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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Steve Awodey (Managing Editor), John Burgess, Mark Colyvan, Anuj Dawar, Steffen Lempp, Rahim Moosa, Ernest Schimmerling, Alex Simpson, Kai Wehmeier, and Matthias Wille. Authors and publishers are requested to send, for review, copies of books to *ASL, Box 742, Vassar College, 124 Raymond Avenue, Poughkeepsie, NY 12604, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

Motivic integration and its interactions with model theory and non-Archimedean geometry, Volumes I and II, edited by Raf Cluckers, Johannes Nicaise and Julien Sebag, London Mathematical Society Lecture Note Series, 383 and 384. Cambridge University Press, Cambridge, 2011, 346 and 262 pp.

Motivic Integration is a name for a whole collection of new techniques that originated from a talk by M. Kontsevich in 1995. As in any fast-developing subject, the literature is difficult to track, and there is hardly any exposition of the newer aspects of motivic integration. In this situation, these two volumes that resulted from a conference held in Edinburgh in 2008 provide an invaluable resource.

In fact, Volume I reads not so much as a conference proceedings, but as a book—it is a very thoughtful collection of high-level introductory articles, aimed at the exposition of some of the theories of motivic integration, explaining the links between them, and providing the necessary background. Remarkably, it features efficient exposition of the lesser-known aspects of the theory, while avoiding repetitions and overlaps with the prior introductory literature. It also includes some developments that happened after 2008 (since the contributors to these volumes are among the main contributors to the development of the subject).

Volume II, more in the conference-proceedings style, is a far-reaching collection of articles related to some of the fields where the philosophy of motivic integration found applications, such as Arakelov Geometry, cf. Chapter 9 "*Heights and measures on analytic spaces. A survey*

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of recent results, and some remarks" by Antoine Chambert-Loir; and Number Theory, cf. Chapter 13 "*A survey of algebraic exponential sums and some applications*" by Emmanuel Kowalski, and Chapter 12 "*Triangulated motives over Noetherian separated schemes*" by Florian Ivorra. It also discusses some finer questions related to the techniques that make motivic integration work, such as desingularization, cf. Chapter 15, "*Absolute desingularization in characteristic zero*" by Michael Temkin; C-minimality, cf. Chapter 10, "*C-minimal structures without density assumption*" by Françoise Delon; and the structure of definable sets over $\mathbb{Z}_p$, cf. Chapter 11 "*Trees of definable sets in $\mathbb{Z}_p$*" by Immanuel Halupczok.

So what is motivic integration, why "motivic", and how does it relate to model theory, algebraic geometry and other areas where it found applications? The Introduction to Volume 1 by the editors, R. Cluckers, J. Nicaise and J. Sebag, gives succinct answers to these questions.

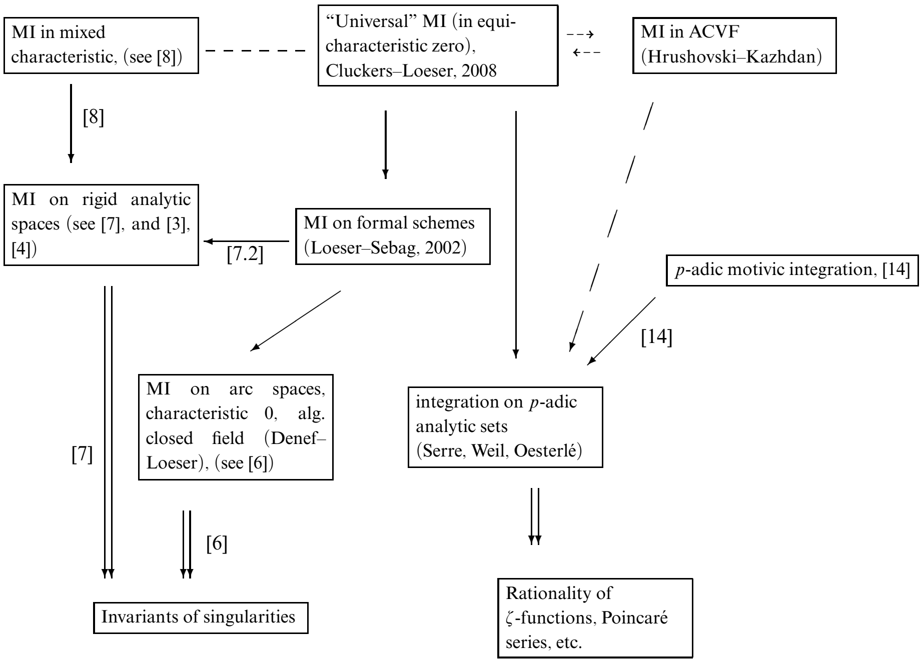
It all started with the work of V. Batyrev "Birational Calabi–Yau n-folds have equal Betti numbers" (around 1995), where he used integration over p-adic manifolds to prove a result about resolution of singularities of *complex* algebraic varieties. At a lecture in Orsay in 1995, M. Kontsevich introduced the notion of the *arc space* of a complex algebraic variety, and a notion of "integration" over the arc space that mimicked the construction of the integrals over p-adic numbers, making an actual transition between the complex world and the p-adic world unnecessary, and generalizing Batyrev's theorem. This was the original idea of motivic integration. It was quickly developed by J. Denef and F. Loeser in a series of papers in 1995–1999, and at that time it belonged to the world of algebraic geometry (over an algebraically closed field of characteristic zero). Chapter 6 "*A short course on geometric motivic integration*" by Manuel Blickle is an excellent introduction to this theory.

