a page; see Definition 8.4.2.)

We now apply the Peter-Paul contactomorphism to realize $(Y_0 \cup Y_1, \eta)$ as a transverse slice (i.e., transverse to the characteristic foliation) on Σ; *we then blur the distinction between Y and $Y_0 \cup Y_1$ and write $Y = Y_0 \cup Y_1$*. The price that we pay is that we lose control of the z-height. (Recall that the z-height restricts the thickness of the contact handlebody of the mushroom that we want to grow.)

To remedy this we apply quantitative stabilization (Lemma 6.2.1) with $k \gg 0$ and $0 < \epsilon \ll \epsilon'$ to Y_0 to obtain $\pi_k : Y_0 \to S^1$. After a C^∞-small perturbation of Y_1 whose size depends on k and which we still denote by Y_1, there is a smooth extension $\pi_k : Y \to S^1$ which agrees with $(t, x) \mapsto kt$ on Y_1.

Choose a finite number, say $N = 5$, and a small constant $0 < \epsilon'' < \epsilon'$. Then, for $j = 0, \dots, 4$, let H'_j be the sector $\pi_k^{-1}([\frac{2\pi j}{5}, \frac{2\pi(j+1)}{5}])$. Assuming we chose a suitable extension of π_k, H'_j is a (complete) contact handlebody of thickness $\frac{2\pi}{k}$. Next let H_j be a slight modification of H'_j, $j = 0, \dots, 4$, obtained by removing an $\frac{\epsilon''}{2}$-neighborhood $\pi_k^{-1}(\{z \leq \frac{\epsilon''}{2}\}) \cap Y_0$ of the binding B_k and thickening the contact handlebody by flowing forward and backward by $\frac{\epsilon''}{k}$ in the Reeb direction.

Finally, we slightly (i.e., in a C^∞-small manner) modify η by "shifting the binding" B_k away from H_0 using a contactomorphism ϕ and construct the analogous contact handlebody $\phi(H_0)$ of thickness $\frac{2\pi + 2\epsilon''}{k}$ such that $\phi(H_0)$ contains the ϵ''-neighborhood of B_k, as follows:

Brief explanation of "shifting the binding". By flowing along the characteristic foliation of each page, one can normalize α_{std} on a small neighborhood $B_k \times D^2$ of the binding B_k as

$$\alpha_{std} = f\beta + g(xdy - ydx),$$

where D^2 is a disk with Euclidean/polar coordinates (x, y)/(r, θ) and small radius; $\beta = \alpha_{std}|_{B_k}$; $f = f(r)$ and $f(0) = 1$; $g = g_0 r^2 + O(r^3)$ and g_0 is a nonvanishing function on B_k; and *the pages are still $\theta = const$.* Letting

$$\alpha := f^{-1}\alpha_{std} = \beta + h(xdy - ydx), \quad \text{where} \quad h = f^{-1}g,$$

and $X = \frac{\partial}{\partial x} - hyR_\beta$, where R_β is the Reeb vector field of β, we compute:

$$d\alpha = d\beta + dh \wedge (xdy - ydx) + 2hdxdy,$$
$$i_X d\alpha = 2hdy + O(r), \quad d(\alpha(X)) = -2hdy + O(r),$$

and hence $\mathscr{L}_X \alpha = O(r)$. Therefore, the translation by aX, $a < 0$ small, on $B_k \times D^2$ (where defined), is close to a contactomorphism and the modification needed to make it into a contactomorphism ϕ is on the order of $a \cdot O(r)$, which is an order of magnitude smaller. Hence $\phi(H_0)$ has the property of being close to the $x \mapsto x + a$-translate of H_0 with error much smaller than a; in particular $\phi(H_0)$ contains the ϵ''-neighborhood of B_k. Finally, one may adjust the contact form on $\phi(H_0)$ by multiplying by a function that is close to 1 so that $\phi(H_0)$ becomes a contact handlebody.

In what follows we abuse notation and refer to $\phi(H_0)$ by H_0.

The pre-plug $\widetilde{U}$ consists of mushrooms Z_j, $0 \le j \le 4$, with bases

$$B_j \coloneqq [\tfrac{8-2j}{10}, \tfrac{9-2j}{10}]_s \times H_j;$$

see Figure 7.3.1. (Strictly speaking, the base B_0 is obtained by shifting $B_0' = [\frac{8}{10}, \frac{9}{10}] \times H_0$ by the map $(s,x) \mapsto (s + f_0(x), x)$, where f_0 is a C^0-small smooth function on H_0. We will assume that we started with a slightly smaller $B_0' = [\frac{8}{10} + \delta, \frac{9}{10} - \delta] \times H_0$ and that the base for the contact handlebody H_0 *contains* B_0.) We also require the damping of $Z_1, \dots, Z_4$ to occur inside the $\frac{3\epsilon''}{4}$-neighborhood of B_k so that the damping regions of $H_1, \dots, H_4$ are contained in H_0 and the damping region of H_0 to occur inside H_2.

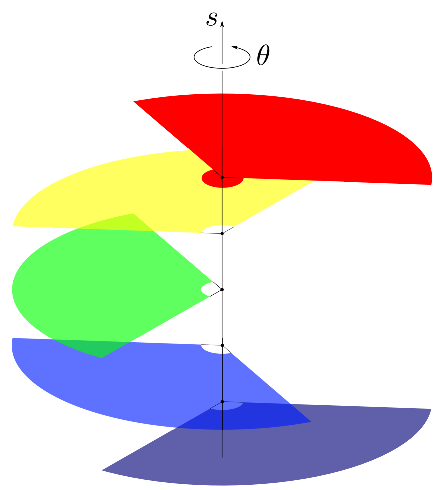

FIGURE 7.3.1. $H_0, \dots, H_4$, from top to bottom.

Verification of (3') in Definition 7.1.2. This is a direct consequence of Proposition 5.5.2 and our choice of the ordering of the mushrooms in the s-direction.

We first give names to regions of $\partial_- U$: Viewing H_j as a subset of $\partial_- U$, let $\widetilde{H}_j$ be the closure of the union of H_j and the set of points such that the holonomy from $s = \frac{8-2j}{10} - \delta$ to $\frac{9-2j}{10} + \delta$ (for $\delta > 0$ small) is not trivial or does not exist; note that $\widetilde{H}_j$ is contained in a small neighborhood of H_j. We denote the portion of $\widetilde{H}_j$ that closely approximates it and acts as a sink by $H_{j,\mathrm{in}}$ and $\widetilde{H}_j \setminus H_{j,\mathrm{in}}$ by $H_{j,\partial}$, and the corresponding products with $[\frac{8-2j}{10}, \frac{9-2j}{10}]$ by $B_{j,\mathrm{in}}$ and $B_{j,\partial}$.

The dynamics of $\widetilde{U}$ in forward time is described as follows: Let $x \in \partial_- U$ and let ℓ_x be the flow line of $\widetilde{U}$ passing through x.

(A) If ℓ_x enters $B_{j,\mathrm{in}}$, $j = 0, \dots, 4$, then ℓ_x converges to a singularity in Z_j.

(B) If ℓ_x enters $B_{j,\partial}$, then ℓ_x exits Z_j at a point near ∂H_j — *recall that as ℓ_x passes through Z_j it flows "from $R_+(\partial H_j)$ to $R_-(\partial H_j)$" in the sense of (Z5) of Proposition 5.5.2 with possibly large nontrivial components in the Reeb direction of $\Gamma_{\partial H_j}$ on the damping region* — and one of the following will happen:

(1) ℓ_x follows ∂_s until $\{s = 1\}$ and exits the plug; the holonomy map is obtained by following a small perturbation of $(\partial Y)_{\ker \eta}$;
(2) ℓ_x follows ∂_s until $B_{i,\text{in}}$ for $i < j$; we then apply (A) after letting the new j equal i;
(3) ℓ_x follows ∂_s until $B_{i,\partial}$ for $i < j$; we then apply (B) after letting the new j equal i.

Note that H_0 was chosen so that Z_0 captures all the trajectories that enter $\partial_- U$, "survives to" $s = \frac{8}{10}$, and is not close to ∂Y. A similar analysis can be applied to the dynamics of $\widetilde{U}$ in backward time and Definition 7.1.2(3') then holds.

In light of Proposition 2.2.6 and Lemma 2.3.2, we conclude that $\widetilde{U}$ is Morse.

7.4. **ϵ-short hypersurfaces.** In this subsection we strengthen Theorem 1.2.3 in a quantitative way. Namely, in addition to requiring that Σ_ξ be Morse$^+$ or 1-Morse$^+$, we also require all the smooth flow lines of Σ_ξ to be short.

Definition 7.4.1. *A closed hypersurface $\Sigma \subset (M, \xi)$ is ϵ-*short *if the length of any smooth flow line of Σ_ξ is shorter than ϵ with respect to the induced metric on Σ.*

Observe that any closed Morse$^+$ hypersurface Σ is ϵ-short for a sufficiently large ϵ which depends on Σ. For our purposes we take $\epsilon > 0$ to be a small number which is independent of the choice of convex hypersurface. Theorem 1.2.3 can be strengthened as follows:

Theorem 7.4.2. *Given $\epsilon > 0$, any closed hypersurface Σ in a contact manifold can be C^0-approximated by an ϵ-short and Morse$^+$ one. Moreover, if Σ is Weinstein convex, then there exists a t-invariant neighborhood $[-\delta, \delta]_t \times \Sigma$ of $\{0\} \times \Sigma$ and a 1-parameter family of pairwise disjoint embeddings $\phi_t : \Sigma \to [-\delta, \delta] \times \Sigma$, $t \in [-\delta, 0]$, such that:*

(1) $\phi_{-\delta}(\Sigma) = \{-\delta\} \times \Sigma$,
(2) $\phi_t(\Sigma)$ is C^0-close to $\{t\} \times \Sigma$ for all $t \in [-\delta, 0]$,
(3) $\phi_0(\Sigma)$ is 1-Morse$^+$ and ϵ-short, and
(4) $\phi_t(\Sigma)$ have the same number and type of singular points for all $t \in [-\delta, 0)$ and the whole family $\phi_t(\Sigma)$, $t \in [-\delta, 0]$ is Weinstein convex.

In words, what (3) and (4) are saying is that the singular points of the characteristic foliation of $\phi_t(\Sigma)$ remain the same for $t < 0$ and all the singular points — necessarily of birth-death type — are created at the same time when t reaches 0.

Theorem 7.4.2 holds in dimension 3 by §4.3, together with a slightly more careful analysis of the characteristic foliation when installing a mushroom. This will be the base case of our inductive argument.

7.5. **Construction of the plug.** The goal of this subsection is to prove the following:

Theorem 7.5.1. *Let Y be the standard Darboux ball of dimension $2n-1$ with convex boundary. Then for any $\epsilon > 0$ small there exists a Y-shaped plug with parameter ϵ, provided Theorem 7.4.2 holds for contact manifolds of any dimension $\leq 2n-1$.*

We adopt a simplification due to Eliashberg, Fauteux-Chapleau, and Pancholi, used with their permission, while retaining certain elements of the original version of the paper.

Proof. Given $\epsilon > 0$ small, choose $0 < \epsilon_0 \ll \epsilon$. Let Σ be the pushoff of ∂Y towards $\partial N_{\epsilon_0}(Y)$ such that Σ is the *standard Weinstein convex sphere*, which we take to satisfy:

(Sph1) $\Sigma \simeq S^{2n-2}$ and $\Gamma(\Sigma) \simeq S^{2n-3}$, where $\simeq$ means "diffeomorphic";
(Sph2) $R_\pm(\Sigma) \simeq D^{2n-2}$ is a Weinstein domain with precisely one singular point, and the singular point on $R_+(\Sigma)$ (resp. $R_-(\Sigma)$) has index 0 (resp. index $2n-2$).

Applying Theorem 7.4.2, there exists a foliation by $\Sigma'_t := \phi_t(\Sigma)$ for $t \in [-\delta, 0]$ satisfying (1)–(4). We are starting with $\Sigma'_{-\delta}$ which is the standard Weinstein convex sphere by (1); for $t \in [-\delta, 0)$, Σ'_t remains the standard Weinstein convex sphere by (4); and for $t = 0$, Σ'_0 is ϵ-short and Weinstein convex with birth-death singular points by (3).

Setup for $\widetilde{U}$. The Y-shaped plug $\widetilde{U}$ consists of two mushrooms Z_0, Z_2 with contact handlebody profiles H_0, H_2; a pre-plug Z_1 with profile H_1; and bases $[\frac{2j}{6}s_0, \frac{2j+1}{6}s_0] \times H_j$ (i.e., Z_0 has the smallest s-value, followed by Z_1, and then by Z_2), such that the following hold:

(P1) $N_{\epsilon_0}(Y) \supset H_0 \cup H_1 \cup H_2 \supset Y$ and $H_0 \cap H_2 = \varnothing$;
(P2) the damping region of H_0 is a subset of $H_{1,\mathrm{in}}$ and the damping region of H_2 is a subset of $H_{1,\mathrm{out}}$.

Note that H_1 does not need a damping region. See Figure 7.5.1. Recall that on the damping regions the characteristic foliation may have large components in the Reeb direction of the dividing set. Since these can potentially create trouble with the ϵ-shortness, we require (P2).

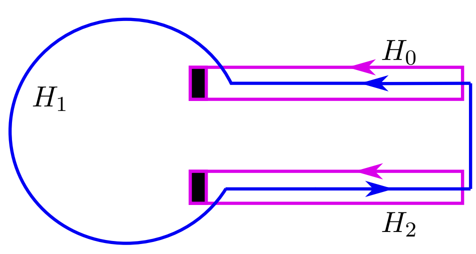

FIGURE 7.5.1. A schematic diagram of the plug, with the s-direction projected out (the boundaries on the right-hand side do not actually exist). The arrows indicate the direction of the flow along the characteristic foliations of the handlebodies (i.e., from the positive side to the negative side). The circular region represents $N(\Gamma)$ and the shaded regions are the damping regions.

Analysis of the dynamics of $\widetilde{U}_\xi$. We will specify H_0, H_1, H_2 later, but for the moment we analyze the effect of the plug $\widetilde{U}$ when we stack the mushrooms as above.

(1) Given a flow line of $\widetilde{U}_\xi$ that enters Z_0, either it flows to a negative singularity of Z_0 or passes through Z_0 after flowing parallel to the characteristic foliation of ∂H_0 as given by (Z5) of Proposition 5.5.2.
(2) Given a flow line of $\widetilde{U}_\xi$ that enters Z_1 (including one that exits Z_0 in the previous step), either it flows to a negative singularity of Z_1 or passes through Z_1 after flowing parallel to the characteristic foliation of ∂H_1. Notice that all the potentially problematic orbits that had large components in the Reeb direction in Z_0 (i.e., those that flowed out of the damping region of H_0 or near $R_-(\partial H_0)$) flow to a negative singularity of Z_1 by (P2).
(3) A similar consideration holds for a flow line of $\widetilde{U}_\xi$ that enters Z_2 (including one that exits Z_1 in the previous step).
(4) Backwards flow lines can be analyzed similarly.

The orbits that pass through $\widetilde{U}$ flow along the characteristic foliations of $\partial H_0, \partial H_1$, and ∂H_2 without ever entering damping regions; this means they flow closely along $\partial(H_0 \cup H_1 \cup H_2)$.

Description of H_0, H_1, H_2. Let us write $\Gamma = \Gamma(\Sigma_0')$ and let $N(\Gamma) = \Gamma \times D^2_{\rho,\theta}$ be a tubular neighborhood of Γ with contact form $\beta_\Gamma + C(\frac{\rho^2}{2} d\theta)$ for $C > 0$ small and such that the thickness in the D^2-direction is $\leq \epsilon''$ for $\epsilon'' > 0$ small. We take H_0 (resp. H_2) to be a thin contact handlebody over an ϵ'''-retraction of $R_+(\Sigma_0')$ (resp. $R_-(\Sigma_0')$), where $0 < \epsilon''' \ll \epsilon''$, and take the damping region to be small. There exists $\delta_0' > 0$ small such that

$$R_+(\Sigma'_{-\delta_0}) \setminus N(\Gamma) \subset H_{0,\mathrm{out}} \quad \text{and} \quad R_-(\Sigma'_{-\delta_0}) \setminus N(\Gamma) \subset H_{2,\mathrm{in}}.$$

We then take H_1 to be union of $N(\Gamma)$ and the region of $N_{\epsilon_0}(Y)$ bounded by $\Sigma'_{-\delta_0}$. There is some flexibility in choosing the dividing set and we take $\Gamma_{\partial H_1}$ and also the damping region to be a subset of $H_{2,\mathrm{in}}$. (P1) and (P2) hold by construction and also $\partial(H_0 \cup H_1 \cup H_2)$ is ϵ-short.

Lemma 7.5.2. *H_1, after a C^∞-small perturbation, is contactomorphic to a standard Darboux ball with convex boundary.*

Proof of Lemma 7.5.2. Let $H_1' := [0,1]_t \times D$, where $D = D^{2n-1}$, and let $D_t := \{t\} \times D$. We sketch the proof that if α is a contact form on H_1' such that:

(1) on a neighborhood of $[0,1] \times \partial D_t$, $\alpha = dt + \beta$, where β is independent of t and is the symplectization of a standard contact form on ∂D.
(2) the characteristic foliation on each D_t consists of a positive elliptic singularity e_t and all its trajectories go from e_t to ∂D_t;

then $(H_1', \ker\alpha)$ is contactomorphic to $(H_1', \ker(dt + \beta))$ for some extension of β to D. The assumptions imply that $\alpha = f_t dt + \beta_t$, where f_t is a function on D_t and β_t is a 1-form on D_t. We slightly perturb α so that the $(N(e_t), \beta_t|_{N(e_t)})$

are all diffeomorphic, where $N(e_t) \subset D_t$ is a neighborhood of e_t, and apply a 1-parameter family of diffeomorphisms to straighten the characteristic foliation and make β_t independent of t, which we now write as β'. The contact condition implies that $f_t > 0$; applying the Reeb flow gives the normalization $f dt + \beta'$; and we divide by f.

The lemma then follows modulo adjusting/rounding corners. □

In view of Lemma 7.5.2 and the construction of a pre-plug from §7.3, $\widetilde{U}$ is a Y-shaped plug with parameter ϵ.

□

8. Bifurcations of characteristic foliations and bypass attachments

The goal of this section is to relate certain codimension 1 degenerations of Morse$^+$ hypersurfaces to bypass attachments introduced in [HH]. Such a correspondence is fundamental in bridging the more dynamical approach [Gir91, Gir00] and the more combinatorial approach [Hon00] of convex surface theory in dimension 3. Unfortunately the details have never existed in the literature.

To this end, we slightly repackage the Morse theory on Morse hypersurfaces from Section 2 in terms of *folded Weinstein hypersurfaces*.

8.1. Definitions and examples.

In this subsection we define folded Weinstein hypersurfaces and examine a few examples.

Definition 8.1.1. *An oriented hypersurface $\Sigma \subset (M, \xi)$ is a* folded Weinstein hypersurface *if the characteristic foliation Σ_ξ satisfies the following properties:*

(FW1) There exist pairwise disjoint closed codimension 1 submanifolds $K_i \subset \Sigma$, $i = 1, \dots, 2m-1$, which cut Σ into $2m$ pieces, i.e.,

$$\Sigma = W_1 \cup_{K_1} \cdots \cup_{K_{2m-1}} W_{2m},$$

where W_i are compact with boundary $\partial W_i = K_i \cup K_{i-1}$. Here we are setting $K_0 = K_{2m} = \varnothing$. We call K_i the folding loci *of Σ.*

(FW2) The singular points of Σ_ξ in each W_i have the same sign, and the sign changes when crossing K_i. We assume the singular points in W_1 are positive and that each W_i has at least one singular point.

(FW3) There exists a Morse function f_i on each W_i such that K_{i-1} and K_i are regular level sets and $(W_i)_\xi$ is gradient-like with respect to f_i. In particular, Σ_ξ is transverse to all the K_i.

Observe that if $\Sigma = W_1 \cup \cdots \cup W_{2m} \subset (M, \xi)$ is a folded Weinstein hypersurface, then there exists a contact form α for ξ whose restriction to the interior of each W_i defines a Weinstein cobordism (the argument is similar to that of Proposition 2.3.3). Moreover, the orientation on W_i given by the Weinstein structure agrees with (resp. is opposite to) the orientation inherited from Σ if the singular points of $(W_i)_\xi$ are positive (resp. negative). We say a folding locus K_i is *maximal* (resp. *minimal*) if the Liouville vector fields on W_i and W_{i+1} are pointing towards (resp. away from) K_i.

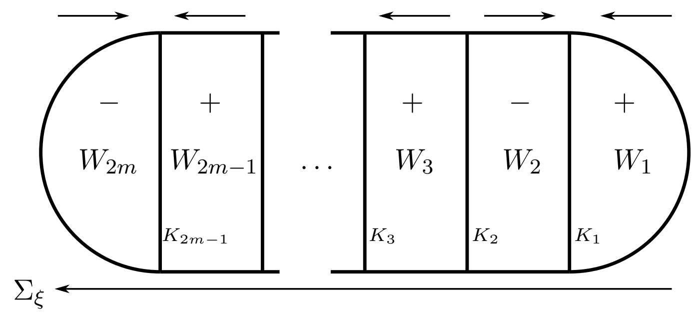

FIGURE 8.1.1. A schematic picture of a folded Weinstein hypersurface Σ. The top arrows indicate the direction of the Liouville vector fields on the W_i and the bottom arrow indicates the direction of the characteristic foliation.

Note that by definition any folded Weinstein hypersurface is Morse, and any Morse hypersurface can be equipped with the structure of a folded Weinstein hypersurface.

We now give examples of folded Weinstein hypersurfaces and explain why they are called "folded".

Example 8.1.2. If Σ is a convex hypersurface such that $R_\pm(\Sigma)$ are Weinstein manifolds, then Σ can be given the structure of a folded Weinstein hypersurface where the folding locus coincides with the dividing set Γ_Σ and is maximal. On the other hand, a folded Weinstein hypersurface is not always convex because there may exist trajectories of Σ_ξ from a negative singularity to a positive one. Nevertheless, since a C^∞-small perturbation of a Morse hypersurface is Morse$^+$ by Lemma 2.3.2, any folded Weinstein hypersurface is C^∞-generically convex by Proposition 2.3.3.

Example 8.1.3. Consider $(\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$ with contact form $\alpha = dz + \sum_{i=1}^n r_i^2 d\theta_i$. The unit sphere S^{2n} is convex with respect to the contact vector field $2z\partial_z + \sum_{i=1}^n r_i \partial_{r_i}$.

We slightly generalize this example as follows, which motivates our definition of a "folded" Weinstein hypersurface: We refer the reader to Definition A.3.1 for the definition of a *v-folded hypersurface* and a *seam*, where v is a vector field; for example, the graph of $y = x^2$ is a ∂_x-folded hypersurface in $\mathbb{R}^2$. Taking v to be $R_\alpha = \partial_z$, we consider closed R_α-folded hypersurfaces $\Sigma \subset \mathbb{R}^{2n+1}$ with seam C and decomposition $\Sigma \setminus C = \Sigma_+ \cup \Sigma_-$ such that R_α is positively (resp. negatively) transverse to $\Sigma_\pm$. It follows that $\Sigma_\pm$ are naturally exact symplectic manifolds with symplectic forms $d\alpha|_{\Sigma_\pm}$.

Now consider the following nontrivial condition:

(WC) Each component of $\Sigma_\pm$ is a (completed) Weinstein cobordism.

For example (WC) holds if each component of C is contained in $\{z = \mathrm{const}\} \cong \mathbb{R}^{2n}$ and is transverse to the radial vector field. Moreover, $\Sigma_\pm$ are graphical over $\mathbb{R}^{2n}$ and are Weinstein homotopic to subdomains of $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$.

Any R_α-folded hypersurface satisfying (WC) is clearly folded Weinstein with the folding locus equal to C, and hence can be made convex by a C^∞-small perturbation. Note, however, that if Σ happens to be convex, $R_\pm(\Sigma) \neq \Sigma_\pm$ in general.

This explains our terminology but at the same time raises a hard problem:

Question 8.1.4. *Characterize or classify convex hypersurfaces (e.g., spheres) in a Darboux chart.*

Any answer to this question will be of fundamental importance in understanding contact manifolds. See [Eli92] for a complete answer to this question in the case $S^2 \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$.

8.2. **Normalization of contact structure near a folded Weinstein hypersurface.** Recall that if $\Sigma \subset (M,\xi)$ is a convex hypersurface, then there exists a "standard" tubular neighborhood $U(\Sigma) \simeq \mathbb{R}_t \times \Sigma$ of Σ such that $\xi|_{U(\Sigma)} = \ker(f dt + \beta)$, where $f \in C^\infty(\Sigma)$ and $\beta \in \Omega^1(\Sigma)$.

The goal of this subsection is to generalize this to folded Weinstein hypersurfaces, namely associate to a folded Weinstein surface $\Sigma \subset (M, \xi = \ker\alpha)$ a *standard tubular neighborhood*, i.e., a tubular neighborhood $U(\Sigma)$ of Σ and a contactomorphism $\phi : U(\Sigma) \xrightarrow{\sim} ((-\epsilon,\epsilon)_t \times \Sigma, \ker\alpha_\Sigma)$, where α_Σ is a normalized contact form (specified below), $\epsilon > 0$ is sufficiently small, and $\phi(\Sigma) = \Sigma_0$. Here $\Sigma_t := \{t\} \times \Sigma$.

