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THE FOUNDATIONAL PROBLEM OF LOGIC

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Abstract. The construction of a systematic philosophical foundation for logic is a notoriously difficult problem. In Part One I suggest that the problem is in large part methodological, having to do with the common philosophical conception of "providing a foundation". I offer an alternative to the common methodology which combines a strong foundational requirement (veridical justification) with the use of non-traditional, holistic tools to achieve this result. In Part Two I delineate an outline of a foundation for logic, employing the new methodology. The outline is based on an investigation of why logic requires a veridical justification, i.e., a justification which involves the world and not just the mind, and what features or aspect of the world logic is grounded in. Logic, the investigation suggests, is grounded in the formal aspect of reality, and the outline proposes an account of this aspect, the way it both constrains and enables logic (gives rise to logical truths and consequences), logic's role in our overall system of knowledge, the relation between logic and mathematics, the normativity of logic, the characteristic traits of logic, and error and revision in logic.

It is an interesting fact that, with a small number of exceptions, a systematic philosophical foundation for logic, a foundation for logic rather than for mathematics or language, has rarely been attempted.[1] In this essay I aim to understand why this is the case, utilize this understanding to develop an appropriate foundational methodology, and use this methodology to construct an outline of a philosophical foundation for logic. The notion of a philosophical foundation will be clear to some readers, but due to the diverse readership of this journal it would be useful to briefly spell out and motivate the kind of philosophical foundation I have in mind.

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[1]One recent exception is Maddy [2007, Part III], which differs from the present attempt in being thoroughly naturalistic. Another psychologically oriented attempt is Hanna [2006]. Due to limitations of space and in accordance with my constructive goal, I will limit comparisons and polemics to a minimum.

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By a philosophical foundation for logic I mean in this paper a substantive philosophical theory that critically examines and explains the basic features of logic, the tasks logic performs in our theoretical and practical life, the veridicality of logic—including the source of the truth and falsehood of both logical and meta-logical claims,[2] the grounds on which logical theories should be accepted (rejected, or revised), the ways logical theories are constrained and enabled by the mind and the world, the relations between logic and related theories (e.g., mathematics), the source of the normativity of logic, and so on. The list is in principle open-ended since new interests and concerns may be raised by different persons and communities at present and in the future. In addition, the investigation itself is likely to raise new questions (whether logic is similar to other disciplines in requiring a grounding in reality, what the distinctive characteristics of logical operators are, etc.).

A foundational theory of this kind does not purport to be infallible. Like all other human theories it is subject to standards appropriate to its field (in the present case, philosophy and meta-logic), and it is open to criticisms and improvements. The foundation it seeks is a foundation for logic in a broad sense—the discipline of logic rather than a specific logical theory—but it should provide us with tools for criticizing, justifying, evaluating, constructing, and improving specific theories. These elements—critical examination, veridical justification, epistemic evaluation, and so on, as well as creating theoretical tools for these tasks—are the main elements of what I call "grounding" in this paper.

The motivation for engaging in a foundational project of this kind is both general and particular, both intellectual and practical, both theoretical and applicational. Partly, the project is motivated by an interest in providing a foundation for knowledge in general—i.e., a foundation both for human knowledge as a whole and for each branch of knowledge individually (logic being one such branch). Partly, the motivation is specific to logic, and is due to logic's unique features: its extreme "basicness", generality, modal force, normativity, ability to prevent an especially destructive type of error (logical contradiction, inconsistency), ability to expand all types of knowledge (through logical inference), etc. In both cases the interest is both intellectual and practical. Finally, our interest is both theoretical and applicational: we are interested in a systematic theoretical account of the nature, credentials, and scope of logical reasoning, as well as in its applications to specific fields and areas.

Given the broad scope of the foundational project of logic, there is no question of encompassing its full range in this essay. What I am looking for is a fruitful standpoint from which to approach this project and a constructive

[2]From specific object-language claims like "No object is both round and not round" to general meta-linguistic claims, e.g., that certain inference-forms are logically valid or invalid.

investigation that addresses some of its key questions in a unified manner. Such an investigation would serve as a starting point for a more complete foundation and, just as importantly, as a catalyst for further theoretical discussion of the foundations of logic.

Logic, however, is a very broad discipline, and the present investigation does not purport to apply to all its branches. Instead, it focuses on that branch which in our time is often referred to as "mathematical logic" and in earlier times took the forms of syllogistic logic, Fregean logic, and type-theoretic logic. And even here it is largely concerned with finitistic versions of this branch. These and other self-imposed restrictions will enable us to be more specific on the questions we address in this paper and will give us the space to discuss several issues of interest to mathematical as well as philosophical logicians. These include, for example, Feferman's criticism of what he called the Tarski–Sher thesis (Feferman [1999], [2010]), the relation between logic and mathematics, the possibility of extending structuralism from mathematics to logic, and topics of relevance to model-theoretic logic.

