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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

BRUNO COURCELLE and JOOST ENGELFRIET. ***Graph structure and monadic second-order logic.*** Encyclopedia of Mathematics and Its Applications, vol. 138. Cambridge University Press, 2012, xiv + 728 pp.

This book gives a comprehensive overview of the work of Courcelle and his coauthors on the use of monadic second-order logic in graph theory. The main ideas can be summarised as follows. Fixing a graph algebra, i.e., an algebra of graphs and graph operations, one considers the class of all graphs that are denoted by a term in that algebra. Using the association of such graphs with their denoting terms, one can reduce graph-theoretic problems to problems concerning trees. This way we can use, for instance, tree automata to develop fixed-parameter tractable algorithms for many graph-theoretic properties. Logic enters the picture with the realisation that, for certain graph algebras, the evaluation function (which maps a term to the graph it denotes) can be expressed in a suitable variant of monadic second-order logic. This means that the above reduction from graphs to trees transfers to the logical context. A famous application of this connection is Courcelle's Theorem stating that every monadic second-order definable graph property is fixed-parameter tractable over graphs of bounded tree-width.

This theory has evolved over several decades in a long series of papers. For the first time it has now been collected into a single text. I predict that it will become the standard reference in the field. The writing style is characterised by a careful presentation and attention to

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detail. Thanks to that it should be equally accessible to both logicians and graph theorists, and it is well suited for graduate students.

In mathematical writing one has always to balance between being sufficiently exact and detailed, and not obscuring the greater picture by excessive verbosity and irrelevant details. The book solves this predicament by starting with a whole chapter containing a high-level overview of the remainder of the book. Using this as a guide the reader can follow the detailed presentation in the following chapters without becoming lost.

Chapter 2 sets up the machinery of graph operations and related graph complexity measures like tree-width and clique-width. After a short introduction to selected parts of Universal Algebra and Graph Theory, the authors introduce the two main graph algebras studied in the book: the HR-algebra corresponding to hyperedge replacement grammars and the VR-algebra corresponding to vertex replacement grammars. The first one is related to tree decompositions and tree-width, while the latter one can be used to define clique-width.

Chapter 3 introduces the notion of an equational set and a recognisable set and develops their theory in the context of an arbitrary algebra. Equational sets are given by a system of equations and generalise the notion of a context-free language. Recognisable sets are given by a finite congruence and generalise regular languages. In Chapter 4 the notions of a equational set and a recognisable set are considered in the special case of the HR-algebra and VR-algebra. The chapter contains numerous examples of well-known concepts from Graph Theory and Discrete Mathematics that can be viewed from this point of view.

Chapter 5 begins the study of monadic second-order logic. After introducing several variants of this logic and giving examples of graph properties expressible in them, it concentrates on the link between definability and recognisability. One of the main results is the fact that all definable classes of graphs are recognisable. The chapter concludes with the introduction of several technical tools to study logical properties of graphs.

Chapter 6 turns to algorithmic applications of the logical machinery. The main result is Courcelle's Theorem stating that evaluating a monadic second-order formula is fixed parameter tractable for graphs of bounded tree-width. The corresponding algorithm has two phases: it first constructs a tree decomposition of the input graph and then runs a tree automaton on the decomposition tree. Consequently, the main themes of the chapter are (i) the parsing problem (finding decompositions of a given graph) and (ii) the translation of monadic second-order formulae into tree automata.

Chapter 7 deals with the important topic of transductions, graph operations defined by logical formulae. Their main property is the fact that each transduction induces a translation of formulae such that, instead of evaluating a formula on the image under the transduction, one can evaluate the translated formula on the preimage. This means that transductions provide a notion of reduction for logical theories. Since a class of graphs has bounded clique-width if, and only if, it is the image of a set of trees under a transduction and, similarly, it has bounded tree-width if, and only if, the corresponding class of incidence graphs is such an image, it follows that we can reduce the evaluation of monadic second-order formulae on graphs of bounded tree-width or bounded clique-width to the evaluation of such formulae on classes of trees. In the same way we can characterise HR-equational and VR-equational classes of graphs as the images of regular classes of trees.

