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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Steve Awodey (Managing Editor), John Burgess, Mark Colyvan, Anuj Dawar, Steffen Lempp, Rahim Moosa, Ernest Schimmerling, Alex Simpson, Kai Wehmeier, and Matthias Wille. Authors and publishers are requested to send, for review, copies of books to *ASL, Box 742, Vassar College, 124 Raymond Avenue, Poughkeepsie, NY 12604, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

Ernst Zermelo: Collected Works. Gesammelte Werke. Volume I: Set Theory, Miscellania. Mengenlehre, Varia, edited by H.-D. Ebbinghaus and A. Kanamori, Springer, Berlin and Heidelberg 2010, xxiv + 654 pp.

This is the first volume of a two volume set of ***Collected Works*** of Ernst Zermelo (1871–1953), devoted especially to set theory and miscellania. A second volume will present work on the calculus of variations, applied mathematics and physics.

This first volume includes papers and other documents on set theory and foundations, but also isolated papers on algebraic analysis or number-theoretic topics, and selected items of Zermelo's correspondences with K. Gödel, R. Baer and P. Bernays. The reader will also find excerpts from publications of E. Landau and D. König which were, according to these authors, written by Zermelo. An unexpected item might be Zermelo's translation of book V of Homer's ***Odyssey*** (in the shortened version published in 1930, and the unabridged manuscript from the literary remains). It presents Zermelo as a learned, humanistically well-educated scholar.

The edition satisfies highest editorial standards, following the model of the edition of K. Gödel's ***Collected Works*** (5 Vols., Oxford University Press, Oxford etc., 1986–2003). The edition is bilingual with original and English translation on pages facing each other. Mathematical symbolism is slightly modernized, but only in the English versions. The texts are accompanied by introductory notes provided by a team of experts including O. Deiser, J. Elstrodt, U. Felgner, M. Hallett, A. Henrichs, P. B. Larson, C. Parsons, R. G. Taylor, D. van

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Dalen, and D. Wolke. The notes discuss the historical contexts of the documents, and they give background material for readers not entirely familiar with the discussions on set theory and foundations in the first decades of the 20th century. Among the introductory material the reader will find H.-D. Ebbinghaus' biographical essay "Ernst Zermelo: A Glance at His Life and Work" (pp. 3–36, with photographs, pp. 37–41, and a chronological curriculum vitae, pp. 42–51), based on his seminal full-scale biography ***Ernst Zermelo: An Approach to His Life and Work*** (Springer, Berlin etc., 2007).

Ebbinghaus stresses two factors crucial for considering Zermelo's life: "First, he had a confrontational personality: he was polemical in his approach to mathematics, especially against those who disagreed with him, and this appeared in his writing and partly stimulated it. Second, he was very ill for a significant part of his middle life, and this slowed down his mathematical activities and ended his academic career when he was in his mid-forties" (p. 3). This led, despite his path-breaking contributions in different fields of research, to a life in frustration. This frustration is expressed in the letter Zermelo wrote to P. Bernays on October 1, 1941: "As far as I have been able to ascertain, *none* of the presentations and discussions [at the congress *Les fondements et la méthode des sciences mathématiques* in December 1938 in Zurich] refer to or consider any of my papers published since 1904 [...]. The questionable contributions of a Skolem and a Gödel, on the other hand, have been expatiated upon at great length. [...] Ever since, I have lost all interest in speaking publicly on foundational matters. This apparently is the lot of anyone who has no 'school' or clique behind him. But perhaps a time will come when even my papers will be rediscovered and read again" (p. 609). This sober self-perception might surprise, given the observation that Zermelo was a legend already during his life time. On the other hand, he three times unsuccessfully attempted to publish his collected works himself with different publishers. This might have supported his perception. With this edition the editors fulfilled Zermelo's dream and saved his heritage in a perfect manner.

