The Bulletin of Symbolic Logic
Volume 19, Number 4, Dec. 2013

ANALYTIC EQUIVALENCE RELATIONS AND THE FORCING METHOD

JINDŘICH ZAPLETAL

Abstract. I describe several ways in which forcing arguments can be used to yield clean and conceptual proofs of nonreducibility, ergodicity and other results in the theory of analytic equivalence relations. In particular, I present simple Borel equivalence relations E, F such that a natural proof of nonreducibility of E to F uses the independence of the Singular Cardinal Hypothesis at $\aleph_\omega$.

§1. Introduction.

The theory of analytic equivalence relations on Polish spaces [2, 6, 5] has experienced a great activity surge in the last two decades. The equivalence relations are quasi-ordered by Borel reducibility: E is Borel reducible to F if there is a Borel function $h\colon \mathrm{dom}(E) \to \mathrm{dom}(F)$ such that for every $x, y \in \mathrm{dom}(E)$, $x\ E\ y \leftrightarrow h(x)\ F\ h(y)$. The main task of the theory is to find informative features of this quasi-order, and place existing equivalence relations in it. In a line of major successes [18, 1, 11, 3], the workers in the field were able to evaluate many classification programs in various fields of mathematics in terms of this quasi-order.

One of the critical challenges of the theory is finding ways of showing that one equivalence relation is not Borel reducible to another one. The main point of this article is to show that many Borel nonreducibility results between equivalence relations have very efficient and informative forcing proofs. Forcing should not be understood as a tool for obtaining consistency results here; rather, it is a way of conveniently manipulating models of set theory. Models of set theory in turn provide an environment stable with respect to various reducibility features of equivalence relations in question. In each section, I will introduce only the most illustrative concepts and theorems. The full details of proofs as well as further ramifications of the methods must be left to other publication venues.

Received April 28, 2013.
2010 *Mathematics Subject Classification.* 03E40.
Partially supported by NSF grants DMS 0801114 and 1161078 and Institutional Research Plan No. AV0Z10190503 and grant IAA100190902 of GA AV ČR.

1079-8986/13/1904-0002/$2.80
DOI:10.2178/bsl.1904020

§**2. Canonizing analytic equivalence relations.** In a joint book with Vladimir Kanovei and Marcin Sabok [7], we extended the concerns of the canonical Ramsey theory on finite or countable sets to analytic equivalence relations on Polish spaces. Recall the original Erdős–Rado canonization theorem on countable domain:

FACT 2.1. *If E is an equivalence relation on unordered pairs of natural numbers, there is an infinite set $a \subset \omega$ such that E restricted to pairs of numbers from a attains one of the following four canonical forms*:

1. *for all* $b, c \in [a]^2$, $b\ E\ c$ *holds*;
2. $b\ E\ c$ *iff* $\min(b) = \min(c)$;
3. $b\ E\ c$ *iff* $\max(b) = \max(c)$;
4. $b\ E\ c$ *iff* $b = c$.

Now, if I is a σ-ideal on a Polish space X and E is an analytic equivalence relation on X, is there a Borel I-positive set $B \subset X$ such that $E \restriction B$ is significantly simpler than E itself, perhaps even $E \restriction B$ attaining one of finitely or countably many prescribed forms? The answer to this question greatly depends on the Borel reducibility properties of E as well as the forcing properties of I. A varied and rich landscape of both positive and negative results appears.

In some cases, we obtain the most comprehensive forms of canonization possible.

DEFINITION 2.2. A σ-ideal I on a Polish space X has *total canonization* if for every Borel I-positive set $B \subset X$ and every analytic equivalence relation E on B there is a Borel I-positive set $C \subset B$ consisting either of pairwise equivalent elements, or pairwise inequivalent elements. The ideal has the *Silver property* if for every Borel equivalence relation E on X, exactly one of the following occurs:

1. either there is a Borel I-positive set consisting of pairwise E-unrelated elements,
2. or X can be covered by countably many equivalence classes and a set in I.

One satisfactory general result of this kind is the following:

DEFINITION 2.3. [8] A σ-ideal I of closed sets on a Polish space X is *calibrated* if for every closed set $C \notin I$ and all closed sets $\{D_n : n \in \omega\}$ in I there is a closed I-positive set included in $C \setminus \bigcup_n D_n$.

THEOREM 2.4. [7] *Let I be a σ-ideal on a Polish compact space X σ-generated by a coanalytic downwards hereditary collection of compact sets. Suppose that I is calibrated. Then I has total canonization and the Silver property.*

PROOF. First, an involved fusion-type argument combining the ideas of [16] and [14], is used to prove the *free set property* for I: namely, whenever

$B \subset X$ is a Borel I-positive set and $D \subset B \times B$ is an analytic set whose vertival sections are in I, then there is a Borel I-positive set $C \subset B$ such that $D \cap (C \times C)$ is a subset of the diagonal.

The free set property implies the total canonization immediately. Whenever B is a Borel I-positive set and E is an analytic equivalence relation on B, then either E has an I-positive class which then has a Borel I-positive subset $C \subset B$ by a theorem of Solecki [15], or else the application of the free set property with $D = E$ yields a Borel I-positive set $C \subset B$ consisting of pairwise E-unrelated elements.

The upgrade to the Silver property is still nontrivial. First, a uniformization argument shows that I has the *uniformization property*: whenever $D \subset 2^\omega \times X$ is a Borel set with I-positive vertical sections, then D contains a graph of a Borel function with an I-positive range. Now let E be a Borel equivalence relation on X. There are two cases. Either there are only countably many I-positive E-classes. In this case, the set $B = \{x \in X : [x]_E \in I\}$ is Borel. If $B \in I$ then we are in the second case of the Silver property, and if $B \notin I$ then the total canonization of E on B sends us to the first clause of the Silver property. Or, there are uncountably many I-positive E-classes. Then the standard Silver dichotomy yields a perfect set of E-unrelated points with I-positive classes, and the uniformization property shows that it is possible to find an I-positive Borel set $C \subset X$ which visits each of these I-positive classes in at most one point. The set C consists of pairwise E-unrelated elements, and we are again in the first clause of the Silver property. $\dashv$

The strong canonization results may seem desirable from the point of view of Ramsey theory. However, it seems that the negative results are more significant. They can be formulated in a way that seems to have no counterpart in finite or countable realm.

DEFINITION 2.5. Let I be a σ-ideal on a Polish space X, and let E be an analytic equivalence relation. Say that E is in the *spectrum* of I if there is a Borel I-positive set $B \subset X$ and an equivalence relation F on B such that for every Borel I-positive $C \subset B$, $F \restriction C$ is Borel bireducible with E.

It is not difficult to show that the spectrum of the meager ideal and the Lebesgue null ideal is very rich. Finding equivalence relations in spectra of other ideals is more challenging; it typically corelates with identification of the forcing properties of the quotient poset P_I of Borel I-positive subsets of X ordered by inclusion. For example, let E_{K_σ} be the equivalence on $\mathbb{R}^\omega$ connecting x, y if the set of numbers $x(n) - y(n)$ is bounded.