I have said that systematic attempts to construct a philosophical foundation for logic have been rare. But was not the period between the late 19th-century and the early 20th-century a period of "foundational studies in logic and mathematics", indeed a period of extraordinary growth and remarkable breakthroughs in this area? The answer is "Yes" with a caveat. Yes, there were foundational investigations and groundbreaking developments, but for the most part they aimed at a foundation for *mathematics*, with logic playing a mostly instrumental, if crucial, role. Frege, for example, developed a logical system that would provide a foundation for mathematics, but aside from a few hints, did not attempt to provide a systematic philosophical foundation for logic itself. Russell improved and further developed Frege's logicism, but although he appreciated the need to provide a systematic philosophical explanation of logic itself—one that would answer such questions as: "In virtue of what are logical propositions true?"—he despaired of accomplishing this task. Thus he says:

> The fundamental characteristic of logic, obviously, is that which is indicated when we say that logical propositions are true in virtue of their form. ... I confess, however, that I am unable to give any clear account of what is meant by saying that a proposition is "true in virtue of its form". Russell [1938, xii]

Indeed, many of the momentous discoveries in meta-logic (by Hilbert, Gödel, Turing, and others) are commonly designated as contributions to "meta-mathematics". These epochal achievements, however, are not irrelevant to the foundational problem of logic. On the contrary, by giving rise to a sophisticated logical framework and establishing its mathematical properties they created a fertile ground for a theoretical foundation for logic. It is all the more surprising, therefore, that few 20th- and 21st-century philosophers have taken up the challenge. Many have believed that a substantive, theoretical foundation for logic is impossible, some have considered it superfluous, quite a few have been content to simply say that logic is obvious, others have viewed logic as conventional, hence not in need of a foundation, and so on.

This tendency to avoid a philosophical engagement with the foundational problem of logic is not limited to the recent past. We can see it in the great philosophical systems of the 17th and 18th century. Take Kant, for example. Without purporting to offer scholarly exegesis of Kant's philosophy of logic, we may note that Kant's approach to logic is quite different from his approach to other disciplines. While Kant set out to provide a foundation for human knowledge in its entirety, he took formal logic largely as given. Logic, Kant emphasizes, has not required a major revision since Aristotle, and although there is room for clarifications and adjustments, there is no need for establishing the "certainty" of logic:

> That logic has already, from the earliest times, proceeded upon this sure path is evidenced by the fact that since Aristotle it has not required to retrace a single step, unless, indeed, we care to count as improvements the removal of certain needless subtleties or the clearer exposition of its recognised teaching, features which concern the elegance rather than the certainty of the science.
> Kant [1781/7, Bviii]

The scarcity of attempts to provide a theoretical foundation for logic is especially notable in light of epistemologists' recognition that logic has a special standing in knowledge. Compare logic and physics, for example. It is quite common to say that physics is bound by the laws of logic but logic is not bound by the laws of physics, that a serious error in logic might undermine our physical theory, but a serious error in physics would not undermine our logical theory. And it stands to reason that the more general, basic and normative a given field of knowledge is, the more important it is to provide it with a foundation. Nevertheless a theoretical foundation for logic, and in particular a non-trivializing foundation, has rarely been attempted. Why?

Clearly, the failure to attempt such a foundation is not due to neglect, oversight, or intellectual limitations. The extraordinary advances in logic and meta-logic on the one hand, and the wealth of attempts to construct a philosophical foundation for mathematics and science on the other, suggest that neither neglect nor intellectual handicaps are the problem. In my view, the source of the problem is methodological. Certain features of the customary foundational methodology make it very problematic to construct a philosophical foundation for logic, and the first step in confronting the foundational problem of logic is, therefore, dealing with the methodological difficulty.

I. Methodology.

1. The illusion of foundationalism. Since the inception of philosophy in ancient Greece, the dominant method of providing a foundation for knowledge has been the *foundationalist method*. Foundationalism seeks to establish all human knowledge on a solid foundation of (i) *basic knowledge*, and (ii) *knowledge-extending procedures*. Commonly, basic knowledge is required to be indubitable, and the knowledge-extending procedures incontrovertible. Speaking in terms of beliefs, the idea is that a given belief constitutes knowledge iff (if and only if) it is either basic or obtained from basic beliefs by an absolutely reliable procedure. In terms of a system of knowledge (a partially idealized notion of a collection of disciplines that constitute our integrated body of theoretical knowledge), the view is that a proper system of knowledge has units of two kinds—basic and non-basic ("derivable"); each basic unit is indubitably true, and each non-basic unit "inherits" the truth of some basic units through highly reliable procedures. Foundationalism purports to provide a foundation for knowledge in a simple and straightforward manner:

(i) Basic items of knowledge are grounded in reality (the world),[3] or whatever else they might be grounded in, directly, through direct experience, rational intuition, convention, etc.

(ii) Non-basic items of knowledge are grounded indirectly, through reliable knowledge-extending procedures (deductive, inductive, and possibly others).

A salient feature of foundationalist systems is their *strict ordering*. Foundationalism imposes a non-trivial ordering requirement on our system of knowledge, reflected in metaphors like the *tree* and the *pyramid*. This requirement says that paradigmatically the grounding relation (i) is irreflexive, asymmetric,[4] and transitive, (ii) has an absolute base consisting of minimal (initial, atomic) elements, and (iii) connects each non-minimal element to one or more minimal elements by a finite chain. This salient feature of foundationalist epistemology is both an asset and a handicap—both a source of its considerable attraction, and a cause of its ultimate failure. On the one hand, foundationalism is capable of reducing the unmanageable task of grounding our entire system of knowledge in reality to the (seemingly) manageable task of grounding only its basic constituents in reality. On the other hand, foundationalism has no resources for grounding the basic constituents of knowledge. This is the *basic-knowledge predicament* of foundationalist epistemology. The same strict ordering which made foundationalism so attractive in the first place creates a formidable obstacle to the

[3]I use "reality" and "world" as synonyms in this paper.

[4]Specifically, asymmetry means that the grounding of two distinct items of knowledge, a and b, cannot take the form "a is justified (licensed) by b and b is justified (licensed) by a". Although asymmetry is implied by irreflexivity and transitivity, I prefer to state it explicitly.

realization of its promise. Since the basic units of knowledge lie at the bottom of the foundationalist hierarchy, no unit (or combination of units) is suitably situated to produce the requisite resources for grounding the basic units.