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Wadge degrees and projective ordinals. The Cabal Seminar, Volume II, edited by A. S. Kechris, B. Löwe, and J. R. Steel, Lecture Notes in Logic, vol. 37. Association for Symbolic Logic and Cambridge University Press, Cambridge, 2012, xxii + 526 pp.—therein:

A. ANDRETTA AND A. LOUVEAU. *Wadge degrees and pointclasses. Introduction to Part III*. Pp. 3–24.

R. VAN WESEP. *Wadge degrees and descriptive set theory*. Pp. 24–43.

A. KECHRIS. *A note on Wadge degrees*. Pp. 43–47.

A. LOUVEAU. *Some results in the Wadge hierarchy of Borel sets*. Pp. 47–74.

A. LOUVEAU AND J. SAINT-RAYMOND. *The strength of Borel Wadge determinacy*. Pp. 74–102.

J. R. STEEL LOUVEAU. *Closure properties of pointclasses*. Pp. 102–118.

A. KECHRIS, R. SOLOVAY AND J. R. STEEL. *The axiom of determinacy and the prewellordering property*. Pp. 118–141.

S. JACKSON AND D. A. MARTIN. *Pointclasses and wellordered unions*. Pp. 141–154.

H. S. BECKER. *More closure properties of pointclasses*. Pp. 154–160.

J. R. STEEL. *More measures from AD*. Pp. 160–166.

W. W. WADGE. *Early investigations of the degree of Borel sets*. Pp. 166–199.

S. JACKSON. *Projective ordinals. Introduction to Part IV*. Pp. 199–270.

A. KECHRIS. *Homogenous trees and projective scales*. Pp. 270–304.

A. KECHRIS. *AD and projective ordinals*. Pp. 304–346.

R. M. SOLOVAY. *A Δ^1_3 coding of the subsets of ω_ω*. Pp. 346–364.

S. Jackson. *AD and projective ordinals*. Pp. 364–484.

D. Martin. *Projective sets and cardinal numbers: some questions related to the continuum problem*. Pp. 484–509.

S. Jackson. *Regular cardinals without weak partition property*. Pp. 509–519.

Descriptive set theory is the study of definable sets of reals that grew out of analysis. An important theorem from the classical period of the subject is that every uncountable closed set has a perfect subset. This says that closed sets have the Perfect Set Property (PSP). A corollary is that uncountable closed sets have the same cardinality as the set of all reals. In other words, closed sets do not provide counterexamples to the Continuum Hypothesis (CH). Recall that a subset of $\mathbb{R}$ is analytic if and only if it is the projection to $\mathbb{R}$ of a closed subset of $\mathbb{R} \times \mathbb{R}$. Souslin proved that a set is Borel if and only if both it and its complement are analytic. And, he proved that analytic sets have the PSP. Other so-called regularity properties from analysis such as Lebesgue Measurability (LM) and the Baire Property (BP) were also shown to hold for analytic sets during the classical period. By the middle of the 20th century, the subject saw several key contributions from logic. One observation was that little or nothing was lost by thinking about the Baire space ${}^{\omega}\omega$ rather than $\mathbb{R}$. Another was the recognition that analytic sets are exactly those which are $\undertilde{\Sigma}^1_1$ definable. In other words, analytic sets are those of the form $\{x \in {}^{\omega}\omega \mid \exists y \in {}^{\omega}\omega \, \forall n \in \omega \, R(x \restriction n, y \restriction n, z \restriction n)\}$ for some computable predicate R and $z \in {}^{\omega}\omega$. It follows that co-analytic sets are exactly those which are $\undertilde{\Pi}^1_1$. Also, Souslin's theorem says that Borel and $\undertilde{\Delta}^1_1$ coincide. This led to speculation on how results about analytic sets generalize to higher levels of the projective hierarchy ($\undertilde{\Sigma}^1_n$, $\undertilde{\Pi}^1_n$ and $\undertilde{\Delta}^1_n$ for $n < \omega$). And it drew a strong connection between analysis and computability. Logicians also isolated several concepts that were implicit in classical proofs including the Prewellordering Property (PWO) and Scale Property. They also added the Axiom of Determinacy (AD) to the mix. Beginning in the late 1960s, certain logicians, including a large group in California, succeeded in answering many of the natural conjectures about the projective hierarchy in the theory ZF + DC + AD. Their results were reported in the original Cabal Seminar volumes.