THEOREM 2.6. *Let I be the σ-ideal on ω^ω associated with Laver forcing. Then*

1. *I has total canonization for equivalence relations classifiable by countable structures*;
2. *E_{K_σ} is in the spectrum of I.*

PROOF. The first item is proved via an association between equivalence relations and models of set theory. If E is an analytic equivalence relation on a Polish space X and $x \in X$ is a point generic over V, one can define a model $V[x]_E \subset V[x]$, which depends only on the equivalence class of x. Roughly speaking, $V[x]_E$ is the collection of all sets hereditarily definable in all further forcing extensions from the equivalence class $[x]_E$ and elements of V. The Laver forcing is proper and adds a minimal forcing extension [4]. Therefore, if x is P_I-generic over V, the model $V[x]_E$ must be equal to either V or $V[x]$. This abstractly implies total canonization for I for equivalence relations classifiable by countable structures and in fact for trim equivalence relations as well, see Definition 3.5.

To place E_{K_σ} in the spectrum of I, consider the equivalence relation E on ω^ω defined by $x \mathrel{E} y$ if there is n such that for all $m \in \omega$, $\max_{i<m} x(i) \leq \max_{i<m+n} y(i)$ and $\max_{i<m} y(i) \leq \max_{i<m+n} x(i)$. It is not difficult to show that for every Laver tree T, $E \restriction [T]$ is bireducible with E_{K_σ}. ⊣

Interesting behavior is exhibited by a number of benchmark equivalence relations. Each of them seems to be a unique obstacle to canonization in its own way, as explained by the following results. Let E_0 be the modulo finite equality on 2^ω.

THEOREM 2.7. *Let E be an analytic equivalence relation on 2^ω. There is a Borel set $B \subset 2^\omega$ such that $E_0 \restriction B$ is still non-smooth, and the relation $E \restriction B$ is either equal to the identity, or to E_0, or it connects every two points of the set B.*

Let E_1 be the equivalence relation on $(2^\omega)^\omega$ connecting x, y if the set $\{n \colon x(n) \neq y(n)\}$ is finite.

THEOREM 2.8. *Let E be a hypersmooth equivalence relation on $(2^\omega)^\omega$. There is a Borel set $B \subset (2^\omega)^\omega$ such that $E_1 \restriction B$ is still bireducible with E_1, and the relation $E \restriction B$ is either equal to the identity, or to E_1, or it connects every two points of the set B.*

Let E_2 be the summable equivalence on 2^ω, connecting points x, y if the sum $\sum\{\frac{1}{n+1} \colon x(n) \neq y(n)\}$ is finite. For an ideal J on ω, write $=_J$ for the equivalence relation on 2^ω connecting x, y if $\{n \colon x(n) \neq y(n)\}$ belongs to the ideal J. In particular, E_2 is identical with $=_J$ for the summable ideal on ω, which happens to be an F_σ P-ideal.

THEOREM 2.9. *Let E be an analytic equivalence relation on 2^ω Borel reducible to $=_J$ for an F_σ P-ideal on ω. There is a Borel set $B \subset 2^\omega$ such that $E_2 \restriction B$ is still Borel bireducible with E_2, and the relation $E \restriction B$ is either equal to the identity, or to E_2, or it connects every two points of the set B.*

For most σ-ideals, the precise status of canonization for various equivalence relations remains unknown, generating a large class of challenging open questions. The positive answer to the following question would completely characterize canonization properties of definable σ-ideals σ-generated by closed sets in terms of a basic forcing property.

QUESTION 2.10. Let I be a σ-ideal on a Polish space X σ-generated by a coanalytic hereditary collection of closed sets. Must one of the following happen?

1. The quotient poset P_I of Borel I-positive sets ordered by inclusion adds a Cohen real;
2. I has total canonization and the Silver property.

§**3. Ergodicity results.** Forcing methods are particularly useful for obtaining ergodicity results between various equivalence relations. In a straightforward generalization of the concept from the theory of dynamical systems, define:

DEFINITION 3.1. [5, Section 3.1] Let E, F be analytic equivalence relations on respective Polish spaces X, Y. Say that E is *F-generically ergodic* if for every Borel homomorphism $h : X \to Y$ of E to F there is an F-equivalence class with a comeager preimage. If ν is a Borel probability measure on X, say that E is *F-ν-ergodic* if for every Borel homomorphism of E to F there is an F-equivalence class with a preimage of full ν-measure.

Clearly, if E is an equivalence relation with meager or null classes, the ergodicity means that E cannot be Borel reducible to F. A strong ergodicity result was a motivation for the development of Hjorth's concept of turbulence:

DEFINITION 3.2. [5, Section 3.2] Let $G \curvearrowright X$ be a continuous action of a Polish group on a Polish space. The action is

1. *turbulent* at a point $x \in X$ if for every open $U \subset G$ containing 1 and every open $O \subset X$ containing x, the *local orbit* of x, which is the set $\{y \in O\colon$ there are elements $\{h_i : i \in n\}$ of U such that for every $m \leq n$ the point $y_m = h_{m-1} \dots h_1 h_0 \cdot x$ is in O and $y_n = y\} \subset X$, is somewhere dense;
2. *generically turbulent* if it is turbulent at a comeager set of points, and orbits are meager and dense.

FACT 3.3. [5, Theorem 3.21] *If $G \curvearrowright X$ is a generically turbulent action, E its orbit equivalence relation, and F is an equivalence relation classifiable by countable structures, then E is F-generically ergodic.*

To motivate the techniques introduced in this section, I will start with a clean forcing restatement of turbulence. Let $G \curvearrowright X$ be a continuous Polish group action, and let E be its orbit equivalence. Write P_G, P_X for the posets of nonempty open subsets of G, X respectively, ordered by inclusion. These are just Cohen posets, introducing new elements $\dot{g}_{\text{gen}} \in G, \dot{x}_{gen} \in X$ whose genericity is characterized by belonging to every Borel comeager set coded in the ground model. In the product $P_G \times P_X$, write $\dot{x}_{\text{left}} = \dot{x}_{gen}$ and

$\dot{x}_{\mathrm{right}} = \dot{g}_{\mathrm{gen}} \cdot \dot{x}_{\mathrm{left}}$. These are E-related points of X, each of them separately P_X-generic over the ground model. Now, the following holds:

THEOREM 3.4. *If X is a Polish G-space with dense and meager orbits, the following are equivalent*:

1. *$G \curvearrowright X$ is a generically turbulent action*;
2. $P_G \times P_X \Vdash V[\dot{x}_{\mathrm{left}}] \cap V[\dot{x}_{\mathrm{right}}] = V$.

PROOF. Suppose first that (1) holds, and τ, σ are P_X-names for sets of ordinals and $\langle p, q\rangle \in P_G \times P_X$ is a condition forcing $\sigma/\dot{x}_{\mathrm{left}} = \tau/\dot{x}_{\mathrm{right}}$. I will find a strengthening $q' \leq q$ which decides the membership of all ordinals in σ, concluding that $\langle p, q'\rangle \Vdash \sigma/\dot{x}_{\mathrm{left}} \in V$. This will prove (2).