We may put this in inference-form as follows:

1. It is a central principle of foundationalist epistemology that to ground X we can only use resources more basic than those generated by X.
2. It is a central principle of foundationalist epistemology that the grounding of our system of knowledge is reduced to the grounding of the basic units.
3. It is a central principle of foundationalist epistemology that our system of knowledge cannot generate more basic resources than those generated by the basic units.

Conclusion 1: No unit of knowledge can produce resources for grounding the basic units.

Conclusion 2: No unit of knowledge can produce resources for grounding our system of knowledge.

Since, due to its special properties (especially its high generality, intuitive basicness, normative force, etc.) logic is categorized as a "basic discipline" by foundationalism, foundationalism is incapable of providing a foundation for logic. Placing logic at the base means that while logic can provide (or partake in providing) a foundation for other sciences, no science (or combination of sciences) can provide a foundation for logic. Yet, since a serious error in logic will undermine our entire system of knowledge, a foundation for logic is imperative. Such a foundation could, perhaps, ground logic in something other than reality (conceivably, the mind); but for foundationalism to endure, a solid foundation for logic must be provided. Must and cannot. Having postulated (i) that any resource for founding logic must be more basic than the resources produced by logic itself, and (ii) that there are no resources more basic than those produced by logic, foundationalism is committed to the inability of our system of knowledge to construct a foundation for logic. This is the basic-knowledge predicament as it applies to logic.

This predicament leaves foundationalism with two alternatives: (a) show that logic does not require a foundation after all, or (b) show that it is possible to provide a foundation for logic without using any resources produced by our system of knowledge.

Alternative 1: No foundation for logic. Foundationalists might try to justify the "no foundation" approach by arguing that since it is impossible (for anyone, foundationalist or non-foundationalist) to provide a foundation for logic, it is unfair to fault the foundationalist for not providing one. Or they might argue that no matter what methodology one chooses, grounding must stop at some point; why not at logic? Neither argument, however, would stand. What follows from the impossibility of providing a foundation for logic is the unviability of foundationalism, not its inculpability. If, to perform its job, foundationalism has to accomplish an impossible feat, then foundationalism must be renounced, not excused. And granting the practical necessity of stopping the grounding process at some point, not all points are equal as far as foundationalism goes. Leaving a higher area ungrounded would have few ramifications for the overall structure, but leaving a lower area, and especially a widely connected lower area, ungrounded could have serious consequences. If logic is ungrounded, then, due to its position in the foundationalist hierarchy, the entire system of knowledge is ungrounded. It is a structural predicament of foundationalism that leaving the higher disciplines ungrounded would undercut its raison d"etre, while leaving the basic disciplines ungrounded would undermine its integrity.

Alternative 2: Foundation without resources produced by our system of knowledge. This appears the solution of choice to the basic-knowledge predicament. It is inherent in the foundationalist method, many of its adherents would say, that the foundation of the basic units is different in kind from that of the other units. The former utilizes no knowledge-based resources, and in this sense it is free-standing—a foundation "for free", so to speak. Three contenders for a free-standing foundation of logic are: (a) pure intuition, (b) common-sense obviousness, and (c) conventionality. All, however, are highly problematic. From the familiar problems concerning Platonism to the fallibility of "obviousness" and the possibility of introducing error through conventions, it is highly questionable whether these contenders are viable. We cannot rule out the feasibility of internal revision in these or other attempts to overcome the inherent impediments of foundationalism, but the severity of these impediments suggests that a search for a new methodological strategy is more promising.

2. The "Foundation without Foundationalism" strategy. If the bulk of our criticisms is correct, the traditional foundationalist strategy for constructing a foundation for logic (and for our system of knowledge in general) should be rejected. It is true that for a long time the foundationalist strategy has been our only foundational strategy, and as a result many of its features have become entangled in our conception of a foundation, but this entanglement can and ought to be unraveled. Indeed, it has already been challenged by 20th-century *holistic*, or *anti-hierarchical*, strategies. "Holism", as it is used in this paper, has its roots in Quine's use of this term, and it emphasizes the existence of a large, non-hierarchical, network of connections between various items of knowledge, along the lines delineated in Section 6 of Quine [1951].[5]

[5] In the philosophical literature this type of holism is sometimes referred to as "confirmation holism", but I will not use this term here. In the course of this paper I will develop a type of holism that is suitable for foundational investigations and I will distinguish it from other types of holism. At his point, however, I use "holism" in a more generic way.

But the holistic route has often led to a total abandonment of the foundational project, and I am reluctant to follow it in this respect. My goal is an epistemic strategy that is both free of the unnecessary encumbrances of the foundationalist strategy and strongly committed to the grounding project. Following Shapiro [1991], I will call such a strategy a *foundation without foundationalism*.

The key to a foundational yet not foundationalist method lies in freeing ourselves from the rigid ordering-requirement of the foundationalist methodology. If we demand that the grounding relation be rigidly ordered, then the only route open to us is the foundationalist route (or something like it). But why should the grounding relation be required to have this specific formal structure? Why should the image of "foundation" underlying past epistemologies control our search for a foundation today? Granted, logic itself provides an example of a rigidly-ordered method of justification, but doing logic and providing a philosophical foundation for logic are two different things.