As explained above many questions concerning graph classes can be reduced to questions concerning their decompositions. To a lesser extent the same is true for operations on graph classes. Chapter 8 studies operations on classes of trees that are defined either by tree transducers or transductions. The main result states that these two formalisms have the same expressive power.

The final chapter generalises the theory presented for graphs to classes of arbitrary relational structures. In many cases this generalisation is straightforward, but when dealing with edge-set quantification some thought is required. Consequently, the focus of the chapter is on tree-width and variants of monadic second-order logic with edge-set quantification. One of the main theorems is the result that, for uniformly sparse structures, edge-set quantification can be eliminated. A second topic where generalisation to relational structures is non-trivial is the definition of VR-algebras and a corresponding notion of clique-width in this setting.

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MARK COLYVAN. ***An introduction to the philosophy of mathematics.*** Cambridge Introductions to Philosophy. Cambridge University Press, Cambridge, UK, 2012, 198 pp.

The past 150 years in the philosophy of mathematics have divided (very roughly) into three periods. Beginning with Frege's attempts to put mathematics on a more precise and rigorous footing, the first period—which we might call the *period of mathematical rigour*—sought the correct foundational theory for mathematics. Beginning with Benacerraf's seminal papers in the 1960s, the second period—which we might call the *period of mathematical objects*—explored the metaphysical nature of the discipline's subject matter, and tried to account for our epistemic access to it. In both cases, philosophers were concerned to defend and understand the unique status of mathematics as the most certain of the sciences. That preoccupation has faded into the background during the third period—which we might call the *period of mathematical practice*. In this period, philosophers of mathematics have started to address more local questions raised by the practice of mathematics itself, rather than by philosophical reflection on its place in our understanding of the world.

In this introductory undergraduate textbook on the philosophy of mathematics, Mark Colyvan surveys all three periods, though at very different levels of detail. In Chapter 1, he conducts a whistlestop tour of the period of mathematical rigour, covering intuitionism, logicism, and formalism; and he prepares us for the slightly longer survey of the period of mathematical objects that he provides two chapters later: Chapter 3 is devoted to the realist positions of structuralism and platonism; Chapter 4 covers the anti-realist positions of fictionalism and nominalism. In between, in Chapter 2, Colyvan includes a useful informal survey of the main metamathematical results that constrain philosophical accounts of mathematics: the Löwenheim–Skolem theorems, Gödel's first Incompleteness Theorem, and the independence of the Continuum Hypothesis.

In Chapters 5–8, Colyvan turns to the period of mathematical practice. At this point the book really begins, moving from a fairly standard, if admirably clear, presentation of well-worn material to an exciting exploration of nascent topics on which there is still relatively little literature. Due to the paucity of published material on these topics, Colyvan's style of presentation departs from the usual style of philosophy textbooks: instead of presenting fully formulated theses and arguments, together with objections, responses, and counter-responses, he presents an array of examples and uses them to illustrate observations that are yet to be subsumed under any single overarching account. In a research monograph, this would be frustrating; but my feeling is that students using this textbook will find it engaging. Colyvan hereby extends an invitation to those students to engage in the philosophical project of understanding the phenomena that his examples illustrate, rather than merely learning what others have said about them.

In Chapter 5, Colyvan considers explanation in mathematics. He presents Euclid's proof that there is no greatest prime number, a proof of Rolle's theorem that proceeds by cases, and the standard inductive proof that the sum of the first n naturals is $\frac{1}{2}n(n+1)$. He invites us to agree that they do not explain the facts that they prove. If these are archetypes of non-explanatory proofs—and, in fact, I disagree that the first two are—the discussion would benefit from archetypes of explanatory proofs: many constructive proofs of $\forall\exists$ theorems have this property, such as the proof of Cayley's Theorem or Cantor's Theorem. The student would be better equipped to explore the distinction between explanatory and non-explanatory proofs if she had clear examples from both sides of the divide.

In Chapter 6, Colyvan turns to the applicability of mathematics. Again, his approach is to raise puzzles rather than to provide exhaustive summaries of the debates. Maxwell formulated the Maxwell–Ampère law on the basis of a mathematical analogy with Newtonian gravitational theory; yet it made novel empirical predictions that were confirmed. Colyvan uses this example to raise the question of the unreasonable effectiveness of mathematics.