Let $g \in G$, $x \in X$ be generic points below the condition $\langle p, q\rangle$. The turbulence assumption shows that the action is turbulent at x. Let $U \subset G$ be an open neighborhood of 1 such that whenever $h \in U$ then $gh^{-1} \in p$. For every ground model element $h \in U$, $hx \in q$ implies $\sigma/x = \sigma/hx$: since $\langle gh^{-1}, hx\rangle$ is a $P_G \times P_X$-generic pair meeting the condition $\langle p, q\rangle$, it must be the case that $\sigma/x = \tau/gx = \tau/gh^{-1} \cdot hx = \sigma/hx$ by the forcing theorem. It follows that the value σ/y is the same for all y in the following version c of the local orbit of the point x: $c = \{y \in q\colon$ there is a finite sequence $\{h_i : i \in n\}$ of elements of the ground model in U such that all points $y_m = h_{m-1} \dots h_1 h_0 \cdot x$ are in q and $y_n = y\}$. Use the turbulence of the action at the point x to find a nonempty open set $q' \subset q$ in which this version of local orbit is dense. It is easy to verify that q' works.

Suppose now that (2) holds and (1) fails. Then the set of points where turbulence fails is nonmeager, and it is possible to find a nonturbulent point $x \in X$ P_X-generic over V. Let $U \subset G$, $O \subset X$ be basic open sets witnessing nonturbulence of $x \in O$, let $U' \subset U$ be a symmetric basic open neighborhood of 1 such that $hx \in O$ for all $h \in U'$, and let $g \in U'$ be a point P_G-generic over $V[x]$. Then the nowhere dense U, O-local orbits of the points x, gx are the same, since gx is on the orbit of x and vice versa. The collection W of basic open sets disjoint from this local orbit is in both models $V[x]$ and $V[gx]$ and so it is in V by (2). The closed set $X \setminus \bigcup W$ is nowhere dense, coded in V, and contains the point x P_X-generic over V. This is a contradiction. $\dashv$

This restatement of turbulence invites many generalizations as well as new ergodicity results based on the following concept.

DEFINITION 3.5. An analytic equivalence relation E on a Polish space X is *trim* if for every sufficiently large cardinal θ, for a closed unbounded set of countable elementary submodels M of H_θ, for every pair N_0, N_1 of forcing extensions of M and every pair $x_0 \in N_0$, $x_1 \in N_1$ of E-equivalent points in X,

1. either there is $x \in M$ such that $x \mathrel{E} x_0 \mathrel{E} x_1$;
2. or $N_0 \cap N_1 \neq M$.

The definition is clearly very flexible. One can for example require the generic extensions to come from posets with a prescribed forcing property Φ, yielding Φ-*trimness*. Another way is to allow countably many generic extensions as opposed to just two, yielding σ-*trimness*. Still other possibilities obtain if one considers collections of models with more complicated structure, such as infinite towers etc. All of these variations in fact distinguish some existing analytic equivalence relations. A brief outline of basic properties of the trimness concept follows.

THEOREM 3.6. 1. *All variations of trimness are preserved under Borel reducibility.*
2. *Strengthening the forcing property Φ gives a larger class of equivalence relations.*
3. $\aleph_1$-*preserving trimness is preserved under the Friedman–Stanley jump; trimness itself is not.*
4. *The universal S_∞-orbit equivalence relation is $\aleph_1$-preserving trim.*

Recall that the *Friedman–Stanley jump* of an equivalence relation E on a space X is the equivalence relation E^+ on X^ω connecting x, y if $[\mathrm{rng}(x)]_E = [\mathrm{rng}(y)]_E$.

PROOF. I will prove the most instructive item (3). Suppose that E is an analytic equivalence relation on a Polish space X, and E is $\aleph_1$-preserving trim. Let E^+ be the equivalence relation on X^ω defined by $x \; E^+ \; y$ if $[\mathrm{rng}(x)]_E = [\mathrm{rng}(y)]_E$. To show that E^+ is $\aleph_1$-preserving trim, let M be a countable elementary submodel of a large structure containing E and X, let N_0, N_1 be proper forcing extensions of M with $N_0 \cap N_1 = M$, and let $x_0 \in N_0$ and $x_1 \in N_1$ be E^+-related points; I must find a point $x \in M$ E^+ related to x_0, x_1. As x_0, x_1 are E^+-related, for every $n \in \omega$ there is $m \in \omega$ such that $x_0(n) \; E \; x_1(m)$, and then $\aleph_1$-preserving trimness of E shows that there must be a point $y \in M \cap X$ E-related to $x_0(n)$. Let a be the set of those E-classes represented in the model M that contain some entry of x_0. Then $a \in N_0 \cap N_1$ and so $a \in M$. Since the set a is countable in an $\aleph_1$-preserving extension N_0 of M, it must be countable already in M. Let $x \in M$ be any point in X^ω which visits exactly all E-classes listed in the set a. It is clear that $x \; E^+ \; x_0$ as required.

To see why trimness is not preserved under the Friedman–Stanley jump, note that trimness implies the pinned property and F_2 is not pinned. The offending partial order is the collapse of $\mathfrak{c}$ to ω. ⊣

An important point is that there are many equivalence relations which are trim and still not classifiable by countable structures. Finding such examples is a rewarding endeavor, here I will only mention one. Let J be the ideal of nowhere dense subsets of $2^{<\omega}$:

$$J = \{a \subset 2^{<\omega} : \forall s \in 2^{<\omega} \, \exists t \supset s \, \forall u \supset t \; u \notin a\}.$$

Let $X = 2^{2^{<\omega}}$, and let $=_J$ be the usual equality modulo J, a Borel equivalence relation on X. Then $=_J$ is trim.

The following theorem strengthens the motivating Hjorth's result and shows the new possibilities.

THEOREM 3.7. *If $G \curvearrowright X$ is a generically turbulent action, E its orbit equivalence relation, and F a $\aleph_1$-preserving trim equivalence relation, then E is F-generically ergodic.*

PROOF. Let $h\colon X \to \mathrm{dom}(F)$ be a Borel homomorphism of E to F. Let M be a countable elementary submodel of a large structure containing h, E, F, X. Let x_0, x_1 be elements of X separately Cohen-generic over the model M such that $x_0\ E\ x_1$ and $V[x_0] \cap V[x_1] = V$ as obtained in Theorem 3.4. Then $h(x_0) \in N_0$ and $h(x_1) \in N_1$ are F-related as h is a homomorphism. By the $\aleph_1$-preserving-trimness of F, there must be $y \in \mathrm{dom}(F) \cap M$ such that $y F h(x_0)$. Then, $h^{-1}[y]_F$ is an E-invariant analytic set coded in M, containing the Cohen-generic point x_0 as an element. Therefore, $h^{-1}[y]_F$ must be nonmeager, and since it is E-invariant, it must be in fact comeager as desired. ⊣

The turbulence method is heavily dependent on the notion of Baire category, and it is not suitable for obtaining ergodicity results in measure. Ergodicity in measure indeed requires more sophisticated tools. In many cases it turns out to be directly connected with concentration of measure computations. One satisfactory general result is the following.