The volumes under review contain the main body of research carried out by the logicians in California from late sixties to mid eighties. Most of the material in these volumes was presented at the Cabal Seminar at UCLA and Caltech and some of the results appeared in a series of papers published in the Springer Lecture Notes in Mathematics series: Cabal Seminar 76–77 (vol. 689), Cabal Seminar 77–79 (vol. 839), Cabal Seminar 79–81 (vol. 1019), Cabal Seminar 81–85 (vol. 1333). The four books have been out of print. However, the results contained in them are as relevant today as they were when they first appeared. It is a wonderful news that ASL decided to reprint them as it was becoming increasingly difficult to own a copy of these books. The new volumes are organized by topic rather than dates which makes it far more easier to navigate. Moreover, some of the new volumes contain papers that never appeared in the old volumes but are from that period. The new volumes also contain contributions that came after the original four books appeared. In particular, the next new volume, ***The Cabal Seminar Volume III***, will contain several papers dealing with modern applications of inner model theory into descriptive set theory.

The book under review is the second in the planned series of four volumes. The book contains two parts that focus on respectively the*Wadge degrees* and the study of *projective ordinals*. The two parts are somewhat confusingly enumerated as III and IV. Nevertheless, such numbering creates a sense of natural continuation as ***The Cabal Seminar Volume I*** had two parts numbered as I and II. The first part begins with a paper by Andretta and Louveau titled *Wadge degrees and pointclasses. Introduction to Part III.* The objective of the paper is to give the reader a taste of what to expect from the papers in the first part of the book. The paper ends with a discussion of several recent developments and it also contains several conjectures that are currently a topic of active research. One particularly interesting and important conjecture is the following: Assuming $V = L(\mathbb{R})$, *the semi-linear ordering principle* (SLO), which is just the statement that the conclusion of *Wadge's Lemma holds*, implies AD.

Wadge's lemma is the starting point of the first part of the book. It states that under AD, given any two sets of reals A and B, either A is a continuous preimage of B or B is a continuous preimage of the complement of A. The fact that A is a continuous preimage of B is captured by the notation $A \leq_W B$. Thus, Wadge's lemma states that AD implies that for every $A, B \subseteq \omega^\omega$ either $A \leq_W B$ or $B \leq_W A^c$. Wadge proved his lemma by devising a very simple yet elegant game which, if determined, gives the desired conclusion. The game is so simple that can be stated here: given two sets of reals A and B, the Wadge game $W_{A,B}$ is the game in which players I and II alternatively play integers producing two reals x and y respectively and I wins the game if $x \in A \iff y \in B$.

While *SLO* has a very simple statement and proof, its consequences are rather far reaching. It implies a hierarchical stratification of sets of reals according to their *Wadge ranks*. Given a set of reals A and B, we write $A \equiv_W B$ if $A \leq_W B$ and $B \leq_W A$. The equivalence class of A under $\equiv_W$, $[A]_W$, is called the Wadge degree of A. Assuming AD, Martin showed that the Wadge degrees form a semi-linear well-ordering, i.e., there is no infinite decreasing sequence of Wadge degrees. This then implies that to each sets of reals A we can associate an ordinal $\alpha = |A|_W$ which is just the rank of $[A]_W$ in the semi-linear well-ordering of Wadge degrees. It can be shown that $|A|_W = |A^c|_W$. The results mentioned in this paragraph can be found in the second paper of the book, *Wadge degrees and descriptive set theory* by Van Wesep. This paper is highly recommended to all beginners as it contains many arguments and ideas that are used throughout descriptive set theory.