In Chapters 7 and 8, Colyvan leads us into even less well charted territority. In Chapter 7, he asks how we ought to reconstruct *actual* mathematical reasoning, rather than our *idealised* textbook versions of it. He cites naïve set theory and the early calculus as examples of inconsistent theories in which mathematicians reasoned successfully, and which may therefore be better reconstructed using paraconsistent logics than using the more standard classical systems.

In Chapter 8, he highlights the importance of mathematical notation as a catalyst for further mathematical discoveries. At this point, I wondered whether there was a genuine philosophical issue here, rather than a phenomenon of interest primarily to sociologists, linguists, and psychologists.

The book concludes with a list of Colyvan's twenty 'desert island' theorems. I would have preferred fewer theorems and more proofs or applications, particularly in light of the philosophical questions that mathematical results tend to raise. But the list is certainly a valuable resource for a student entering the philosophy of mathematics without a strong mathematical background.

Inevitably, Colyvan has had to omit some central topics even from the current period in the philosophy of mathematics: category theory is barely mentioned; reverse mathematics not at all; and the question of automated reasoning and computer-aided proofs is absent. But no textbook can cover all topics, and one might feel that these belong more naturally to a more advanced course in the subject.

In sum: while some of the book might lack a little of the detail and rigour I'd like future students of the topic to value, Colyvan has written the first textbook that initiates the student into the current period in the philosophy of mathematics. Students will find this volume extremely engaging; and it will provoke them to formulate their own philosophical hypotheses and arguments more readily than more traditional textbooks. In short, it will provide exactly the sort of philosophical education we ought to be providing. Furthermore, I predict that it will draw into the philosophy of mathematics students of applied areas of the subject, such as mechanics or dynamical systems. After a century of pure mathematicians and logicians dominating the subject, this is to be welcomed.

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L. A. Levin. *Some theorems on the algorithmic approach to probability theory and information theory* (1971 *Dissertation directed by A. N. Kolmogorov*). ***Annals of Pure and Applied Logic***, vol. 162 (2010), pp. 224–235.

Algorithmic information theory has enjoyed an enormously fruitful two decades. Since the early 90s, open borders and the rise of email and internet have created a truly international community of researchers working on questions connected to Kolmogorov's and Solomonoff's pioneering work on a computational foundation of information theory. During that time, origins have been revisited, notions and contributions have been compared, presentations have been streamlined. Two substantial monographs have appeared recently: A. Nies, ***Computability and Randomness***, Oxford University Press, 2009, and R. Downey and D. Hirschfeldt, ***Algorithmic randomness and complexity***, Springer, 2010. Moreover, the third edition of the classic text by M. Li and P. Vitányi, ***An introduction to Kolmogorov complexity and its applications***, Springer, was published in 2008.

Leonid A. Levin's work in the early 70s is undoubtedly one of the most important contributions to the subject. It is the reviewer's belief that this body of work is still not completely known, let alone understood. There are many factors responsible for this, but one is arguably the fact that many of Levin's papers were until recently not available online, while others could be found only in Russian.

The publication of Levin's 1971 PhD thesis in the ***Annals of Pure and Applied Logic*** closes an important gap in this regard. It appeared in a volume in honor of N. A. Shanin, who was one of the referees of Levin's thesis, and whose referee report is also included in the volume. Despite his remarkable work, Levin was denied a doctorate degree at the time.

Arguably the central notion of algorithmic information theory is *Kolmogorov complexity*. To put it briefly, the Kolmogorov complexity of a finite binary string σ is the length of the shortest program p so that a fixed universal Turing machine M on input p outputs σ, $M(p) = \sigma$. Kolmogorov complexity is now commonly denoted by C, but Levin, following Kolmogorov's original paper, used K.

Levin's thesis is remarkable for the scope of new ideas it presents on Kolmogorov complexity. As Shanin remarks in his referee report, "the author used a wide variety of methods—from introducing new generalizing schemas to specific ingenious constructions". Some of the new concepts, such as optimal semimeasures, have become indispensable cornerstones of the field. Other results, such as the relation between the entropy of dynamical systems and the asymptotic behavior of Kolmogorov complexity, have laid the groundwork for research that is still ongoing, and has seen a recent surge in contributions.