DEFINITION 3.8. A sequence $\mathfrak{X} = \langle X_i, d_i, \mu_i, \varepsilon_i, \delta_i : i \in \omega\rangle$ has *concentration of measure* if for every $i \in \omega$, X_i is a finite set with metric δ_i and a probability measure μ_i, for every set $A \subset X_i$ of μ_i-measure $> \varepsilon_i$ the δ_i-neighborhood of A has μ_i-measure $> 1 - \varepsilon_i$, and $\varepsilon_i > 4\sum_{j>i} \varepsilon_j$.

For a sequence $\mathfrak{X}$ as above, let $X = \prod_i X_i$, let $\mu = \prod_i \mu_i$, and let $E_{\mathfrak{X}}$ be the equivalence relation on X defined by $x\ E_{\mathfrak{X}}\ y$ if there is $n \in \omega$ such that for every $i \in \omega$, $d_i(x(i), y(i)) < n\delta_i$.

THEOREM 3.9. *Let $\mathfrak{X}$ be a sequence with concentration of measure. Whenever F is $\aleph_1$-preserving trim equivalence relation, then $E_{\mathfrak{X}}$ is F-μ-ergodic.*

PROOF. The attentive reader certainly sees that the point is in forcing two $E_{\mathfrak{X}}$-related points $x_0, x_1 \in X$ separately μ-random over the ground model, so that $V[x_0] \cap V[x_1] = V$. Once this is achieved, the argument follows the lines of Theorem 3.7.

For the notation, let T_{ini} be the tree of all finite sequences t such that for every $i \in \mathrm{dom}(t)$, $t(i) \in X_i$. For a tree $T \subset T_{ini}$ write $[T]$ for the set of its infinite branches; so $X = [T_{ini}]$. Two points $x, y \in X$ are *close* if $\{i \in \omega : x(i) \neq y(i)\}$ is finite and $\forall i\ d_i(x(i), y(i)) \leq \delta_i$; the same terminology applies to nodes in T_{ini}. For $x \in X$ and $t \in T_{ini}$ write x rew t for the point $y \in X$ obtained by replaicing an initial segment of x with t; for $B \subset X$ let B rew $t = \{x \text{ rew } t : x \in B\}$.

Let P be the poset of all pairs $p = \langle B_p, t_p \rangle$ where $B_p \subset X$ is a Borel set of μ-positive measure, $t_p \in T_{ini}$, and each $x \in B_p$ is close to t_p. The ordering is given by $q \leq p$ if $B_q \subset B_p$, $t_q \subset t_p$, and B_q rew $t_q \subset B_p$ rew t_p. The point $\dot{x}_{\text{left}}$ is obtained as the unique point in $\bigcap\{B_p \colon p$ in the generic filter$\}$, and $\dot{x}_{\text{right}}$ is obtained as the unique point in $\bigcap\{B_p$ rew $t_p \colon p$ in the generic filter$\}$. It is easy to verify that the points $\dot{x}_{\text{left}}, \dot{x}_{\text{right}}$ are separately μ-random generic over V, and for every $i \in \omega$, $d_i(\dot{x}_{\text{left}}(i), \dot{x}_{\text{right}}(i)) \leq \delta_i$; therefore, $\dot{x}_{\text{left}} \, E_{\mathfrak{X}} \, \dot{x}_{\text{right}}$.

The verification of $V[\dot{x}_{\text{left}}] \cap V[\dot{x}_{\text{right}}] = V$ is more difficult and it uses the concentration of measure assumptions. The main claims are stated without the use of the forcing relation:

Claim 3.10. *If $i \in \omega$ and $A \subset X_i$ is a set with $\mu(A) > 4\varepsilon_i$, then there is a set $A' \subset A$ which is δ_i-connected and $\mu_i(A \setminus A') < \varepsilon_i$.*

Proof. Divide the set A into δ_i-connectedness classes. If one of these classes has measure $> \varepsilon_i$ then in fact it has measure $> \mu_i(A) - \varepsilon_i$, since its δ_i-neighborhood has μ_i-measure greater than $1 - \varepsilon_i$. Thus, it is enough to derive contradiction from the assumption that all the classes have measure $\leq \varepsilon_i$. In such case, it would be possible to divide A into two disjoint pieces A_0, A_1 invariant under δ_i-connectedness, such that both $\mu_i(A_0), \mu_i(A_1) > \varepsilon_i$. The concentration of measure assumption shows that there are points $u \in A_0$, $v \in A_1$ such that $d(u, v) \leq \delta_i$, which contradicts the assumption that the sets A_0, A_1 were invariant under δ_i-connectedness. ⊣

Claim 3.11. *For every $s \in T_{ini}$ and Borel sets $C_0, C_1 \subset [s]$ of relative measure $> \varepsilon_{|s|}$ there are points $x_0 \in C_0$ and $x_1 \in C_1$ which are close.*

Proof. Let $D_0 \subset C_0$, $D_1 \subset C_1$ be closed sets of relative measure $> \varepsilon_{|s|}$ consisting only of points of density of C_0, C_1. By induction on $j \geq |s|$ build nodes s^0_j and s^1_j of the tree T_{ini} so that:

- $s = s^0_j \subset s^1_j \subset \cdots$, $|s^0_j| = j$, D_0 has relative measure $> 2\varepsilon_j$ in $[s^0_j]$. Similarly for s^1_j;
- s^0_j, s^1_j are close.

The induction step is possible to perform since the sets $a_0 = \{u \in X_j \colon D_0$ has relative measure $> 2\varepsilon_{j+1}$ in $[(s^0_j)^\frown u]\}$ and $a_1 = \{v \in X_j \colon D_1$ has relative measure $> 2\varepsilon_{j+1}$ in $[s^1_j)^\frown v]\} \subset X_j$ have both μ_j-measure greater than ε_j and so there are $u \in a_0$ and $v \in a_1$ such that $d_j(u, v) \leq \delta_j$ by the concentration of measure assumption. In the end, let $y_0 = \bigcup_j s^0_j$ and $y_1 = \bigcup_j s^1_j$. These are points of density of the sets C_0, C_1, so there must be a number j such that the sets $C_0 \cap [s^0_j]$ and $C_1 \cap [s^1_j]$ have both relative measure greater than $1/2$. Clearly, these two sets then contain points x_0 and x_1 that are close. ⊣

Let $C \subset B \subset X$ be Borel sets. Call the set C *relatively saturated* in B if with all its points it contains also all points of B that are close to them. Say that a Borel set $B \subset X$ is *ergodic* if for every relatively saturated Borel set $C \subset B$ is of μ-measure zero or $\mu(B)$.