In the following three steps we construct the normalized contact form α_Σ on $\mathbb{R} \times \Sigma$ such that $\alpha_\Sigma|_\Sigma = \alpha|_{\Sigma_0}$, up to rescaling by a positive function. Lemma 2.1.3 then gives the desired contactomorphism.

Following Definition 8.1.1, we write $\Sigma = W_1 \cup_{K_1} \cup \cdots \cup_{K_{2m-1}} W_{2m}$. Choose a tubular neighborhood $U(K_i)$ for each K_i and identify it with $[-1,1]_\tau \times K_i$ such that Σ_ξ is directed by ∂_τ on $U(K_i)$. (In particular, this means that $\{-1\} \times K_i \subset W_i$ and $\{1\} \times K_i \subset W_{i+1}$.)

STEP 1. *Construct the contact form on* $\mathbb{R} \times (\Sigma \setminus \cup_{i=1}^{2m-1} U(K_i))$.

Let $W_i^\circ := W_i \setminus (U(K_{i-1}) \cup U(K_i))$. After possibly rescaling α by a positive function as in Proposition 2.3.3, we may assume that $\beta_i := \alpha|_{W_i^\circ}$ is Liouville for all i. (Here $d\beta_i > 0$ for i odd and $d\beta_i < 0$ for i even.) Moreover, we can arrange so that the Liouville vector field X_{β_i} equals $\partial_\tau/(2\tau)$ near $\partial U(K_i)$ if i is even, and equals $-\partial_\tau/(2\tau)$ if i is odd. (This is a purely technical arrangement which makes the gluing of contact forms below easier.) We define

$$\alpha_\Sigma := (-1)^{i+1} dt + \beta_i \tag{8.2.1}$$

on $\mathbb{R} \times (\Sigma \setminus \cup_{i=1}^{2m-1} U(K_i))$.

STEP 2. *Construct the contact form on* $\mathbb{R} \times U(K_i)$ *for i even.*

In this case K_i is minimal. Assume without loss of generality that $\alpha|_{U(K_i)} = e^{\tau^2}\lambda$, where λ is a contact form on K_i. We will choose α_Σ of the form

$$\alpha_\Sigma = -f(\tau)dt - tg(\tau)d\tau + e^{\tau^2}\lambda \tag{8.2.2}$$

on $\mathbb{R} \times U(K_i)$. Clearly $\alpha_\Sigma|_{U(K_i)} = \alpha|_{U(K_i)}$. A straightforward computation shows that α_Σ is positively contact if and only if

$$f' - 2\tau f - g > 0. \tag{8.2.3}$$

We choose f to be a decreasing odd function which equals ± 1 when τ is close to ∓ 1, and then choose g to be a nonpositive even function which equals 0 when τ is close to ± 1, subject to (8.2.3); see Figure 8.2.1.

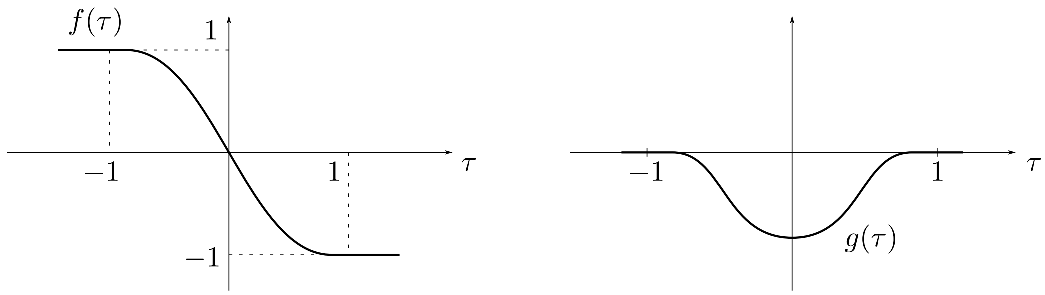

FIGURE 8.2.1. The graph of functions used in the contact form given by Eq. (8.2.2).

For later use, note that α_Σ restricts to the Liouville form $\beta_{i,t} = -tg(\tau)d\tau + e^{\tau^2}\lambda$ on $\{t\} \times (U(K_i) \setminus K_i)$ for any $t \in \mathbb{R}$. We compute the Liouville vector fields

$$X_{\beta_{i,t}} = 1/(2\tau)(\partial_\tau + te^{-\tau^2} g(\tau) R_\lambda),$$

where R_λ denotes the Reeb vector field on (K_i, λ).

It follows that

$$U(K_i)_{t,\xi} := (\{t\} \times U(K_i))_\xi = \partial_\tau + te^{-\tau^2} g(\tau) R_\lambda. \tag{8.2.4}$$

STEP 3. *Construct the contact form on $\mathbb{R} \times U(K_i)$ for i odd.*

In this case K_i is maximal. This step is analogous to the construction of the contact form on $\Gamma \times [-1, 1]$ in the proof of Proposition 2.3.3. Assume without loss of generality that $\alpha|_{U(K_i)} = e^{-\tau^2}\lambda$. We define the 1-form

$$\alpha_\Sigma = f(\tau)dt + e^{-\tau^2}\lambda \tag{8.2.5}$$

on $\mathbb{R} \times U(K_i)$. Since α_Σ is positively contact if and only if $-f' - 2\tau f > 0$, taking $f(\tau)$ as in Figure 8.2.1 suffices.

We compute the Liouville vector fields $X_{\beta_{i,t}} = -1/(2\tau)\partial_\tau$ on $\{t\} \times (U(K_i) \setminus K_i)$, and note that it is independent of t. In fact $U(K_i)$ is convex with respect to the contact vector field ∂_t.

Combining (8.2.1), (8.2.2), and (8.2.5), we obtain a contact form α_Σ on $\mathbb{R} \times \Sigma$ such that $\alpha_\Sigma|_\Sigma = \alpha|_{\Sigma_0}$, up to rescaling by a positive function.

Remark 8.2.1. A crucial difference between the normal forms of contact structures near a convex hypersurface and a folded Weinstein hypersurface is the following: For convex hypersurfaces, since ∂_t is a transverse contact vector field, any small neighborhood of Σ is in fact contactomorphic to the entire $\mathbb{R}_t \times \Sigma$. On the other hand, the above-constructed α_Σ is not t-invariant, and hence only specified by the data on Σ for $|t|$ sufficiently small.

8.3. **Bypass attachment as a bifurcation.** Let ξ be a contact structure on $\Sigma \times [0,1]$ such that $\Sigma \times \{0,1\}$ are Weinstein convex, or equivalently Morse$^+$. Then by Proposition 2.3.3, Σ_t is Morse$^+$ for all $t \in [0,\epsilon) \cup (1-\epsilon, 1]$ with $\epsilon > 0$ small. It will be shown in Section 9 that after a boundary-relative isotopy, Σ_t is 1-Morse for all $t \in [0,1]$. However, the 1-Morse$^+$ condition may fail at isolated instances. In particular, there may exist a first instance $t_0 > 0$ such that Σ_t is convex for any $t \neq t_0$ sufficiently close to t_0 but there exists a "retrogradient" flow line of $(\Sigma_{t_0})_\xi$ from a negative index n singularity to a positive one. Such a phenomenon is called a *bifurcation* of the characteristic foliation in [Gir00].[2] As we will see, crossing such t_0 corresponds precisely to a bypass attachment as introduced in [HH].

To set up the "bypass–bifurcation correspondence", it is convenient to reformulate the bypass attachment in the language of folded Weinstein hypersurfaces.

8.3.1. *Bypass attachments.* We briefly review bypass attachments from [HH], leaving the details of contact handle attachments and Legendrian (boundary) sums to [HH].

Let Σ be a Weinstein convex hypersurface with the usual decomposition $\Sigma \setminus \Gamma = R_+ \cup R_-$. A *bypass attachment data* $(\Lambda_\pm; D_\pm)$ is given as follows: Let $D_\pm \subset R_\pm$ be Lagrangian disks with cylindrical ends which are regular in the sense of [EGL18], i.e., the complement in $R_\pm$ of a standard neighborhood of $D_\pm$ is still Weinstein. Let $\Lambda_\pm = \partial \overline{D}_\pm$ be Legendrian spheres in Γ equipped with the contact form $\alpha|_\Gamma$, which we assume have a unique $\xi|_\Gamma$-transversal intersection point (i.e., they intersect transversely when projected to $\xi|_\Gamma$) .

Next we discuss Reeb pushoffs. If Λ is a Legendrian submanifold of Γ, then let Λ^ϵ be the Reeb pushoff of Λ in the Reeb direction by ϵ. Clearly Λ^ϵ is embedded for $|\epsilon|$ sufficiently small. Moreover, if Λ bounds a Lagrangian disk D in some Weinstein filling, then there exists a corresponding Lagrangian D^ϵ in the same filling with $\partial D^\epsilon = \Lambda^\epsilon$.

We now explain how to attach a bypass to Σ using the bypass attachment data $(\Lambda_\pm; D_\pm)$ to obtain a contact structure on $\Sigma \times [0,1]$. The bypass attachment is a smoothly canceling pair of contact handle attachments in the middle dimensions. The first is a contact n-handle attachment to Σ_0 along the Legendrian sphere $\Lambda_- \uplus \Lambda_+ \subset \Gamma$ obtained by Legendrian sum. This step produces a new convex hypersurface S. It turns out the pushoffs $\Lambda_\pm^{\mp\epsilon}$ of $\Lambda_\pm$ become Legendrian isotopic when viewed on Γ_S. Hence we can attach a contact $(n+1)$-handle to S along the Legendrian sphere that we denote by $D_+^{-\epsilon} \cup D_-^{+\epsilon}$ and is obtained by gluing $D_+^{-\epsilon}$ and $D_-^{+\epsilon}$ via the Legendrian isotopy.

[2]Since Morse-Smale vector fields are considered in [Gir00], there exists a different kind of bifurcation where a pair of periodic orbits appear or disappear. This phenomenon does not occur here since we are dealing with Morse gradient vector fields.

Remark 8.3.1. It is not necessary to assume that $D_\pm$ are regular in the definition of a bypass attachment. It is an outstanding, and of course hard, problem to even find an irregular Lagrangian disk in any Weinstein domain. One consequence of our work in this paper is that, as far as convex hypersurface theory and open book decompositions are concerned, one can completely stay in the world of Morse theory, e.g., avoid using any irregular Lagrangian disks, regardless of their very existence, without losing any generality. Compare this with the work of Lazarev [Laz20] which proves the existence h-principle for regular Lagrangians.

Let $(\Sigma \times [0,1], \xi)$ be the contact manifold resulting from the bypass attachment. Write $\Sigma_t \coloneqq \Sigma \times \{t\}$, where $\Sigma = \Sigma_0$. We have the usual decomposition $\Sigma_i \setminus \Gamma_i = R_+^i \cup R_-^i$, $i = 0, 1$. Then by [HH, Theorem 5.1.3]:

- R_+^1 is obtained from R_+^0 by removing a standard neighborhood of $D_+^{-\epsilon}$ and attaching a Weinstein handle along $\Lambda_- \uplus \Lambda_+$.
- R_-^1 is obtained from R_-^0 by removing a standard neighborhood of $D_-^{+\epsilon}$ and attaching a Weinstein handle along $\Lambda_- \uplus \Lambda_+$.
- Γ_1, viewed as the boundary of R_+^1, is obtained from Γ_0 by a contact $(+1)$-surgery along $\Lambda_+^{-\epsilon}$ and a contact (-1)-surgery along $\Lambda_- \uplus \Lambda_+$. Γ_1, viewed as the boundary of R_-^1, is obtained from Γ_0 by a contact $(+1)$-surgery along $\Lambda_-^{+\epsilon}$ and a contact (-1)-surgery along $\Lambda_- \uplus \Lambda_+$. These two presentations of Γ_1 are canonically identified by a handleslide.

8.3.2. *Folded Weinstein description.* We will now describe the Morse$^+$ hypersurface Σ as a folded Weinstein hypersurface.

Let $W_1 \subset R_+$ be the Weinstein subdomain obtained by digging out a standard neighborhood of $D_+^{-\epsilon}$. Then $D_+^{-\epsilon}$ is the unstable manifold of an index n critical point q_+ with respect to the Liouville flow on R_+ and R_+ can be viewed as the concatenation $W_1 \cup W_3'$, where W_3' is a Weinstein cobordism with a unique critical point q_+. Similarly, let $W_4 \subset R_-$ be the Weinstein subdomain such that R_- is the concatenation of W_4 and a Weinstein cobordism W_2' with a unique index n critical point q_-, whose unstable manifold is D_-^ϵ with respect to the Liouville flow on R_-. Since $\Lambda_\pm^{\mp\epsilon} = \partial D_\pm^{\mp\epsilon}$ are disjoint, we can shuffle the critical values of $q_\pm$ to obtain the following decomposition

$$\Sigma = W_1 \cup_{K_1} W_2 \cup_{K_2} W_3 \cup_{K_3} W_4,$$

where W_2, W_3 are Weinstein cobordisms (slight variants of W_2', W_3') associated with the critical points q_-, q_+, respectively. See Figure 8.3.1.

In particular we have:

(FBP1) As contact manifolds, K_1, oriented as ∂W_1 (and also as ∂W_2), is obtained from Γ by a contact $(+1)$-surgery along $\Lambda_+^{-\epsilon}$; K_3, oriented as ∂W_3 (and also as ∂W_4), is obtained from Γ by a contact $(+1)$-surgery along Λ_-^ϵ; and K_2, oriented as $-\partial W_2$ (and also as $-\partial W_3$), is obtained from Γ by contact $(+1)$-surgeries along $\Lambda_+^{-\epsilon}$ and Λ_-^ϵ.

(FBP2) Let $(D_+^{-\epsilon})^\dagger$ and $(D_-^\epsilon)^\dagger$ be the stable manifolds of q_+ and q_- in W_3 and W_2 with respect to the Liouville flows. Then $(D_+^{-\epsilon})^\dagger \cap K_2 = \Lambda_+^{-\epsilon}(K_2)$ and

$(D_-^\epsilon)^\dagger \cap K_2 = \Lambda_-^\epsilon(K_2)$, where we write $\Lambda_+^{-\epsilon}(K_2)$ and $\Lambda_-^\epsilon(K_2)$ for the core Legendrians of the contact $(+1)$-surgeries.

(FBP3) The $\xi|_\Gamma$-transverse intersection point between Λ_+ and Λ_- turns into a short Reeb chord $\gamma \subset K_2$ from $\Lambda_+^{-\epsilon}(K_2)$ to $\Lambda_-^\epsilon(K_2)$.

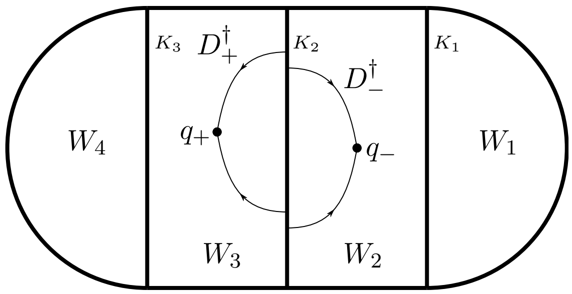

FIGURE 8.3.1. A Morse$^+$ hypersurface viewed as a folded Weinstein hypersurface.

8.3.3. *Bypass-bifurcation correspondence.*

Proposition 8.3.2 (Bypass–bifurcation correspondence)**.** *Let* $(\Sigma\times[-\delta,\delta],\alpha_\Sigma)$ *with* $\delta > 0$ *small be a standard neighborhood of a* $2n$*-dimensional folded Weinstein hypersurface*

$$\Sigma_0 = W_1 \cup_{K_1} W_2 \cup_{K_2} W_3 \cup_{K_3} \cup W_4, \tag{8.3.1}$$

where we are writing $\Sigma_t := \Sigma \times \{t\}$ *as before, such that:*

(F1) W_2 *and* W_3 *are Weinstein cobordisms associated with* $\mathrm{ind} = n$ *negative and positive critical points* q_- *and* q_+*. Let* $D^\dagger_{-,0}$ *and* $D^\dagger_{+,0}$ *(resp.* $D_{-,0}$ *and* $D_{+,0}$*) be the stable (resp. unstable) manifolds of* q_- *and* q_+ *in* W_2 *and* W_3 *with respect to the Liouville flows.*

(F2) $\Lambda'_- := \partial D^\dagger_{-,0}$ *and* $\Lambda'_+ := \partial D^\dagger_{+,0}$ *are Legendrians which intersect at a unique* $\xi|_{K_2}$*-transversal point.*

Then the contact manifold $(\Sigma\times[-\delta,\delta],\ker\alpha_\Sigma)$ *is contactomorphic, relative boundary, to the bypass attachment to* $\Sigma_{-\delta}$ *along a quadruple* $(\Lambda_\pm; D_\pm)$*.*

Under the conditions of Proposition 8.3.2,

(Fa) Σ_t is convex for all $t \neq 0$ by (8.2.1), (8.2.4), and (8.2.5);

(Fb) Σ_t for $t < 0$ (resp. $t > 0$) has a folded Weinstein structure which satisfies (8.3.1) and (F1) with stable and unstable manifolds $D^\dagger_{-,t}$, $D^\dagger_{+,t}$, $D_{-,t}$, $D_{+,t}$, and $\partial D^\dagger_{-,t}$ is a positive (resp. negative) Reeb pushoff of $\partial D^\dagger_{-,0}$ and $\partial D^\dagger_{+,t}$ is a negative (resp. positive) Reeb pushoff of $\partial D^\dagger_{+,0}$.

Moreover, comparing with the notation from §8.3.2, $\Sigma = \Sigma_{-\delta}$, $D_+^{-\epsilon} = D_{+,-\delta}$, $D_-^\epsilon = D_{-,-\delta}$, $(D_+^{-\epsilon})^\dagger = D^\dagger_{+,-\delta}$, $(D_-^\epsilon)^\dagger = D^\dagger_{-,-\delta}$, and $\Lambda_\pm^{\mp\epsilon} = \partial D_\pm^{\mp\epsilon}$. We then take $\Lambda_\pm$ and $D_\pm$ so that $\Lambda_\pm^{\mp\epsilon}$ and $D_\pm^{\mp\epsilon}$ are pushoffs of $\Lambda_\pm$ and $D_\pm$. We will use both sets of notations interchangeably.

Since the proof of Proposition 8.3.2 is somewhat complicated, we start by explaining the key ideas involved and also highlight the difference between the usual 3-dimensional strategy and the higher-dimensional approach.

First note that the bypass attachment is a local operation, i.e., the hypersurface is only affected in a neighborhood of $D_+ \cup D_-$. Let $B \subset \Sigma$ be a small neighborhood of $D_+ \cup D_-$ which is diffeomorphic to a ball. The question is then reduced to understanding the contact structure on $B \times I$ given by the bypass attachment, where $I = [-\delta, \delta]$.

At this point, two "miracles" happen in dimension 3 (i.e., $\dim \Sigma = 2$) which greatly simplify the 3-dimensional proof. The first is that one can take ∂B to be Legendrian using the *Legendrian realization principle* (see [Hon00, Theorem 3.7]). This gives us good control over the contact structure near $\partial B \times I$. The second, and more significant, miracle is Eliashberg's theorem (see [Eli92, Theorem 2.1.3]) on the uniqueness of tight contact structures on the 3-ball. Using these two facts, one can prove Proposition 8.3.2 in dimension 3 by arguing that both the bifurcation and the bypass attachment produce tight contact structures on the 3-ball $B \times I$ with the same boundary conditions and hence must coincide.

Unfortunately, both of the above-mentioned miracles fail in dimension > 3: the first one fails for dimensional reasons and the second one fails by results of [Eli91, Ust99a]. Nevertheless, the proof of Proposition 8.3.2 follows the same general outline as in dimension 3 by replacing the Legendrian boundary condition on ∂B by a transverse boundary condition and Eliashberg's theorem by a direct proof that both the bifurcation and the (trivial) bypass attachment produce the standard ball in a Darboux chart.

Proof of Proposition 8.3.2. The proof follows the above outline and consists of several steps.

STEP 1. *Localizing the problem to B.*

By (FBP1), the contact manifold K_2 is obtained from Γ by a contact $(+1)$-surgeries along $\Lambda_+^{-\epsilon}$ and Λ_-^{ϵ}; we are also viewing K_2 as a submanifold of Σ_t for all $t \in I$. Let $\Lambda'_\pm := \partial D^\dagger_{\pm,0} \subset K_2 \subset \Sigma_0$ be the Legendrian spheres which $\xi|_{K_2}$-transversely intersect at one point and let $D^\dagger_{\pm,t} \subset \Sigma_t$ be the Lagrangian disks in W_3 and W_2 from (Fb) with boundary $\Lambda'_{\pm,t}$.

We now describe a small closed neighborhood B of $D^\dagger_{+,0} \cup D^\dagger_{-,0}$ in Σ_0. We take $B \cap K_2$ to be a small contact handlebody neighborhood $C_2 = [-\kappa, \kappa]_z \times A_2$, $\kappa > 0$ small, of $\Lambda'_+ \cup \Lambda'_-$, where A_2 is the plumbing of two copies of disk bundles $\mathbb{D}^* S^{n-1}$ with the canonical Liouville form and $\Lambda'_\pm$ are the 0-sections of the corresponding $\mathbb{D}^* S^{n-1}$ in $\{0\} \times A_2$. The restriction of B to a tubular neighborhood $[-1,1]_\tau \times K_2$ with the 1-form $e^{\tau^2}\lambda$ is $[-1,1] \times C_2$. Then B is obtained from $[-1,1] \times C_2$ by attaching Weinstein handles along $\{-1\} \times \Lambda'_-$ and $\{1\} \times \Lambda'_+$. The boundary decomposes as $\partial B = C_1 \cup C_h \cup C_3$, where C_1 (resp. C_3) is the compact contact manifold obtained from C_2 by a contact (-1)-surgery along $\{-1\} \times \Lambda'_-$ (resp.

$\{1\} \times \Lambda'_+$) and $C_h = [-1,1] \times \partial C_2$. We are viewing $B \subset W_2 \cup W_3$, $C_1 \subset K_1$, and $C_3 \subset K_3$.

Note that the $C_i, i = 1,2,3$, are all contactomorphic since applying a contact (-1)-surgery along $\{0\} \times \Lambda'_+$ to $[-\kappa,\kappa] \times T^*\Lambda'_+$ still yields $[-\kappa,\kappa] \times T^*\Lambda'_+$. The case for $\{0\} \times \Lambda'_-$ is identical.

The characteristic foliation B_ξ is inward-pointing along C_1, outward-pointing along C_3, and tangent to C_h. By slightly tilting C_h, we may assume that B_ξ is outward-pointing along $C_3 \cup C_h$ and inward-pointing along C_1. This results in a fold-type tangency roughly along ∂C_1.

Moreover, for $\delta > 0$ sufficiently small, we can construct parallel copies $B_t \subset \Sigma_t, t \in I$, of $B = B_0$ such that the characteristic foliation is t-invariant near ∂B_t. The copies B_t are obtained from $[-1,1] \times A_2$ by attaching cores of the handles along $\{-1\} \times \Lambda'_{-,t}$ and $\{1\} \times \Lambda'_{+,t}$, and $\delta > 0$ small ensures that we can attach handles (i.e., the thickened cores) to $\{\pm 1\} \times C_2$ in a manner that varies smoothly with t.

By using certain folding techniques similar to (and in fact simpler than) those in Section 5, one can reverse the direction of the characteristic foliation on C_1 through an isotopy of B in a suitably wiggled Σ such that Σ_ξ is everywhere outward-pointing along ∂B. This will be achieved in Step 3. The folding technique is called the *Creation Lemma* which in dimension 3 is the converse of the usual *Elimination Lemma* (see [Gei08, §4.6.3]). This is described in Step 2.

STEP 2. *The Creation Lemma.*

In this step, we describe the effect of applying a C^0-small perturbation called a *box-fold*. This is the content of the Creation Lemma, which we do not state formally.

We closely follow the discussion of Section 5, except that we replace t by $\tilde{t}$ here, since we are already using t to parametrize the hypersurfaces Σ_t. Consider $\mathbb{R}^3_{z,s,\tilde{t}} \times V$ equipped with the contact form $\alpha = dz + e^s(d\tilde{t} + \lambda)$, where (V,λ) is a complete Weinstein manifold. Let $F := \{z = 0\}$ be the hypersurface on which we will create singularities. Clearly $F_\xi = \partial_s$ where $\xi = \ker \alpha$.

Fix $z_0, s_0, \tilde{t}_0 > 0$. Let $\Pi^3 \subset \mathbb{R}^3_{z,s,\tilde{t}}$ be a surface obtained from the flat $\mathbb{R}^2_{s,\tilde{t}}$ by growing a box with base $\square := [0,s_0] \times [0,\tilde{t}_0]$ and height z_0; see Figure 8.3.2. Of course, as in the construction of Z in §3.2, one needs to smooth the corners of Π^3 and "Morsify" the resulting characteristic foliation Π^3_ξ. These operations are suppressed from the notation. We say Π^3 is obtained from $\mathbb{R}^2_{s,\tilde{t}}$ by a 3-dimensional *box-fold*.