All of the results from Sections 2 and 3 as well as Theorem 7 from Section 1 are included in the seminal paper by Zvonkin and Levin, ***Russian Mathematical Surveys***, vol. 25 (1970), no. 6. These results include the important material on optimal semimeasures and mutual information. The Zvonkin–Levin paper was translated into English and hence made some of Levin's work accessible to a wider community early on. It is still one of the standard references for the subject.

This review will thence focus on the part of Levin's thesis which has hitherto not received wide attention—Levin's unifying treatment of complexity notions in Section 1.

Besides the original notion of Kolmogorov complexity (now usually called "plain complexity"), many alternative measures of complexity for finite strings have been introduced. The paper by Uspensky and Shen, ***Mathematical Systems Theory***, vol. 29 (1996), no. 3, gives a good overview. Each notion tries to overcome a technical or conceptual drawback of plain complexity. For example, prefix-free complexity eliminates the problem that a binary string codes information not only through its bits but also through its length (and hence a shortest program does not truly represent the information content of a string).

Levin's unified framework is quite remarkable in that it naturally integrates plain complexity and closely related notions such as relativized complexity, but also captures other, measure based notions such as a priori (KM-) complexity. Moreover, for the two notions that today occupy the extreme ends of the spectrum of complexity notions, prefix-free and a priori complexity, it becomes clear *why* they are at the extreme ends. This is an insight not present in the modern literature.

Complexity notions are usually defined via some kind of *effective optimization*. For instance, a priori complexity is the negative logarithm of an optimal enumerable semimeasure on $2^{\mathbb{N}}$. The optimal object is required to be itself effective, which usually means left- or right-enumerable.

Levin introduces the concept of a *volume restriction* and a *majorant* as a common framework for complexity notions. A *finitary function* is a mapping from $2^{<\mathbb{N}}$ to $\mathbb{N} \cup \{\infty\}$ that takes finite values on at most finitely many strings. A *volume restriction* is a recursively enumerable family V of finitary functions so that

(1) if $f \geq g$ (pointwise) and $g \in V$, then $f \in V$, and
(2) there exists a constant c so that for all $f, g \in V$, $\min\{f, g\} + c \in V$.

The motivation behind these notions appears to be the following. A complexity function assigns to each string a natural number, its *complexity*. $f \in V$ is an effective function that has at some stage assigned finitely many strings a complexity (those σ for which $f(\sigma) < \infty$). From such complexity assignments we can create new ones: If we increase the complexities, this is a valid complexity function, too (1). However, if we want to combine two complexity functions into a new one by optimizing (i.e., taking minima), we have to offset this by adding c (2). This reflects the idea that objects with low complexities should somehow be *restricted*.

For example, for plain complexity C, for any length n, there are at most $2^n - 1$ programs of length less than n that could give a string of length n a shorter description. Accordingly, the family V_{C} of finitary functions with

$$f \in V_{\mathrm{C}} \Leftrightarrow \forall a\, |\{\sigma : f(\sigma) < a\}| < 2^a$$

is a volume restriction.

Given a volume restriction V, a function F qualifies as a possible complexity associated with V if its graph is enumerable from above and it has a "no gap" property with respect to V: If $g \geq F$, then $g \in V$. Levin calls such functions *V-majorants*. He then goes on to show (Theorem 1) that for any volume restriction there exists a least (up to an additive constant) V-majorant. This function can then be considered as the *complexity induced by V*.

Plain complexity is thus obtained as the complexity induced by V_{C} above. Other algorithmic complexities Levin explicitly obtains this way are *conditional complexity* and *decision complexity* due Loveland and Markov (Theorem 4).

But also measure theoretic complexity notions are covered. The volume restriction V_M given by

$$f \in V_M \Leftrightarrow \text{for all finite, prefix-free sets } P \subseteq 2^{<\mathbb{N}},\ \sum_{\sigma \in P} 2^{-f(\sigma)} \leq 1$$

(where we put $2^{-\infty} = 0$) induces a priori complexity KM.