CLAIM 3.12. *Every μ-positive Borel set contains an ergodic μ-positive Borel subset.*

PROOF. Let $B \subset X$ be a Borel set of positive μ-measure. Using Claim 3.10, it is elementary to find a tree $T \subset T_{ini}$ and a node $t \in T$ so that

- $[T] \subset B$ and every node of T is compatible with t;
- for every node $s \supseteq t$ in T, writing $|s| = i$ it is the case that the relative μ-measure of $[T \restriction s]$ in $[s]$ is at least $4\varepsilon_i$ and the set $\{v \in X_i : s^\frown v \in T\}$ is δ_i-connected.

I claim that the set $[T]$ with the above properties must be ergodic. First, by induction on $j \geq |t|$ show that every two nodes of the tree T of length j can be connected by a finite walk consisting of nodes of T only, such that each two successive nodes on the walk are close. Then, suppose for contradiction that $[T] = C_0 \cup C_1$ is a partition into two Borel relatively saturated sets of positive μ-measure. Find nodes $s_0, s_1 \in T$ of the same length i such that C_0 has relative measure $> 1/2$ in $[T \restriction s_0]$ and C_1 has relative measure $> 1/2$ in $[T \restriction s_1]$. Considering the finite walk from s_0, s_1, it is clear that s_0, s_1 can be chosen to be close. Claim 3.11 then shows that there must be points $x \in C_0 \cap [s_0]$ and $y \in C_1 \cap [s_1]$ which are close, contradicting the assumed saturation of the sets C_0, C_1. ⊣

Now suppose that $p \in P$ is a condition and σ, τ are names in the random forcing for sets of ordinals such that $p \Vdash \sigma/\dot{x}_{\text{left}} = \tau/\dot{x}_{\text{right}}$. For simplicity of notation assume $t_p = 0$. Thin out the set B_p to make it ergodic. I claim that B_p decides the membership of every ordinal in the set σ; this will conclude the proof. Suppose that α is an ordinal and C is the Boolean value of the statement $\check{\alpha} \in \sigma$.

CLAIM 3.13. *The relative saturation of C in B has the same measure as C.*

PROOF. Otherwise, there would have to be $u \in T_{ini}$ such that the set $D = \{x \in C : x \text{ rew } u \notin C\}$ has μ-positive measure. Find μ-positive set D' such that the set D' decides the statement $\check{\alpha} \in \tau$, say forcing $\check{\alpha} \notin \tau$. Then the condition $\langle D', u\rangle \leq p$ forces in P $\check{\alpha} \in \tau/\dot{x}_{\text{left}}$ and $\check{\alpha} \notin \sigma/\dot{x}_{\text{right}}$, contradicting the initial assumption on p. ⊣

By the ergodicity of B_p, this means that the set C has either μ-measure zero or $\mu(B_p)$, which is to say that the condition B_p decides the membership of every ordinal in σ as required. ⊣

The main point of the previous theorem is that a good number of equivalence relations contain some $E_{\mathfrak{X}}$ as a subset, which leads to measure-ergodicity results for them.

COROLLARY 3.14. *Let E_2 be the summable equivalence relation on 2^ω and let μ be the usual product measure on 2^ω. Then for every $\aleph_1$-preserving-trim equivalence relation F, E_2 is F-μ-ergodic.*

PROOF. If I_i are suitably fast increasing finite subintervals stacked end-to-end, $X_i = 2^{I_i}$, μ_i the usual product measure on X_i, d_i the summable distance on X_i given by $d_i(x, y) = \sum\{\frac{1}{j+1} : j \in I_i \wedge x(j) \neq y(j)\}$, $\delta_i = 2^{-i}$ for $i > 0$ and ε_i is a suitable sequence of reals, then the sequence has concentration of measure. This is a nontrivial fact; it follows from the concentration of measure computations in [13, Theorem 4.2.5]. Its associated equivalence relation E is a subset of E_2. Thus, as E is F-μ-ergodic, E_2 must be F-μ-ergodic as well. ⊣

A great number of other ergodicity results proved in the same vein can be found in [19]. I will state the most obvious open question:

QUESTION 3.15. Is there a largest trim analytic equivalence relation in the sense of Borel reducibility?

§4. The pinned cardinal.

Greg Hjorth introduced the notion of pinned equivalence relation:

DEFINITION 4.1. [6, Chapter 17] Let E be an analytic equivalence relation on a Polish space X. If P is a forcing and τ is a P-name for an element of X then τ is (E-)*pinned* if $P \times P \Vdash \tau_{\text{left}}\ E\ \tau_{\text{right}}$. Here, $\tau_{\text{left}}, \tau_{\text{right}}$ are the $P \times P$-names for sets obtained by the evaluation of the name τ using the left or right generic filter on P, respectively. The equivalence relation is *pinned* if for every poset P and every E-pinned name τ in P there is $x \in X$ such that $P \Vdash \check{x}\ E\ \tau$.

Since the class of pinned equivalence relations is closed downwards in the Borel reducibility order, this is a useful concept for proving nonreducibility theorems. For example, let F_2 be the equivalence relation on $(2^\omega)^\omega$ connecting x, y if $\text{rng}(x) = \text{rng}(y)$; then F_2 is not reducible to E_{K_σ} since E_{K_σ} is pinned while F_2 is not. This result led to a conjecture [6, Question 17.6.1] that F_2 is the smallest Borel unpinned equivalence relation in the Borel reducibility order.

In this section, I will introduce a cardinal invariant which can be used to further discern between unpinned equivalence relations.

DEFINITION 4.2. If E is an analytic equivalence relation on a Polish space X, P, Q are posets and τ, σ are P- and Q-names for an element of X respectively, write $\langle P, \tau\rangle\ \bar{E}\ \langle Q, \sigma\rangle$ if $P \times Q \Vdash \tau\ E\ \sigma$. $\kappa(E)$ is the smallest cardinal κ such that every E-pinned name is $\bar{E}$-equivalent to a name on a poset of size $< \kappa$ if such κ exists; otherwise $\kappa(E) = \infty$. If E is pinned then write $\kappa(E) = \aleph_1$.

It is not difficult to see that the relation $\bar{E}$ on the class of E-pinned names is an equivalence relation again. The cardinal $\kappa(E)$ is a reducibility invariant, as spelled out in the following:

THEOREM 4.3. *Suppose that E, F are analytic equivalence relations on respective Polish spaces X, Y. If E is Borel reducible to F then $\kappa(E) \leq \kappa(F)$.*

PROOF. Let $h\colon X \to Y$ be a Borel reduction of E to F; by Shoenfield's absoluteness, h is still a reduction of E to F in every forcing extension. Let P be a poset and τ an E-pinned P-name; I must produce a poset Q of size $< \kappa(F)$ and a Q-name τ' $\bar{E}$-related to τ. Consider the P-name $\sigma = \dot{h}(\tau)$ for an element of the space Y. As h is a reduction in the $P \times P$-extension, σ is an F-pinned name. Find a poset Q of size $< \kappa(F)$ and a Q-name σ' which is $\bar{F}$-related to σ. Since $P \times Q \Vdash \exists x \in X\, \dot{h}(x)\, F\, \sigma'$ (consider $x = \tau$), the Mostowski absoluteness between the $P \times Q$-extension and Q-extension implies that $Q \Vdash \exists x \in X\, \dot{h}(x)\, F\, \sigma'$. Let τ' be a Q-name for such x and observe that $\langle Q, \tau' \rangle\, \bar{E}\, \langle P, \tau \rangle$ as desired. ⊣

This result must be complemented with tools for evaluating or manipulating the cardinals $\kappa(E)$. I will start with several basic observations.