Relaxing the foundationalist ordering requirement by itself, however, will not automatically lead to a better theory. A prime example of a holistic doctrine free of the rigid ordering injunction is *coherentism*—a view that regards the coherence of, or internal relations between, our various beliefs and theories as the main factor in their justification.[6] In radical forms of coherentism grounding-in-reality plays no role in justifying knowledge, while in others its role varies, depending on the particular version of coherentism at hand. Limiting ourselves to radical coherentism (for the purpose of pursuing our line of reasoning), we may say that this coherentism is not just anti-foundatio*nalist* but also anti-foundatio*nal* in our sense.[7] Foundationalism and (radical) coherentism, indeed, mark two extremes of the foundation—no-foundation divide. We can characterize these two methodologies as follows:

	Strict Ordering of the Grounding Relation	**Use of Knowledge-Based Resources in the Grounding Process**	**Substantial Veridical Grounding of Knowledge**
Foundationalism	required	restricted to the grounding of non-basic knowledge	required
Coherentism	not required	unrestricted	not required.

[6] See, e.g., Kvanvig [2007].

[7] For non-radical forms of coherentism see, e.g., Lehrer [1974, 1990] and BonJour [1985]. Since it is not clear, however, whether they put as much emphasis on the groundedness of

It is readily seen, however, that foundationalism and coherentism do not exhaust the whole array of possible configurations of the above parameters. One configuration left out is:

Strict Ordering	**Use of Knowledge-Based Resources**	**Substantial Veridical Grounding**
not required	unrestricted	required.

This configuration permits us to employ all the resources available to us, including resources generated by our system of knowledge, throughout the grounding process, yet it does not relax the requirement of a substantial veridical grounding for all disciplines. The key idea is that there is no inherent connection between the requirement of veridical grounding (grounding or justification centered on truth) and the requirement that some types of knowledge be grounded without use of any resources produced by any part of our system of knowledge. The two will be connected if we assume that all grounding relations must satisfy the foundationalist ordering requirement, but this assumption is unwarranted. It is sometimes argued that strict ordering is needed to avoid infinite regress and circularity, but it is widely acknowledged that regress and circularity are not always vicious. Indeed, it is not clear that recoil from *all* forms of regress and circularity is not the outcome, rather than the cause, of an unquestioning acceptance of the rigid ordering requirement.

The new methodology, it is readily seen, is both *holistic* and *foundational*. In rejecting foundationalism's rigid ordering requirement it is a holistic methodology; in being committed to a (substantial) veridical grounding of knowledge it is a foundational methodology. I will call it "foundational holism". *Foundational holism* is a foundation-without-foundationalism methodology. It shares foundationalism's commitment to a strong grounding, but it says that we need not encumber ourselves with unreasonable restrictions, and it grants us maximum freedom in designing and carrying out the grounding project. Unlike foundationalism, it does not determine in advance either the formal structure of, or the resources used in, each stage of the grounding process; and unlike radical coherentism, its does not give up on, or in any way compromise, the application of robust veridicality standards to our system of knowledge (logic included).

It should be noted that while holism is sometimes identified with the idea that we can only consider our system of knowledge as a *whole* (as one unit)

knowledge—all knowledge, including logical knowledge—in reality as I do, I prefer to leave the relation between their conception of knowledge and mine an open question. Accordingly, I use a different title for my position—"foundational holism"—one that is also more informative.

and not as a structure of differentiable units, the present conception of holism emphasizes the *interconnectedness* of significantly differentiated units of knowledge. In this sense, ours is a *relational* or a *structural* holism rather than a "one unit" holism. Foundational holism conceives of our corpus of knowledge as a highly-structured composite system, with multiple significant constituents standing in multiple significant relationships. This rich structure produces manifold avenues for obtaining, transmitting, and justifying knowledge, setting the ground for a non-foundationalist foundation for knowledge.

A well-known metaphor that fits our idea of foundational holism is "Neurath's boat" (Neurath [1921, 1932]). Our system of knowledge, according to this metaphor, is a boat in the sea, and one of the central questions we are facing is: "What do we, its sailors, do when a hole opens at the bottom of the boat?". On the foundationalist account we take the boat to the shore, dock it, and standing on firm ground repair the boat. The anti-foundationalist account is more complex. There is no Archimedean standpoint, no firm ground for us to stand on. To solve the problem we find a temporary foothold in some relatively sound area of the boat, use the available tools (or build new tools using available resources), repair the hole (as best we can), and continue sailing. Once we have mended one area of the boat, we use it as a temporary position for investigating the soundness of, and making repairs in, other areas. We may use these areas to create new tools for re-patching (or better patching) the original hole, and so on. In this way any section of the boat can (in principle) be repaired, and any section can (in principle) be used as a foothold.

The Neurath-Boat metaphor is sometime viewed as a coherentist metaphor. But while it is possible (and, depending on your reading of Neurath himself, possibly historically accurate) to interpret this metaphor as representing coherentism, it is also possible to interpret it as representing foundational holism. Indeed, the metaphor itself suggests the latter interpretation. In the first place, the boat exists in a real sea, part of the real world, and is affected by real forces: winds, rain, waves, underwater currents, and so forth. Its sailors, therefore, must take these forces into consideration in choosing a method for mending the boat. (They cannot use water-soluble materials to patch the boat.) In the second place, Neurath Boat is not drifting haphazardly in the sea, nor is it a recreation boat. The boat, as representing our system of knowledge, is on a mission to explore the world (sea) and has to face whatever difficulties are associated with this goal. Finally, where do the boat's sailors obtain their resources but from the sea and its elements, i.e., from the world. Neurath's Boat, thus, is heavily invested in, constrained by, and directed at the world, and as such is an apt metaphor for foundational holism.