Comparing Figure 8.3.2 with Figure 3.1.1, we note that the key difference is that Π^3_ξ admits only positive singularities: one source and one saddle. See Figure 8.3.3. This is the content of the Creation Lemma in dimension 3.

In higher dimensions, we consider the hypersurface $\Pi^3 \times V^c$, where $V^c \subset V$ is the compact domain, i.e., $V \setminus V^c \cong [0,\infty)_\tau \times \partial V^c$ is symplectomorphic to a half-symplectization of ∂V^c. Following the strategy from Section 5, define the general *box-fold* Π to be the hypersurface which extends $\Pi^3 \times V^c$ by damping out

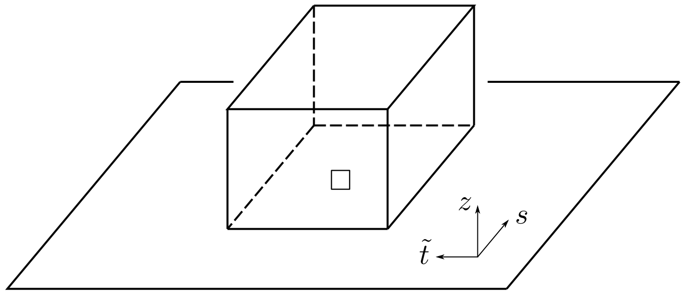

FIGURE 8.3.2. The PL model of Π^3.

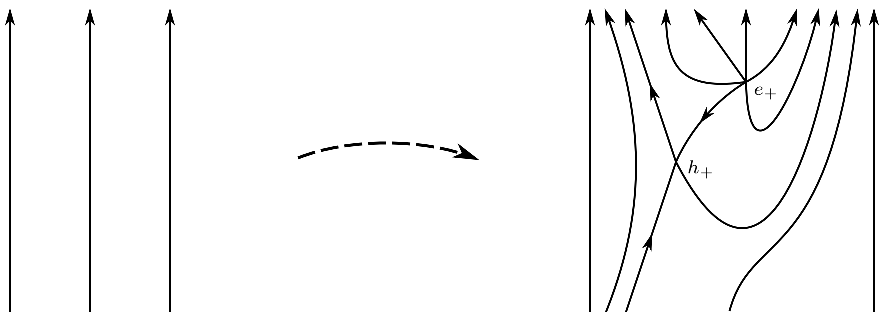

FIGURE 8.3.3. The characteristic foliation before and after the box folding.

the Π^3-factor on $\mathbb{R}^3 \times [0, \tau_1] \times \partial V^c$ as τ increases. Then Eq. (5.2.1) implies that Π_ξ is Morse with a pair of canceling critical points for each one in V^c. In particular, let $D \subset \Pi^3$ be a disk containing the source e_+ such that Π^3_ξ is transverse to ∂D. Then Π_ξ is everywhere outward-pointing along $\partial(D \times V^c)$. Note that instead of creating a pair of canceling critical points as in dimension 3 (see [CE12, Proposition 12.21] for the higher-dimensional version), our Creation Lemma produces many pairs of canceling critical points at once, in fact as many as the number of critical points of V.

STEP 3. *Modification from B to $\widehat{B}$.*

In Step 1 we constructed the family $B_t \subset \Sigma_t$, $t \in I$, such that $\Sigma_{t,\xi}$ is inward-pointing along C_1. The goal of this step is to modify B_t (and $\Sigma_{t,\xi}$) to $\widehat{B}_t$ so that $\Sigma_{t,\xi}$ is outward-pointing along $\partial \widehat{B}_t$. Write $C_1 = [-\kappa, \kappa]_{\tilde{t}} \times A_2$, where $\kappa > 0$ is sufficiently small. Since Σ_ξ points into B along C_1, we can choose $t_0, s_0 > 0$ so that there exists an embedding

$$U(C_1) \coloneqq I_{t_0} \times [0, s_0] \times C_1 \subset M, \quad I_{t_0} = [-t_0, t_0],$$

such that $t_0 \ll \delta$, $U(C_1) \cap B_t = \{(t, s_0)\} \times C_1$, and $\{t\} \times [0, s_0] \times C_1 \subset W_1$. Write the contact form as $\alpha|_{U(C_1)} = dt + e^s(d\tilde{t} + \lambda)$, where λ is the standard Liouville form on A_2. *Here we are identifying the z-coordinate from the previous step with t.*

We then apply the Creation Lemma with $V = A_2$ to install/uninstall a smoothed box-fold along $[0, s_0] \times C_1$, which we are assuming is contained in W_1. This is the higher-dimensional analog of the procedure in Section 4.4 (see Figure 4.4.1). More precisely, we modify the foliation of $\Sigma \times I$ by leaves Σ_t to obtain a C^0-close foliation by leaves still denoted by Σ_t such that the following hold:

(i) $\Sigma_{t,\xi}$ is unchanged for $t \in \partial I$ and on $W_2 \cup W_3 \cup W_4$ for $t \in I$.
(ii) The box-fold is installed along $[0, s_0] \times C_1$ for $t \in [-\delta, -t_0]$ and is uninstalled along $[0, s_0] \times C_1$ for $t \in [t_0, \delta]$. The characteristic foliation on W_1 is t-invariant for $t \in I_{t_0}$. (This is possible since the smoothed box-folds can be taken to be graphical.)
(iii) For $t \in I_{t_0}$, W_1 contains a subdomain symplectomorphic to $D \times A_2$, where $D \subset \mathbb{R}^2$ is a disk containing e_+ as in Step 2 and there exists an arc $\mu \subset \partial D$ such that $\mu \times A_2$ is identified with $\{s_0\} \times C_1$; see Figure 8.3.4.

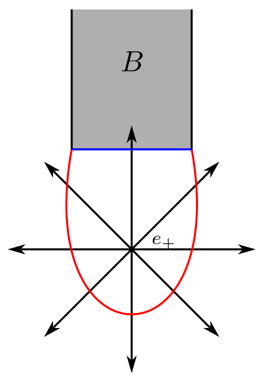

FIGURE 8.3.4. A schematic picture for isotoping B to $\widehat{B}$ so it encompasses e_+. The blue arc represents μ and the red arc represents $\partial D \setminus \mu$.

In order to achieve the transversal boundary condition on $B_t, t \in I_{t_0}$, it remains to isotop μ through D to $\partial D \setminus \mu$ and use the fact from Step 2 that $\Sigma_{t,\xi}$ is everywhere transverse to $\partial(D \times A_2)$, to obtain the new $\widehat{B}_t \subset \Sigma_t$ such that $\Sigma_{t,\xi}$ is everywhere outward-pointing along $\partial \widehat{B}_t$. In particular $\partial \widehat{B}_t$, $t \in I_{t_0}$, are contact submanifolds of M.

Remark 8.3.3. Similar ideas will be exploited in greater generality in Section 11.

Claim 8.3.4. *$\Sigma_{t,\xi}$ is 1-Morse for all $t \in I$ and 1-Morse$^+$ (hence Σ_t is convex) for $t \neq 0$.*

Proof of Claim 8.3.4. By the folded Weinstein structure for Σ_t and the fact that installing/uninstalling the box-fold induces a Weinstein homotopy on W_1, it follows that $\Sigma_{t,\xi}$ is 1-Morse for all $t \in I$. As for the 1-Morse$^+$ property, it suffices to consider the stable manifold of the unique singular point in W_3. There is a stable flow line that comes from a negative singularity precisely when $t = 0$ (this is the same as the situation before Σ_t was perturbed). The convexity of Σ_t, $t \neq 0$, then follows from Proposition 2.3.3. □

Hence we may restrict attention to the new Σ_t, $t \in I_{t_0}$.

STEP 4. *Triviality of the contact structure on $I_{t_0} \times \widehat{B}$ for $t_0 > 0$ small.*

Let $S := \partial(I_{t_0} \times \widehat{B}) = \widehat{B}_{-t_0} \cup \widehat{B}_{t_0} \cup (I_{t_0} \times \partial\widehat{B})$. Suppose $t_0 > 0$ is small.

Claim 8.3.5. *After corner rounding, S is convex and $R_\pm(S)$ are Weinstein homotopic to the standard ball with a unique critical point.*[3]

Proof of Claim 8.3.5. For $R_+(S)$ we describe the positive critical points and the stable manifolds of these critical points; the situation for $R_-(S)$ is similar.

The critical points of $R_+(S)$ are as follows:

(1) sitting over e_+ in $\widehat{B}_{t_0}$ are one index 0 critical point q_0 and two index $(n-1)$ critical points $q_\pm$ corresponding to $\Lambda'_\pm \subset A_2$; and
(2) the critical points p_+ on $\widehat{B}_{t_0} \cap W_3$ and p_- on $\widehat{B}_{-t_0} \cap W_2$ have index n, where $+$ indicates being on the "top sheet" $\widehat{B}_{t_0}$.

We denote the analogous critical points of $R_-(S)$ by $q'_0, q'_\pm, p'_\pm$.

We denote the stable manifold of a critical point p by $\mathscr{W}_p$. For definiteness, we assume that there exists $\epsilon' > 0$ small such that

(i) $\mathscr{W}_{p_+}, \mathscr{W}_{p'_+}$ intersect $C_2 \subset \widehat{B}_{a_\gamma+t_0}$ along the pushoffs $(\Lambda'_+)^{2\epsilon'}$, $(\Lambda'_-)^{-2\epsilon'}$;
(ii) $\mathscr{W}_{p_-}, \mathscr{W}_{p'_-}$ intersect $C_2 \subset \widehat{B}_{a_\gamma-t_0}$ along the pushoffs $(\Lambda'_-)^{\epsilon'}$, $(\Lambda'_+)^{-\epsilon'}$.

By (i), $\mathscr{W}_{p_+}$ intersects $K_1 = \partial W_1$ along $(\Lambda'_+)^{2\epsilon'}$ and therefore limits to $\Lambda'_+ \subset A_2$ over e_+; moreover, there is a unique flow line from p_+ to q_+. Next, $\mathscr{W}_{p_-}$ intersects C_3 along a Legendrian which is isotopic to a positive pushoff of Λ'_-, continues inside $\widehat{B}_{a_\gamma+t_0}$ to a Legendrian isotopic to a positive pushoff of Λ'_- on K_1, and limits to $\Lambda'_- \subset A_2$ over e_+. Moreover, there is a unique flow line from p_- to q_-. This implies that S_ξ is Morse$^+$ and convex, with Weinstein structures on $R_\pm(S)$ just described.

The index $(n-1)$ and index n critical points cancel in pairs and $R_\pm(S)$ are Weinstein homotopic to the standard ball with a unique critical point. □

Remark 8.3.6. The reader might find it instructive to consider the $n = 1$ (i.e., $\dim M = 3$) case, where we have three index 0 critical points "sitting over e_+".

Now consider the 1-parameter family of pairwise disjoint homotopy spheres $S_{\widetilde{t}}$, $\widetilde{t} \in [0, \frac{3t_0}{2}]$, that are slight perturbations of $\partial([-t_0 + \widetilde{t}, t_0] \times \widehat{B})$ and such that the region G bounded by $S_{3t_0/2}$ is t-invariant.

Claim 8.3.7. *The homotopy spheres $S_{\widetilde{t}}$, $\widetilde{t} \in [0, \frac{3t_0}{2}]$, can be made simultaneously convex after a small perturbation.*

[3] For the moment we only know that S is a homotopy sphere. Presumably S is a standard sphere, although we will not need this.

Proof of Claim 8.3.7. We use the argument of Claim 8.3.5, but this time there are two moments t_0' and t_0 with $t_0' < t_0$, where a bifurcation occurs as explained in the next paragraph. There is still a unique flow line from p_+ to q_+ for all $\widetilde{t}$.

Next we describe the trajectories of $\mathscr{W}_{p_-}$ for $\widetilde{t} \in [0, \frac{3t_0}{2}]$. When $\widetilde{t} = t_0'$, all the trajectories of $\mathscr{W}_{p_-}$ reach C_3 but one continues to the critical point of $\widehat{B}_{t_0} \cap W_3$; when $\widetilde{t} = t_0$, a flow line of $\mathscr{W}_{p_-}$ limits to the critical point of $\widehat{B}_{a_\gamma} \cap W_3$. We have:

(1) For $\widetilde{t} < t_0'$, $\mathscr{W}_{p_-} \cap K_1$ is Legendrian isotopic to a positive pushoff of Λ_-'.
(2) For $\widetilde{t} > t_0$, $\mathscr{W}_{p_-} \cap K_1$ is Legendrian isotopic to a negative pushoff of Λ_-'.
(3) For $t_0' < \widetilde{t} < t_0$, $\mathscr{W}_{p_-} \cap K_1$ is Legendrian isotopic to $\Lambda_+' \uplus \Lambda_-'$.
(4) For $\widetilde{t} = t_0', t_0$, $\overline{\mathscr{W}}_{p_-} \cap K_1$ is Legendrian isotopic to $\Lambda_+' \cup \Lambda_-'$ intersecting at a point and $\mathscr{W}_{p_-} \cap K_1$ corresponds to $\Lambda_-' \setminus \Lambda_+'$.

Here K_1 is understood to be on $t = t_0$. (3) is a consequence of corner-rounding along C_3, which has the effect of introducing a slight negative Reeb flow along the corner as we go from the bottom sheet to the top. (4) is the limiting configuration of (1)–(3). In all the cases, there is a unique flow line from p_- to q_-, although there may be trajectories from p_- to q_+ for $t_0' < \widetilde{t} < t_0$.

The claim then follows from the usual Elimination Lemma (see [CE12, Proposition 12.22]) and a trick from [Hua13, Lemma 3.3]: By a C^0-small perturbation one can simultaneously eliminate the pairs (p_+, q_+) and (p_-, q_-) on $S_{\widetilde{t}}$ for all $\widetilde{t} \in [0, \frac{3t_0}{2}]$ (since the trajectories from p_+ to q_+ and p_- to q_- vary continuously with respect to $\widetilde{t}$), which in turn implies that all the $S_{\widetilde{t}}$ are convex. $\square$

Finally we observe that the bypass attachment to Σ along $(\Lambda_\pm; D_\pm)$ restricts to the trivial bypass attachment to S in the sense of [HH, Definition 6.1.1]. It follows from [HH, Proposition 8.3.2] that the contact structure on $(I_{t_0} \times \widehat{B}) \setminus G$ given by a trivial bypass attachment is standard. By Claim 8.3.7, ξ on $(I_{t_0} \times \widehat{B}) \setminus G$ is standard, hence is equivalent to a bypass attachment. This finishes the proof of the proposition. $\square$

8.4. **Bypass attachment in terms of partial open book decompositions.**

The goal of this subsection is to summarize the main constructions and results from [HH, Section 8]. The reader is referred to the original paper for details.

8.4.1. *Partial open book decompositions.*

Definition 8.4.1. *Given a Weinstein domain S, a* cornered Weinstein subdomain *$W \subset S$ is a (possibly empty) codimension 0 submanifold with corners which satisfies the following properties:*

(CW1) There exists a decomposition $\partial W = \partial_{\text{in}} W \cup \partial_{\text{out}} W$ such that
- *(1) $\partial_{\text{in}} W$ and $\partial_{\text{out}} W$ are compact manifolds with smooth boundary that intersect along their boundaries;*
- *(2) $\partial(\partial_{\text{in}} W) = \partial(\partial_{\text{out}} W)$ is the codimension 1 corner of ∂W; and*
- *(3) $\partial_{\text{out}} W = W \cap \partial S$ and is a proper subset of each component of ∂S.*

(CW2) The Liouville vector field X_λ on S is inward-pointing along $\partial_{\text{in}} W$ and outward-pointing near $\partial_{\text{out}} W$.

(CW3) $S \setminus W$ *is a Weinstein domain after smoothing.*

A particularly useful class of cornered Weinstein subdomains consists of regular neighborhoods of Lagrangian cocore disks in S.

Definition 8.4.2. *A* Weinstein partial open book decomposition (POBD) *is a pair* $([0,1] \times S, \phi : W_1 \xrightarrow{\sim} W_0)$*, where:*

(1) $[0,1]_t \times S$ *is a generalized contact handlebody (with* $S_t := \{t\} \times S$ *Weinstein);*
(2) $W_0 \subset S_0$ *and* $W_1 \subset S_1$ *are cornered Weinstein subdomains; and*
(3) ϕ *is a partial monodromy map which preserves the Liouville forms and restricts to the identity map* $\partial_{\mathrm{out}} W_1 \xrightarrow{\sim} \partial_{\mathrm{out}} W_0$.

By taking the quotient of $[0,1] \times S$ by the map ϕ and filling in $D^2 \times \partial S$ as in the closed case (note that this yields a concave sutured contact manifold and we need to round the concave sutures; see [CGHH11, Section 4.2] for details) we obtain a compact contact manifold with Weinstein convex boundary.

8.4.2. *Bypass attachment in terms of POBDs.* We follow the recipe from [HH, Section 8.3] to interpret a bypass attachment in terms of POBDs.

Lemma 8.4.3 ([HH], Proposition 8.3.1)**.** *Let* (M, ξ, Γ) *be a compact contact manifold with Weinstein convex boundary associated to the Weinstein POBD* $([0,1] \times S, \phi : W_1 \to W_0)$*. If* $(M^\flat, \xi^\flat, \Gamma^\flat)$ *is the contact manifold obtained by attaching a bypass to* M *along* ∂M *with data* $(\Lambda_\pm; D_\pm)$*, where* $D_\pm \subset R_\pm$ *are regular in the sense of* [EGL18]*, then* $(M^\flat, \xi^\flat, \Gamma^\flat)$ *is associated to the Weinstein POBD* $([0,1]_t \times S^\flat, \phi^\flat : W_1^\flat \to W_0^\flat)$ *satisfying:*

(PO1) $S^\flat$ *is obtained from* S *by attaching a Weinstein handle along* $\Lambda_- \uplus \Lambda_+$*; this is done independently of* t*;*
(PO2) $W_1^\flat = W_1 \sqcup N_{\epsilon/2}(D_+^{-\epsilon}) \subset S^\flat$*, where* $\epsilon > 0$ *is small and* $N_{\epsilon/2}(D_+^{-\epsilon})$ *is a cornered Weinstein subdomain which is a standard* $\epsilon/2$*-neighborhood of the Lagrangian disk* $D_+^{-\epsilon}$*;*
(PO3) $\phi^\flat : W_1^\flat \to W_0^\flat$ *is determined by specifying the Lagrangian disk* $\phi^\flat(D_+^{-\epsilon}) \subset S_0^\flat$*: Start with the Lagrangian disk* $D_-^\epsilon \subset S_0^\flat \setminus W_0^\flat$ *with Legendrian boundary* $\Lambda_-^\epsilon \subset \partial S_0^\flat$*. We can slide* Λ_-^ϵ *in the negative Reeb direction across the Weinstein handle along* $\Lambda_- \uplus \Lambda_+$ *so it precisely matches* $\Lambda_+^{-\epsilon} = \partial D_+^{-\epsilon}$*. The sliding is induced by a Weinstein isotopy* $\tau_s : S_0^\flat \xrightarrow{\sim} S_0^\flat$*,* $s \in [0,1]$*, with* $\tau_0 = \mathrm{id}$*. Then we set* $\phi^\flat(D_+^{-\epsilon}) = \tau_1(D_-^\epsilon)$*.*

Note that Lemma 8.4.3 is a direct consequence of the interpretation of a bypass attachment as a smoothly canceling pair of contact handle attachments. If $\dim M = 2n+1$, then the handles have indices n and $n+1$.

8.4.3. *Contact Morse functions and vector fields.* It is helpful, although not technically necessary in this paper, to interpret the contact handle attachments in terms of *contact Morse functions*. This is the contact-topological analog of the correspondence between handle decompositions and Morse function presentations of a given smooth manifold. See Sackel [Sac] for a more thorough discussion of contact Morse functions and a careful construction of contact handles. Sackel also works out the dictionary between contact Morse functions and open book decompositions and our proofs of Corollaries 1.3.1 and 1.3.2 follow the same line of reasoning.

Recall that a vector field v on (M, ξ) is a *contact Morse vector field* if v is gradient-like for some Morse function $f : M \to \mathbb{R}$ and the flow of v preserves ξ, i.e., $\mathscr{L}_v \alpha = g\alpha$ where $\xi = \ker \alpha$ and $g \in C^\infty(M)$. The Morse function f is called a *contact Morse function*. The zeros of v are precisely the critical points of f and hence it makes sense to refer to the (Morse) indices of the zeros of v.

If $(\Sigma \times [0,1], \xi)$ is the contact manifold corresponding to a bypass attachment as above, then there exists a contact Morse vector field v on $\Sigma \times [0,1]$ satisfying the following properties:

(BM1) v is inward-pointing along Σ_0 and outward-pointing along Σ_1.
(BM2) v has exactly two zeros — p of index n and q of index $n+1$ — which are connected by a unique flow line of v.
(BM3) For any $x \in \Sigma \times [0,1]$, the flow line of v passing through x either converges to a zero of v or leaves $\Sigma \times [0,1]$ in both forward and backward time.
(BM4) The unstable manifold of p intersects Σ_0 along the Legendrian $\Lambda_- \uplus \Lambda_+ \subset \Gamma_0$, and the stable manifold of q intersects Σ_1 along the Legendrian $\Lambda_+^{-\epsilon} \subset \Gamma_1$, viewed as the boundary of R_+^1.

Note, however, that in general Σ_0, Σ_1 are not regular level sets of f since contact vector fields are not stable under rescaling by positive functions.

9. C^0-APPROXIMATION BY CONVEX HYPERSURFACES

In this section we complete the proofs of Theorems 7.4.2 and 1.2.5. The main technical ingredient is the higher-dimensional plug constructed in Section 7. In fact our proofs are basically the same as those for the 3-dimensional case discussed in Section 4.

Proof of Theorem 7.4.2. The proof is by induction on the dimension of M.

Fix a Riemannian metric on M. Given a closed hypersurface $\Sigma \subset (M, \xi)$, we may assume that the singularities of Σ_ξ are isolated and Morse after a C^∞-small perturbation. By Theorem 4.2.3 a finite barricade $B_I = \{B_i = [0, s_i] \times Y_i\}_{i \in I}$ exists for Σ_ξ, where each Y_i is a standard $(2n-1)$-dimensional Darboux ball.

Choose $\epsilon' > 0$ much smaller than the sizes of the Y_i. We then replace each B_i with a Y_i-shaped plug with parameter ϵ' constructed in Section 7; note that the construction of the Y_i-shaped plug uses the inductive step of Theorem 7.4.2 for $\dim(M) - 2$. Let $\Sigma^\vee$ be the resulting hypersurface. A trajectory that passes near Y_i is either trapped by Y_i or has holonomy at most ϵ'. Since ϵ' is small, the trajectory that continues is close to the original trajectory on Σ_ξ and will be trapped by some other Y_j by the positioning of the barricade. Hence $\Sigma_\xi^\vee$ satisfies Conditions (M1)–(M3) of Proposition 2.2.6. A further C^∞-small perturbation of $\Sigma^\vee$ will make it convex by Proposition 2.3.3.

The ϵ-Morse$^+$ property is guaranteed if all the $B_i, i \in I$, have diameters $< \frac{\epsilon}{3}$ and all the trajectories of $\Sigma_\xi \setminus (\cup_{i \in I} B_i)$ have lengths $< \frac{\epsilon}{3}$.

Finally, the second statement (for Σ already Weinstein convex) follows from observing that all singularities of all the Y_i-shaped plugs can be "turned on simultaneously"; see Remark 3.2.2 which also holds for higher dimensions. □

Proof of Theorem 1.2.5. Let $(\Sigma \times [0,1], \xi)$ be a contact manifold such that the hypersurfaces Σ_i, $i = 0, 1$, are Weinstein convex. The proof from §4.5 carries over almost verbatim with "surface" replaced by "hypersurface". We use Lemma 4.2.4 to construct refinements of barricades so that we can install/uninstall two sets of barricades.

Suppose for simplicity that the barricade consists of one flow box $B = [0, s_0] \times D^{2n-1}$. Even though the folded hypersurface $\Sigma^\vee$ can be made convex, the intermediate hypersurfaces appearing in the procedure of installing and uninstalling a D^{2n-1}-shaped plug need *not* be Morse. To remedy this defect, we take a cover $D^{2n-1} = \cup_{1 \le i \le K} U_i$ by a finite number of balls of small diameter $\frac{1}{N}$ for which there exists a partition

$$\mathscr{P} : \{1, \dots, K\} \to \{1, \dots, 2n\}$$

such that $U_i \cap U_j = \varnothing$ if $\mathscr{P}(i) = \mathscr{P}(j)$. Now we choose $2n$ pairwise distinct values in $(0, s_0)$ and position U_i along $[0, s_0]$ such that all the U_i with the same $\mathscr{P}$-value have the same s-value. When we install/uninstall all the U_i-shaped plugs, the interior discrepancy for B goes to zero as $N \to \infty$ by the analog of Lemma 4.4.1 and the fact that the ϵ that appears in Theorem 7.5.1 is one order of magnitude smaller than the sizes of U_i.