THEOREM 4.4. *Let E be an analytic equivalence relation on a Polish space X.*

1. *If E is unpinned then $\kappa(E) > \aleph_1$.*
2. *If $\kappa(E) < \infty$ then $\kappa(E)$ is not greater than the first strong[1] cardinal.*
3. *If E is Borel of rank $\alpha \in \omega_1$ then $\kappa(E) \leq \beth_\alpha^+$.*
4. *If E is the isomorphism of binary relations then $\kappa(E) = \infty$.*
5. $\kappa(F_2) = \mathfrak{c}^+$.

Here, a cardinal κ is *strong* if for every λ there is an elementray embedding $j\colon V \to M$ to a transitive model M which contains V_λ as a subset. The existence of strong cardinals is a more substantial hypothesis than the existence of measurable cardinals; in particular, ZFC does not prove that strong cardinals exist.

The cardinal $\kappa(E)$ also has a rather predictable behavior with respect to the usual operations on equivalence relations; I will avoid going into further details here.

PROOF. For (1), it is enough to show that every pinned name in the Cohen forcing is forced to be E-equivalent to an element of the ground model. Let τ be the name, let M be a countable elementary submodel of a large enough structure including τ, let $H \in V$ be a Cohen-generic filter over M and let $x = \tau/H$. I claim that the Cohen forcing forces $\tau\, E\, \check{x}$. Indeed, if G is a filter Cohen generic over V, then $G \times H$ is a product generic filter over M, so $M[G, H] \models \tau/G\, E\, \tau/H$, and by the Mostowski absoluteness $V[G] \models \tau/G\, E\, x$.

[1]Added in proof. Measurable suffices.

For (2), let κ be a strong cardinal and assume that $\kappa(E) < \infty$. Let $j : V \to M$ be an elementary embedding to a transitive model M such that $j(\kappa) > \kappa(E)$ and $V_{\kappa(E)} \subset M$. In M, let P be a poset and τ an E-pinned name. The Mostowski absoluteness between the $P \times P$ extensions of V and M shows that τ is an E-pinned name even in V. Thus, there is an E-pinned name σ in some poset $Q \in V_{\kappa(E)}$ which is $\bar{E}$-related to τ. Then $\tau \in M$ and one more use of Mostowski absoluteness shows that τ is $\bar{E}$-related to τ even in M. Since τ is a name on a poset of size $< j(\kappa)$ in the model M, the elementarity of j shows that $\kappa(E) \leq \kappa$ as desired.

For (3), suppose that E is Borel of rank $\alpha \in \omega_1$ and let τ be a pinned P-name; I must produce a $\mathrm{Coll}(\omega, \beth_\alpha)$-name σ which is $\bar{E}$-related to τ. Note that $[\tau]_E$ is a P-name for a Borel set of rank $\leq \alpha$. As is the case for every name for a Borel set, [17, Corollary 2.9] shows that in the $\mathrm{Coll}(\omega, \beth_\alpha)$ extension $V[G]$ there is a Borel code for a Borel set $B \subset X$ such that in every further forcing extension $V[G][H]$ and every $x \in X \cap V[G][H]$ in that extension, $x \in B$ if and only if $V[x] \models P \Vdash \check{x} \in [\tau]_E$. Note that if $H \subset P$ is generic over $V[G]$, then the set B is nonempty in $V[G][H]$, containing the point τ/H; this follows from the fact that τ is E-pinned. Thus, the set B is nonempty already in $V[G]$ by the Mostowski absoluteness between $V[G]$ and $V[G][H]$. Back in V, let σ be any $\mathrm{Coll}(\omega, \beth_\alpha)$-name for an element of the set B. This clearly works.

For (4), let E be isomorphism of binary relations on ω. For every transitive set a, let τ_a be the $\mathrm{Coll}(\omega, |a|)$-name for a relation on ω which is isomorphic to the $\in$-relation on a. This is an E-pinned name which is not $\bar{E}$-related to any name on a poset of size $< |a|$. The reason is the Mostowski collapse theorem: if $V[G]$ is any extension in which $|a|$ is uncountable and R is a relation on ω in $V[G]$, then either R fails the axiom of extensionality or it is not wellfounded (and then can never be isomorphic to a), or it is isomorphic to a unique transitive set, which cannot be a since the set a is uncountable in $V[G]$.

For (5), whenever P is a poset and τ is an F_2-pinned name, then a standard product forcing argument shows that there is a (perhaps uncountable) ground model set $a \subset 2^\omega$ such that $P \Vdash \mathrm{rng}(\tau) = \check{a}$. Then τ is $\bar{F}_2$-equivalent to any $\mathrm{Coll}(\omega, \mathfrak{c})$-name for an enumeration of a in ordertype ω. $\dashv$

A somewhat bewildering array of analytic equivalence relations with all kinds of values of the cardinal $\kappa(E)$ appears in the study of isomorphism relation between wellfounded models of set theory.

Definition 4.5. Let L be a finite signature including a binary relation $\in$, and let ϕ be a first order sentence in L which implies the axiom of extensionality for $\in$. The equivalence relation E^w_ϕ connects L-models x, y on ω if either both of them fail to be wellfounded models of ϕ or they are isomorphic.

THEOREM 4.6. *$\kappa(E^w_\phi)$ is the least cardinal κ such that ϕ has no wellfounded model of size κ.*

PROOF. First, assume that τ is an E^w_ϕ-pinned name in some poset P for a wellfounded model of ϕ, and let $\dot{M}$ be the name for its unique transitive isomorph.

CLAIM 4.7. *$P \Vdash \dot{M}$ belongs to the ground model.*

PROOF. Write $U(M)$ for the universe of M; this is a transitive set. First argue that $P \Vdash U(M) \cap V \in V$: if there were conditions $p, q \in P$ deciding a membership of some set $b \in V$ in $U(M)$ differently, then $\langle p, q\rangle$ forces in the product $P \times P$ that $\tau_{\text{left}} \, E^w_\phi \, \tau_{\text{right}}$ fails, as the unique transitive isomorphs of the two are not the same–one contains b and the other does not. This contradicts the assumption that τ is a pinned name.

Now argue that $U(M) \in V$. To see this, let b be the transitive set in V such that $P \Vdash U(M) \cap V = \check{b}$, as obtained in the previous paragraph. I must show that in fact $P \Vdash U(M) = \check{b}$. Suppose not and let $p \in P$ be some condition that forces $U(M) \neq \check{b}$ and $\dot{c}$ be a name for an $\in$-minimal element of $U(M) \setminus \check{b}$. Then $\dot{c}$ is forced to be a subset of the ground model, which does not belong to the ground model. The product forcing theorem shows that $P \times P$ forces $\dot{c}_{\text{left}} \notin V[\dot{G}_{\text{right}}]$, and so it forces the unique transitive isomorphs of τ_{left} and τ_{right} to be distinct, as the universe of the transitive isomorph of the former contains $\dot{c}_{\text{left}}$ and the latter does not. This contradicts the assumption that τ is a pinned name again.