The foundational holistic methodology, as it is reflected in the Neurath-boat metaphor, considerably increases the resources available to us in carrying out the foundational project. First, it introduces a dynamic element into the grounding process: the grounding of knowledge proceeds in steps, where at each juncture we are free to move in different directions, including a back-and-forth movement that is ruled out by the strict ordering of the foundationalist method. Second, we can approach the grounding project from multiple perspectives and use the resources produced by multiple units in multiple areas of knowledge. And third, we can in principle forge multiple routes from the different branches of knowledge to reality, including new routes of reference and correspondence. The interweaving of resources, back-and-forth movement, and new routes of reference (correspondence) will be on display later on, e.g., in the discussion of mathematics and its dynamic interaction with logic.[8]

One of the distinctive features of foundational holism as a Neurath-Boat methodology is its ability to deal with *circularity* in a sensible yet flexible manner. All forms of circularity are banned by foundationalism, but this is neither necessary nor desirable. It is not necessary because not all forms of circularity are destructive, and it is not desirable because (i) it prevents us from engaging in perfectly rational and fruitful endeavors, and (ii) it deprives us of the use of powerful cognitive tools. Let me elaborate.

By circularity I understand, in this paper, the use of X, or some constituents of X, in critically studying X, developing a foundational theory of X, justifying X, etc. Now, there is no question that we need to use other tools besides logical tools in providing a theoretical foundation for logic. The question is whether we can legitimately use some logical tools in this enterprise. Given the basicness of logic, this is unavoidable: we cannot make any step in theorizing about anything without using some logic; in particular we cannot theorize about logic itself without using some logic.

But is the use of circularity compatible with the foundational project? Foundational holism's answer is "Yes". While it is true that brute circularity ("P; therefore P") has an unacceptable trivializing effect, in other cases there are ways of minimizing this effect. One important device for avoiding trivialization is *bringing diverse elements into the foundational mix*. To provide a foundation for X, we may use some constituents of X in combination with other things, external to X, and possibly new combinations of things involving (related to) X. Circularity, then, becomes *partial*. At each stage we use only part of X, at different stages we use different parts of X, we use other things in addition to constituents of X, and we are always open to the possibility of revising X. This is characteristic of the foundational holistic method. In the same way that we can use a patch in a boat as a standpoint for collecting resources to create a new, better patch to replace it, so we can use components of our current logical theory to create resources for grounding, finding flaws in, replacing, or improving that same logic. Below we will see

[8]For a more general discussion of *routes of reference and correspondence* see Sher [forthcoming].

how, using elements of our current logical theory in conjunction with other things, we will be able to investigate logic's grounding in reality, develop a criterion of logicality, study the relations between logic and mathematics, create tools for critically evaluating specific logical theories (including the one we are using), and so on. An especially fruitful use of this process is in grounding two disciplines in tandem, or at least understanding how they develop in tandem, each drawing resources from the other in a continuous step-by-step process. Our account of the intertwined development of logic and mathematics below is a case in point.

Some might worry about circularity's ability to introduce error into the foundational study, and those who are haunted by the destructive paradoxes of the late 19th- and early 20th-century might be especially attuned to this danger. They might worry that circularity, broadly construed as encompassing self-reference, impredicativity, and similar phenomena, would give rise to paradoxes like Russell's and the Liar. It is clear, however, that not all circularity is destructive in this way. After all, our paradigm of circular reasoning, "P; therefore P", is logically valid, and as such does not introduce error into any theory. This does not make it suitable for a foundational study (it is also a paradigm of triviality), but it shows that circularity per se does not introduce *error* into our theories.

Another reason people might cite for prohibiting circularity in foundational studies is that it supposedly precludes the *discovery of error*. It might be thought, for example, that we cannot hope to discover errors in our logical principles given that we have to use them in the discovery process. I believe this is not the case. Of course, a careless use of circularity would be detrimental to the discovery of error, but this is not the case with careful uses. Using logic does not mean being blinded by one's use of logic. Take, for example, Russell's discovery of a paradox in Frege's logic. In discovering this paradox Russell had to use some logic. Which logic did he use? Clearly, he had to use a quite powerful logic, a logic more powerful than, say, sentential logic. But neither type-theoretic logic (which he developed only *after* discovering the paradox, and largely as a means of avoiding it) nor standard 1st-order logic plus axiomatic set theory were available at the time. In all likelihood, he used something on the order of Frege's logic, but he used it flexibly, sparingly, partially, dynamically, critically, and intelligently—holding off some parts, switching from part to part, and so on—so the paradox could come to light. A similar, perhaps more perspicuous, case might be the discovery of the "heterological" paradox, which in all likelihood was done using a language susceptible to semantic paradoxes. Likewise, the discovery of the Liar paradox was done in a language that is not immune to such paradoxes.

In fact, a careful use of circularity might enhance our cognitive powers. This is famously demonstrated in meta-logic, where Gödel's method of representing syntax by means of (the same) syntax testifies to the considerable advantage that a critical use of circularity can gain us. I will call circularity used to further our epistemic goals "constructive circularity".[9] Constructive circularity, indeed, is a constituent of many philosophical methods: Rawls' [1971] method of reflective equilibrium, Glymour's [1980] bootstrap method, Gupta & Belnap's [1993] revision method, and others.[10]

It is important to note, however, that although foundational holism sanctions some uses of circularity (in particular, constructive circularity), it does not give circularity a blanket endorsement. Foundational holism recognizes both the dangers of circularity and its advantages. It demands that as theorists we are constantly on guard to avoid destructive circularity, yet it also encourages us to make use, and to search for new forms, of constructive circularity. Furthermore, to avoid destructive circularity it allows us to accept certain compromises (e.g., the customary limitation of models to universes consisting of proper sets), and most importantly, it demands that we use circularity critically and with caution.