Furthermore, the volume restriction V_m given by

$$f \in V_m \Leftrightarrow \sum_{\sigma \in 2^{<\mathbb{N}}} 2^{-f(\sigma)} \leq 1,$$

is the most "narrow" among the "reduced" volume restrictions, a normalized subclass of all volume restrictions. (This is shown in Theorem 2, an easy consequence of (2)). Its induced complexity function (optimal majorant) is the largest possible complexity function. It is the logarithm of an optimal, enumerable semimeasure on $\mathbb{N}$. In Theorem 3 it is shown that this "universal" majorant is within a logarithmic term of plain complexity.

Levin later showed (via what is now known as the *coding theorem*) that this universal majorant coincides (up to an additive constant) with prefix-free complexity. Levin's thesis is hence, to the reviewer's knowledge, the first time a concept equivalent to prefix-free complexity was introduced in the literature, and Theorem 3 is the first result relating plain and prefix-free complexity. It should also be noted that related versions of the results described here appeared in L. A. Levin, ***Doklady Akademii Nauk SSSR***, vol. 227 (1976), no. 4, (in Russian).

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Uri Andrews. *A new spectrum of recursive models using an amalgamation construction*. ***The Journal of Symbolic Logic***, vol. 73 (2011), no. 3, pp. 883–896.

Bakhadyr Khoussainov and Antonio Montalbán. *A computable $\aleph_0$-categorical structure whose theory computes true arithmetic*. ***The Journal of Symbolic Logic***, vol. 72 (2010), no. 2, pp. 728–740.

Many classical proofs in algebra and model theory are algorithmic in nature. For example, when we show that every field is contained in an algebraically closed field, we usually *construct* the closure. Another illustrative example is the proof of the Completeness Theorem for a first-order language via the Henkin construction. Although these proofs are typically given via "algorithms", the algorithmic nature of these and similar results became fully accessible only after Mal'cev and Rabin (early 1960's) independently came up with the following definition:

Definition. A countably infinite algebraic structure $\mathcal{A}$ is *computable* or *constructive* if there exists a bijection $\nu\colon \mathbb{N} \to \mathcal{A}$ such that the ν-preimages of operations and relations on $\mathcal{A}$ become (uniformly) computable sets of natural numbers. The map ν is called a computable presentation (or computable copy) of $\mathcal{A}$.

The definition generalizes the notion of a recursively presented group in which the word problem is solvable, and also the notion of an explicitly given field suggested by van der Waerden and formalized by Fröhlich and Shepherdson. A computable model $\mathcal{M}$ is *decidable* or *strongly constructive* if there is a constructivization ν of $\mathcal{M}$ such that every ν pre-images of definable sets are uniformly computable.

The effective content of the Henkin construction is rather straightforward: A complete consistent first-order theory T is decidable if and only if T has a decidable model (folklore). The algorithmic nature of the closure theorem for fields is more interesting: Every computable field F can be embedded into its computable algebraic closure $\overline{F}$ via a computable isomorphism, but locating the image of F within $\overline{F}$ does not have to be an algorithmically decidable problem (Rabin, based on the work of Fröhlich and Shepherdson).

The two basic examples discussed above belong to the field known as *computable model theory* which examines the algorithmic nature of algebraic structures and their theories. One of the central topics in computable model theory is the study of relations between algorithmic complexity of a theory and its models. In this review we concentrate on model-theoretic restrictions and consider $\aleph_0$- and $\aleph_1$-categorical theories and their models.

The first paper under review studies computable models of $\aleph_1$-categorical theories. Recall that a first-order theory is κ-categorical if all its models of cardinality κ are isomorphic. In 1965, Morley showed that if a theory in a countable language is categorical in some uncountable cardinality, then it is categorical in all uncountable cardinalities. Baldwin and Lachlan (1971) proved that countable models of an uncountably categorical theory (which is not countably categorical) can be listed in a chain $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \cdots \preceq \mathcal{A}_\omega$ of proper elementary embeddings, where $\mathcal{A}_0$ is the prime model and $\mathcal{A}_\omega$ is the saturated model of the theory. Which models from the Baldwin–Lachlan elementary chain have constructivizations? Starting from Goncharov (1978), there has been a line of study into the effective content of the Baldwin and Lachlan theorem. For a uncountably categorical theory T which is not countably categorical, its *spectrum* is the collection of i corresponding to computable $\mathcal{A}_i$ in the Baldwin–Lachlan chain. Various authors including Goncharov, Kudaibergenov, Hirschfeldt, Khoussainov, Nies, Semukhin and Shore showed that $\{0\}$, $\{0, \dots, n\}$, $\{\omega\}$, ω, $\omega + 1 \setminus \{0\}$ and $\{1, \dots, \alpha\}$ for every $\alpha \in (1, \dots, \omega]$, can be realized as spectra.