Finally, proceeding as in §4.5, we can foliate $\Sigma \times [0,1]$ by hypersurfaces of the form Σ_t, which we may assume are 1-Morse by the smallness of the interior discrepancy and Step 1 of §4.5. The only obstruction to convexity occurs when $(\Sigma_t)_\xi$ fails to be Morse$^+$, which occurs at isolated moments by the description of a standard neighborhood of a non-Morse$^+$ folded Weinstein hypersurface in §8.2. The theorem then follows from Proposition 8.3.2. □

10. The existence of (partial) open book decompositions

The goal of this section is to prove Corollary 1.3.1, a stronger/more precise version of Corollary 1.3.2, and Corollary 1.3.3. The proofs are, again, essentially the same as the proofs in the 3-dimensional case; see [Gir02] for the absolute case and [HKM09] for the relative case.

Proof of Corollary 1.3.1. Let (M, ξ) be a closed connected contact manifold of dimension $2n+1$. Choose a generic self-indexing Morse function $f : M \to \mathbb{R}$. Then the regular level set $\Sigma := f^{-1}(n + \frac{1}{2})$ is a smooth hypersurface which divides M into two connected components $M \setminus \Sigma = Y_0 \cup Y_1$. It follows that $Y_i, i = 0, 1$, deformation retracts (along $\pm \nabla f$) to the skeleton $\mathrm{Sk}(Y_i)$, which is a finite n-dimensional CW-complex.

Writing Y for either of Y_0 or Y_1, we now construct $N(\mathrm{Sk}(Y))$ as a compact contact handlebody. There exists a neighborhood of the 0-cells of Y that can be written as a contact handlebody $H_0 = [-1,1] \times W_0$, where W_0 is Weinstein. Arguing by induction, assume that a neighborhood of the k-skeleton of $\mathrm{Sk}(Y)$ can be realized as a contact handlebody $H_k = [-1,1] \times W_k$, where W_k is Weinstein and $\Gamma_k = \{0\} \times \partial W_k$ is the dividing set of ∂H_k. We explain how to attach the $(k+1)$-handles to ∂H_k, where $k+1 \leq n$, in a contact manner. Write K for the core disk of a $(k+1)$-handle. Then $\dim \partial K = k$ and by dimension reasons $\partial K \subset \partial H_k$, after possible perturbation, can be isotoped into Γ_k using the Liouville flow on W_k. Let v (resp. w) be a nonvanishing (TM-valued) vector field along Γ_k that is tangent to ξ, transverse to ∂H_k (resp. transverse to Γ_k and tangent to ∂H_k), and is symplectically orthogonal to $\xi|_{\Gamma_k}$.

Next we would like to isotop ∂K to an isotropic submanifold $\partial K'$ in Γ_k (it may be Legendrian if $k+1 = n$) and then isotop K to an isotropic submanifold $K' \subset Y \setminus \mathrm{int}\, H_k$ with boundary $\partial K'$, using Gromov's h-principle [Gro86, p. 339] for isotropic submanifolds in a contact manifold. To this end we show that:

Claim 10.0.1. *There is a formal homotopy $\phi_t : TK \to TM$, $t \in [0,1]$, such that:*

(1) ϕ_0 is the derivative of the inclusion map;
(2) ϕ_t is an injective bundle map for all $t \in [0,1]$;
(3) the fibers of $\phi_1(TK)$ are ξ-isotropic;
(4) the fibers of $\phi_t(TK)$ for all $t \in [0,1]$ along ∂K have the form $\mathbb{R}\langle v\rangle$ times a plane of $T\Gamma_k$; and
(5) when $t = 1$, the planes in (4) are $\xi' := \xi \cap T\Gamma_k$-isotropic.

Proof of Claim 10.0.1. We will explain the Legendrian (i.e., $k+1 = n$) case, which is the hardest case. Since K is a disk, it is clearly formally Legendrian inside its disk neighborhood $N(K)$. The key point is to make ∂K formally isotropic as well.

Let τ be a trivialization of $\xi|_{N(K)}$. Projecting out the Reeb direction and using the trivialization τ, the embedding $K \hookrightarrow N(K)$ can be converted into the map $\phi_0^\flat : K \to G(n, 2n)$, where $G(n, 2n)$ is the Grassmannian of n-planes in $\mathbb{R}^{2n}$. Since K is a disk, $\phi_0^\flat$ is homotopic to $\phi_{1/2}^\flat : K \to \mathscr{L}_n$, where $\mathscr{L}_n \subset G(n, 2n)$ is the Lagrangian Grassmannian. Next, we would like to further homotop $\phi_{1/2}^\flat$ to $\phi_1^\flat : K \to \mathscr{L}_n$ such that $v(x) \in \phi_1^\flat(x)$ for all $x \in \partial K$. At this point we note that over ∂K the trivial complex bundle ξ satisfies

$$\xi \simeq \mathbb{R}\langle v, w\rangle \oplus \xi'$$

and that the classification of complex vector bundles over $\partial K \simeq S^{n-1}$ is given by $\pi_{n-2} U \simeq 0$ or $\mathbb{Z}$. In the former case, ξ' is a trivial complex vector bundle; in the latter case, ξ' is classified by its $(n-1)/2$nd Chern class, but then $\mathbb{C} \oplus \xi'$ is trivial, so the Chern class must vanish and ξ' must be trivial. Hence we can view the desired map $\phi_1^\flat$ as restricting to $\partial K \to \mathscr{L}_{n-1} \subset \mathscr{L}_n$, corresponding to a standard inclusion $\mathbb{R}^{2n-2} \hookrightarrow \mathbb{R}^{2n}$.

We now claim that $\pi_{n-1}\mathscr{L}_{n-1} \to \pi_{n-1}\mathscr{L}_n$ is surjective. Using the fact that $\mathscr{L}_n = U(n)/O(n)$, we have:

$$\begin{tikzcd}
\pi_{n-1}U(n) \arrow[r] & \pi_{n-1}\mathscr{L}_n \arrow[r] & \pi_{n-2}O(n) \arrow[r] & \pi_{n-2}U(n) \\
\pi_{n-1}U(n-1) \arrow[r] \arrow[u, "a"] & \pi_{n-1}\mathscr{L}_{n-1} \arrow[r] \arrow[u, "b"] & \pi_{n-2}O(n-1) \arrow[r] \arrow[u, "c"] & \pi_{n-2}U(n-1) \arrow[u, "d"]
\end{tikzcd}$$

Using the homotopy exact sequences for $U(n)/U(n-1) = S^{2n-1}$ and $O(n)/O(n-1) = S^{n-1}$, it follows that a, c are surjective and d is injective. The claim then follows from the five lemma.

The claim then allows us to homotop $\phi^\flat_{1/2}$ to $\phi^\flat_1$ such that, when we view $\phi^\flat_t$, $t \in [0,1]$, as family of maps $\phi_t : TK \to TM$, $t \in [0,1]$, Conditions (1)–(3) and (5) hold and (4) holds for $t = 0, 1$. It remains to modify ϕ_t so that (4) holds for all $t \in [0,1]$: Observe that $Z := \sqcup_{t\in[0,1]}\phi_t(TK|_{\partial K})$ has dimension $2n$. Since ϕ_t may be taken to be generic, $Z \pitchfork \mathbb{R}\langle w\rangle$ and projecting to $\mathbb{R}\langle v\rangle \oplus T\Gamma_k$ along $\mathbb{R}\langle w\rangle$ is an isomorphism for each $\phi_t(T_xK)$, $t \in [0,1]$, $x \in \partial K$. Hence (4) is also satisfied and Claim 10.0.1 follows. □

Hence a neighborhood of the $(k+1)$-skeleton of $\mathrm{Sk}(Y)$ can be realized as a contact handlebody $H_{k+1} = [-1,1] \times W_{k+1}$ and $N(\mathrm{Sk}(Y_0)) \cup N(\mathrm{Sk}(Y_1))$ can be realized as compact contact handlebodies with sutured convex boundary and its complement in M has sutured concave boundary.

Now identify $M \setminus (N(\mathrm{Sk}(Y_0)) \cup N(\mathrm{Sk}(Y_1)))$ with $\Sigma \times [0,1]$ such that if we write $\Sigma_t := \Sigma \times \{t\}$, then $\Sigma_i = \partial N(\mathrm{Sk}(Y_i))$, $i = 0, 1$, are convex with dividing sets corresponding to the sutures. By Theorem 1.2.5, $\xi|_{\Sigma\times[0,1]}$ is given by a finite sequence of bypass attachments, which can be further turned into a sequence of modifications of the trivial Weinstein POBD of

$$N(\mathrm{Sk}(Y_0)) = [-1,1] \times W_{0,n},$$

according to Lemma 8.4.3 (here $W_{0,n}$ is W_n for Y_0). In this way we obtain a 1-Weinstein POBD of $M \setminus N(\mathrm{Sk}(Y_1))$, viewed as a contact manifold with sutured concave boundary. It remains to fill in $N(\mathrm{Sk}(Y_1))$ in the obvious manner to get a compatible 1-Weinstein OBD. (Alternatively, we can attach all the contact n-handles of the bypass attachments to $N(\mathrm{Sk}(Y_0))$, which gives a generalized contact handlebody, and all the contact $(n+1)$-handles of the bypass attachments to $N(\mathrm{Sk}(Y_1))$, which also gives a generalized contact handlebody. Hence we can view the OBD as consisting of two halves and each half is a generalized contact handlebody foliated by 1-Weinstein domains.) □

Next we turn to the relative case, i.e., to contact manifolds with boundary. The following theorem is a more precise version of Corollary 1.3.2 (note the boundary condition of Corollary 1.3.2 was vaguely stated).

Theorem 10.0.2. *If (M, ξ) is a compact contact manifold with sutured concave boundary and $R_\pm(\partial M)$ are Weinstein, then there exists a compatible 1-Weinstein POBD where each of $R_+(\partial M)$ and $R_-(\partial M)$ extends to a page.*

Proof. Choose a generic self-indexing Morse function $f : M \to [\frac{1}{2}, \infty)$ such that $f \equiv \frac{1}{2}$ on ∂M. In other words, f has no index 0 critical points. As in the absolute case, consider the hypersurface $\Sigma := f^{-1}(n + \frac{1}{2})$ which divides M into two components $Y_i, i = 0, 1$, such that Y_0 contains all the critical points of index at most n and Y_1 contains all the critical points of index at least $n+1$. By the handle attachment discussion in the proof of Corollary 1.3.1, we can turn the critical points in Y_0 into isotropic handles attached to ∂M along the suture which we still denote by $N(\mathrm{Sk}(Y_0))$ although it is no longer a contact handlebody, and the critical points in Y_1 into the handle decomposition of a contact handlebody $N(\mathrm{Sk}(Y_1))$ with sutured convex boundary, as in the closed case. The rest of the proof proceeds as in the closed case. □

Proof of Corollary 1.3.3. We modify the proof of Corollary 1.3.1 as follows: In the decomposition of M into contact handlebodies Y_0 and Y_1, we first take a standard contact neighborhood $N(\Lambda)$ of the Legendrian Λ and a corresponding contact Morse function f on $N(\Lambda)$ that is constant on $\partial N(\Lambda)$. We then extend f arbitrarily to a Morse function which is self-indexing on $M \setminus N(\Lambda)$, realize the k-handles with $k \leq n$ as contact handles, and attach them to $N(\Lambda)$ to obtain Y_0. The rest of the proof is the same as that of Corollary 1.3.1. □

11. Applications to contact submanifolds

In this section, we apply the techniques developed so far to prove Corollaries 1.3.6, 1.3.7, and 1.3.9.

Recall that Ibort, Martínez-Torres, and Presas [IMTP00] constructed contact submanifolds Y of (M, ξ) as the zero loci of "approximately holomorphic" sections of a complex line or vector bundle over M. Our strategy for constructing contact submanifolds is rather different and relies on the following key observation: if $\Sigma \subset M$ is a hypersurface which contains a codimension 2 submanifold Y such that the characteristic foliation Σ_ξ is transverse to Y, then $Y \subset (M, \xi)$ is a contact submanifold.[4]

11.1. **Models for pushing across mushrooms.** We first explain the idea of the proofs of Corollaries 1.3.6 and 1.3.7 in the $n = 1$ case. We apply the operation of "pushing across a mushroom", as given in Figure 11.1.1. (Also refer to Figure 3.2.1.) There are three variants of this operation: The initial step is given by Figure 11.1.1(a), where the curve Y (in purple) is pushed across the shaded region (in light blue). The result is a curve Y' which intersects the characteristic foliation Z_ξ of the mushroom Z with opposite sign on the blue interval between the dots. (The purple portion represents where Y' intersects Z_ξ with the sign that we want to reverse and the blue portion represents where Y' intersects Z_ξ with the desired sign.) Now assuming that there is another mushroom placed just to the right of the first one (i.e., with smaller t-coordinate) but with larger s-coordinate, we can push Y' across that mushroom (or more precisely the shaded region given in Figure 11.1.1(b)) to obtain Y'' with a "larger" blue portion. Finally, after several steps

[4]We can view this as the contact analog of symplectic reduction.

of type (b), we complete the "sign reversal" by pushing across the shaded region given by Figure 11.1.1(c).

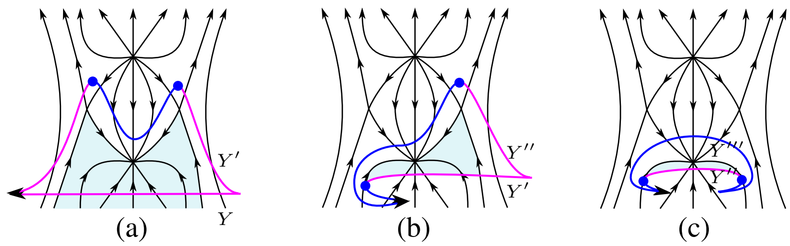

FIGURE 11.1.1. Pushing across mushrooms.

We now discuss the general case. We start with the ambient contact manifold

$$(M = \mathbb{R}^3_{z,s,t} \times W, \xi = \ker(dz + e^s(dt + \lambda)))$$

as before, where (W, λ) is a Weinstein manifold and the hypersurface before perturbation is $\Sigma = \{z = 0\}$. Let $H = [0, t_0] \times W^c_{\tau_0}$ (refer to Eq. (5.1.1)) be a contact handlebody, Z a mushroom with contact handlebody profile H, and H_{in} the subset of $\mathbb{R}_t \times W$ given by (5.5.2) and satisfying Proposition 5.5.2. By Definition 5.5.1, Z and Σ agree outside of a small neighborhood $N([0, s_0] \times [0, t_0] \times W^c_{\tau_0})$ of $[0, s_0] \times [0, t_0] \times W^c_{\tau_0}$ whose size depends on the smoothing parameters.

By taking an isotopy $\mathcal{I}$ from Z to Σ, which is supported outside of

$$\Sigma \setminus N([0, s_0] \times [0, t_0] \times W^c_{\tau_0})$$

and assumed to satisfy (I1)–(I3) below, and carrying the germ of the contact structure with it, we will assume in this section that $Z = \{z = 0\} = \mathbb{R}_s \times \mathbb{R}_t \times W$. The isotoped contact structure will still be called ξ.

Let $\rho = \rho_W : W \setminus \mathrm{Sk}(W) \to \mathbb{R}$ be the function such that $\rho(\partial W^c) = 0$ and $d\rho(X_\lambda) = 1$, and let $\tilde{\tau} = \tilde{\tau}_W : W \to \mathbb{R}_{\geq 0}$, where $\tilde{\tau} = e^\rho$ on $W \setminus \mathrm{Sk}(W)$ and $\tilde{\tau} = 0$ on $\mathrm{Sk}(W)$. The function $\tilde{\tau}$ is a variant of τ which appears in Section 5.4. Consider the map

$$\Phi = \Phi_W : Z = \mathbb{R}_s \times \mathbb{R}_t \times W \to \mathbb{R}_{\geq 0} \times \mathbb{R}^2, \quad (s, t, x) \mapsto (\tilde{\tau}(x), s, t). \tag{11.1.1}$$

In view of Eq. (5.2.1) and Eq. (5.4.1), we may choose the isotopy $\mathcal{I}$ from Z to Σ so that $Z_\xi = Z^A_\xi + Z^B_\xi$, where:

(I1) $Z^A_\xi(s, t, x)$ is the component in the $\mathbb{R}^2_{s,t}$-direction and depends only on $(\tilde{\tau}(x), s, t)$;
(I2) $Z^A_\xi(s, t, x) = Z^3_\xi(s, t)$ for $\tilde{\tau}(x) \leq 1/2$, where Z^3_ξ is the characteristic foliation of the 3-dimensional mushroom; and
(I3) $Z^B_\xi(s, t, x)$ has the form $f(\tilde{\tau}(x), s, t)X_\lambda(x) + g(s, t, x)R_\eta$ away from $\mathbb{R}^2 \times \mathrm{Sk}(W)$, where R_η is the Reeb vector field of $\lambda|_{\partial W^c}$.

Claim 11.1.1. *Away from $\mathbb{R}^2 \times \mathrm{Sk}(W)$, $\Phi_*(Z_\xi)$ is a well-defined (i.e., single-valued) vector field on $\mathbb{R}_{>0} \times \mathbb{R}^2$.*

Proof. This follows immediately from (I1)–(I3). □

Given a codimension 0 submanifold S of $\{z = s = 0\}$, let $U(S) \subset \Sigma$ be the set of points p for which there is a forward smooth flow line of Z_ξ from some $q \in S$ to p and let $\overline{U}(S)$ be its closure. We orient S using the normal orientation ∂_s which agrees with Z_ξ.

Remark 11.1.2. For our submanifolds S of interest, Claim 11.1.1 reduces the problem of determining $\overline{U}(S)$, pushing S across $\overline{U}(S)$ to S', and determining the set of points where S' is positively/negative transverse to Z_ξ, to a 3-dimensional one on $\mathbb{R}_{\geq 0} \times \mathbb{R}^2$.

Recall that the zeros of Z_ξ are $e^x_-, e^x_+, h^x_-, h^x_+$, where e_-, e_+, h_-, h_+ are zeros of the 3-dimensional mushroom Z^3 and $\lambda(x) = 0$. They are all contained in $Z' = Z^3 \times W^c$; see §5.3 and Figure 3.2.1.

Lemma 11.1.3. *$\overline{U}(H_{\rm in})$ is a compact manifold diffeomorphic to $H_{\rm in} \times [0,1]$, after smoothing Z_ξ on a sufficiently small neighborhood of $E_- := \{e_-\} \times \mathrm{Sk}(W)$. Here $H_{\rm in} = H_{\rm in} \times \{0\}$.*

The smoothing is generally necessary even in the $n = 1$ case, as the shaded region of Figure 11.1.1(a) will usually have a corner at e_-. Note that E_- is the closure of the union of all the unstable trajectories of e^x_- as we range over all the x satisfying $\lambda(x) = 0$.

Proof. This is immediate from Proposition 5.5.2. □

To prepare for the next lemma let us identify $\Phi(\overline{U}(H_{\rm in}))$ with $V \times [0,1]$, where $V = \Phi(H_{\rm in} \times \{0\})$ is a half-disk $\{z \in \mathbb{C} \mid |z| \leq 1, \mathrm{Im}\, z \geq 0\}$. Let V^δ be the slight enlargement $\{z \in \mathbb{C} \mid |z| \leq 1 + \delta, \mathrm{Im}\, z \geq 0\}$ of V with $\delta > 0$. Then the back face $V^\delta \times \{1\}$ and the bottom face $(V^\delta \times [0,1]) \cap \{\tilde{\tau} = 0\}$ are as shown in Figure 11.1.2.

The following lemma explains how to modify three types of S. Figures 11.1.2(a), (b), (c) correspond to (a), (b), (c) in the lemma.

Lemma 11.1.4.

(a) If S is a slight enlargement of $H_{\rm in}$ such that $S \setminus H_{\rm in}$ is a collar neighborhood of S, then there is a submanifold S' obtained by isotoping S across $\overline{U}(H_{\rm in})$ rel ∂S in Z such that S' decomposes into $S'_+ \cup S'_-$ and:

(i) Z_ξ is positively (resp. negatively) transverse to S' along $\mathrm{int}(S'_+)$ (resp. $\mathrm{int}(S'_-)$).

(ii) S'_+ is a pushoff of $(\partial H_{\rm in}) \times [0,1]$ and S'_- is a pushoff of $H_{\rm in} \times \{1\}$, using an outward-pointing vector field to $H_{\rm in} \times [0,1]$ after rounding the corner along $\partial H_{\rm in} \times \{1\}$.

(iii) Along the folding locus $S'_+ \cap S'_-$, Z_ξ points into S'_+.

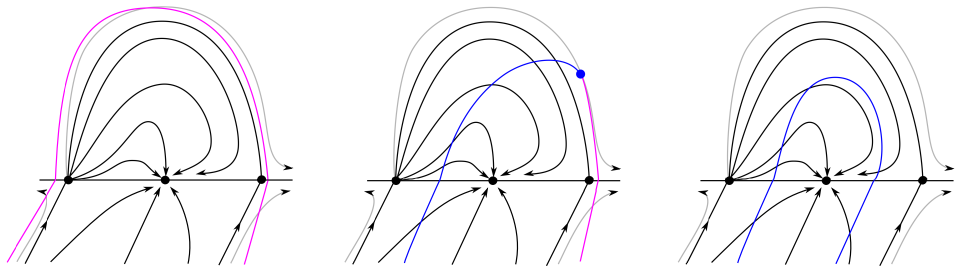

FIGURE 11.1.2. The image $V \times [0,1] = \Phi(\overline{U}(H_{\text{in}}))$ of the sink $\overline{U}(H_{\text{in}})$, after smoothing. Shown are the back face $V^\delta \times \{1\}$ and the bottom face $(V^\delta \times [0,1]) \cap \{\tilde{\tau} = 0\}$. The black dots represent the images of h_+, e_-, h_+ times $\mathrm{Sk}(W)$. The purple and blue arcs are the intersections of $\Phi(S')$ with the back and bottom faces and are subsets of $\Phi(S'_+)$ and $\Phi(S'_-)$, respectively. Figure 11.1.2(a), (b), (c) represent the higher-dimensional analogs of Figure 11.1.1(a), (b), (c).

(b) Let W'_+ (resp. W'_-) be a possibly empty Weinstein subdomain of $W^c \times \{t_0\}$ (resp. $W^c \times \{0\}$). If S is obtained from H_{in} by perturbing ∂H_{in} to ∂S so that $\partial S \pitchfork \partial H_{\text{in}}$ and either (α) $\partial S \setminus H_{\text{in}}$ is a submanifold parallel to W'_+ or (β) $S \cap \partial H_{\text{in}} = W'_-$, then there is a submanifold S' obtained by isotoping S across $\overline{U}(S \cap H_{\text{in}})$ rel ∂S in Z such that S' decomposes into $S'_+ \cup S'_-$ and:

(i) Z_ξ is positively (resp. negatively) transverse to S' along $\mathrm{int}(S'_+)$ (resp. $\mathrm{int}(S'_-)$).

(ii) Let $\partial_-\overline{U}$ be the closure of the subset of $\partial\overline{U}(S \cap H_{\text{in}})$ for which there is a smooth flow line from some $y \in S \cap \partial H_{\text{in}}$ and let $\partial_+\overline{U}$ be the closure of $\partial\overline{U}(S \cap H_{\text{in}}) \setminus \partial_-\overline{U}$. Then S'_+ (resp. S'_-) is a pushoff of $\partial_+\overline{U}$ (resp. $\partial_-\overline{U}$).

(iii) Along the folding locus $S'_+ \cap S'_-$, Z_ξ points into S'_+.

(c) If S is a slight retraction of H_{in} such that $H_{\text{in}} \setminus S$ is a collar neighborhood of H_{in}, then there is a submanifold S' obtained by isotoping S across $\overline{U}(S^{\text{sh}})$ rel ∂S in Z such that Z_ξ is negatively transverse to S'. Here S^{sh} is a slight retraction of S.

The smoothing of Z_ξ can be done inside a sufficiently small neighborhood $N(E_-)$ of E_-, and hence the resulting S' can be made to avoid $N(E_-)$. Also note that in Figure 11.1.1 S'_+ (resp. S'_-) corresponds to the purple (resp. blue) portion of Y'.

Remark 11.1.5. The directions of the folding locus given by (a)(iii) and (b)(iii) are essential when iterating the moves given by (a)–(c).

Proof. (a) The submanifold S' is constructed using the map Φ as follows: Let $\mathscr{S}$ be the union of the side $\mathscr{S}_+ = \partial\Phi(S) \times [0, 1+\epsilon']$ and the back $\mathscr{S}_- = \Phi(S) \times \{1+\epsilon'\}$

of $\partial(\Phi(S)\times[0,1+\epsilon'])$, with $\epsilon'>0$ small. Then S' is the preimage of the set $\Phi(S')$ obtained by rounding the corner and perturbing $\mathscr{S}$ so that:

- $\Phi(S')\cap\{\tilde\tau\in[0,\epsilon]\}$ with $\epsilon>0$ small agrees with $Y'\times\{\tilde\tau\in[0,\epsilon]\}$, where Y' is as given in Figure 11.1.1(a).
- $\Phi(S'_+)$ is a perturbation of $\mathscr{S}_+$, is positively transverse to $\Phi_*(Z_\xi)$, and intersects the back and bottom faces in the purple arcs given in Figure 11.1.2(a).
- $\Phi(S'_-)$ is a perturbation of $\mathscr{S}_-$ and is negatively transverse to $\Phi_*(Z_\xi)$.