With the foundational holistic methodology at hand, we are ready to proceed to the second, constructive, part of our solution to the foundational problem of logic. Our next task is to construct an actual foundation for logic using the new methodology. Since this methodology does not *determine* an actual foundation, let alone a unique foundation, we will have to engage in a series of investigations to arrive at such a foundation. In attending to this task I will treat the foundational problem of logic as a *theoretical problem*, analogous to other theoretical problems investigated in science, mathematics, logic, and philosophy, rather than as a problem of spelling out our pre-theoretical intuitions on logic. As a result, the investigation might lead to a critical outlook on some constituents of the prevalent conception of logic and suggest revisions of some of its components. This, I believe, is as it should be. I should also note that my focus, in searching for a foundation for logic, will *not* be on logic's use in natural language, as were many philosophical discussions of logic in the 20th-century. Instead I will focus on the veridicality of logic, its contribution to our overall system of knowledge, and its relation to mathematics.[11]

[9] "Constructive" here is used simply as an antonym of "destructive". No connection to "constructivity" as a term-of-art in meta-logic/mathematics or the philosophy of logic/mathematics is intended.

[10] Rawls' method emphasizes *back and forth* movement between particular judgments and general principles, where the former instantiate the latter and the latter generalize the former, using each to check the other, until reaching a "reflective equilibrium". Glymour's method sanctions the use of some hypotheses of a given scientific theory to aid in the confirmation of that same theory (or some of its parts). And Gupta and Belnap's method enables us to make sense of, and deal with, circular concepts by making use of a revision process associated with such concepts.

[11] This contrast is reflected in my giving greater weight to prescriptive than descriptive considerations in my explanation of logic, my giving smaller weight to considerations like "It seems natural/unnatural to say (in our language) that inference X is logically valid" than to

II. An Outline of a Foundation.

1. Logical consequence and its veridicality. Let us begin with some of the roles that logic plays, or is intended to play, in our system of knowledge. It is reasonable to expect that due to our biological, cognitive, and other limitations, we are in need of a *powerful universal instrument for expanding our knowledge*. And one effective way of fulfilling this need is by constructing a theory (method, system) of *inference* or *consequence*. Such a theory will single out a type of consequence that transmits truth from sentences to sentences in all fields of knowledge and with an especially strong modal force.[12] This role or task (or a major part of it) is assigned to logic.

Another role that logic plays has to do with prevention of error. Being fallible creatures, we require methods for discovering and correcting error. Now, while some errors are limited to a specific area of knowledge, others can occur in any field. And while some errors are relatively innocuous, others are pernicious. The occurrence of *errors that are both pernicious and universal* calls for a powerful and universal instrument for removing and preventing errors (contracting and constraining our body of beliefs).[13]

Apart from providing the above two (and related) services to our system of knowledge, logic is itself a branch of knowledge: a *theoretical discipline with its own subject matter*. One of its important jobs, therefore, is to provide theoretical knowledge about this subject-matter.[14]

The view that logic is a genuine branch of knowledge is contested by those who view it as a mere practical instrument. Logic, according to their view, does not have a distinct subject-matter of its own, and therefore it is not in the business of providing theoretical knowledge. This view often appeals to the "topic neutrality of logic": to be neutral to every topic, so the thought goes, is to have no topic at all.

I think this view is based on a confusion. Logic is indeed topic neutral, but being topic-neutral is not the same thing as not having a subject-matter of its own. Logic does have a subject-matter of its own. Its subject-matter is logical inference, logical inconsistency, logical truth, etc., where these are

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considerations like "Given the intended role of logic in our system of knowledge, it makes sense to consider X logically valid", and so on.

[12] At this point I use "modal force" as a general term of our language, not as a philosophical term-of-art associated with, say, Kripke semantics. As we progress, I will make this term more specific and explain what it does and does not mean.

[13] Since such errors arise from violation of specially strong and universal principles, principles of the kind needed to license especially strong inferences, a single method might be able to perform both roles.

[14] In thinking of logic as a branch of knowledge I do not draw a sharp distinction between what is commonly labeled "logic" and what is commonly labeled "meta-logic", or at least some parts thereof. For example, the semantic and proof theoretic definitions of logical consequence technically belong to meta-logic, but here I regard them as parts of the discipline whose foundation we are investigating.

very different from the subject-matters of other disciplines. Logic studies the special conditions under which an inference is logically valid, a sentence is logically true, a theory is logically consistent. It tells us whether specific inferences, sentences, and theories satisfy these conditions. And it develops a systematic theory of these things, one that views them from several perspectives, including a proof-theoretic perspective and a semantic perspective.

In spite of having a definite subject-matter, logic is topic neutral. Its topic neutrality consists in the fact that it *applies* the same tests of logical validity, logical truth, etc., to inferences and sentences in *all* area of discourse, regardless of *their* subject matter.

Having these theoretical and instrumental tasks to perform, logic must be subjected to high standards of truth and instrumental success. Epistemically, this means that logic is in need of a foundation, and in particular its claims to truth and success require a critical justification and substantive explanation. Here, however, we seem to be pulled in opposite directions. To the extent that logic's subject-matter is linguistic (conceptual, mental), logic requires a grounding in language, concepts, or more broadly the mind. But to the extent that logic has to work in the world and has to be factually true, it requires a grounding in the world (reality, fact). I.e., to the extent that logic is an instrument for expanding *knowledge of the world* and preventing incorrect depiction of the world by theory (theoretical error), and to the extent that it is charged with saying true things about its subject-matter, it requires a grounding in reality. In my view, the apparent conflict between the need to ground logic in the mind and the need to ground it in the world is just that: apparent. Logic, like all other branches of knowledge, requires a grounding both in the mind and in the world. And since here we are primarily concerned with a *veridical* grounding of logic, it is its grounding in the world that primarily concerns us.