Andrews showed that there is an uncountably categorical theory whose only constructive models are the prime and the saturated model. Thus, set $\{0, \omega\}$ is a spectrum. The proof of the theorem which can be found in the reviewed paper uses an alteration of the Hrishovski construction. The Hrushovski construction is essentially the Fraisse limit construction in which *strong embeddings* are used in place of usual embeddings to control categoricity of the theory. Using his construction, Hrushovski (1991) obtained a new example of a strongly minimal theory which does not fall into three classes defined by Zilber (1986), thus refuting Zilber's conjecture. The formal definition of a strong embedding is technical and can be found in the paper under review.

The very rough idea of the proof can be described as follows. Let $(\mathcal{M}_e)_{e\in\omega}$ be an effective listing of partial computable structures inherited from the enumeration of all partial computable functions. Using an alternation of the Hrushovski construction, Andrews builds a computable saturated model $\mathcal{M}$ so that for every $k > 0$ and e, if $\mathcal{M}_e$ has a transcendental k-element basis, then $\mathcal{M}_e \not\preceq \mathcal{M}$. The strategy of satisfying $\mathcal{M}_e \not\preceq \mathcal{M}$ can be viewed as an infinite game. Andrews describes a strategy which guarantees that either $\mathcal{M}_e$ has no finite basis, or is not embeddable. To make the prime model computable, Andrews names by a constant every element algebraic over $\emptyset$, thus making the signature of $\mathcal{M}$ infinite. The specific properties of the Hrushovski construction are crucial for the verification.

The second paper under review studies $\aleph_0$-categorical theories. It is well-known that a theory is $\aleph_0$-categorical if and only if the number of complete n-types is finite for every $n \in \mathbb{N}$. Since all countable models of such a theory are isomorphic, the theory T is decidable if and only if its only countable model is decidable.

Various algorithmic aspects of countably categorical theories have been investigated by Schmerl, Lerman, Knight, and others. For instance, we could consider a computable model whose theory is countably categorical and ask for the complexity of this theory. Khoussainov and Goncharov (2004) showed that for every $n \in \mathbb{N}$ this complexity can be at least at the Δ^0_{n+1} level of the Arithmetical Hierarchy. Using Marker's extensions, Khoussainov and Goncharov came up with an elegant coding of $0^{(n)}$, the canonical Δ^0_{n+1}-complete set, into the countably categorical first-order theory of a computable structure.

It has been a long standing problem if the result of Khoussainov and Goncharov can be improved to the coding of $Th(\mathbb{N}; 0, S, +, \times)$, the highest possible upper bound. Khoussainov and Montalban have resolved this problem by constructing an $\aleph_0$-categorical theory T whose only countable model is computable such that T computes the theory of true first order arithmetic. The result shows that even under very strong model-theoretic restrictions on $\mathcal{M}$, the theory of a computable model could be as hard as possible.

The proof of the above stated theorem uses a generalization of the random graph to higher dimensions. The constructed computable structure consists of "levels", each level is *either* isomorphic to the n-dimensional analog of the random graph, *or* to a graph which extends the random n-graph. For any given Σ^0_n-sentence ϕ_n in the signature of true arithmetic, Khoussainov and Montalban produce a computable n-graph G_n which is "random" if $\neg\phi_n$ holds, and which is isomorphic to the extension mentioned above, otherwise. The n-graphs G_n can be glued together so that the resulting model is countably categorical, and so that the satisfyability of ϕ_n is encoded into its Σ^0_n-theory. They ensure that the theory computes the true arithmetic by choosing an appropriate sequence of ϕ_n. Although various codings of Σ^0_n information are common in computable model theory, the use of generalized random graphs is certainly a fresh idea.

It is worth mentioning that Andrews has recently announced another proof of the result of Khoussainov and Montalban discussed above. The new proof uses a direct amalgamation construction combined with the coding of the true arithmetic rather than a generalization of the random graph.

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