(i)–(iii) are immediate from the construction.

(c) is similar to (a). In this case $\mathscr{S}$ is the union of the side $\partial\Phi(S)\times[0,1+\epsilon']$ and the back $\Phi(S)\times\{1+\epsilon'\}$ of $\partial(\Phi(S)\times[0,1+\epsilon'])$, with $\epsilon'>0$ small, and $\Phi(S')$ satisfies:

- $\Phi(S')\cap\{\tilde\tau\in[0,\epsilon]\}$ with $\epsilon>0$ small agrees with $Y'\times\{\tilde\tau\in[0,\epsilon]\}$, where Y' is as given in Figure 11.1.1(c).
- $\Phi(S')$ is negatively transverse to $\Phi_*(Z_\xi)$ and intersects the back and bottom faces in the purple arcs given in Figure 11.1.2(c).

(b) is similar to (a) and (c) in the case where (β) holds and $W'_-\subset W^c$ is a retraction of W^c along the Liouville flow. In this case $\mathscr{S}$ is again the union of the side $\partial\Phi(S)\times[0,1+\epsilon']$ and the back $\Phi(S)\times\{1+\epsilon'\}$ of $\partial(\Phi(S)\times[0,1+\epsilon'])$, with $\epsilon'>0$ small. $\mathscr{S}$ decomposes into $\mathscr{S}_+$ and $\mathscr{S}_-$ along the seam consisting of the blue dot on the back face of Figure 11.1.2(b) times $[0,1+\epsilon']$, followed by the extension along $\partial\Phi(S)\times\{1+\epsilon'\}$ to the point with $\tilde\tau=0$ and $t\le 0$ (which is analogous to the blue dot on Y'' in Figure 11.1.1(b)). (i)–(iii) are immediate from the construction.

For (β) in general, i.e., when $W'_-\subset W^c$ is an arbitrary Weinstein subdomain, we first write S as the union of S_1 of type (c) and S_2 a subset of

$$\{s=0\}\times\{t\in[-\epsilon'',\epsilon'']\}\times W'_-,$$

such that $S_1\cup S_2$ has smooth boundary and $\epsilon''>0$ is small. (In particular, S_2 has a rounded corner at $\{-\epsilon''\}\times\partial W'_-$ and a cusp corner at $\{\epsilon''\}\times\partial W'_-$.) We then apply (c) to push S across S_1. Next, let $\rho_{W'_-}: W'_-\setminus \mathrm{Sk}(W'_-)\to\mathbb{R}$ be the function such that $\rho_{W'_-}(\partial W'_-)=0$ and $d\rho_{W'_-}(X_\lambda)=1$, and $\tilde\tau_{W'_-}: W'_-\to\mathbb{R}_{\ge 0}$ satisfy $\tilde\tau_{W'_-}=e^{\rho_{W'_-}}$ on $W'_-\setminus\mathrm{Sk}(W'_-)$ and $\tilde\tau_{W'_-}=0$ on $\mathrm{Sk}(W'_-)$, and

$$\Phi_{W'_-}:\mathbb{R}\times[-\epsilon'',\epsilon'']\times W'_-\to\mathbb{R}_{\ge0}\times\mathbb{R}\times[-\epsilon'',\epsilon''],$$
$$(s,t,x)\mapsto(\tilde\tau_{W'_-},s,t),$$

as in Eq. (11.1.1). We then push S_2 — viewed as a subset of S from the previous paragraph — across using $\Phi_{W'_-}$. The seam with respect to the vector field $(\Phi_{W'_-})_*(Z_\xi|_{\mathbb{R}\times[-\epsilon'',\epsilon'']\times W'_-})$ is analogous to that of the previous paragraph, where the blue dot on the back face of Figure 11.1.2(b) times $[0,1+\epsilon']$ can be viewed as the rounded corner near $\Phi_{W'_-}(\{(0,-\epsilon'')\}\times\partial W'_-)\times[0,1+\epsilon']$.

The proof of (α) follows the same general reasoning and is omitted. □

11.2. **Existence h-principles.** We use the models from §11.1 (and specifically Lemma 11.1.4(a), (b), (c)) to prove Corollaries 1.3.6 and 1.3.7 as well as the existence h-principle for contact structures in this subsection.

Let Y be a closed codimension 2 submanifold of (M, ξ). For the moment assume that $Y \subset (M, \xi)$ has a trivial normal bundle. Any even-codimensional submanifold with trivial normal bundle is almost contact by [BCS14, Lemma 2.17].

Proof of Corollary 1.3.7 assuming trivial normal bundle. Let $Y \subset (M, \xi)$ be a contact submanifold. By the trivial normal bundle condition there exists a hypersurface $\Sigma := Y \times [-1, 1]_s \subset M$ such that $Y = Y_0$ and $\Sigma_\xi = \partial_s$, where $Y_s := Y \times \{s\}$.

By Corollary 1.3.1, Y admits a strongly Weinstein OBD (B, π) and an adapted contact form α. Referring to Appendix B, by Lemma B.2.1, Lemma B.3.1 and Proposition B.0.1, for $\delta > 0$ small there exists a strongly Weinstein and damped OBD (B', π') of (M, α'), where α' is C^1-close to α with α' strongly adapted and damped and $A(B', \pi', \alpha') < \delta$.

Fix a large integer $N > 0$. For $\delta > 0$ small, we can cover Y by a finite number $H_1, \dots, H_N$ of contact handlebodies over the Weinstein pages (arranged in order of increasing θ) of total thickness at most δ. We then install mushrooms with profiles $H_1, \dots, H_N$, arranged as in §7.3 and Figure 7.3.1, on a small neighborhood of $Y \times \{\frac{1}{2}\}$. The mushroom corresponding to H_N will be of type (a) from Lemma 11.1.4, those corresponding to $H_{N-1}, \dots, H_2$ will be of type (b), and the last one corresponding to H_1 will be of type (c). Pushing Y across the mushrooms and applying Lemma 11.1.4 in order yields a contact submanifold with orientation reversed. □

Proof of Corollary 1.3.6 assuming trivial normal bundle. As before, by the trivial normal bundle condition there exists a hypersurface $\Sigma := Y \times [-1, 1]_s \subset (M, \xi)$ such that $Y = Y \times \{0\}$, but now Σ_ξ is not necessarily transverse to Y.

By Proposition A.3.2, we can assume that Y is Σ_ξ-folded with folding locus $C \subset Y$. We then have the decomposition $Y = Y_+ \cup_C Y_-$ such that Σ_ξ is positively transverse to $\mathrm{int}(Y_+)$ and negatively transverse to $\mathrm{int}(Y_-)$. Using Theorem 1.2.3, we perturb C into a convex hypersurface. We then decompose $C = C_+ \sqcup C_-$, where C_+ (resp. C_-) is the set of points where Σ_ξ points out of (resp. into) Y_-.

Next we perturb Y on a neighborhood of C and relative to C and convert C from a fold to a cusp, keeping the same notation C, Y_+, Y_-. For more details see Appendix A. Note that Y_- is the "lower sheet" near the cusp C_+ and the "upper sheet" near the cusp C_-.

Since we can deal with the connected components of Y_- one at a time, assume Y_- is connected.

(i) Suppose for the moment that $\partial Y_- = C_+$ or $\partial Y_- = C_-$. Assuming $\partial Y_- = C_+$ (the case $\partial Y_- = C_-$ is similar), there exist a collar neighborhood $N_\epsilon(Y_-) = Y_- \times [-\epsilon, 0]_s$ of $Y_- = Y_- \times \{0\}$ and a piecewise smooth approximation $Y^\star$ of Y such that:

- $\Sigma_\xi|_{N_\epsilon(Y_-)} = \partial_s$;
- $Y^\star \cap N_\epsilon(Y_-) = (Y_- \times \{-\frac{\epsilon}{2}\}) \cup (C_+ \times [-\frac{\epsilon}{2}, 0]) \cup (N_\epsilon(C_+) \times \{0\})$, where $N_\epsilon(C_+) \subset Y_-$ is a collar neighborhood of the boundary; and

- $Y^\star \setminus N_\epsilon(Y_-)$ is positively transverse to Σ_ξ.

Informally, $Y^\star$ is sutured with respect to Σ_ξ, where the suture is $C_+ \times [-\frac{\epsilon}{2}, 0]$.

We now apply the proof of Corollary 1.3.7 above, with POBDs in place of OBDs. By Theorem 10.0.2 $Y^\star$ admits a compatible POBD. In particular we have the following data:

(1) a contact 1-form α on $Y^\star$, the binding B, and Weinstein pages S_θ, $\theta \in [0, 2\pi]$;
(2) an identification $\sim$ of Weinstein cobordisms $S'_0 \subset S_0$ and $S'_{2\pi} \subset S_{2\pi}$ such that $\partial S'_0 \supset \partial S_0$ and $\partial S'_{2\pi} \supset \partial S_{2\pi}$; and
(3) an identification of $Y^\star \setminus B$ with $\sqcup_{\theta \in [0,2\pi]} S_\theta / \sim$.

Applying the relative version of Proposition B.0.1, for $\delta > 0$ small there exists a strongly Weinstein POBD of $Y^\star$, where the new contact form α' is C^1-close to the old one, α' is strongly adapted and damped, and $\sup_{\gamma \in \mathscr{R}(\alpha')} A(\gamma) < \delta$ (defined as in Definition B.1.1). We can cover Y by a finite number $H_1, \dots, H_N$ of contact handlebodies over the new Weinstein pages S'_θ (arranged in order of increasing θ) of total thickness at most δ; here H_1 is a contact handlebody over S'_0 and H_N is a contact handlebody over $S'_{2\pi}$. We then install mushrooms with profiles $H_1, \dots, H_N$ as before. The mushrooms corresponding to $H_N, \dots, H_2$ will be of type (b) from Lemma 11.1.4 and the last one corresponding to H_1 will be of type (c). We then push $Y^\star$ across the mushrooms and replace $Y^\star \cap N_\epsilon(Y_-)$ by a hypersurface which is positively transverse to $\Sigma_\xi^\vee$ and has the same boundary.

(ii) Next suppose that ∂Y_- has components in C_+ and C_-. In this case we perturb Y to create extra folds (or equivalently cusps) near C_- as in Figure 11.2.1 so that the new Y is $Y'_- \cup A_+ \cup A_- \cup Y'_+$, where $A_\pm \simeq C_- \times [-1, 1]$.

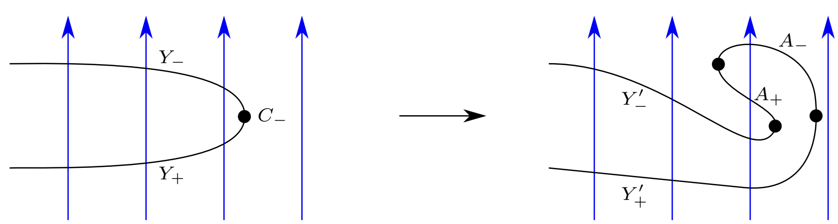

FIGURE 11.2.1. Modification near C_-. The actual folds are obtained by multiplying by C_- and the dots represent the folding loci. The arrows indicate Σ_ξ.

Note that after the modification each of the negative regions Y'_- and A_- satisfies the conditions of (i). Hence we can now apply (i) to conclude the proof. □

Completion of proofs of Corollaries 1.3.6 and 1.3.7. We explain how to remove the requirement that $Y \subset M$ have trivial normal bundle. Let $D \subset Y$ be a closed ball around some point in Y. Then $Y \setminus D$ is an open manifold. If Y is an almost contact manifold, then so is $Y \setminus D$. By Gromov's h-principles for open contact manifolds (see [EM02, 10.3.2]) and open isocontact embeddings (see [EM02, 12.3.1]), we may assume that $Y \setminus D \subset (M, \xi)$ is an open contact submanifold. Hence the problem reduces to an extension problem over the ball D, which clearly has a trivial normal bundle in M. Observe that the proof of Corollary 1.3.6 is essentially a relative extension problem. Corollary 1.3.6 then immediately implies Corollary 1.3.7 since if Y is contact then $-Y$ is almost contact (just reverse the orientation of the line field transverse to $TY \cap \xi$). $\square$

Proof of Corollary 1.3.9. We refer the reader to Uebele [Ueb16, Section 4.2] and the references therein for a useful summary of almost contact structures and their homotopy classes.

Let M^{2n-1} be a closed manifold with an almost contact structure $\zeta = (H_\zeta, J_\zeta)$, where H_ζ is a hyperplane field and J_ζ is an almost complex structure on H_ζ. We view ζ as a section $\mathfrak{s}_\zeta$ of the fiber bundle $P \to M$ with fiber $SO(2n-1)/U(n-1)$ associated to TM. Let $D \subset M$ be a closed ball in M. By Gromov's h-principle (see [EM02]) there exists a homotopy of $\mathfrak{s}_\zeta$ to $\mathfrak{s}_{\zeta'}$ (as sections of $P \to M$) such that $\zeta'|_{M \setminus int(D)}$ is a genuine contact structure.

Since TD is trivial, we may view $\mathfrak{s}_{\zeta'}|_{S^{2n-2}=\partial D}$ as a map

$$S^{2n-2} \to SO(2n-1)/U(n-1);$$

the map is homotopically trivial since it extends over D. The set of homotopy classes of extensions of $\mathfrak{s}_{\zeta'}|_{S^{2n-2}}$ to D is given by $\pi_{2n-1}(SO(2n-1)/U(n-1))$. Now

$$SO(2n-1)/U(n-1) \simeq SO(2n)/U(n)$$

(see [Ueb16, Section 4.2]), so their homotopy groups agree.

Next we stabilize $\zeta'|_D = (H_{\zeta'}, J_{\zeta'})|_D$ to the almost contact structure

$$\eta = (H_{\zeta'} \times \mathbb{R}^2, J_{\zeta'} \times j)|_{D \times \mathbb{R}^2},$$

where j is the standard almost complex structure on $\mathbb{R}^2$. Again by Gromov's h-principle there exists a formal homotopy on $D \times \mathbb{R}^2$ from η to a genuine contact structure η' relative to $N(\partial D) \times \{0\}$, where $N(\partial D) \subset D$ is a neighborhood of ∂D. We then apply Corollary 1.3.6 to obtain a contact submanifold of $(D \times \mathbb{R}^2, \eta')$ which is C^0-close to $D \times \{0\}$, isotopic to $D \times \{0\}$, and agrees with $D \times \{0\}$ on $N(\partial D) \times \{0\}$. The induced contact structure ξ on D is stably homotopic to $\zeta'|_D$ relative to ∂D.

It remains to find contact structures in each homotopy class of almost contact structures on S^{2n-1} to connect sum with so that we can adjust the homotopy class of the almost contact structure.

When $n \equiv 1$ or $3 \bmod 4$ (i.e., the manifold has dimension $4k+1$ where $k \in \mathbb{Z}^+$), we claim that each homotopy class can be represented by connected sums of canonical contact structures $\xi_{a_0,\dots,a_n}$ on Brieskorn spheres $\Sigma(a_0, \dots, a_n)$ that are diffeomorphic to standard spheres. In our case we have $\pi_{2n-1}(SO(2n)/U(n)) \simeq \mathbb{Z}/d$ by Massey [Mas61]. By Morita [Mor75] there is a map ac that assigns the value

$$ac(\xi_{a_0,\dots,a_n}) = \tfrac{1}{2} \textstyle\prod_{j=1}^{n} (a_j - 1) \in \mathbb{Z}/d.$$

In particular, we can take $(a_0, \dots, a_n) = (7, 2, \dots, 2)$ or $(9, 2, \dots, 2)$ (these were studied by Ustilovsky [Ust99b] and are diffeomorphic to the standard spheres).

Since

$$ac(\xi_{(7,2,\ldots,2)}) = 3 \quad \text{and} \quad ac(\xi_{(9,2,\ldots,2)}) = 4$$

are relatively prime they generate $\mathbb{Z}/d$. $\square$

APPENDIX A. WRINKLED AND FOLDED EMBEDDINGS

The technique of wrinkled maps and wrinkled embeddings, developed by Eliashberg and Mishachev in the series of papers [EM97, EM98, EM00, EM09], is extremely powerful in dealing with homotopy problems of smooth maps between manifolds. The goal of this appendix is to give a brief overview of their theory and prove a technical result, Proposition A.3.2, which is only used in Section 11.

This appendix is organized as follows. First we review several fundamental definitions and results in the theory of wrinkled embeddings following Eliashberg and Mishachev. Then we use the wrinkling technique to put a generic hypersurface in a "good" position with respect to a nonvanishing vector field.

A.1. **Wrinkled and cuspidal embeddings.**

In this subsection we review the main results of [EM09].

Let $f : \Sigma \to M$ be a smooth map between smooth manifolds. In this subsection we assume that $\dim \Sigma = \dim M - 1 = k$, unless otherwise specified. All the results in this subsection are also valid whenever $\dim \Sigma < \dim M$. If $\dim \Sigma \leq \dim M$, then the *singular set* $\mathrm{Sing}(f)$ *of* $f : \Sigma \to M$ is the set of points in Σ where df is not injective.

Definition A.1.1 (Wrinkled embedding)**.** *A smooth map* $f : \Sigma \to M$ *is a* wrinkled embedding *if:*

(WE1) f *is a topological embedding.*

(WE2) $\mathrm{Sing}(f)$ *is diffeomorphic to a disjoint union of spheres* $S_i \cong S^{k-1}$*, each of which bounds a* k*-disk in* Σ*. Each such* S_i *is called a* wrinkle *of* f*.*

(WE3) The map f *can be written locally near each wrinkle as:*

$$\mathscr{O}p_{\mathbb{R}^k}(S^{k-1}) \to \mathbb{R}^{k+1},$$

$$(y, z) \mapsto \left(y, z^3 + 3(|y|^2 - 1)z, \int_0^z (z^2 + |y|^2 - 1)^2 dz \right),$$

where $(y, z) = (y_1, \ldots, y_{k-1}, z)$ *denotes the Cartesian coordinates on* $\mathbb{R}^k$ *such that* $S^{k-1} = \{|y|^2 + z^2 = 1\}$ *is the unit sphere.*

Let $f : \Sigma \to M$ be a wrinkled embedding. Consider a wrinkle $S \cong S^{k-1}$ of f given in the local model specified by (WE3). Let $S' := \{z = 0\} \subset S$ be the equator of S. By identifying the wrinkled map f with its image in M which we also denote by Σ, we say that Σ has *cusp* singularities along $S \setminus S'$ and *unfurled swallowtail* singularities along S'. See Figure A.1.1.

Remark A.1.2. Although a wrinkled embedding $f : \Sigma \to M$ is in general not a smooth embedding, it follows from (WE3) that the image $f(\Sigma)$ has a well-defined k-dimensional tangent plane everywhere. We shall denote by $Gdf : \Sigma \to \mathrm{Gr}_k(M)$

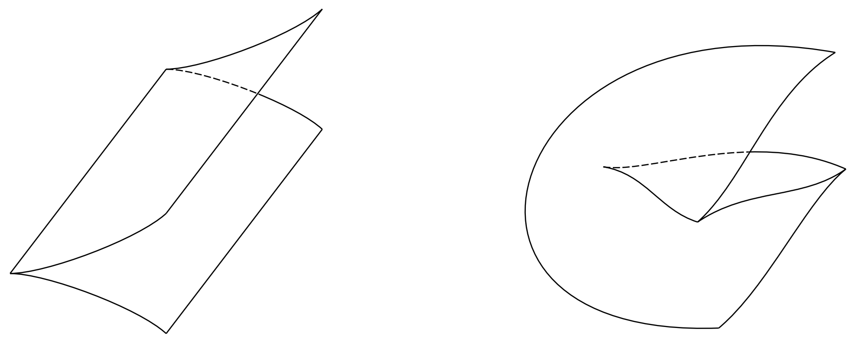

FIGURE A.1.1. Left: cusp singularity; Right: unfurled swallowtail singularity.

the corresponding "Gauss map", where $\pi : \mathrm{Gr}_k(M) \to M$ is the tangent k-plane bundle on M.

According to [EM09], the significance of wrinkled embeddings is that they satisfy an h-principle with respect to *tangential rotations*.

Definition A.1.3 (Tangential rotation)**.** *Given a smooth embedding $f : \Sigma \to M$, a* tangential rotation *is a smooth homotopy $G_t : \Sigma \to \mathrm{Gr}_k(M)$, $t \in [0,1]$, such that $G_0 = Gdf$ and $f = \pi \circ G_t$.*

The following theorem was proved by Eliashberg and Mishachev in [EM09, Theorem 2.2]. Although we are only interested in codimension 1 submanifolds $\Sigma \subset M$, the theorem holds for embedded submanifolds of any codimension.

Theorem A.1.4 (Wrinkled approximation of a tangential rotation)**.** *Let $G_t : \Sigma \to \mathrm{Gr}_k(M)$ be a tangential rotation of a smooth embedding $f : \Sigma \to M$. Then there exists a homotopy of wrinkled embeddings $f_t : \Sigma \to M$ with $f_0 = f$ such that $Gdf_t : \Sigma \to \mathrm{Gr}_k(M)$ is arbitrarily C^0-close to G_t. If the rotation G_t is fixed on a closed set $K \subset \Sigma$, then the homotopy f_t can also be chosen to be fixed on K.*

Here a homotopy of wrinkled embeddings allows birth-death type singularities.

Remark A.1.5. The wrinkles that appear in Theorem A.1.4 can be made arbitrarily small.

It turns out that the unfurled swallowtail singularities in a wrinkle can be eliminated by a C^0-small operation called *Whitney surgery*. Whitney surgery involves first choosing an embedded $(k-1)$-disk D in the wrinkled Σ such that $\partial D = S'$ for some wrinkle $S \subset \Sigma$ and the interior of D is disjoint from the wrinkles. (The existence of such a disk D is immediate.) Then one removes the unfurled swallowtail singularities along S' and adds a family of zigzags along D as in Figure A.1.2. The formal treatment of Whitney surgery can be found in [EM09, §2.10].

The resulting hypersurface has only cusp singularities along spheres and we formalize it in the following definition.

Definition A.1.6 (Cuspidal embedding)**.** *A smooth map $f : \Sigma \to M$ is a* spherically cuspidal embedding *(or simply a* cuspidal embedding*) if the following hold:*

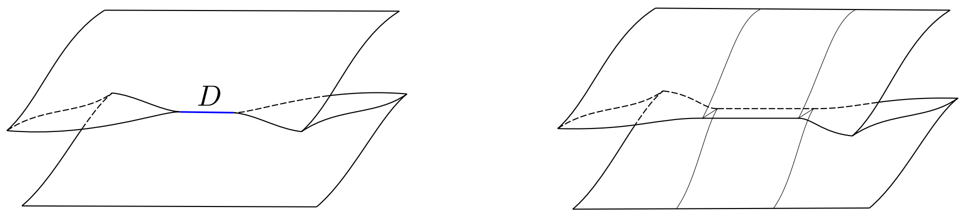

FIGURE A.1.2. Left: before the Whitney surgery; Right: after the Whitney surgery. The vertical sides are identified in these pictures.

(CE1) *f is a topological embedding.*
(CE2) $\mathrm{Sing}(f)$ *is a finite disjoint union of smoothly embedded spheres $S_i \cong S^{k-1}$, called* cusp edges*, in Σ.*
(CE3) *The map f can be written locally on a tubular neighborhood $S_i \times (-\epsilon, \epsilon)$ of S_i in Σ as:*

$$S^{k-1} \times (-\epsilon, \epsilon) \to S^{k-1} \times \mathbb{R}^2,$$
$$(y, z) \mapsto (y, z^2, z^3).$$

Remark A.1.7. As in the case of wrinkled embeddings, the image of a cuspidal embedding $f : \Sigma \to M$ also has well-defined tangent planes everywhere. We denote by $Gdf : \Sigma \to \mathrm{Gr}_k(M)$ the corresponding Gauss map.

Remark A.1.8. Our cuspidal embeddings are called *folded embeddings* in [EM09], where the cusp edges are not necessarily diffeomorphic to the sphere. The reason we use the terminology "cuspidal embedding" is that a "folded embedding" means something else in this paper. See Definition A.3.1.

The following result follows immediately from Theorem A.1.4 and the Whitney surgery on wrinkles discussed above.

