Stochastic Methods for Low-Rank Tensor Recovery

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Motivation: Low-rank Matrix Approximation

A Common Problem

Given matrix $A \in \mathbb{R}^{n \times d}$, find a rank-$k$ matrix B where $||A - B||$ is small.

- Motivation in CS: sparse vectors can be recovered from incomplete measurements
- Usually uses nuclear norm minimization (convex relaxation)

$$\min_{\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}} ||\mathbf{X}||_* \qquad \text{s. t.} \qquad \mathcal{A}(\mathbf{X}) = \mathbf{y}$$

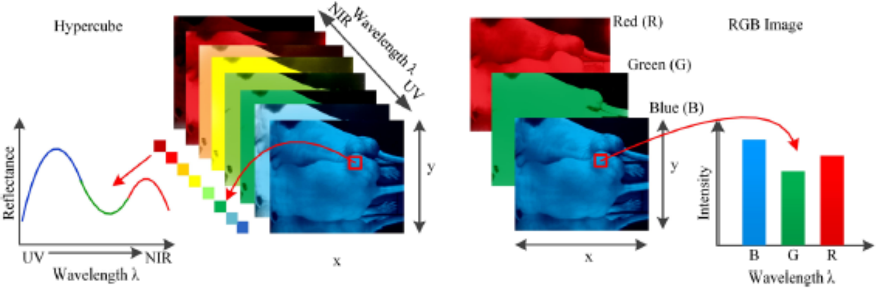
where $||\mathbf{X}||_* = \text{tr}\left((\mathbf{X}^*\mathbf{X})^{1/2}\right)$.

Problem

Question

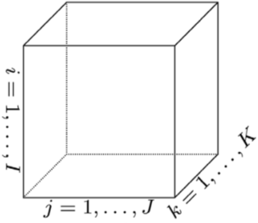
How do we deal with large dimensional data sets?

Example:



What is a tensor?

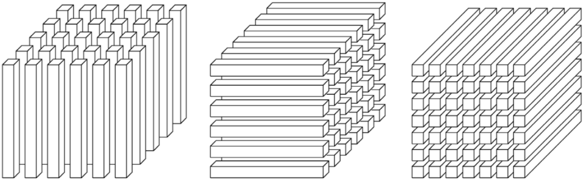
- High-dimensional extension of vectors and matrices:
  $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$.



- Examples: hyperspectral imaging, video processing, signal processing

Tensor Operations

- Matricize/Unfolding: $\mathbf{X}^{\{i\}} \in \mathbb{R}^{n_i \times (n_1 n_2 \cdots n_{i-1} n_{i+1} \cdots n_d)}$



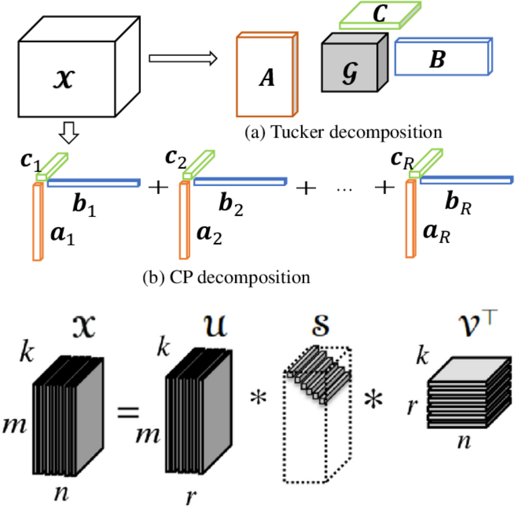
(a) Mode-1 (column) fibers: $\mathbf{x}_{:jk}$ (b) Mode-2 (row) fibers: $\mathbf{x}_{i:k}$ (c) Mode-3 (tube) fibers: $\mathbf{x}_{ij:}$

- Vectorize: $\text{vec}(\mathbf{X}) = \text{vec}(\mathbf{X}^{\{i\}})$.
- Multiplication: mode-i (matrix) product $(\mathbf{X} \times_i \mathbf{B})$ product of tensor and matrix along i-th mode
- Inner Product: $\langle \mathbf{X}_1, \mathbf{X}_2 \rangle := \text{vec}(\mathbf{X}_2)^\intercal \text{vec}(\mathbf{X}_1)$
- (induced) Frobenius norm: $||\mathbf{X}||_F := \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle}$

Images from [KB09]

Tensor Decomposition

$$\mathbf{X} \approx \mathbf{G} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C}$$

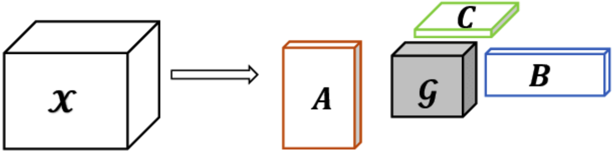


(c) Others (tubal)