The same argument as in the first paragraph shows that all relations and functions of $\dot{M}$ must belong to the ground model, which proves the claim. ⊣

Now suppose that $\kappa < \kappa(E^w_\phi)$ is a cardinal. There is a pinned name which is not $\bar{E}^w_\phi$-equivalent to any pinned name on a forcing of size $< \kappa$; therefore, the unique transitive isomorph of M (which is in V by the claim) has size at least κ. It follows that there are wellfounded models of ϕ in every infinite cardinal below $\kappa(E^w_\phi)$.

On the other hand, if M was a wellfounded model of ϕ of size $\geq \kappa(E^w_\phi)$, then consider the poset $\text{Coll}(\omega, |M|)$ and any name τ for an isomorph of M with universe ω. Clearly, this is a pinned name. Also, it is not equivalent to any pinned name on a poset of size smaller than κ just like in the proof of Theorem 4.4(4), contradicting the definition of $\kappa(E^w_\phi)$. Thus, there are no wellfounded models of ϕ of size $\geq \kappa(E^w_\phi)$ and the theorem follows. ⊣

In the examples below, ZFC− denotes some large finite fragment of ZFC and f an extra symbol for a binary relation. If $\langle M, \in, f\rangle$ is a model and x is an element of M definable without parameters in $\langle M, \in\rangle$, the sentence $\theta(x)$ says that f is a graph of a bijection between the elements of x and the universe of M.

EXAMPLE 4.8. Let ϕ be the conjunction of ZFC$-$ and the statement $\theta(\aleph_{\omega_1})$. Then $\kappa(E^w_\phi) = \aleph_{\omega_1+1}$. If ψ is the conjunction of ZFC$-$ and the statement θ(first inaccessible cardinal), then $\kappa(E^w_\psi)$ is the successor of the first weakly inaccessible cardinal.

EXAMPLE 4.9. Let ϕ be the conjunction of ZFC$-$ with $2^{\aleph_0} = \aleph_2$ and $\theta(\aleph_3)$. If CH holds then $\kappa(E^w_\phi) \leq \aleph_3$, if $2^{\aleph_0} = \aleph_2$ holds then $\kappa(E^w_\phi) = \aleph_4$. Similarly, if ψ be the conjunction of ZFC$-$ with $2^{\aleph_1} = \aleph_3$ and $\theta(\aleph_3)$, then the value of $\kappa(E^w_\psi)$ is equal to or smaller than $\aleph_4$ depending on the size of $2^{\aleph_1}$. Manipulating the cardinal arithmetic by forcing, we conclude that E^w_ψ, E^w_ϕ are not Borel reducible to one another.

It is clear that the above examples just scratch the surface; various independence results, including statements about cardinal arithmetic, the square principle, thin-very tall Boolean algebras or partition relations with countable ordinals on the right hand side of the arrow, yield irreducibility for isomorphism of wellfounded models of various extensions of ZFC.

The search for Borel equivalence relations E with interesting values of the cardinal $\kappa(E)$ is a more sophisticated affair, and connections with combinatorics on small uncountable cardinals and stability theory show up. I will limit the exposition to two simple examples.

THEOREM 4.10. *For every countable ordinal $\alpha > 0$ there is a Borel equivalence relation E_α such that (in all forcing extensions) $\kappa(E_\alpha) = \aleph_\alpha$.*

PROOF. First, by induction on $\alpha \in \omega_1$ build $L_{\omega_1\omega}$ sentences ϕ_α such that ϕ_α has a model of size $\aleph_\alpha$ but not of size $\aleph_{\alpha+1}$. Start with an $L_{\omega_1\omega}$ sentence ϕ_0 describing the ordering ω. If ϕ_α has been obtained, add to its language two unary predicates A, B, a binary predicate $<$ and a ternary predicate F and let ϕ_α say "A, B partition the universe, A is a model of ϕ_α, $<$ is a linear ordering on B and F is a graph of a function from $A \times B$ to B such that for every $x, y \in B$ with $y < x$ there is $z \in A$ such that $F(z, x, y)$". In other words, a model of $\phi_{\gamma+1}$ consists of a model of ϕ_α (which is of size at most $\aleph_\alpha$) and a linear order whose proper initial segments have size at most $\aleph_\gamma$, and so has size at most $\aleph_{\alpha+1}$. If α is limit then add new unary predicates R_β for $\beta \in \alpha$ and let ϕ_α say "the predicates $\{R_\beta : \beta \in \alpha\}$ partition the universe and for every $\beta \in \alpha$ the set $\{x : \langle x\rangle \in R_\beta\}$ is a model of ϕ_β". It is easy to check that this works.

Now, let $E_{\alpha+1}$ be the isomorphism relation on the structures $\langle M, \in, \dots\rangle$ such that the universe of M is ω, $\in$ is a wellfounded two-place relation satisfying the axiom of extensionality, of rank $\leq \omega + \alpha$, and the remaining relations satisfy the sentence ϕ_α. Since the rank of the structures is bounded, the domain of the equivalence relation E_α is Borel. The models of its domain are all rigid by the Mostowski collapse theorem, and so the equivalence relation is Borel. If α is limit then let E_α be the disjoint union of equivalence relations E_β for $\beta \in \alpha$.

For the evaluation of the cardinal $\kappa(E_{\alpha+1})$, fix a transitive set a of rank $\leq \omega + \alpha$ and size $\aleph_\alpha$, equip it with relations that turn it into a model of ϕ_α, and let τ be a $\mathrm{Coll}(\omega, \aleph_\alpha)$-name for some isomorph of a with universe ω. As before, this is a pinned name that is not $\bar{E}_{\alpha+1}$-related to a name on a poset of smaller size; thus, $\kappa(E_{\alpha+1}) \geq \aleph_{\alpha+1}$. For the other inequality, suppose that τ is an E_α-pinned name on a poset P. As in the proof of Theorem 4.6, there is a ground model transitive set a with additional predicates such that $P \Vdash \tau$ is isomorphic to $\check{a}$. The choice of the sentence ϕ_α shows that $|a| \leq \aleph_\alpha$ and so τ is $\bar{E}_{\alpha+1}$-related to the $\mathrm{Coll}(\omega, \aleph_\alpha)$-name for some isomorph of a with universe ω. Ergo, $\kappa(E_{\alpha+1}) \leq \aleph_{\alpha+1}$ and the proof is complete. The easy evaluation of $\kappa(E_\alpha)$ for α limit is left to the reader. $\dashv$

Corollary 4.11. *For $\alpha > 2$, F_2 is not Borel reducible to E_α and E_α is not Borel reducible to F_2. Thus, the F_2 conjecture fails.*

Proof. Suppose for example that h is a Borel reduction of F_2 to E_α. Pass to a forcing extension $V[G]$ in which $\mathfrak{c} > \aleph_{\omega_1}$. By the Shoenfield absoluteness, h is still a reduction in $V[G]$. At the same time, $V[G] \models \kappa(F_2) > \kappa(E_\alpha)$, contradicting Theorem 4.3. Existence of reduction from E_α to F_2 is disproved by passing to a model of CH instead. $\dashv$

The last specific example in this paper illustrates a somewhat disturbing possibility: existence of two Borel equivalence relations such that the natural proof of nonreducibility of one to the other passes through the proof of independence of the Singular Cardinal Hypothesis on $\aleph_\omega$.