Model theory provides an ideal framework for transferring results between different fields, and starting from the work of Denef and Loeser "Definable sets, motives, and p-adic integrals" (1999) model theory has been playing an increasingly central role in motivic integration. There now exist two motivic integration theories based on model theory—by R. Cluckers and F. Loeser, and by E. Hrushovski and D. Kazhdan. These volumes do not include expositions of either, except for a summary and references in the Introduction to Volume I. However, Chapter 2 "*Introductory notes on the model theory of valued fields*" by Zoe Chatzidakis, provides an essential introduction to the required model theory, filling a gap in the motivic integration literature, and creating a wonderful resource for non-model theorists who find themselves studying motivic integration, like the author of this review.

Let us introduce the other chapters via a partial and informal "map of the subject". In the diagram, solid arrows indicate existence of a *specialization*, dashed arrows indicate an analogy (or possibly less understood connection), and double arrows indicate applications. The relevant chapter numbers are listed in brackets, and "MI" stands for "Motivic integration". One notable omission from this diagram (also not represented in these volumes, except for the introduction) is motivic integration on stacks, developed by T. Yasuda, and independently, by C. Balwe.

The two central features common in the early theories of motivic integration were the arc spaces, and Grothendieck rings. The arc space of an algebraic variety is constructed as an inverse limit of the spaces whose points parametrize "germs of degree n maps into the variety". If X is a variety over an algebraically closed field k of characteristic zero, the k-points of the arc space of X are in bijection with the $k[[t]]$-points of X. Motivic volume was defined for certain "cylindrical" subsets of the arc space. What made motivic volume different from all the classical theories of integration was the ring of its values.

As a first approximation, one could say that the values of the motivic volume are isomorphism classes of varieties over the base field k. More precisely, one needs to be able to take formal sums of these isomorphism classes (i.e., form a relevant Grothendieck ring), and to invert the class of the affine line. To highlight the relationship with p-adic integration, note that if one could take k to be a finite field $\mathbb{F}_p$, then the points of the arc space of X



would be in bijection with $X(\mathbb{F}_p[\![t]\!])$ (and thus with $X(\mathbb{Q}_p)$ as well). In this case the values of the motivic volume would involve varieties over the finite field, which, by counting their $\mathbb{F}_p$-points, *specialize* to integers. Inverting p, and allowing geometric series (in order to make the volume countably-additive), we could recover integration on $X(\mathbb{Q}_p)$ as defined by A. Weil in the 1970s. This was the analogy that spurred motivic integration into existence, and it remained just an analogy until model theory tools were introduced, since the problems of replacing an algebraically closed field of characteristic zero with a finite field could not be overcome otherwise.

Chapter 5 "*The Grothendieck ring of varieties*" by J. Nicaise and J. Sebag introduces Grothendieck rings that serve as the rings of values for the various versions of motivic volume; in particular, the Grothendieck rings of algebraic varieties, of Chow motives, and of definable sets. One important feature of all these Grothendieck rings is existence of the *realization maps* (counting points over $\mathbb{F}_p$ is an example of such a map). Ultimately, the realization maps is what allows to specialize motivic integration to p-adic integration, or to derive information about invariants of singularities arising in algebraic geometry from the relevant motivic volumes. This chapter also provides a state-of-the-art summary of what is known about the structure of these Grothendieck rings (e.g., the questions related to zero divisors) and their completions. In particular, the issues related to the characteristic of the ground field are emphasized.