Tucker Rank

One method: Higher-Order SVD (HOSVD)

$$\mathbf{X} = \mathbf{G} \times_1 \mathbf{U}^{(1)} \times_2 \cdots \times_d \mathbf{U}^{(d)}$$



Tucker rank:

$$\mathbf{r} = (r_1, \ldots, r_d), \qquad r_i = \text{rank}(\mathbf{X}^{\{i\}})$$

Tensor Recovery

Linear measurements $\mathbf{y} = \mathcal{A}(\mathbf{X}) \in \mathbb{R}^m$, $\mathcal{A} : \mathbb{R}^{n_1 \times n_2 \times \cdot \times n_d} \to \mathbb{R}^m$, linear sensing operator and

$$\mathbf{y}(i) = \mathcal{A}_i(\mathbf{X}) = \langle \mathbf{A}_i, \mathbf{X} \rangle, \qquad i = 1, \ldots, m$$

Goal:

$\min_{\mathbf{X}} F(\mathbf{X})$ subject to $\operatorname{rank}(\mathbf{X}) \leq r$ where

$$F(\mathbf{X}) = \frac{1}{2m} ||\mathbf{y} - \mathcal{A}(\mathbf{X})||_2^2 = \frac{1}{M} \sum_{i=1}^{M} \sum_{i=1}^{M} f_i(\mathbf{X})$$

$\mathcal{A}$ measurement operator.

Define functions f_i using tensor inner product:

$$f_i(\mathbf{X}_t) := \frac{1}{2b} \sum_{j=(i-1)b+1}^{ib} (y(j) - \langle \mathbf{A}_j, \mathbf{X} \rangle)^2$$

Extension to Tensors

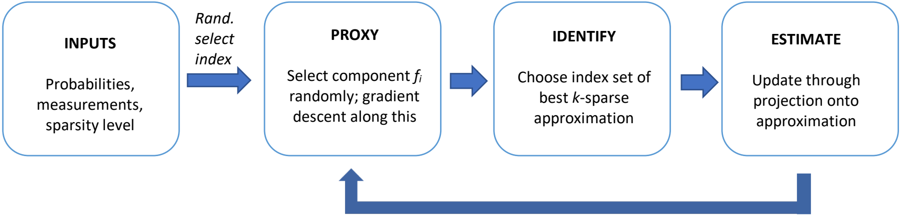
Extension of low-rank matrix recovery algorithms to tensors not obvious

- Computation of nuclear norm for tensors NP-hard
- Computation of rank of tensor NP-hard

Stochastic IHT Algorithm

Used for applications in Compressed Sensing and low-rank matrix recovery:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{M}\sum_{i=1}^{M} f_i(\mathbf{x}), \qquad \text{subject to} \qquad ||x||_{0,\mathcal{D}} \leq k.$$



Tensor IHT

- First, extended IHT to tensors by using sum nuclear norm of matricizations [GRY11, MHWG14] $\rightarrow$ nonoptimal
- Rauhut et al [RSS17] extend to TIHT:

$$\begin{aligned} \tilde{\mathbf{X}}^t &= \mathbf{X}^t + \mu\mathcal{A}^*(\mathbf{y} - \mathcal{A}(\mathbf{X}^t)) \\ \mathbf{X}^{t+1} &= \mathcal{H}_r(\tilde{\mathbf{X}}^t) \end{aligned}$$

- $\mathcal{H}_r$ computes (quasi-best) rank-$\mathbf{r}$ approx using successive HOSVD.

Tensor RIP

Let $\mathcal{A} : \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d} \to \mathbb{R}^m$. For a fixed tensor Tucker decomposition and a corresponding Tucker rank r, we say $\mathcal{A}$ and $\mathcal{A}_{b_i}$ satisfy the TRIP if there exists a tensor restricted isometry constant δ_r such that

$$(1 - \delta_r)\|\mathbf{X}\|_F^2 \leq \|\mathcal{A}(\mathbf{X})\|_2^2 \leq (1 + \delta_r)\|\mathbf{X}\|_F^2$$

hold for all tensors $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ of Tucker-rank at most r.

Tensor IHT

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- $\mathcal{H}_r$ computes (quasi-best) rank-**r** approx using successive HOSVD.
- Provable convergence guarantees under TRIP and mild assumptions

This may ring some bells...