Theorem 4.12. *There are Borel equivalence relations E, F such that (provably in ZFC) $\kappa(E) = (\aleph_\omega^{\aleph_0})^+$ and $\kappa(F) = \max(\mathfrak{c}, \aleph_{\omega+1})^+$.*

Proof. For the construction of E, I will first find an $L_{\omega_1\omega}$ sentence ϕ which has models of size $\aleph_\omega^{\aleph_0}$ but not larger. Just let ϕ_0, ϕ_ω be sentences found in the proof of Theorem 4.10 which have models of respective sizes $\aleph_0$ and $\aleph_\omega$ but not larger, and let ϕ be a sentence in language with three additional unary relational symbols A, B, C and an additional binary functional symbol f which says "the predicates A, B, C partition the universe, A satisfies ϕ_0, B satisfies ϕ_ω, and $f\colon A \times C \to B$ is a function such that for distinct $x, y \in C$, $\mathrm{rng}(f(\cdot, x))$ and $\mathrm{rng}(f(\cdot, y))$ are distinct". It is clear that ϕ works. Now let E be the isomorphism of structures on ω for an appropriate language including the language of ϕ and a binary relation $\in$, for which $\in$ is a wellfounded relation of rank $\leq \omega + \omega + 2$ satisfying the axiom of extensionality, and the remaining relations satisfy ϕ. The same argument as in Theorem 4.10 shows that $\kappa(E) = (\aleph_\omega^{\aleph_0})^+$ as required.

For the construction of F, let $G_0 = F_2$ and $G_1 = E_{\omega+1}$ as found in Theorem 4.10. The equivalence relation $F = G_0 \times G_1$ works. $\dashv$

Corollary 4.13. *If suitable large cardinals exist, then E is not Borel reducible to F.*

PROOF. Suppose for contradiction that $h\colon \operatorname{dom}(E) \to \operatorname{dom}(F)$ is a Borel reduction. Pass to a forcing extension $V[G]$ in which SCH fails at $\aleph_\omega$, say $2^{\aleph_0} = \aleph_1$ and $\aleph_\omega^{\aleph_0} > \aleph_{\omega+1}$. Such an extension exists by [12]. Now, in $V[G]$, h is still a Borel reduction of E to F by the Shoenfield absoluteness, while $\kappa(E) > \kappa(F)$. This contradicts Theorem 4.3. $\dashv$

Mathematically significant examples of Borel equivalence relations E with interesting values of $\kappa(E)$ can be obtained from the work of Komjáth and Shelah [9, 10], and other partition results. For most unpinned Borel equivalence relations, the value of the cardinal $\kappa(E)$ remains unknown, generating a number of open questions. For example:

QUESTION 4.14. Let E be the conjugacy equivalence relation on the unitary group. What is $\kappa(E)$?

REFERENCES

[1] ILIJAS FARAH, ANDREW TOMS, and ASGER TÖRNQUIST, *Turbulence, orbit equivalence, and classification of C^* algebras*, ***Journal für die reine und angewandte Mathematik***, to appear.

[2] SU GAO, ***Invariant descriptive set theory***, CRC Press, Boca Raton, 2009.

[3] SU GAO and ALEXANDER S. KECHRIS, *On the classification of Polish metric spaces up to isometry*, ***Memoirs of the American Mathematical Society***, vol. 161 (2000), p. 78.

[4] MARCIA GROSZEK, *Combinatorics of ideals and forcing with trees*, ***The Journal of Symbolic Logic***, vol. 52 (1987), pp. 582–593.

[5] GREG HJORTH, ***Classification and orbit equivalence relations***, Mathematical Surveys and Monographs, American Mathematical Society, Providence, 2000.

[6] VLADIMIR KANOVEI, ***Borel equivalence relations***, University Lecture Series 44, American Mathematical Society, Providence, 2008.

[7] VLADIMIR KANOVEI, MARCIN SABOK, and JINDŘICH ZAPLETAL, ***Canonical Ramsey theory on Polish spaces***, Cambridge University Press, 2013, to appear.

[8] ALEXANDER KECHRIS and ALAIN LOUVEAU, ***Descriptive set theory and the structure of sets of uniqueness***, Cambridge University Press, Cambridge, 1989.

[9] PETER KOMJATH and SAHARON SHELAH, *On uniformly antisymmetric functions*, ***Real Analysis Exchange***, vol. 19 (1993–1994), pp. 218–225.

[10] ———, *Coloring finite subsets of uncountable sets*, ***Proceedings of the American Mathematical Society***, vol. 124 (1996), pp. 3501–3505.

[11] ALAIN LOUVEAU and CHRISTIAN ROSENDAL, *Complete analytic equivalence relations*, ***Transactions of the American Mathematical Society***, vol. 357 (2005), pp. 4839–4866.

[12] MENACHEM MAGIDOR, *On the singular cardinals problem. I*, ***Israel Journal of Mathematics***, vol. 28 (1977), pp. 1–31.

[13] VLADIMIR PESTOV, ***Dynamics of infinite-dimensional groups***, University Lecture Series 40, American Mathematical Society, Providence, 2006.

[14] ROMAN POL and PIOTR ZAKRZEWSKI, *On Borel mappings and σ-ideals generated by closed sets*, ***Advances in Mathematics***, vol. 231 (2012), pp. 651–663.

[15] SLAWOMIR SOLECKI, *Covering analytic sets by families of closed sets*, ***The Journal of Symbolic Logic***, vol. 59 (1994), pp. 1022–1031.

[16] SLAWOMIR SOLECKI and OTMAR SPINAS, *Dominating and unbounded free sets*, ***The Journal of Symbolic Logic***, vol. 64 (1999), pp. 75–80.

[17] Jacques Stern, *On Lusin's restricted continuum problem*, ***Annals of Mathematics***, vol. 120 (1984), pp. 7–37.

[18] Simon Thomas, *The classification problem for torsion-free abelian groups of finite rank*, ***Journal of the American Mathematical Society***, vol. 16 (2003), pp. 233–258.

[19] Jindřich Zapletal, *Generalizations of turbulence*, preprint, 2013.

INSTITUTE OF MATHEMATICS
ACADEMY OF SCIENCES
ŽITNÁ 25
11567 PRAHA 1, CZECH REPUBLIC
and
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF FLORIDA
GAINESVILLE, FL 32611, USA
E-mail: zapletal@math.ufl.edu