Theorem A.1.9 (Cuspidal approximation of tangential rotation)**.** *Let $G_t : \Sigma \to \mathrm{Gr}_k(M)$ be a tangential rotation of a smooth embedding $f : \Sigma \to M$. Then there exists a homotopy of cuspidal embeddings $f_t : \Sigma \to M$ with $f_0 = f$ such that $Gdf_t : \Sigma \to \mathrm{Gr}_k(M)$ is arbitrarily C^0-close to G_t. If the rotation G_t is fixed on a closed set $K \subset \Sigma$, then the homotopy f_t can be chosen to be fixed on K.*

We conclude this subsection with a smoothing operation which turns a cuspidal embedding into a smooth embedding. Suppose $f : \Sigma \to M$ is a cuspidal embedding with cusp edges S_i. Using the local model near cusps given by (CE3), the smoothing operation amounts to replacing each fiber $\{(y_0, z^2, z^3) \mid z \in (-\epsilon, \epsilon)\}$ at $y_0 \in S_i$ by $\{(y_0, z^2, z^{\nu(z)}) \mid z \in (-\epsilon, \epsilon)\}$. Here $\nu : (-\epsilon, \epsilon) \to [1, 3]$ is a smooth function which equals 1 near 0, equals 3 near $\pm\epsilon$, is nondecreasing on $[0, \epsilon)$, and is nonincreasing on $(-\epsilon, 0]$. We denote the resulting smooth embedding by $\mathrm{Sm}(f) : \Sigma \to M$ and the image by $\mathrm{Sm}(\Sigma)$.

A.2. **Cuspidal embeddings of a disk.** In the previous subsection, we saw that any tangential rotation of a smooth embedding can be C^0-approximated by a homotopy of wrinkled or cuspidal embeddings. However, for our purposes, we also need to change the homotopy class of the tangential distribution, and ask if it can be approximated by cuspidal embeddings. This was done in great generality by Eliashberg and Mishachev in [EM00]. In this subsection we review their work in a special case.

Let D^k be the unit disk in $\mathbb{R}^k$ and let $f : D^k \to \mathbb{R}^k \times \mathbb{R}_s$ be a smooth embedding such that f is positively transverse to ∂_s on a neighborhood of ∂D^k. We identify D^k with its image in $\mathbb{R}^{k+1}$ and coorient D^k by declaring that ∂_s is positively transverse to it near ∂D^k. Using the Euclidean metric on $\mathbb{R}^{k+1}$, let $\mathbf{n}$ be the positive unit normal vector field along D^k.

Remark A.2.1. Since f is a codimension 1 embedding, specifying an oriented hyperplane distribution along D^k is equivalent to specifying a nonvanishing vector field along D^k.

Let $C \subset \text{int}(D^k)$ be an embedded codimension 1 submanifold which divides D^k into two parts $D^k \setminus C = D_+ \sqcup D_-$ such that $\partial D^k \subset D_+$ and the sign switches when we cross C. Identify a small tubular neighborhood $N(C) \subset D^k$ of C with $C \times [-\epsilon, \epsilon]$. Choose a decomposition $C = C_+ \cup C_-$ and define a vector field v on $N(C)$ such that v points into $D_\pm$ along $C_\pm$.

Given $\mathbf{n}$ and $C = C_+ \cup C_-$ as above, we define a nonvanishing vector field $\mathbf{n}(C)$ along D^k as follows:

- $\mathbf{n}(C) = \mathbf{n}$ along $D_+ \setminus N(C)$.
- $\mathbf{n}(C) = -\mathbf{n}$ along $D_- \setminus N(C)$.
- Along each fiber $\{y\} \times [-\epsilon, \epsilon] \subset C \times [-\epsilon, \epsilon] = N(C)$, $\mathbf{n}(C)$ rotates counterclockwise from $\mathbf{n}$ to $-\mathbf{n}$ in the oriented 2-plane spanned by $(\mathbf{n}, v)$.

Roughly speaking, C_+ becomes a convex suture and C_- becomes a concave suture with respect to $\mathbf{n}(C_+, C_-)$.

We state the following result [EM00, Theorem 1.7], adapted to our special case; see also [Eli72].

Theorem A.2.2. *Suppose the manifolds C_+ and C_- are nonempty and the vector field $\mathbf{n}(C)$ is homotopic to ∂_s rel ∂D^k. Then there exists a cuspidal embedding $f' : D^k \to \mathbb{R}^k \times \mathbb{R}$ that is everywhere transverse to ∂_s, such that $f' = f$ near ∂D^k and $\text{Sm}(f')$ is C^0-small isotopic to f rel ∂D^k.*

Remark A.2.3. In fact a stronger result is given in [EM00], i.e., one can further arrange so that the cusp edges of f' coincide with C. This fact, however, is not needed in this paper.

Corollary A.2.4. *Given any smooth embedding $f : D^k \to \mathbb{R}^k \times \mathbb{R}$ which is positively transverse to ∂_s on a neighborhood of ∂D^k, there exists a cuspidal embedding $f' : D^k \to \mathbb{R}^k \times \mathbb{R}$ that is everywhere transverse to ∂_s, such that $f' = f$ near ∂D^k and $\text{Sm}(f')$ is C^0-small isotopic to f rel ∂D^k.*

Sketch of proof. Let $f : D^k \to \mathbb{R}^k \times \mathbb{R}$ be a smooth embedding which is positively transverse to ∂_s on a neighborhood of ∂D^k.

We claim there exists $C = C_+ \cup C_-$ such that $\mathbf{n}(C)$ is homotopic to ∂_s rel ∂D^k, where $C_\pm$ can be taken to be spherical boundaries of small neighborhoods of points in D^k. For generic D^k, there exists a finite set of points $\{x_i\}_{i\in I}$ in D^k where $\mathbf{n} = -\partial_s$. Let $D_i \subset D^k$ be a small disk neighborhood of x_i and let $S_i = \partial D_i$. We then take $C = \cup_i \partial S_i$ and construct $\mathbf{n}(C)$ as appropriate (depending on the local degree of the Gauss map on D_i) so that $\mathbf{n}(C)$ is homotopic to ∂_s. (It is not hard to add extra components of $C_\pm$ if necessary without affecting the homotopy type of $\mathbf{n}(C)$.)

Corollary A.2.4 then follows from Theorem A.2.2. □

A.3. **Folding hypersurfaces.** Using the techniques reviewed in §A.1 and §A.2, we show in this subsection how to "fold" a generic hypersurface with respect to a nonvanishing vector field.

Let $\Sigma \subset M$ be a closed cooriented hypersurface and v a nonvanishing vector field defined on a neighborhood of Σ. In general it is not possible to find a C^0-small isotopy $\phi_t : M \xrightarrow{\sim} M$ with $\phi_0 = \mathrm{id}_M$ such that Σ_1 is everywhere transverse to v, where $\Sigma_t := \phi_t(\Sigma)$. However, if we allow Σ_t to have cusp singularities (here we are implicitly allowing birth-death type singularities), then there exists a cuspidal embedding $\Sigma_1 \subset M$ which is everywhere transverse to v, and whose smoothing $\mathrm{Sm}(\Sigma_1)$ is C^0-small isotopic to Σ — this is the content of Proposition A.3.2.

The smoothing $\mathrm{Sm}(\Sigma_1)$ is a *v-folded hypersurface* in the following sense:

Definition A.3.1 (v-folded hypersurface)**.** *Let $\Sigma \subset M$ be a closed, cooriented hypersurface. If v is a nonvanishing vector field defined on a tubular neighborhood of Σ, then Σ is* v-folded *if there exists a codimension 1 submanifold $C(\Sigma) \subset \Sigma$ such that:*

(1) $\Sigma \setminus C(\Sigma) = \Sigma_+ \sqcup \Sigma_-$, where v is positively (resp. negatively) transverse to Σ_+ (resp. Σ_-) with respect to the coorientation of Σ and the sign switches when we cross $C(\Sigma)$.

(2) For each connected component C of $C(\Sigma)$, there exists an orientation-preserving diffeomorphism from $C \times \mathbb{R}^2_{x_1,x_2}$ to a tubular neighborhood U of C in M such that $\Sigma \cap U$ is identified with $C \times \{x_1 = x_2^2\}$, C is identified with $C \times \{0\}$, and $v|_U$ is identified with ∂_{x_2}.

The submanifold $C(\Sigma)$ is called the v-seam *(or the* seam *if v is understood) of Σ. Then $C(\Sigma) = C_+(\Sigma) \cup C_-(\Sigma)$, where a component C of $C(\Sigma)$ belongs to $C_+(\Sigma)$ (resp. $C_-(\Sigma)$) if, in the local model described in (2) above, $\Sigma_+ \cap U$ is identified with $C \times \{x_1 = x_2^2, x_2 > 0\}$ (resp. $\Sigma_+ \cap U$ is identified with $C \times \{x_1 = x_2^2, x_2 < 0\}$).*

Proposition A.3.2. *Given a closed cooriented hypersurface $\Sigma \subset M$ and a nonvanishing vector field v defined on a tubular neighborhood of Σ, there exists a C^0-small isotopy $\phi_t : M \xrightarrow{\sim} M$ with $\phi_0 = \mathrm{id}_M$ such that $\Sigma_1 = \phi_1(\Sigma)$ is v-folded.*

Proof. Fix a Riemannian metric on M such that v has unit length. Let $\mathbf{n}$ be the positive unit normal vector field along Σ. We note that since v and $\mathbf{n}$ may not be homotopic, some extra steps are needed to apply the wrinkling h-principle.

As in the proof of Corollary A.2.4, for generic Σ, there exists a finite set of points $\{x_i\}_{i\in I}$ in Σ where $v = -\mathbf{n}$. Let $D_i \subset \Sigma$ be a small disk neighborhood of x_i and let $S_i = \partial D_i$. Choose nested tubular neighborhoods $S_i \subset N_\epsilon(S_i) \subset N_{2\epsilon}(S_i)$ of S_i in Σ such that $x_i \notin N_{2\epsilon}(S_i)$.

It is not hard to see that there exists a homotopy $\mathbf{n}_t, t \in [0,1]$, of nonvanishing vector fields along Σ with $\mathbf{n}_0 = \mathbf{n}$ such that:

(1) $\mathbf{n}_t = \mathbf{n}$ on $\cup_{i\in I}(D_i \setminus N_{2\epsilon}(S_i))$;
(2) $\mathbf{n}_1 = v$ on the complement of $\cup_{i\in I}(D_i \setminus N_\epsilon(S_i))$.

Now we apply Theorem A.1.9 to the tangential rotation induced by $\mathbf{n}_t$ to obtain a C^0-approximation of Σ by a cuspidal hypersurface Σ'', whose smoothing is v-folded on the complement of $\cup_{i\in I}(D_i \setminus N_\epsilon(S_i))$ and such that the v-seam is disjoint from S_i for all $i \in I$. To see this, first apply Theorem A.1.4 to obtain a C^0-approximation Σ' of Σ by a wrinkled hypersurface whose Gauss map is close to (the orthogonal complement of) $\mathbf{n}_1$. Since the wrinkles can be made arbitrarily small by Remark A.1.5, we can C^0-small isotop $S_i \subset \Sigma$ so that they avoid all the wrinkles. The Whitney surgery of the wrinkles can also be made disjoint from S_i, so we obtain a cuspidal C^0-approximation Σ'' of Σ' such that the cusp edges are disjoint from S_i. The smoothing of Σ'' converts the cusp edges to the v-seam.

Finally we apply Corollary A.2.4 to each D_i to conclude the proof. □

APPENDIX B. QUANTITATIVE STABILIZATIONS OF OPEN BOOK DECOMPOSITIONS IN GENERAL

The goal of this appendix is to explain (see Section B.1 for the definitions of the terms involved) and prove the following generalization of Lemma 6.2.1:

Proposition B.0.1 (Quantitative stabilization). *Let α be a contact form on M^{2n+1} which is strongly adapted to the strongly Weinstein OBD (B, π) and damped. Choose $\delta > 0$ small. Suppose $\alpha|_B$, the restriction to the binding B, is strongly adapted to the strongly Weinstein OBD (B_1, π_1), damped, and with action*

$$A(B_1, \pi_1, \alpha|_B) < \delta.$$

Then there exists a strongly Weinstein OBD (B', π') of (M, α') with a strongly adapted contact form α' C^1-close to α which is damped and with action

$$A(B', \pi', \alpha') < \delta.$$

While Lemma 6.2.1 suffices for the purposes of constructing a plug, we need the much more technical Proposition B.0.1 for proving Corollaries 1.3.6, 1.3.7, and 1.3.9.

B.1. **Definitions.** In this subsection we define the terms involved in Proposition B.0.1. In what follows let (B, π) be an α-compatible strongly Weinstein OBD on M. Its pages are denoted by $S_\theta = \pi^{-1}(e^{i\theta})$.

Definition B.1.1. *The* action $A(B, \pi, \alpha)$ *is* $\sup_{\gamma\in\mathscr{R}(\alpha)} A(\gamma)$*, where $\mathscr{R}(\alpha)$ is the set of Reeb chords γ in $M \setminus S_0$ whose closures have endpoints on S_0 and $A(\gamma)$ is the action $\int_\gamma \alpha$.*

Suppose M admits a decomposition $M = N_\epsilon(B) \cup T_\phi$, where $N_\epsilon(B)$ and T_ϕ are glued along their boundary and:

(SA1) $N_\epsilon(B) = D^2(\epsilon) \times B$, where $\epsilon > 0$ is small, $D^2(\epsilon)$ is the open disk $\{(r,\theta) \mid r < \epsilon\}$, and (r,θ) are polar coordinates, is a tubular neighborhood of the binding $\{0\} \times B$, with contact form

$$\alpha|_{N_\epsilon(B)} = (1 - c_1 r^2)\lambda + c_2 r^2 d\theta \tag{B.1.1}$$

where $c_1, c_2 > 0$ are constants.

(SA2) On $N_\epsilon(B)$, S_{θ_0} restricts to $\{((r,\theta),x) \mid \theta = \theta_0\}$ for all $e^{i\theta_0} \in S^1$.

(SA3) T_ϕ is the mapping torus of $(S^\star, \phi)$, where $S^\star = S_0^\star$ is a truncated page $S_0 \cap (M \setminus N_\epsilon(B))$ and $\phi|_{\partial S_0^\star}$ is a positive-time flow of the Reeb vector field R_λ.

On $N_\epsilon(B)$ the Reeb vector field is given by

$$R_\alpha = R_\lambda + (c_1/c_2)\partial_\theta. \tag{B.1.2}$$

Hence all the Reeb chords γ in $N_\epsilon(B) \setminus S_0$ have the same action $A(\gamma) = 2\pi c_2/c_1$.

Definition B.1.2. *A contact form α adapted to the strongly Weinstein (B,π) is* strongly adapted to (B,π) *if there exists $\epsilon > 0$ such that (SA1)–(SA3) hold.*

Given a strongly adapted contact form which takes the form of (B.1.1) near the binding, one can easily rescale the r-coordinate such that $c_2 = 1$, which we assume to be the case from now on.

Definition B.1.3. *A strongly adapted contact form α adapted to (B,π) is* damped *if its total infinitesimal variation (see Definition 6.1.2) satisfies $V < 11/10$.*[5]

Given a damped contact form α compatible with the OBD (B,π), one can assume that ρ from Eq. (6.1.1) is almost constant.

Definition B.1.4. *Given a Riemannian manifold M, two contact forms α_0, α_1 are* C^1-close *if both $\|\alpha_0 - \alpha_1\|_{C^0}$ and $\|d\alpha_0 - d\alpha_1\|_{C^0}$, measured with respect to the Riemannian metric, are sufficiently small.*

Our notion of C^1-closeness given in Definition B.1.4 is strictly weaker than the usual notion of C^1-closeness since we take the exterior derivatives rather than all the partial derivatives.

B.2. **Perturbing to a strongly adapted contact form.**

Lemma B.2.1. *Let (B,π) be a strongly Weinstein OBD of (M^{2n+1},ξ) with an adapted contact form α. Then there exists a contact form α' which is C^1-close to α such that α' is strongly adapted to (B,π).*

Proof. By assumption $B \subset (M,\xi)$ is a contact submanifold and there exists a tubular neighborhood $N_\epsilon(B) = D^2(\epsilon) \times B$ such that $\pi|_{N_\epsilon(B)}$ is the projection onto

[5] Here the precise constant slightly larger than 1 is not important, but we choose one for definiteness.

the $D^2(\epsilon)$-factor. It follows that $B_p := \{p\} \times B$ is contact for any $p \in D^2(\epsilon)$ if ϵ is sufficiently small.

The characteristic foliation $\mathscr{F}_\theta$ on $S_\theta \cap N_\epsilon(B)$ has a nonzero ∂_r-component for any $e^{i\theta} \in S^1$. Hence through any point $x \in B = B_0$ there exists a 2-disk D_x such that $D_x \cap S_\theta$ is tangent to $\mathscr{F}_\theta$ for all $e^{i\theta} \in S^1$, and the family of disks $D_x, x \in B$, varies smoothly in B. Using the disks D_x we can reparametrize $N_\epsilon(B)$ such that the characteristic foliation on $S_\theta \cap N_\epsilon(B)$ is parallel to ∂_r and $\alpha|_{S_\theta \cap N_\epsilon(B)}$ has the form $F\lambda$, where $\lambda = \alpha|_B$ and F only depends on r. This implies that

$$\alpha|_{N_\epsilon(B)} = F\lambda + Gd\theta, \tag{B.2.1}$$

where $F, G : N_\epsilon(B) \to \mathbb{R}_{\geq 0}$ satisfy (i) $F|_B = 1$, $G|_B = 0$, (ii) F only depends on r, and (iii) $\lim_{r\to 0} \frac{\partial_r G}{r} > 0$. Here (iii) follows from the contact condition $\alpha \wedge (d\alpha)^n > 0$.

We compute

$$d\alpha|_{N_\epsilon(B)} = \partial_r F dr \wedge \lambda + F d\lambda + d_B G \wedge d\theta + \partial_r G dr \wedge d\theta$$

and claim that $R_\alpha|_{N_\epsilon(B)}$ has the form $\phi(\frac{\partial_r G}{r} R_\lambda - \frac{\partial_r F}{r} \partial_\theta + X)$, where ϕ is a positive function and $X \in \mathrm{Span}(\partial_r, \ker \lambda)$. Indeed we calculate

$$\begin{aligned}
i_{R_\lambda} d\alpha|_{N_\epsilon(B)} &= -\partial_r F dr + d_B G(R_\lambda) d\theta, \\
i_{\partial_\theta} d\alpha|_{N_\epsilon(B)} &= -d_B G - \partial_r G dr, \\
i_{\partial_r} d\alpha|_{N_\epsilon(B)} &= \partial_r F \lambda + \partial_r G d\theta, \\
i_Y d\alpha|_{N_\epsilon(B)} &= F i_Y d\lambda + d_B G(Y) d\theta,
\end{aligned}$$

where $Y \in \ker \lambda$. The vanishing of coefficient of the dr term implies that $R_\alpha|_{N_\epsilon(B)}$ is parallel to $\frac{\partial_r G}{r} R_\lambda - \frac{\partial_r F}{r} \partial_\theta + C_0 \partial_r + Y$ for some C_0 and Y. Since α is adapted by assumption, R_α must have a positive ∂_θ-component for $r > 0$. It follows that $\partial_r F < 0$ for $r > 0$.

Finally, to obtain the strongly adapted α', it suffices to pick $0 < \epsilon' \ll \epsilon$ and write $\alpha'|_{N_\epsilon(B)} = F'\lambda + G'd\theta$ such that

- $F' = (1 + c_0) - c_1 r^2$ and $G' = \frac{1}{2} r^2$ for $r < \epsilon'$ and some $c_0, c_1 > 0$;
- $F' = F$ and $G' = G$ for r close to ϵ;
- $\partial_r F' < 0$ and $\partial_r G' > 0$ for all $0 < r < \epsilon$.

It is straightforward to check that α' is C^1-close to α, is contact, and is strongly adapted. □

A key feature of C^1-close contact forms is the following:

Lemma B.2.2. *If α, α' are two C^1-close contact forms on M, then there exists a C^1-diffeomorphism $\phi : M \xrightarrow{\sim} M$ isotopic to the identity and a function $f \in C^\infty(M)$ which is C^0-close to the constant function 1 such that $\phi^*(\alpha') = f\alpha$.*

The proof is a standard application of the proof of Gray's theorem and is omitted.

B.3. **Damped OBD.** In order to construct a damped contact form adapted to an OBD, we work with *abstract OBDs*, whose data consists of a Weinstein domain (S, η) and an exact symplectomorphism $\phi : S \xrightarrow{\sim} S$ such that $\phi = \mathrm{id}$ near ∂S and $\phi^*(\eta) = \eta + dF$ for some $F \in C^\infty(S)$ which vanishes near ∂S.

We construct the compatible contact structure as follows: Let $\mathbb{R}_t \times S$ be the contactization of (S, η) with contact form $\alpha = dt + \eta$. Choose a constant $C > 0$ such that $F + C > 0$. Let $T_{\phi,C}$ be the mapping torus

$$T_{\phi,C} := \{(t, x) \in \mathbb{R} \times S \mid 0 \le t \le F(x) + C\} / (0, \phi(x)) \sim (F(x) + C, x).$$

Then $\partial T_{\phi,C} = \mathbb{R}/C\mathbb{Z} \times \partial S$. We extend α to $N(B) = D^2 \times B$, where $D^2 = \{(r, \theta) \mid r \le 1\}$ (in polar coordinates), by

$$\alpha|_{N(B)} = f(r)\lambda + g(r)d\theta, \tag{B.3.1}$$

where λ is a contact form on B and f, g satisfy the following conditions:

(NB1) There exists $\epsilon > 0$ small such that $\alpha|_{N_\epsilon(B)}$ satisfies (SA1)–(SA3);
(NB2) The contact condition $(f', g') \cdot (-g, f) > 0$ holds for $r > 0$;
(NB3) $f(r)\lambda = \eta$ and $g(r) \equiv \frac{C}{2\pi}$ for r close to 1. In particular $\theta = \frac{2\pi t}{C}$ along $\{r = 1\}$.

The resulting contact manifold $N(B) \cup T_{\phi,C}$ will be denoted by $M_{(S,\phi),C}$.

The mapping torus $T_{\phi,C}$ admits a foliation by

$$S_t := \mathrm{graph}(h_t), t \in [0, C],$$

where $h_t : S \to \mathbb{R}$ varies smoothly with respect to t and satisfies the following:

(1) $h_0 = 0, h_C = F + C$;
(2) $h_{t_0}(x) < h_{t_1}(x)$ for any $t_0 < t_1$ and $x \in S$;
(3) $h_t \equiv t$ near ∂S.

Then $S_t, t \in [0, C]$, are the truncated pages of a compatible OBD of $M_{(S,\phi)}$; see Figure B.3.1. Each S_t is naturally a Liouville domain with Liouville form $\eta + dh_t$.

We will impose a strongly Weinstein OBD condition which guarantees that there exist S_t, $t \in [0, C]$, that are all Weinstein.

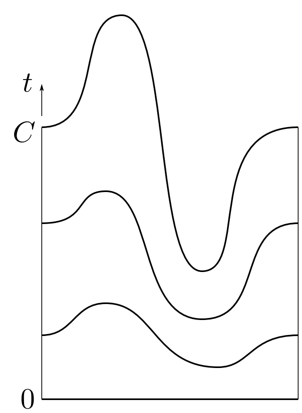

FIGURE B.3.1. Foliation of $T_{\phi,C}$ by the pages S_t.

Lemma B.3.1. *Suppose α is a contact form for (M, ξ) which is strongly adapted to the strongly Weinstein OBD (B, π). Then α can be isotoped to the contact form α' which is damped in addition.*

Proof. Let $((S, \eta), \phi)$ be an abstract OBD representing (B, π). We first discuss the mapping torus part of the open book. Starting with the fundamental domain

$$T^{\bullet}_{\phi,C} = \{(t, x) \in \mathbb{R} \times S \mid 0 \le t \le F(x) + C\} \subset \mathbb{R} \times S$$

foliated by Weinstein domains $S_t = \text{graph}(h_t), t \in [0, C]$, the trick is to thicken $T_{\phi,C}$ by inserting $[0, C'] \times S$ with $C' \gg 0$. More precisely, take $k \gg 0$ and consider a new angular variable $\tau \in [0, kC]$. Then the graph of

$$H_\tau := h_{\tau/k} + \tfrac{k-1}{k}\tau, \quad \tau \in [0, kC],$$

defines a foliation on the mapping torus

$$T_{\phi,kC} = \{(t, x) \in \mathbb{R} \times S \mid 0 \le t \le F + kC\} / (0, \phi(x)) \sim (F(x) + kC, x).$$

Since the Liouville form on each $S^k_\tau := \text{graph}(H_\tau)$ coincides with that on $S_{\tau/k}$, all the pages S^k_τ are Weinstein by assumption. Now observe that

$$\dot{H}_\tau = \tfrac{1}{k}\dot{h}_{\tau/k} + \tfrac{k-1}{k} \to 1$$

uniformly as $k \to \infty$, where the dot means τ-derivative. Hence we can choose a large k such that $T_{\phi,kC}$ is damped with respect to the contact form $\alpha_k|_{T_{\phi,kC}} := d\tau + \eta$.