In the first part of the book, several papers investigate closure properties of *pointclasses* that are closed under Wadge reducibility or equivalently continuous preimages. Let Γ be a *boldface* pointclass or equivalently, a collection of set of reals closed under continuous preimages. We let $\check{\Gamma} = \{A^c : A \in \Gamma\}$ and $\Delta_\Gamma = \Gamma \cap \check{\Gamma}$. A particularly important closure property of Γ that is considered in the book is the prewellordering property. We let $Pwo(\Gamma)$ be the statement that for every set of reals $A \in \Gamma$ there is a norm $\phi\colon A \to Ord$ such that the resulting prewellorderings on the reals are also in Γ. More precisely, the relations

$$x \leq_\phi y \iff x \in A \wedge [y \notin A \vee \phi(x) \leq \phi(y)],$$
$$x <_\phi y \iff x \in A \wedge [y \notin A \vee \phi(x) < \phi(y)]$$

are in Γ. Notice that if $x \leq_\phi y$ then x is in A and in a sense, appears in A before y. Thus, if $Pwo(\Gamma)$ holds then for any $A \in \Gamma$ there is in a sense a Γ-listing of the elements of A. Such a listing is a crucial ingredient in the proofs of the *separation property* and the *reduction property*. We write $Sep(\Gamma)$ if given any two sets $A, B \in \Gamma$ such that $A \cap B = \emptyset$ then there is $C \in \Delta_\Gamma$ such that $A \subseteq C$ and $B \cap C$. Separation property is a generalization of the well-known theorem that every two disjoint analytic sets can be separated by a Borel sets. We write $Red(\Gamma)$ if given any two different sets of reals $A, B \in \Gamma$ then there are two disjoint sets of reals $C, D \in \Gamma$ such that $C \subseteq A$, $D \subseteq B$ and $C \cup D = A \cup B$. It can be shown that $Pwo(\Gamma)$ implies $Red(\Gamma)$ and $Sep(\check{\Gamma})$. The proof that this is indeed the case is the abstraction of the classical proof that $Red(\underset{\sim}{\Pi}^1_1)$ and $Sep(\underset{\sim}{\Sigma}^1_1)$ hold: the important ingredient of these proofs not known to the founders of descriptive set theory is that $Pwo(\underset{\sim}{\Pi}^1_1)$ holds.

The paper *The axiom of determinacy and the prewellordering property* by Kechris, Solovay and Steel identifies an important equivalent condition, under AD, for $Pwo(\Gamma)$ in terms of $Red(\Gamma)$ and closure under well-ordered unions of Δ_Γ. The authors prove their result under the assumption that Γ is closed under countable unions and intersections and also under either existential or universal quantification but not closed under complementation. As can be seen from the aforementioned theorem, closure under well-ordered unions is an important topic in descriptive set theory. The papers *Pointclasses and well-ordered unions* by Jackson and Martin and *Closure properties of pointclasses* by Steel deal with various aspects of it. For instance the first paper proves that if Γ is non-self dual and closed under both $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$ then either Γ or $\check{\Gamma}$ is closed under well-ordered unions. Becker's paper *More closure properties of pointclasses* investigates when pointclasses are closed under measure and category quantifiers. It is shown that under AD, if $\Gamma \neq \check{\Gamma}$ and Γ is closed under countable intersections and unions then Γ is closed under measure and category quantifiers. Kechris' paper *A note on Wadge degrees* provides a proof from the optimal hypothesis that the length of the wellordering of Wadge degrees of $\underset{\sim}{\Delta}^1_{2n}$ sets of reals has the same length as the sup of all $\underset{\sim}{\Delta}^1_{2n+1}$ prewellorderings.

The three papers *The strength of Borel Wadge determinacy* by Louveau and Saint-Raymond, *Some results in the Wadge hierarchy of Borel sets* by Louveau and *More measures from AD* by Steel are somewhat different from the others. The first, as the title suggests, deals with the strength of *Borel Wadge determinacy*, which is the statement that for every $A, B \in \underset{\sim}{\Delta}^1_1$ the Wadge game $W_{A,B}$ is determined. The authors show, using a work of Friedman, that Borel Wadge determinacy doesn't imply the determinacy of all Borel games. The second paper deals with the "lightface" aspects of the Wadge hierarchy, and Steel's paper constructs measures under AD by avoiding the *Boundedness Lemma*, the usual ingredient in the constructions of measures under AD. The first part ends with a highly recommended paper *Early investigations of the degrees of Borel sets* by Wadge himself. The paper is very well written and is an excellent historical account of the ideas that shaped the subject.