Now, by saying that logic is grounded in the world we do not mean to say that it is grounded in all (or just any) features of the world. Our claim is that there are certain (highly specific) features of the world that logic is grounded in (where "world" is understood in a relatively broad way), and our task is to explain why logic is grounded in the world at all, what specific features of the world it is grounded in, and how these features ground it. In pursuing this task we will use the foundational-holistic method. We will start with the more general aspects of logic's grounding in reality and become more and more specific as we progress.

Viewing logic as oriented toward the world (in one way or another) is not without precedent. Wittgenstein, for example, said that logic "shows" ("displays", "mirrors") the logical form of the world [1921, 4.121. 6.12–6.13]. Quine said that "[l]ogical theory ... is ... world-oriented **rather** than language-oriented" [1970/86, p. 97].[15] And Tarski characterized semantic

[15] Bold words in citations indicate my own emphasis.

notions in general as having to do with the relation between language and the world [1933, p. 252], [1936, p. 401], from which it follows (whether he himself was aware of this or not) that the semantic notions of logic ("logical consequence", "logical inconsistency", etc.) have something—indeed, something significant—to do with the world. Be that as it may, the main issue is whether logic in fact requires a grounding in the world and why.

One straightforward consideration in support of the view that it does is common-sensical: There is a very real sense in which a proposed logical system (theory) either works or does not work in the world. For example, a logical system that contains a law like *affirming the consequent* will normally not work in the world, while a system containing the law of *affirming the antecedent* (Modus Ponens) normally will. This is not because *affirming the antecedent* is more elegant, or more intuitive, or more natural to humans than *affirming the consequent*, but because (under normal conditions) affirming the antecedent is in sync with the world, while affirming the consequent is not. Why some candidates for a logical law are in sync with the world while others are not, and what aspects of the world the former are in sync with, is something I will turn to shortly. But it is clear that some rules that look like logical rules work in the world, while others do not. For that reason, we cannot take it for granted that any logical system we postulate, or any logical system that seems natural to us, would work in the world. To the same extent that using faulty theories of other kinds could cause airplanes to crash, workers to lose their salary, atomic plants to shut down (or explode), cars to stall (or collide), and so on, so using a faulty logical theory can cause all these things. Indeed, what scientific realists say about scientific theories (or the abstract parts of such theories) applies to logical theories as well: It would be a mystery that a logical theory worked in the world (in flying airplanes, computing salaries, etc.) if it were not in tune with the world. In designing a logical system, therefore, the world must be taken into account. This does not mean that there is no room to maneuver in designing such a system, or that all features of the world must be taken into account. But working in the world is a serious constraint.

The view that logic requires a grounding in certain facts is also supported by historical examples of a factual error in logic. The most dramatic example of this kind is the error in Frege's logic discovered by Russell. Frege's logic, Russell showed, was committed to the *existence* of an *object*, a class, that *does not, and cannot, exist*. This commitment led to a fatal paradox, forcing logicians to revise logic.

Less conclusive, but still instructive, are cases where logical systems were criticized for not working under certain significant circumstances. For example, proponents of free-logics, fuzzy-logic, and other "non-standard" logics have argued that classical logic works in some factual contexts but fails in others. The serious attention these arguments have received from logicians and philosophers suggests that factual considerations do play a role in choosing a logic. One might also include the case of quantum logic (Birkhoff and Neumann [1936]), although this is highly controversial.

Finally, it is easy to construct artificial examples of factually erroneous logical laws. Consider, for example, the introduction of the law "$\Phi(x)$; $x \neq y \vdash\sim \Phi(y)$" into a logical theory. The syntactic form of this law is very similar to that of Leibniz's law, and in this respect it appears unobjectionable. But for this law to be valid, a very special condition has to hold in the world: namely, objects in the world must have no common properties (or at least no common definable properties). This, however, is clearly not the case; i.e., this law is factually invalid. There might, of course, be other reasons for rejecting this law, but its conflict with the facts is by itself a sufficient reason for rejecting it. Indeed, the factual conflict in this case is so deep that it renders this law classically inconsistent.

Theoretically, however, the most important considerations connecting logic to reality have to do with *truth* and *skepticism*. Recall Hume's theoretical challenge to science: We can establish that one event (which science says is a cause) occurred shortly before another event (which science says is its effect), but this does not suffice to establish a *causal* connection between the two events, a connection that requires a certain modal force. An analogous challenge to logic says: We can establish that either S_1 is false or S_2 is true, but this does not suffice to establish that S_2 *follows logically* from S_1, a connection that requires an even stronger modal force than causality. In light of this skeptical challenge, a factual grounding of logic is as theoretically important as a factual grounding of science.[16]

Philosophically, the key to logic's grounding (or need for a grounding) in reality is its inherent connection with truth.[17] To see this, let us focus on logical consequence, and as a preliminary let us informally characterize (i) the notion of consequence in general, and (ii) three specific notions of consequence: "material consequence", "nomic consequence", and "logical consequence". Consequence in general, as we understand it in this paper, is a binary relation between two sentences or between a set of sentences and a sentence. The characteristic feature of this relation is *transmission* or

[16](i) Some, of course, have despaired of the foundational project in science. But new work (e.g., Haack [1993]), using a methodology that is in certain respects similar to our foundational holism, opens up new possibilities for such a grounding. (ii) Others too see the modal force of logic as posing an important theoretical challenge to the foundational project. For example, Etchemendy [1990] does, but his approach to this challenge (and, as a result, his conclusions) is (are) different from ours.