Next we extend α_k to $N(B)$. Using $\alpha|_{N(B)} = f(r)\lambda + g(r)d\theta$ as in Eq. (B.3.1) and satisfying (NB1)–(NB3), we can write

$$\alpha_k = f_k(r)\lambda + g_k(r)d\theta,$$

near $\partial D^2 \times B$, where $f_k(r) = f(r)$ and $g_k(r) = kg(r) = \frac{kC}{2\pi}$ for r close to 1. Define $g_k := kg$ for $r \in [0, 1]$, where we are assuming without loss of generality that g is constant for $r \in [\frac{1}{2}, 1]$ and is strictly increasing for $r \in [0, \frac{1}{2}]$. We extend f_k to $r \in [0, 1]$ as a strictly decreasing function in three steps as follows. Fix $\epsilon > 0$ small. First extend f_k to $r \in [\frac{1}{2}, 1]$ arbitrarily; then to $r \in [\frac{1}{2} - \epsilon, \frac{1}{2}]$ such that $f'_k \le -cg'_k$, with equality near $r = \frac{1}{2} - \epsilon$, where we choose a constant $c > 0$ such that $f_k \ll cg_k(1)$ for all $r \in [\frac{1}{2} - \epsilon, 1]$; and finally to $r \in [0, \frac{1}{2} - \epsilon]$ such that $f'_k = -cg'_k$ holds. The reader might find it helpful to note that the curve $\{(f_k(r), g_k(r)) \mid r \in [0, 1]\}$ — for $k \gg 0$ and viewed from sufficiently far away — is close to the line segment connecting $(cg_k(1), 0)$ and $(0, g_k(1))$.

We claim that the total infinitesimal variation on $N(B)$ with respect to $\alpha_k|_{N(B)}$ is less than $11/10$ for ϵ sufficiently small. Indeed, the Reeb vector field is given by

$$R_{\alpha_k}|_{N(B)} = \frac{g'_k R_\lambda - f'_k \partial_\theta}{f_k g'_k - f'_k g_k}.$$

The coefficient of ∂_θ is equal to $\frac{1}{g_k(1)}$ on $r \ge \frac{1}{2}$ and is equal to $\frac{cg'_k}{f_k g'_k + cg'_k g_k} = \frac{c}{f_k + cg_k} \approx \frac{1}{g_k(1)}$ on $r \le \frac{1}{2} - \epsilon$. On $\frac{1}{2} - \epsilon \le r \le \frac{1}{2}$, we can estimate $|f_k g'_k / f'_k| \le |f_k / c| \ll g_k(1)$, which implies that the coefficient $\frac{-f'_k}{f_k g'_k - f'_k g_k}$ of ∂_θ is close to $\frac{1}{g_k(1)}$. The claim then follows.

The contact structure $\ker \alpha' := \ker \alpha_k$ is isotopic to ξ by varying the parameter k. Moreover α' is clearly strongly adapted by construction. □

B.4. **Quantitative stabilization of OBD.** The goal of this subsection is to prove Proposition B.0.1.

Let (B, π) be a compatible strongly Weinstein OBD of (M, ξ) and α be a strongly adapted contact form. Let $N_\epsilon(B) \cong D^2(\epsilon) \times B$ be the ϵ-neighborhood of B such that $\alpha|_{N_\epsilon(B)}$ satisfies (SA1)–(SA3). Here $\epsilon > 0$ is a small constant subject to conditions specified later in the proof.

We construct a map $s : M \to \mathbb{C}$ as follows: First define

$$s|_{N_\epsilon(B)} : N_\epsilon(B) \to D^2(\epsilon) \subset \mathbb{C}$$

as the projection to the first factor $D^2(\epsilon) \subset \mathbb{C}$, and then uniquely extend s continuously to all of M by requiring it to be constant on each $S_\theta \setminus N_\epsilon(B)$, $\epsilon e^{i\theta} \in \partial D^2(\epsilon)$. Hence $\pi = s/|s|$ on $M \setminus B$. Strictly speaking, s is only piecewise smooth, but in what follows we will pretend that s is smooth since a smoothing can easily be constructed. By definition $B = s^{-1}(0)$ and is transversely cut out.

We then consider the map $s^k : M \to \mathbb{C}$ for $k \in \mathbb{Z}_{>0}$. Since 0 is not a regular value of s^k for any $k > 1$, we need to add a small perturbation term coming from B to s^k.

By assumption B also admits a compatible strongly Weinstein OBD (B_1, π_1) such that $\lambda := \alpha|_B$ is strongly adapted. As in (SA1), let $N_{\epsilon_1}(B_1) = D^2_{r_1,\theta_1}(\epsilon_1) \times B_1$ with $\epsilon_1 > 0$ be a neighborhood of B_1 on which

$$\lambda|_{N_{\epsilon_1}(B_1)} = (1 - c_{1,1} r_1^2)\lambda_1 + r_1^2 d\theta_1, \tag{B.4.1}$$

and let $s_1 : B \to \mathbb{C}$ be the associated map defined in the same manner as s with $N_\epsilon(B)$ replaced by $N_{\epsilon_1}(B_1)$. We will be using the convention that the subscript 1 (e.g., B_1, s_1, r_1, $c_{1,1}$) refers to subsets etc. of B that are analogous to those of M (e.g., B, s, r, c_1).

Next let $0 < \epsilon' \ll \epsilon$ and let $\rho : [0, \epsilon] \to \mathbb{R}_{\geq 0}$ be a C^∞-small nonincreasing bump function such that $\rho(0) > 0$, ρ is constant on $[0, \epsilon'/2]$ and ρ is supported on $[0, \epsilon']$. See Figure B.4.1 We then define the map

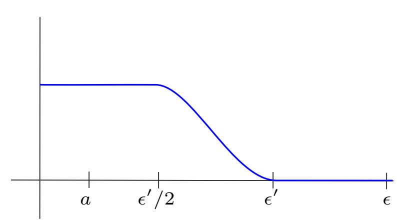

FIGURE B.4.1. The graph of ρ.

$$s_{(k)} := s^k - \rho(r) s_1 : M \to \mathbb{C}, \tag{B.4.2}$$

where s_1 is first extended to $N_\epsilon(B)$ by precomposing with the projection onto the second factor B, and then the cutoff function $\rho(r)$ guarantees that $\rho(r)s_1$ is globally defined on M.

We analyze the OBD $(B_{(k)}, \pi_{(k)})$ given by $s_{(k)}$ and the corresponding Reeb dynamics in steps. Steps 1 and 2 give topological descriptions of the binding $B_{(k)}$ and the page $S_{(k)}$ and the remaining steps describe the compatibility with a suitably C^1-small perturbed α.

Note that besides the trivial case of $\dim M = 1$, the case $\dim M = 3$ is slightly different from and substantially easier than the higher-dimensional cases since $B_1 = \varnothing$. We will point out such differences in the proof when applicable.

STEP 1. *The binding* $B_{(k)} = s_{(k)}^{-1}(0)$.

We can write

$$s(x) = \begin{cases} r(x)e^{i\theta(x)}, & \text{if } x \in N_\epsilon(B), \\ \epsilon e^{i\theta(x)}, & \text{if } x \in M \setminus N_\epsilon(B), \end{cases}$$

where $e^{i\theta(x)} = \pi(x)$ for $x \notin B$. More concisely, we write $s = re^{i\theta}$ with the understanding that $r(x) = \epsilon$ for $x \in M \setminus N_\epsilon(B)$. Similarly we write $s_1 = r_1 e^{i\theta_1}$ on B, where $r_1(y) = \epsilon_1$ for $y \in B \setminus N_{\epsilon_1}(B_1)$. Then

$$s_{(k)} = r^k e^{ik\theta} - \rho(r) r_1 e^{i\theta_1}, \tag{B.4.3}$$

$$B_{(k)} = \{s_{(k)} = r^k e^{ik\theta} - \rho(r) r_1 e^{i\theta_1} = 0\}. \tag{B.4.4}$$

Observe that, since $r^k/\rho(r)$ is strictly increasing, there exists a unique $a \in (0, \epsilon')$ such that $a^k = \rho(a)\epsilon_1$. We may assume that ρ is sufficiently small such that $a \ll \epsilon'/2$ and hence ρ is equal to the constant a^k/ϵ_1 on $[0, \epsilon'/2]$.

We have the following description of $B_{(k)}$:

Claim B.4.1.

(1) $B_{(k)} \subset \{r \le a\}$.
(2) $B_{(k)}$ *is a k-fold branched cover of B with branch locus* $B_1 \subset \{r = 0\}$.
(3) $B_{(k)} \cap \{r = a\}$ *is a k-fold cover of* $B \setminus N_{\epsilon_1}(B_1)$.
(4) $B_{(k)} \cap \{r < a\}$ *is graphical over* $D^2(a) := \{(r, \theta) \mid r < a\}$ *times* B_1.

Proof. (1) This follows from (i) $r_1 \le \epsilon_1$ on B and (ii) $B_{(k)} \cap \{\rho(r) = 0\} = \varnothing$.

(2) Clearly $B_{(k)} \cap B = B_1 \subset \{r = 0\}$. Since $\rho(r) \neq 0$ on $B_{(k)}$ we can write $B_{(k)} = \{r_1 = r^k/\rho(r), e^{i\theta_1} = e^{ik\theta}\}$. This implies that, for each point in $s_1^{-1}(r_1, \theta_1)$ with $r_1 > 0$, there exist k distinct values of (r, θ) for which $s_{(k)} = 0$.

(3) is immediate from (2).

(4) Given a point in $B_{(k)} \cap \{r < a\}$, its (r_1, θ_1)-coordinates are determined by its (r, θ)-coordinates: we have $\theta_1 = k\theta$ (mod 2π) and $r_1 = r^k/\rho(r)$. □

Note that when $\dim M = 3$, we have $\dim B = 1$ and $r_1 \equiv \epsilon_1$. Eq. (B.4.3) implies that the new binding $B_{(k)}$ is the closure of a k-strand braid around B.

STEP 2. *The page* $S_{(k)}$.

It is an easy verification that the map

$$\pi_{(k)} = s_{(k)}/|s_{(k)}| : M \setminus B_{(k)} \to S^1, \tag{B.4.5}$$
$$\pi_{(k)} = \frac{r^k e^{ik\theta} - \rho(r) r_1 e^{i\theta_1}}{|r^k e^{ik\theta} - \rho(r) r_1 e^{i\theta_1}|},$$

is a submersion, and hence induces a smooth fibration.

We analyze the page $S_{(k)} = \overline{\pi_{(k)}^{-1}(1)}$, i.e., examine the solution set to:

$$r^k e^{ik\theta} - \rho(r) r_1 e^{i\theta_1} \in \mathbb{R}_{\geq 0}. \tag{B.4.6}$$

First consider $P := S_{(k)} \cap \{r > a\}$. In this case we always have $r^k > \rho(r) r_1$ since $r_1 \leq \epsilon_1$. We claim that

$$P \cong \cup_{0 \leq j < k} (S_{\theta = 2j\pi/k} \cap \{r > a\}),$$

where the right-hand side is the disjoint union of k copies of the page S and $\cong$ means the left-hand side can be viewed as a graph over the right-hand side. Indeed, if $r \geq \epsilon'$, i.e., $\rho(r) = 0$, then Eq. (B.4.6) holds precisely when $e^{ik\theta} = 1$. Hence

$$S_{(k)} \cap \{r \geq \epsilon'\} = \cup_{0 \leq j < k} (S_{\theta = 2j\pi/k} \cap \{r \geq \epsilon'\}).[6]$$

On the other hand, if $a < r \leq \epsilon'$, then referring to the right-hand side of Figure B.4.2, we have for any fixed r, r_1, θ_1, there exists a unique $e^{ik\theta}$ such that Eq. (B.4.6) holds; the set of allowed values of $e^{ik\theta}$ is drawn in blue. Hence the components of $S_{(k)} \cap \{a < r \leq \epsilon'\}$ can be viewed as graphs over $S_{\theta = 2j\pi/k} \cap \{a < r \leq \epsilon'\}$, where θ is viewed as a function of r, r_1, θ_1. This completes the proof of the claim.

Next consider $Q := S_{(k)} \cap \{r < a\}$. Write $Q = Q_1 \cup Q_2$, where

$$Q_1 := Q \cap \{r^k < \rho(r) r_1\} \quad \text{and} \quad Q_2 := Q \cap \{r^k \geq \rho(r) r_1\}.$$

We first examine Q_1. Referring to the left-hand side of Figure B.4.2, any fixed (r, θ) determines a unique $\theta_1 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ which solves Eq. (B.4.6). Hence

$$Q_1 \cong \cup_{r < a} (S_{1, \theta_1 = \pi} \cap \{r_1 > r^k/\rho(r)\}), \tag{B.4.7}$$

where $S_{1,\theta_1 = \pi}$ is the page of (B_1, π_1) at $\theta_1 = \pi$, and θ_1 is viewed as a function of r, r_1, θ; also the right-hand side of Eq. (B.4.7) deformation retracts to $D^2(a)_{(r,\theta)} \times (S_{1,\theta_1=\pi} \cap \{r_1 = \epsilon_1\})$. Now we turn to Q_2. For each $r < a$, the set of points in M such that $r^k \geq \rho(r) r_1$ is $N_{r^k/\rho(r)}(B_1)$, which is a disk times B_1 and deformation retracts to B_1. Hence

$$Q_2 \cong \cup_{r < a, 0 \leq j < k} (\{\tfrac{2j\pi}{k}\} \times \{r\} \times N_{r^k/\rho(r)}(B_1)),$$

where θ is viewed as a function of r, r_1, θ_1.

Finally consider $R := S_{(k)} \cap \{r = a\}$, which is the union $R_1 \cup R_2$, where

$$R_1 := \{r = a, r_1 = \epsilon, e^{i\theta_1} = e^{ik\theta}\} \subset B_{(k)},$$
$$R_2 \cong \cup_{0 \leq j < k} (\{\tfrac{2j\pi}{k}\} \times \{a\} \times N_{\epsilon_1}(B_1)),$$

[6] We will be writing $S_{\theta = *}$ to avoid confusion with a page S_1 of (B_1, π_1).

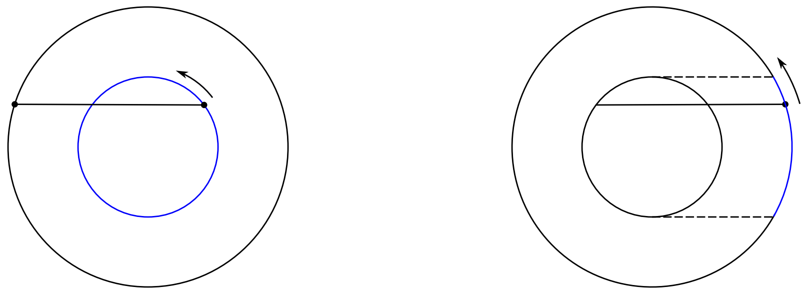

FIGURE B.4.2. The circles have radii r^k and $\rho(r)r_1$ with coordinates $e^{ik\theta}$ and $e^{i\theta_1}$, respectively. On the left we have $r^k < \rho(r)r_1$ and on the right $r^k > \rho(r)r_1$. The blue regions indicate the values of $e^{ik\theta}$ for which there exist $e^{i\theta_1}$ such that Eq. (B.4.6) holds.

where θ is a function of r_1, θ_1.

STEP 3. *Description of modified contact form and Reeb vector field.*

One technical difficulty is that the Reeb vector field given by Eq. (B.1.2) is almost never tangent to $B_{(k)}$ unless the Reeb flow of R_λ on $B \setminus B_1$ takes pages to pages. (Note that this can easily be achieved when $\dim M = 3$ since $\dim B = 1$ and $B_1 = \varnothing$ in this case.) The main task of this step therefore is to "synchronize" the Reeb flows on $M \setminus B$ and $B \setminus B_1$. In particular, we apply a C^1-small perturbation to $\alpha|_{N_\epsilon(B)}$ given by (B.1.1) into the form given by (B.4.11).

Fix $\delta > 0$ small as in the assumption of the proposition. Recall $\lambda = \alpha|_B$, which is strongly adapted to (B_1, π_1) and satisfies Eq. (B.4.1) on $N_{\epsilon_1}(B_1)$. We first choose a splitting of the exact sequence

$$0 \to TS_1^\circ \to T(B \setminus B_1) \to T_{S_1^\circ}(B \setminus B_1) \to 0, \tag{B.4.8}$$

where S_1° denotes the interior of the page: For $r_1 \leq \epsilon_1/2$, the splitting is given by the product structure $N_{\epsilon_1/2}(B_1) = D^2(\epsilon_1/2) \times B_1$. On $B \setminus N_{\epsilon_1}(B_1)$, we choose the splitting $T(B \setminus N_{\epsilon_1}(B_1)) = \mathbb{R}\langle R_\lambda \rangle \oplus TS_1^\star$, where $S_1^\star = S_1 \setminus N_{\epsilon_1}(B_1)$ is the truncated page from before. Finally we interpolate between the two splittings on $\{\epsilon_1/2 \leq r_1 \leq \epsilon_1\}$, e.g., one can use the linear interpolation with suitable smoothing at the endpoints.

We then define the vector field ∂_{θ_1} on $B \setminus B_1$ via the splitting (B.4.8) and recalling Eq. (B.1.2) (applied to λ instead of α) we write:

$$R_\lambda = f(r_1) R_{\lambda_1} + h \partial_{\theta_1}, \tag{B.4.9}$$

where $f(r_1)$ is supported in $\{r_1 < \epsilon_1\}$, $f(r_1) \equiv 1$ for $r_1 \leq \epsilon_1/2$, $h : B \to \mathbb{R}_{>0}$ is constant on $\{r_1 < \epsilon_1\}$, and the functions f and h depend on the splitting (B.4.8). Here *h can be taken to be almost constant since (B_1, π_1) is damped.*

Since α is strongly adapted to (B, π), there exists a small tubular neighborhood $N_\epsilon(B) = D^2(\epsilon) \times B$ of B such that $\alpha|_{N_\epsilon(B)} = (1 - c_1 r^2)\lambda + r^2 d\theta$. There exists

$0 < \epsilon' \ll \epsilon$ and a perturbation of α (with the same name) such that

$$\alpha|_{N_\epsilon(B)} = F(x,r)\lambda + r^2 d\theta, \tag{B.4.10}$$

where $F : N_\epsilon(B) \to \mathbb{R}_{>0}$ depends on $x \in B$ and r and satisfies:

(i) $F(x,r) = 1 - c_1 r^2$ near $r = \epsilon$;

(ii) $F(x,r) = 1 - gr^2$ for $r \leq \epsilon'$, where $g : B \to \mathbb{R}_{>0}$ is h/k and k is a large positive integer such that $A(B,\pi,\alpha)/k < \delta$;[7]

(iii) $\frac{\partial F}{\partial r} < 0$ for any $x \in B$ and $r > 0$.

Here (iii) guarantees that $\alpha|_{N_\epsilon(B)}$ is contact and is adapted to (B,π). (A direct computation shows that R_α is parallel to $R_\lambda - \frac{\partial_r F}{2r}\partial_\theta + v$, where $v \in \eta = \ker \lambda$, and hence is positively transverse to the pages.) We note that

$$\alpha|_{N_{\epsilon'}(B)} = (1 - gr^2)\lambda + r^2 d\theta. \tag{B.4.11}$$

Using Eq. (B.4.9) we compute:

$$d\alpha|_{N_{\epsilon'}(B)} = (1-gr^2)d\lambda - r^2 dg \wedge \lambda - 2grdr \wedge \lambda + 2rdr \wedge d\theta,$$
$$R_\alpha|_{N_{\epsilon'}(B)} = R_\lambda + g\partial_\theta + v = f(r_1)R_{\lambda_1} + h\partial_{\theta_1} + g\partial_\theta + v,$$

where v is the unique vector field tangent to $\eta = \ker \lambda$ solving the equation

$$(1 - gr^2)i_v d\lambda + r^2 d_\eta g = 0. \tag{B.4.12}$$

Here $d_\eta g = dg - dg(R_\lambda)\lambda$. Note that if Eq. (B.4.12) holds, then $dg(v) = 0$ and we have

$$(1-gr^2)i_v d\lambda - r^2(dg(v))\lambda + r^2 d_\eta g = 0.$$

Since both g and $d_\eta g$ are bounded, $|v|$ is small as long as r is sufficiently small. Using the splitting $T(B \setminus B_1) = \mathbb{R}\langle\partial_{\theta_1}\rangle \oplus TS_1$, we can write

$$v = \mu\partial_{\theta_1} + \tilde{v}, \quad \tilde{v} \in TS_1. \tag{B.4.13}$$

We then obtain:

$$R_\alpha|_{N_{\epsilon'}(B)} = f(r_1)R_{\lambda_1} + \tilde{v} + \tilde{h}\partial_{\theta_1} + g\partial_\theta, \quad \text{where} \quad \tilde{h} = h + \mu. \tag{B.4.14}$$

The C^0-norm $\|\tilde{h} - kg\| = \|\mu\| = O(r^2)$ for small r by Eq. (B.4.12); we assume that $\|\tilde{h}\| \gg \|\mu\|$ since we may take $\epsilon' > 0$ to be small.

Moreover, note that $\tilde{h} = kg$ and $\tilde{v} = 0$ on $\{r_1 < \epsilon_1\}$, since h and $g = h/k$ are constant there and $v = 0$ by Eq. (B.4.12). Hence $R_\alpha = f(r_1)R_{\lambda_1} + h\partial_{\theta_1} + \frac{h}{k}\partial_\theta$. One easily computes that $ds_{(k)}(R_\alpha) = ihs_{(k)}$. Hence R_α is tangent to $B_{(k)}$ and transverse to $S_{(k)}$ on $N_{\epsilon'}(B) \cap \{r_1 < \epsilon_1\}$.

Unfortunately, since $\tilde{h} \neq kg$ in general, $R_\alpha|_{N_{\epsilon'}(B)}$ is *not* everywhere tangent to $B_{(k)}$. In fact, the first-order PDE $h = kg - \mu$, whose solution would have solved the problem, has no solutions in g for general h. We will deal with this technical issue in the next few steps.

As a motivation for the above construction, we state the following, which is proved in Step 5:

[7] The choice of k depends only on $A(B,\pi,\alpha)$ and, in particular, not on ϵ.

Claim B.4.2. *There exists a small tubular neighborhood of the stabilized binding $B_{(k)}$, away from which we have*

$$A(B_{(k)}, \pi_{(k)}, \alpha) \leq \max(A(B_1, \pi_1, \lambda), A(B, \pi, \alpha)/k) < \delta. \tag{B.4.15}$$

STEP 4^-. At this point we summarize the order in which we choose the constants. We are given $\delta > 0$ and $\epsilon_1 > 0$. We choose $k > 0$ such that $A(B, \pi, \alpha)/k < \delta$. We then choose $\epsilon > 0$, followed by $\epsilon' > 0$. Finally we choose a small $\rho(r)$ so that $a > 0$ satisfying $a^k = \rho(a)\epsilon_1$ is much smaller than ϵ'. Recall that ρ is constant on $0 \leq r \leq \epsilon'/2$.

STEP 4. *The stabilized binding $B_{(k)}$ is contact.*

The goal of this step is to show that $B_{(k)} \subset (M, \xi)$, as constructed in Step 1, is a contact submanifold. It follows immediately that a small tubular neighborhood of $B_{(k)}$ is foliated by contact submanifolds since the contact condition is open. We then prove Claim B.4.3, which estimates the size of such a neighborhood.

In the following we calculate modulo error terms of order $O(a^2)$; recall that $B_{(k)} \subset \{r \leq a\}$ by Claim B.4.1(1). For α satisfying Eq. (B.4.11) we obtain:

$$\alpha = \lambda + O(a^2), \quad d\alpha = d\lambda - 2grdr\lambda + 2rdrd\theta + O(a^2), \tag{B.4.16}$$

$$\alpha \wedge d\alpha^{n-1} = \lambda \wedge (d\lambda^{n-1} + (n-1)2rdrd\theta d\lambda^{n-2}) + O(a^2). \tag{B.4.17}$$

First consider $B_{(k)} \cap \{r = a\}$, which is a k-fold cover of $B \setminus N_{\epsilon_1}(B_1)$ by Claim B.4.1(3). Since r is constant, Eq. (B.4.17) becomes

$$\alpha \wedge d\alpha^{n-1}|_{B_{(k)} \cap \{r=a\}} = \lambda \wedge d\lambda^{n-1} + O(a^2), \tag{B.4.18}$$

which implies that $B_{(k)} \cap \{r = a\}$ is contact.

Next consider $B_{(k)} \cap \{r < a\}$, which by Claim B.4.1(4) is graphical over $D^2(a) \times B_1$. By expanding Eq. (B.4.17) using $\lambda = (1 - c_{1,1}r_1^2)\lambda_1 + r_1^2 d\theta_1$ and using the fact that $r_1 dr_1 d\theta_1$ can be written as a nonnegative function of r times $drd\theta$ away from $r_1 = 0$,

$$\alpha \wedge d\alpha^{n-1}|_{B_{(k)} \cap \{r<a\}} = \phi(r) rdrd\theta \wedge \lambda_1 \wedge d\lambda_1^{n-2} + O(a^2), \tag{B.4.19}$$

where $\phi(r) \geq 2(n-1)(1 - c_{1,1}\epsilon^2)^{n-1}$. Hence $B_{(k)} \cap \{r < a\}$ is contact.