Chapter 14 "*A motivic version of p-adic integration*" by K. Rökaeus addresses a related question: the problem that the point-counting map does not extend to the natural completion of the Grothendieck ring of k-varieties, when k is a finite field. Presently, this problem is circumvented by Cluckers and Loeser (see the map of the subject above) by using model theory. Chapter 14 suggests a different approach to making a direct link between the integration on arc spaces and p-adic integration.

So far, we have focused on how motivic integration allows the use of traditionally non-Archimedean techniques to obtain results in complex algebraic geometry; rigid geometry was created by J. Tate in 1960s in order to enable an analogue of classical complex analysis over complete non-Archimedean fields. Chapter 3 "*On the definition of rigid analytic spaces*" by S. Bosch is a modern introduction to Tate's construction. Later, M. Raynaud interpreted rigid varieties as the generic fibres of their *formal models*, establishing the connection between rigid geometry and classical algebraic geometry. In Chapter 4 "*Topological rings in rigid geometry*", F. Kato presents a new "relative" approach to rigid geometry, which is similar to making a step from varieties to schemes in the classical algebraic geometry.

Chapters 3 and 4 together provide the background for Chapter 7 "*Motivic invariants of rigid varieties, and applications to complex singularities*" by J. Nicaise and J. Sebag, where the reverse direction of the connection between non-Archimedean and complex geometry (from complex to non-Archimedean) via motivic integration is explained. Chapter 7 is an introduction to motivic integration on formal schemes, and to closely related motivic integration on rigid analytic spaces. The ultimate application of these theories is a new approach to the so-called Motivic Monodromy Conjecture, which is an illuminating generalization of Igusa's Monodromy Conjecture.

In Chapter 8 "*Motivic integration in mixed characteristic with bounded ramification: a summary*", R. Cluckers and F. Loeser outline an approach that allows to introduce Fubini-type theorems into motivic integration on formal schemes (described in Chapter 7 § 2.3), thus opening up the way for new applications.

To conclude, it is probably clear that I am very fond of this volume; perhaps the thing I like the best is that while it is essentially self-contained and expository, it emphasizes some of the details glossed over in much of the other literature; it could be useful for someone just beginning to read about motivic integration, and at the same time a reader can find answers to some of the questions they always wanted to ask (or did not even realize they should ask), having studied the subject for years.

Julia Gordon

Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, BC, V6T 1Z2, Canada. gor@math.ubc.ca.

Patricia A. Blanchette. ***Frege's conception of logic.*** Oxford University Press, 2012, xv + 190 pp.

Blanchette's book broadly revolves around two characteristic—and, as we will see, deeply connected—features of Frege's conception of logic: (*i*) the proper role assigned to conceptual analysis, of which Frege was the first practitioner in the modern sense; and (*ii*) Frege's account of logical entailment, including the extent to which it deviates from the contemporary conception derived from Hilbert and Gödel, and the repercussions of such an account for the possibility of meta-theory. The book also touches on a number of interesting issues, including the problem of "sharp boundaries" of concepts and its connection to the Caesar problem, but we will focus on the two main topics.

Frege's logicism aims to show that arithmetical truths are ultimately grounded in pure logic. Whether arithmetical propositions are in fact true or false is not, of course, open to question. Logicism does not aim to show that arithmetic is in fact true, but that its truth only depends on very basic principles of logic. The main thesis of the book is that establishing that arithmetic is so grounded is "a two-step process involving first a careful elucidation or analysis and second the rigorous proof of the thus-analyzed truths from the simplest possible premises" (p. 11). The issue that occupies roughly the first half of the book is "whether it is in fact reasonable to take the provability of an analysans proposition from some premise P as a guarantee that the original analysandum proposition itself is provable from P" (*ibid.*). The problem is pressing because, in at least a few crucial instances, there is a vast gulf between the relatively simple arithmetical analysandum and the highly complex analysans that corresponds to it. Since it is only the latter that can be rigorously and conclusively proved from purely logical principles, the question arises of how precisely this *transfer of grounding* (to use Blanchette's efficacious expression) from the formal principle to its informal and pre-theoretic counterpart is to be accomplished. Frege often claims that transfer of grounding is made possible by the *shared content* of analysans and analysandum (or, as Frege's would put it in later works, by the fact that the two express the same thought). But there are difficulties with this view, and perhaps Blanchette's main contribution in the book is the proposal, in chapter 4, of specific conditions for the transfer of grounding to obtain.