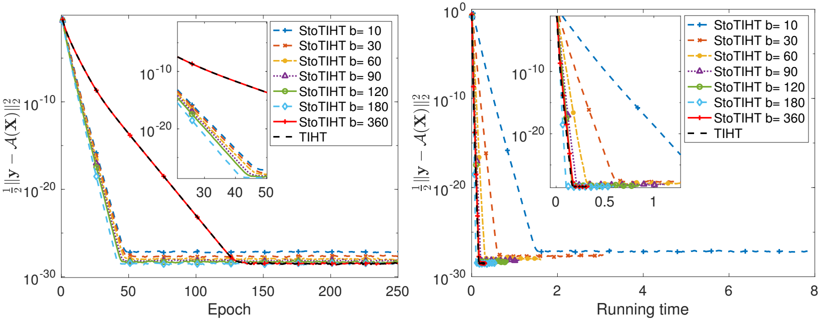
Stochastic TIHT

- Update estimate of $\mathbf{X}$ using the vectorized versions of $\mathbf{X}$ and sensing tensors $\mathbf{A}_i$ using only randomly selected i_t-th block of $\mathbf{A}$:

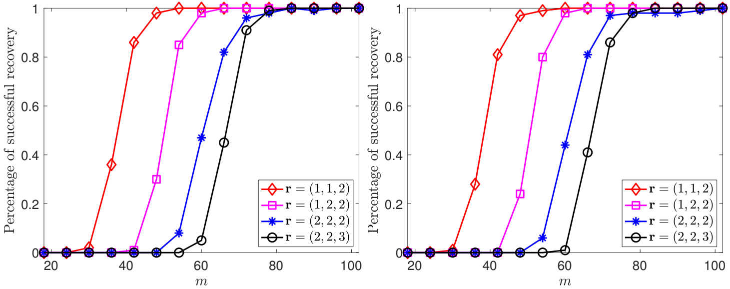
$$\tilde{\mathbf{X}}^t = \mathbf{X}^t - \frac{\mu}{Mp(i_t)} \nabla f_{i+t}(\mathbf{X}^t)$$

  where $f_{i_t} := \sum \frac{1}{2b} \sum_{j=(i_t-1)b+1}^{i_t b} (\mathbf{y}(j) - \langle \mathbf{A}_j, \mathbf{X} \rangle)^2$.
- Linear convergence (using that $\mathcal{A}, \mathcal{A}_{b_i}$ satisfy TRIP)

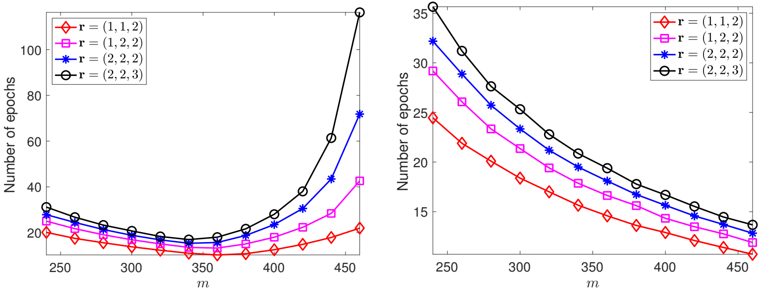
Results (5 × 5 × 6 tensor)



Results Cont

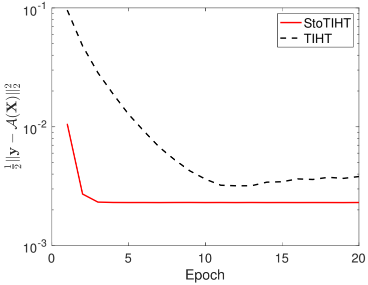


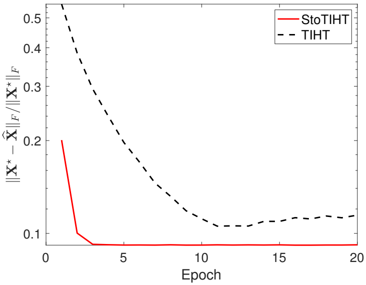
Results Cont



Left: TIHT, Right: StoTIHT

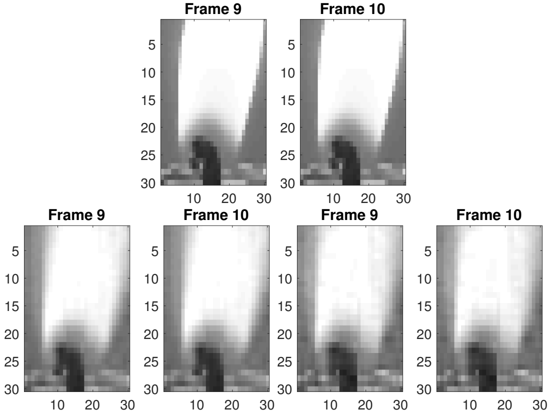
Results Cont





Left: Cost function, Right: Relative error

Results Cont.



The (a) true, (b) StoTIHT, and c) TIHT reconstructions.

Conclusions and Next Steps

- Conclusion: Stochastic version of the TIHT algorithm faster convergence, lower recovery error in large-scale setting
- Working on extending greedy stochastic algorithms for tubal rank
- Apply tensor algorithms to more real-world applications

References I

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