Claim B.4.3. *There exists a small constant $c > 0$ which depends on α but not on a and ρ, for which the tubular neighborhood $N_{ca^k}(B_{(k)}) := s_{(k)}^{-1}(\{|w| \leq ca^k\})$ of $B_{(k)}$ is foliated by contact submanifolds $B_w := \{s_{(k)} = w\}$.*

Proof. Let $|w| \leq ca^k$. We consider $B_w \cap \{r_1 = \epsilon_1\}$. Since $\rho = a^k/\epsilon_1$, $w = r^k e^{ik\theta} - a^k e^{i\theta_1}$. Then r, viewed as a function $B_w \cap \{r_1 = \epsilon_1\} \to \mathbb{R}$, is close to $r = a$, with error that goes to zero as $c \to 0$, where c is independent of a and ρ. The middle term on the right-hand side of Eq. (B.4.17) then goes to zero as $c \to 0$. Hence $B_w \cap \{r_1 = \epsilon_1\}$, $|w| \leq ca^k$, is contact provided $c > 0$ is small.

Next we consider $B_w \cap \{r_1 < \epsilon_1\}$, $|w| \leq ca^k$, which we write as a graph:

$$r_1 e^{i\theta_1} = (1/\rho(r))(r^k e^{ik\theta} - w).$$

By writing $r_1 dr_1 d\theta_1$ in terms of $drd\theta$ as in Eq. (B.4.19) and observing that $\rho = a^k/\epsilon_1$, we see that $B_w \cap \{r_1 < \epsilon_1\}$ is contact. $\square$

STEP 5. *Transversality away from $B_{(k)}$.*

Lemma B.4.4. *For the small constant $c > 0$ from Claim B.4.3, R_α is transverse to $S^\circ_{(k)}$ on $M \setminus N_{ca^k}(B_{(k)})$, provided $a > 0$ is sufficiently small.*

Proof. We fix $S_{(k)}$ to be the page at angle 0. All the other pages can be treated in the same manner.

First consider the restriction of $S_{(k)}$ to $M \setminus N_{\epsilon'}(B)$. Outside of $N_\epsilon(B)$ the contact form α is the original one and R_α is transverse to $S_{(k)}$ since $\rho = 0$. On $N_\epsilon(B) \setminus N_{\epsilon'}(B)$, we still have $\rho = 0$ and by (iii) following Eq. (B.4.10), α is adapted to (B, π) and R_α is transverse to $S_{(k)}$. Recall that $B_{(k)} \subset \{r \leq a\}$, where $a < \epsilon'$.

Next we restrict to $N_{\epsilon'}(B)$, on which R_α is given by Eq. (B.4.14). Observe that R_α is transverse to $S_{(k)}$ when $ds_{(k)}(R_\alpha)$ has positive $i\mathbb{R}$-component. When $r_1 < \epsilon_1$, by the paragraph after Eq. (B.4.14) in Step 3, R_α is tangent to $B_{(k)}$ and transverse to $S^\circ_{(k)}$ on this region.

It remains to consider $N_{\epsilon'}(B) \cap \{r_1 = \epsilon_1\}$. Since $f(r_1) = 0$ on $r_1 = \epsilon_1$ by definition (see the line after Eq. (B.4.9)), $R_\alpha = \tilde{v} + \tilde{h}\partial_{\theta_1} + \frac{\tilde{h}-\mu}{k}\partial_\theta$ by Eq. (B.4.14); here $\tilde{v} \in TS_1$ by Eq. (B.4.13). We compute

$$ds_{(k)}(R_\alpha) = i(kgr^k e^{ik\theta} - \tilde{h}\rho(r) r_1 e^{i\theta_1}) = i\tilde{h}s_{(k)} - i\mu r^k e^{ik\theta}. \tag{B.4.20}$$

By the paragraph after Eq. (B.4.4), $\rho = a^k/\epsilon_1$; hence $s_{(k)} = r^k e^{ik\theta} - a^k e^{i\theta_1}$ by Eq. (B.4.3). By the definition of $N_{ca^k}(B_{(k)})$, for $x \in S_{(k)} \setminus N_{ca^k}(B_{(k)})$ we have $|s_{(k)}(x)| \geq ca^k$. Recall that $\tilde{h}$ is bounded below by a positive constant and $\|\mu\| = O(r^2)$. If $r(x) \leq Ca$, where $C > 0$ is a large constant, then $\|\tilde{h}s_{(k)}\| \geq c' c a^k$ for some $c' > 0$ and $\|\mu r^k e^{ik\theta}\| \leq c'' C^{k+2} a^{k+2}$ for some $c'' > 0$. If a is sufficiently small, then $\|\tilde{h}s_{(k)}\| > \|\mu r^k e^{ik\theta}\|$, and hence $ds_{(k)}(R_\alpha)$ is dominated by the first term $i\tilde{h}s_{(k)}$. If $r(x) > Ca$, then $s_{(k)}$ is dominated by the $r^k e^{ik\theta}$ term whose size is bounded below by $C^k a^k$. This in turn implies that $ds_{(k)}(R_\alpha)$ is dominated by the first term $i\tilde{h}s_{(k)}$ since we may take $\|\tilde{h}\| \gg \|\mu\|$. Hence R_α is transverse to $S_{(k)} \setminus N_{ca^k}(B_{(k)})$ on $N_{\epsilon'}(B) \cap \{r_1 = \epsilon_1\}$. $\square$

Claim B.4.2 now readily follows from the observation that the maximal action of Reeb chords of $(B_{(k)}, \pi_{(k)})$ in $N_\epsilon(B) \setminus N_{ca}(B_{(k)})$ is approximately equal to $A(B_1, \pi_1, \lambda)$.

STEP 6. *Transversality near the binding $B_{(k)}$.*

In this step we modify α on $N_{ca^k}(B_{(k)})$ so that the Reeb vector field becomes compatible with the OBD $(B_{(k)}, \pi_{(k)})$. Since R_α is already transverse to the pages and tangent to the binding for $r_1 < \epsilon_1$ by the proof of Lemma B.4.4, we assume $r_1 = \epsilon_1$ throughout this step.

Let $B^\star_{(k)} := B_{(k)} \cap \{r = a\}$ be the k-fold cyclic cover of $B \setminus N_{\epsilon_1}(B_1)$ and let $S^\star_1 := S_1 \setminus N_{\epsilon_1}(B_1)$ be a truncated page of (B_1, π_1). (It is instructive to keep in mind the $\dim M = 3$ case, i.e., when $S_1 = S^\star_1$ is a point.) We identify

$$B^\star_{(k)} \simeq \{(z,w) \in \mathbb{R} \times S^\star_1 \mid 0 \le z \le C(w)\}/\sim,$$

where $C : S^\star_1 \to \mathbb{R}_{>0}$ is a smooth function such that $C(w)$ is roughly linear in k for each $w \in S^\star_1$ and $(C(w), w) \sim (0, \phi(w))$ for some diffeomorphism ϕ of $S^\star_1$. Moreover we may assume that $\alpha|_{B^\star_{(k)}} = dz + \sigma$ for a Liouville form σ on $S^\star_1$. We then identify

$$N_{ca^k}(B^\star_{(k)}) \simeq D^2_{x,y} \times B^\star_{(k)} \simeq D^2_{x,y} \times \{0 \le z \le C(w)\}/\sim, \tag{B.4.21}$$

so that:

- R_α is parallel to $\partial_z + W + O(a^2)$ on $N_{ca^k}(B^\star_{(k)})$, where W is tangent to $S^\star_1$ and also tangent to $\partial S^\star_1$, by Eq. (B.4.14), and
- $S_{(k)} \cap (\partial D^2 \times B^\star_{(k)})$ is a fiber $\Pi^{-1}(p)$, $p \in \partial D^2$, of the projection $\Pi : N_{ca^k}(B^\star_{(k)}) \to D^2$.

By Claim B.4.3, the fibers of Π are contact submanifolds and, after a diffeomorphism of $N_{ca^k}(B^\star_{(k)})$ which is isotopic to the identity and takes fibers to fibers, α (for the rest of the step we elide $|_{N_{ca^k}(B^\star_{(k)})}$ from the notation) can be written as

$$\alpha = F(dz + \sigma) + \tfrac{1}{2}(xdy - ydx), \tag{B.4.22}$$

where F is a positive function on $N_{ca^k}(B^\star_{(k)})$ such that $\|F - 1\|_{C^1} = O(a^2)$ and we are taking D^2 to be a disk of small radius depending on a.

Writing $R_\alpha \| X := \partial_z + a\partial_x + b\partial_y + W$, where W is tangent to $S^\star_1$, we have

$$a = (1+\sigma(W))F_y, \quad b = -(1+\sigma(W))F_x, \quad F i_W d\sigma = (1+\sigma(W))(d_{S^\star_1}F - F_z\sigma), \tag{B.4.23}$$

where $d_{S^\star_1}$ is the differential in the $S^\star_1$-direction. Indeed we verify that $i_X d\alpha = 0$:

$$\begin{aligned}
d\alpha &= (F_x dx + F_y dy + F_z dz + d_{S^\star_1}F) \wedge (dz + \sigma) + F d\sigma + dxdy,\\
i_{\partial_z} d\alpha &= -(F_x dx + F_y dy + d_{S^\star_1}F) + F_z\sigma,\\
i_{a\partial_x + b\partial_y} d\alpha &= (aF_x + bF_y)(dz + \sigma) + (ady - bdx),\\
i_W d\alpha &= d_{S^\star_1}F(W)(dz+\sigma) - \sigma(W)(F_x dx + F_y dy + F_z dz + d_{S^\star_1}F) + F i_W d\sigma.
\end{aligned}$$

Setting $a = (1 + \sigma(W))F_y$ and $b = -(1 + \sigma(W))F_x$, we can cancel all the dx, dy terms in $i_X d\alpha$ as well as $(aF_x + bF_y)(dz + \sigma)$. Setting $F i_W d\sigma = (1 + \sigma(W))(d_{S^\star_1}F - F_z\sigma)$, we have $d_{S^\star_1}F(W) = F_z\sigma(W)$, and the remaining terms of $i_X d\alpha$ can be canceled.

Observe that as $a \to 0$, $\|dF\|_{C^0}$ becomes small and hence $|\sigma(W)| \ll 1$. Also the component $Y := \partial_z + (1 + \sigma(W))F_y\partial_x - (1 + \sigma(W))F_x\partial_y$ of the vector field X is positively transverse to $\Pi^{-1}(\{\theta = const\})$ near $\{\theta = const\} \cap \partial D^2$, where we are using polar coordinates (r, θ) on D^2. More precisely, by the proof of Lemma B.4.4, $ds_{(k)}(R_\alpha)$ is dominated by the $i\tilde{h}s_{(k)}$ term; hence Y is close to the vector field Y' on $\partial D^2 \times B^\star_{(k)}$ that roughly winds $+k$ times around ∂D^2 while going once around the z-direction. Finally we replace F by the function G such that:

(a) $F = G$ near $\partial N_{ca^k}(B^\star_{(k)})$ and $\|F - G\|_{C^1}$ is small on $N_{ca^k}(B^\star_{(k)})$;
(b) for each z, w, $G(x, y, z, w)$ has the form $G(0, 0, z, w) + C_0 r^2$ near $r = 0$, where C_0 is a negative constant;
(c) $\frac{\partial G}{\partial r} < 0$ for $r > 0$, which guarantees transversality to the pages $S_{(k)}$ on $N_{ca^k}(B^\star_{(k)})$;
(d) α is still contact with F replaced by G.

In view of the above description of the pages $S_{(k)} \cap N_{ca^k}(B^\star_{(k)})$, the new contact form C^1-approximates $\alpha|_{N_{ca^k}(B^\star_{(k)})}$ and is strongly adapted.

STEP 7. *Weinstein structure on the page $S_{(k)}$.*

In this step we describe the Weinstein structure on $S_{(k)}$, which we fix to be the page at angle 0. All the other pages can be treated in a similar manner. We will decompose $S_{(k)} = T_1 \cup T_2 \cup T_3$ into three pieces and study the characteristic foliation on each piece separately. Note that the decomposition of $S_{(k)}$ in this step will be different from, but based on, the one from Step 2.

First let $T_1 := S_{(k)} \cap \{r \geq \epsilon'\} = \cup_{0 \leq j < k}(S_{\theta = 2j\pi/k} \cap \{r \geq \epsilon'\})$. The characteristic foliation on T_1 is Morse since T_1 agrees with restrictions of the Weinstein pages on the region $r \geq \epsilon$ and the modification given by Eq. (B.4.10) does not change the dynamical properties of the characteristic foliation on the region $\epsilon' \leq r \leq \epsilon$.

Next let $T_3 := S_{(k)} \cap (D^2_{r,\theta}(\epsilon') \times N_{\epsilon_1}(B_1))$ and *use the contact form α before the modification done in Step 6,* since the topological conjugacy type of the characteristic foliation on T_3 is the same for both contact forms. On $N_{\epsilon'}(B) = D^2_{r,\theta}(\epsilon') \times B$ we use the contact form $\alpha = \lambda + \frac{r^2}{1-gr^2} d\theta$ (see Eq. (B.4.11)) which is C^1-close to $\lambda + r^2 d\theta$ when ϵ' is small. Restricted to $(D^2_{r,\theta}(\epsilon') \times N_{\epsilon_1}(B_1)$ we have

$$\alpha = (1 - c_{1,1} r_1^2)\lambda_1 + r_1^2 d\theta_1 + \tfrac{r^2}{1-gr^2} d\theta.$$

Let us write $\tilde{r} = r^k/\rho(r)$ and $\tilde{\theta} = k\theta$. Using Cartesian coordinates $(\tilde{x}, \tilde{y}), (x_1, y_1)$ corresponding to the polar coordinates $(\tilde{r}, \tilde{\theta}), (r_1, \theta_1)$, Eq. (B.4.6) becomes $\tilde{x} - x_1 \in \mathbb{R}_{\geq 0}$ and $\tilde{y} = y_1$, and we write

$$\alpha = (1 - c_{1,1} r_1^2)\lambda_1 + (x_1 d\tilde{y} - \tilde{y} dx_1) + \psi(\tilde{r})(\tilde{x} d\tilde{y} - \tilde{y} d\tilde{x}),$$

where $\psi > 0$. We now calculate the characteristic foliation Y.

$$\begin{aligned}
d\alpha &= (1 - c_{1,1} r_1^2) d\lambda_1 - 2c_{1,1}(x_1 dx_1 + \tilde{y} d\tilde{y})\lambda_1 + 2dx_1 d\tilde{y} + 2\phi(\tilde{r}) d\tilde{x} d\tilde{y}, \\
&= (1 - c_{1,1} r_1^2) d\lambda_1 + \tfrac{2c_{1,1}}{1-c_{1,1}r_1^2}(x_1 dx_1 + \tilde{y} d\tilde{y})((x_1 d\tilde{y} - \tilde{y} dx_1) + \psi(\tilde{r})(\tilde{x} d\tilde{y} - \tilde{y} d\tilde{x})) \\
&\qquad + 2dx_1 d\tilde{y} + 2\phi(\tilde{r}) d\tilde{x} d\tilde{y} \\
&= (1 - c_{1,1} r_1^2) d\lambda_1 + \tfrac{2c_{1,1}}{1-c_{1,1}r_1^2}(r_1^2 dx_1 d\tilde{y} + \psi(\tilde{r})(x_1 \tilde{x} dx_1 d\tilde{y} - x_1 \tilde{y} dx_1 d\tilde{x} + \tilde{y}^2 d\tilde{x} d\tilde{y})) \\
&\qquad + 2dx_1 d\tilde{y} + 2\phi(\tilde{r}) d\tilde{x} d\tilde{y},
\end{aligned}$$

where $\phi > 0$ and $\psi(\tilde{r})$ and $\phi(\tilde{r})$ are dominated by a term of the form $C\tilde{r}^{-2+2/k}$ for $\tilde{r} \neq 0$ small. Let us write $Y = Z + \eta R_{\lambda_1}$, where

$$Z = \tfrac{2c_{1,1}}{1-c_{1,1}r_1^2}(-r_1^2\partial_{\tilde{x}} + \psi(\tilde{r})(-x_1\tilde{x}\partial_{\tilde{x}} - x_1\tilde{y}\partial_{\tilde{y}} + \tilde{y}^2\partial_{x_1})) - 2\partial_{\tilde{x}} + 2\phi(\tilde{r})\partial_{x_1}.$$

Then $i_Y d\alpha = 0$ and $\alpha(Y) = 0$ for a suitable choice of function η.

We claim the forward flow of Y limits to $T_3 \cap B_{(k)} = \{\tilde{x} = x_1, \tilde{y} = y_1\} \times B_1$. Assume $\tilde{r} \neq 0$. The case $\tilde{r} = 0$ can be treated separately, and is left to the reader. When $x_1 < 0$, then Y flows to the region $x_1 > 0$ since $r_1 < \epsilon_1$ and the coefficient of ∂_{x_1} is positive and bounded below. When $x_1 > 0$, then the coefficient of ∂_{x_1} is positive and the coefficient of $\partial_{\tilde{x}}$ is less than -2; since $\tilde{x} - x_1 \geq 0$, Y eventually flows to $\tilde{x} = x_1$. The claim then follows.

Finally we consider $T_2 := S_{(k)} \cap (D^2_{r,\theta}(\epsilon') \times (B \setminus N_{\epsilon_1}(B_1)))$. Recall the fibration $\pi_1 : B \setminus N_{\epsilon_1}(B_1) \to S^1_{\theta_1}$ with fiber $S_1^\star$ such that $\lambda|_{B\setminus N_{\epsilon_1}(B_1)} = f_1 d\theta_1 + \beta_1$, where f_1 is a positive function on $B \setminus N_{\epsilon_1}(B_1)$ and β_1 is a θ_1-dependent Liouville form on $S_1^\star$. Let Z_k be the surface in the 3-manifold $D^2_{r,\theta}(\epsilon') \times S^1_{\theta_1}$ satisfying $r^k e^{ik\theta} - \rho(r)\epsilon_1 e^{i\theta_1} \in \mathbb{R}_{\geq 0}$. Then $T_2 = (\mathrm{id}_{D^2_{r,\theta}(\epsilon')}, \pi_1)^{-1}(Z_k)$. Restrict attention to where $\rho(r)$ is constant and write $\tilde{r} = \frac{r^k}{\epsilon_1\rho(r)} = (\frac{r}{a})^k$ and $\tilde{\theta} = k\theta$. Then Eq. (B.4.6) gives $\tilde{r}\sin\tilde{\theta} = \sin\theta_1$. We view $\tilde{\theta}$ as a function of $\tilde{r}, \theta_1$. Without loss of generality we may replace $\frac{r^2}{1-gr^2}$ by r^2 in α, since ϵ' can be taken to be small. Since $\frac{\partial\tilde{\theta}}{\partial\theta_1} = \frac{1}{\tilde{r}}\frac{\cos\theta_1}{\cos\tilde{\theta}}$ and $\frac{\partial\tilde{\theta}}{\partial\tilde{r}} = -\frac{1}{\tilde{r}}\frac{\sin\tilde{\theta}}{\cos\tilde{\theta}}$, we have

$$\begin{aligned}
\alpha &= f_1 d\theta_1 + \beta_1 + r^2 d\theta = (f_1 + \tfrac{r^2}{k\tilde{r}}\tfrac{\cos\theta_1}{\cos\tilde{\theta}})d\theta_1 - \tfrac{r^2}{k\tilde{r}}\tfrac{\sin\tilde{\theta}}{\cos\tilde{\theta}}d\tilde{r} + \beta_1,\\
d\alpha &= d_{S_1}f_1 \wedge d\theta_1 + d_{S_1}\beta_1 + d\theta_1 \wedge \dot{\beta}_1 + \tfrac{2r^2}{k^2\tilde{r}}d\tilde{r} \wedge d\tilde{\theta}\\
&= (d_{S_1}f_1 - \dot{\beta}_1) \wedge d\theta_1 + d_{S_1}\beta_1 + \tfrac{2r^2}{k^2\tilde{r}^2}\tfrac{\cos\theta_1}{\cos\tilde{\theta}}d\tilde{r} \wedge d\theta_1,
\end{aligned}$$

where d_{S_1} is the exterior derivative in the S_1-direction and $\dot{\beta}_1 = \frac{\partial\beta_1}{\partial\theta_1}$. Writing $Y = X + a\partial_{\tilde{r}} + b\partial_{\theta_1}$, where $X \in TS_1$, we solve for Y in $i_Y d\alpha = -(\cos\theta_1)\alpha$. We compute:

$$\begin{aligned}
X \lrcorner\, d\alpha &= i_X d_{S_1}\beta_1 + (d_{S_1}f_1(X) - \dot{\beta}_1(X))d\theta_1,\\
a\partial_{\tilde{r}} \lrcorner\, d\alpha &= a\tfrac{2r^2}{k^2\tilde{r}^2}\tfrac{\cos\theta_1}{\cos\tilde{\theta}}d\theta_1,\\
b\partial_{\theta_1} \lrcorner\, d\alpha &= b(\dot{\beta}_1 - d_{S_1}f_1) - b\tfrac{2r^2}{k^2\tilde{r}^2}\tfrac{\cos\theta_1}{\cos\tilde{\theta}}d\tilde{r}.
\end{aligned}$$

Comparing the coefficients of $d\tilde{r}$ and $d\theta_1$ and the S_1-component, we obtain

$$\begin{aligned}
b &= -\tfrac{k}{2}\tilde{r}\sin\tilde{\theta},\\
i_X d_{S_1}\beta_1 &= -(\cos\theta_1)\beta_1 + \tfrac{k}{2}\tilde{r}\sin\tilde{\theta}(\dot{\beta}_1 - d_{S_1}f_1),\\
a &= \tfrac{k^2\tilde{r}^2}{2r^2}\tfrac{\cos\tilde{\theta}}{\cos\theta_1}(-(\cos\theta_1)f_1 + \dot{\beta}_1(X) - d_{S_1}f_1(X)) - \tfrac{k\tilde{r}}{2}\cos\theta_1.
\end{aligned}$$

By the $b\partial_{\theta_1}$ term in Y, unless $\tilde{r} = 0$, one either flows to the binding or to $\theta_1 = 0$ or π. If $\theta_1 = \pi$, then $\tilde{\theta} = 0$ or π and X is the Liouville vector field of β_1. One then flows to ∂S_1 and hence into T_3. If $\theta_1 = 0$, then $\tilde{\theta} = 0$ and X is minus the Liouville vector field of β_1. One then flows to a zero of X on S_1; then $a = -\frac{k^2\tilde{r}^2}{2r^2} f_1 - \frac{k\tilde{r}}{2} < 0$ and $\tilde{r} \to 1$ along the flow, which means one flows to the binding.

Remark B.4.5. As an aid to understanding T_2, consider the situation when $\dim M = 3$. Then $Z_k \subset S_{(k)}$ and is Morse. The topological determination of $Z_k \subset D^2 \times S^1$ is straightforward and we see that the characteristic foliation on Z_k has one index 0 critical point and k index 1 critical points; see Figure B.4.3 for an illustration in the case $k = 6$.

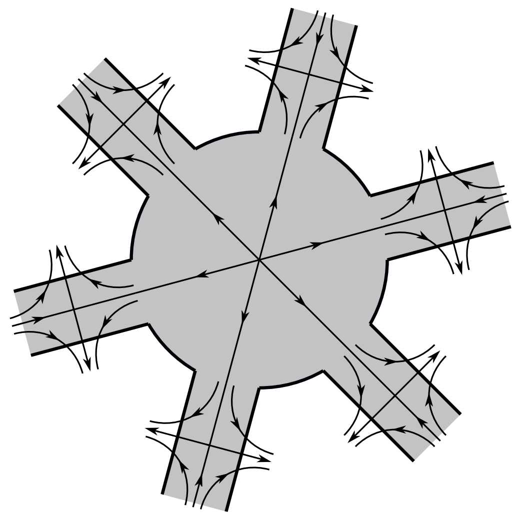

FIGURE B.4.3. The characteristic foliation on Z_k.

STEP 8. *Damping property.*

We finally prove the damping property for $(B_{(k)}, \pi_{(k)})$.

On the region $M \setminus N_{\epsilon'}(B)$, the page $S_{(k)}$ restricts to $\cup_{0 \leq j < k} S_{\theta = 2j\pi/k} \cap \{r \geq \epsilon'\}$. Since (B, π) is damped, the actions of the Reeb chords are close to $\frac{1}{k} A(B, \pi, \alpha) \approx \frac{2\pi}{kc_1}$ in view of Eq. (B.1.2) with the normalization $c_2 = 1$.

On the region $N_{\epsilon'}(B) - N_{ca}(B_{(k)})$, by Eq. (B.4.14), the relevant component of R_α is close to $h\partial_{\theta_1} + \frac{h}{k}\partial_\theta = h(\partial_{\theta_1} + \frac{1}{k}\partial_\theta)$ and the actions of the Reeb chords are close to $2\pi/h$. (We can see this for example from $ds_{(k)}(\partial_{\theta_1} + \frac{1}{k}\partial_\theta) = is_{(k)}$.)

On the region $N_{ca}(B_{(k)})$, in view of the modifications (a)–(c) from Step 6, the actions of the Reeb chords are close to $A(B_1, \pi_1, \lambda) \approx \frac{2\pi}{h}$. Taken together, (B', π') is damped with respect to a C^1-small perturbation of α'.

This completes the proof of Proposition B.0.1.

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