Before we come to this proposal, it is useful to review two of Blanchette's examples of analyses that establish the logical character of certain pre-theoretic notions by transferring logical grounding. The first example is Frege's analysis of the notion "following in a sequence" in terms of the weak ancestral. Famously, Frege proposes to analyze "b follows a in the R-sequence" as "b has all the R-hereditary properties of a" (where a property P is R-hereditary if—as Russell would later put it—it belongs to all R-successors of possessors). In cases such as this, transfer of grounding is, according to Blanchette, by and large unproblematic: the analysans is still close enough to the analysandum that a gapless proof from purely logical principles of some proposition involving the former counts as logical grounding for the corresponding proposition involving the latter as well. This is the kind of reductive analysis that Frege engages with in *Begriffsschrift*, where he shows, for instance, that "following in the R-sequence" is, in virtue of logic alone, a transitive notion.

The second example is derived from Frege's work in *Grundlagen* and, later, in *Grundgesetze*. Blanchette asks us to compare the ordinary arithmetical claim "1 is the successor of 0" with Frege's rendition of this claim, based on his definition of numbers (in *Grundlagen*) as extensions of higher-order predicates under which equinumerous predicates ("concepts," in Frege's terminology) fall. Thus, 0 is defined as the extension of the predicate "concept equinumerous with the empty predicate $x \neq x$," and 1 as the extension of the predicate "concept equinumerous with the predicate $x = 0$." According to the proposed analysis "1 is the successor of 0" is to be construed as:

> There is a concept F and an object a falling under it such that the extension of the predicate "concept equinumerous with Fx" is 1, and the extension of the predicate "concept equinumerous with $Fx \wedge x \neq a$" is 0.

Transfer of grounding appears much more problematic in this case, as is made clear by Blanchette. The connection between "1 is the successor of 0" and the complex statement above is arguably tenuous, and Frege's claim that the analysans captures the sense of (i.e., the "thought expressed by") the analysandum is at the very least worth further scrutiny.

A tempting proposal, which is supported by a certain amount of textual evidence, is that transfer of grounding occurs when the thoughts expressed by analysans and analysandum are the same. This in turn brings up the question of deploying a satisfactory identity criterion for thoughts. Incidentally, the issue is central not just for Fregean scholarship, but also for the neo-Fregean program, which relies on a similar identity of thought-content for the two halves of Hume's Principle, which are regarded as different "carvings" of the same content. Promising as this might sound, the gulf between analysans and analysandum is, in at least some crucial cases, just too wide for the suggestion that they express the same thought to be plausible, a point argued at length by Blanchette. The criterion of individuation of thoughts, whatever it might turn out to be, is just too *fine grained* for the proposal to be viable.

This brings us to Blanchette's own account of the conditions for successful analysis of some content α in terms of content β. Such an analysis requires, she argues, a two-step process. The first step does indeed supply an intermediate content γ that is readily seen to be analytically related to α, perhaps not unlike the *Begriffsschrift*-style analysis of succession mentioned above (although Blanchette is not explicit on this point). The second step is a properly proof-theoretic reduction of γ to β, a reduction that proceeds on the basis of purely logical principles alone, but that has no claim to preservation of thought or content.

This is most clearly seen in the case (which of course also figures prominently in the neo-logicist program) of identities of number. In *Grundlagen* the ordinary arithmetical statement "the number of F equals the number of G" is first re-cast as the claim that there exists a 1–1 correspondence of the F's onto the G's. The second step then provides a proof of the equivalence of the intermediate statement with the identity between the extension of the concept "equinumerous with F" and that of the concept "equinumerous with G." It should be noted that although the process ultimately provides an analysis of a given identity about the numbers assigned to concepts F and G in terms of another identity obtaining between extension of second-order concepts, the intermediate statement is not an identity. But it is only the first step that can lay claim to preservation of content, while the second provides the required reduction to purely logical principles. Notice also that a similar strategy is likewise deployed in *Grundgesetze*, albeit using a slightly different definition of numbers as extensions of first-level concepts. A crucial part of Blanchette's argument is that the possibility of such alternative reductions is essential to a correct understanding of Frege's program (a fact of which it is hard to make sense if successful analysis is predicated upon identity of content). Whether we define the number of F as the extension of the predicate "concept equinumerous with F," or (as in *Grundgesetze*) as the extension of the predicate "extension of a concept equinumerous with F," is irrelevant, as these are just possible choices among many of functions mapping concepts into the objects in such a way that equinumerous concepts are mapped to same object.