The second part of the book deals with projective ordinals. Let $\underset{\sim}{\delta}^1_n$ be the sup of all $\underset{\sim}{\Delta}^1_n$ prewellorderings of the reals. The sequence $(\underset{\sim}{\delta}^1_n : n \in \omega)$ is called the sequence of projective ordinals. It can be shown by rather sophisticated methods due to Moschovakis that under AD the sequence of projective ordinals is a sequence of cardinals. The study of projective ordinals is important because in the context of the Axiom of Choice the fact that for some n, $\underset{\sim}{\delta}^1_n > \aleph_1$ implies a strong failure of CH. In the paper *Projective sets and cardinal numbers: some questions related to the Continuum Problem*, Martin compares the projective ordinals to the sequence of the first ω many cardinals. He proves for instance that in the presence of a measurable cardinal $\underset{\sim}{\delta}^1_2 \leq \aleph_2$. He also shows that under AD, $\underset{\sim}{\delta}^1_3 = \aleph_{\omega+1}$. At the end of the first chapter of the paper, he conjectures that assuming AD for every n, $\underset{\sim}{\delta}^1_{2n+1} = \aleph_{\omega \cdot n+1}$. Locating the exact position of the projective ordinals in the sequence of cardinals while working under AD was one of the most important problems of descriptive set theory at that time. Steve Jackson solved the problem by showing that $\underset{\sim}{\delta}^1_{2n+1} = \aleph_{\tau(2n-1)+1}$ where $\tau(0) = 1$ and $\tau(n+1) = \omega^{\tau(n)}$. Jackson's analysis refutes Martin's aforementioned conjecture.

The second part begins with the paper *Projective ordinals: Introduction to Part IV* by Jackson which contains an outline of his arguments. This paper can also be used as a review of the background material for the papers in the second part of the book. In particular it contains the proof of the fact that under AD all projective ordinals are cardinals. The paper *AD and the projective ordinals* by Jackson contains the full proof of his theorem. The paper *Homogenous trees and projective scales* by Kechris deals with homogeneity properties of various canonical trees used in the analysis of projective sets. The next paper *AD and projective ordinals* by Kechris is a wonderful paper that introduces many of the basic techniques used to study the projective ordinals. It contains a proof of *The Coding Lemma* due to Moschovakis which is one of the most important theorems of descriptive set theory. It also introduces the study of strong partition relations and in particular, gives Martin's proof that under AD, for odd n, $\underset{\sim}{\delta}^1_n$ has the countable exponent partition property, i.e., for every n and for every $\lambda < \omega_1$, $\underset{\sim}{\delta}^1_{2n+1} \to (\underset{\sim}{\delta}^1_{2n+1})^{\lambda}$.

The paper *A* Δ^1_3 *coding of the subsets of* ω_ω by Solovay proves a key coding theorem due to Kunen. This work forms a part of the first inductive step of the aforementioned work of Jackson. The volume ends with the paper *Regular cardinals without the weak partition property* by Jackson in which he gives examples of regular cardinals which do not have the *weak partition property*, i.e., do not satisfy the partition relation $\kappa \to (\kappa)^{<\kappa}$. In contrast, Steel showed that under $AD + V = L(\mathbb{R})$ every regular cardinal is measurable.

The research carried out by the Cabal group provides a strong case for adding *definable determinacy*, such as the determinacy of all projective games (PD) or the determinacy of all games in $L(\mathbb{R})$ $(AD^{L(\mathbb{R})})$, to the axioms of set theory, *ZFC*. In particular, determinacy of definable games gives the best possible explanation behind the intuition acquired by experimentation that the simply definable sets cannot violate *CH*. Moreover, this research forms a basis of further dramatic results due to Martin, Steel and Woodin, that reduce definable determinacy to large cardinals. From the Gödelian prospective the picture emerging from the research carried out by the Cabal group and the subsequent research carried out by inner model theorists is nothing short of a picture of a paradise. To have a full grasp of it one must start with the material presented in these volumes.

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