The second half of the book is dedicated to investigating Frege's conception of logical entailment and the dual notion of consistency, especially in comparison to the alternative, and contemporaneous account given by Hilbert. Whereas Hilbert's notions of implication and consistency, as later developed by Gödel and Tarski, have given rise to the modern model-theoretic notion of logical consequence, it is important to appreciate that for Frege logical entailment is first and foremost a relation between *thoughts* (or between a class of thoughts and an individual thoughts). The two conceptions are on display, famously, in the somewhat one-sided Frege-Hilbert debate on the foundations of geometry.

According to the modern conception, a sentence φ is a logical consequence of a set Σ of sentences if φ is true on every re-interpretation of the extra-logical notation in $\Sigma \cup \{\varphi\}$ on which every sentence in Σ is true. But of course no parallel account can be given in the case of thoughts: for Frege thoughts are either true or false, and the idea of alternative interpretation (even if only of the extra-logical notions) just makes no sense in the case of thoughts. And while, as Blanchette points out, φ's being a logical consequence of Σ is often sufficient for the entailment (in Frege's sense) of the thought expressed by φ by the thoughts expressed by sentences in Σ, the converse fails. Hence, the "gap" between Hilbertian consequence and Fregean entailment. The gap is particularly evident for the dual question of consistency, as it applies, for instance, to the issue of the independence of, say, the commutativity principle from the remaining axioms of arithmetic. For the axioms are *true*, and it makes no sense to ask whether we can reinterpret the theory in such way that commutativity comes out false and the other axioms true (although it makes sense to ask such a question for a set of partially interpreted sentences).

But just what *is* Frege's account of logical entailment? How can we establish that the thoughts expressed by sentences in Σ entail the thought expressed by φ? For, as we have seen, the formal derivability of φ from Σ will not necessarily capture each such instance of entailment. The answer, again, lies within Frege's account of logical analysis. To illustrate the point, Blanchette asks us to consider Frege's example

SUM: If two numbers are multiples of another, so is their sum,

(from the 1881 ms., "Boole's Logical Calculus"). Frege's claim is that the statement is entailed by associativity of addition together with the principle that $n + 0 = n$, but no purely formal (employing only topic-neutral rules) proof of this fact can be given as long as "being a multiple" is a primitive arithmetical notion. Proper conceptual analysis is needed to construe the analysandum "m is a multiple of a" as "m belongs to every concept F closed under addition of a under which 0 falls." The conclusion that $m+n$ is a multiple of a, provided both m and n are as well, then follows by consideration of the concept $G(x) := F(m + x)$. In this case, conceptual analysis brings to the surface previously hidden logical complexity that is needed for the desired proof to go through. It should be pointed out, though, that not every instance of conceptual analysis takes the form of expansion of atomic terms, as in the crucial case of the recasting of "the number of F = the number of G" as "there is a 1–1 correspondence of the F onto the G." In such cases, the Hilbertian notion of consequence as truth on all re-interpretations does not even shed partial light on the corresponding Fregean question of entailment.

Interestingly, Blanchette points out that the Hilbertian question is not ruled out in the Fregean perspective—it just must not be confused with substantive questions of entailment, independence, etc. There is, in Frege's framework, room for investigations that we would consider "meta-theoretic" in the modern sense. We can certainly consider partially interpreted theories, and ask for instance whether sentences that turn out true on all the various interpretations of one such theory are in fact formally deducible from it. But the technical question is philosophically uninteresting from a Fregean point of view, as its connection with the adequacy of, say, arithmetic or geometry is tenuous at best.

Blanchette presents a deep and careful reconstruction of the Fregean notion of conceptual analysis, and she goes to great lengths to extract from the not always coherent Fregean text a viable account. The effort is successful, and her book sheds much needed light on a cluster of thorny issues in Fregean scholarship. But the questions she grapples with have philosophical significance that goes beyond the restricted confines of Fregean, or even neo-Fregean philosophizing. Conceptual analysis of the kind pioneered by Frege is, after all, what philosophers of all stripes are engaged in every day, and the question of the relevance of some proposed conceptual reconstruction to its pre-theoretic counterpart is pressing indeed. Just as for Frege, the twin requirements of the recognizable preservation of content and the derivability from given abstract principles can sometimes be at odds. Blanchette shows us one way to reconcile these potentially conflicting epistemological and logical desiderata in a way that makes the philosophical enterprise both rigorous and fruitful.

G. ALDO ANTONELLI

Department of Philosophy, University of California, Davis, California 95616-8673, USA.
antonelli@ucdavis.edu.