[17]Quine, too, said that the worldly-orientation of logic is due to its connectin with truth [1970/86, p. 97], but he did not elaborate. I prefer not to speculate about the similarity between his (unknown) considerations and mine.

preservation of *truth*:

Sentence σ is a *consequence* of set of sentences Σ iff the *truth* of the sentences of Σ (assuming they are all true) is transmitted to, or preserved by, σ. (C)

Different consequence relations differ in the modal force in which truth is transmitted from Σ to σ:

Sentence σ is a *material* consequence of set of sentences Σ iff the *truth* of the sentences of Σ (assuming they are all true) is *materially* transmitted to, or preserved by, σ.[18] (MC)

An example of a material consequence is:

Barack Obama is president; therefore, Earth has only one moon. (1)

Turning to *nomic* consequence, we characterize:

Sentence σ is a *nomic* consequence of set of sentences Σ iff the *truth* of the sentences of Σ (assuming they are all true) is *nomically* transmitted to, or preserved by, σ (i.e., it is transmitted from Σ to σ with the force of a law of nature; that is, the guarantee of transmission of truth has the modal force of a law of nature).[19] (NC)

An example of a putative nomic consequence is:

The force exerted by a on b is c; therefore, the force exerted by b on a is c. (2)

Now, in the case of *logical* consequence, we have in mind a consequence whose force is stronger than that of *nomic* consequence. However, since we are trying to *understand* the nature of logical consequence (rather than take it as given), it would be better to leave the exact nature of its modal force an open question at this initial stage. (Its modal force is one of the things we

[18] Using the terminology of "possible worlds" discourse—specifically, the notion of "actual world"—we can restate (MC) as:

Sentence σ is a *material* consequence of set of sentences Σ iff it is not the case that in the *actual world* all the sentences of Σ are true and σ is false. (MC*)

We may also say that material consequence is determined by a single row of a truth-table, the one representing the actual world.

[19] Using "possible-worlds" terminology, we may say:

Sentence σ is a *nomic* consequence of set of sentences Σ iff there is no *physically possible world* in which all the sentences of Σ are true and σ is false. (NC*) are trying to figure out.) Speaking in non-specific terms we can say:

Sentence σ is a *logical* consequence of set of sentences Σ iff the *truth* of the sentences of Σ (assuming they are all true) is transmitted to, or preserved by, σ with an *especially strong* modal force, or the guarantee of transmission of truth has an especially strong modal force (one that is stronger than the modal force of a law of nature).[20] (LC)

With these clarifications in place we are ready to explain why logic requires a grounding in reality due to its inherent connection with truth. Focusing on logical consequence, the reasoning underlying our explanation is, roughly, this:

(a) Consequence relations in general hold between sentences due to connections between their truth values. (Consequence relations are relations of transmission or preservation of truth.)
(b) The truth value of a sentence generally depends on how things are. (A sentence is true if things are as the sentence says).
(c) Therefore, consequence relations, including the relation of logical consequence, have to take into account connections (or lack of connections) between the ways things are with respect to the sentences involved. I.e., a sentence σ is a consequence of a set of sentences Σ iff there is an appropriate connection (which ensures truth-preservation with the requisite modal force) between things being as the sentences in Σ say and things being as σ says.

To see the intuitive force of this reasoning, suppose someone comes up with a proposal for an arbitrary logical theory, $\mathcal{L}$. The question we ask is: Under what conditions is $\mathcal{L}$ an acceptable logical theory? In particular: Under what conditions does $\mathcal{L}$ make correct judgments about logical consequences?

Consider a simple claim of logical consequence made by $\mathcal{L}$, a claim that the relation of logical consequence holds between two distinct (non-equivalent) sentences, S_1 and S_2, where S_1 and S_2 are truth-wise simple and unproblematic (in that their truth-conditions straightforwardly concern the way things

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[20] Assuming that "!" indicates an especially broad sense of possibility (the dual of the strong modal force we are after), we can say:

Sentence σ is a *logical* consequence of set of sentences Σ iff there is no !-*possible world* in which all the sentences of Σ are true and σ is false. (LC*)

There is a clear similarity between this characterization and the standard Tarskian definition of logical consequence:

Sentence σ is a *logical* consequence of set of sentences Σ iff there is no *model* in which all the sentences of Σ are true and σ is false. (TC)

This suggests that our question concerning the modal force of logical consequence is related to the question of what the totality of Tarskian models represents.

are in the world). In symbols:

$$(\textit{Level of Logic}) \qquad S_1 \vDash_L S_2.\text{[21]} \tag{3}$$

Assume S_1 is true. Then, for (3) to be true, the truth of S_1 must guarantee that of S_2. I.e., it must be the case that:

$$(\textit{Level of Language}) \qquad T(S_1) \longrightarrow T(S_2). \tag{4}$$

Now, let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be the conditions (situations) required to hold in the world so the two sentences are true:

$$\begin{array}{lcc} (\textit{Level of Language}) & T(S_1) & T(S_2). \\ & \Updownarrow & \Updownarrow \\ (\textit{Level of World}) & \mathfrak{C}_1 & \mathfrak{C}_2. \end{array} \tag{5}$$

Assume the condition $\mathfrak{C}_1$ holds in the world while $\mathfrak{C}_2$ does not. I.e.,

$$(\textit{Level of World}) \qquad \langle \mathfrak{C}_1, \ \textbf{not}\ \mathfrak{C}_2 \rangle. \tag{6}$$

In that case, the claim that S_2 is a logical consequence of S_1—(3) above—is false. Not only is S_2 not a logical consequence of S_1, it is not even a nomic or a material consequence of S_1:

$$(\textit{Level of Logic}) \qquad \textbf{Not:} \quad S_1 \vDash_{M,N,L} S_2.\text{[22]} \tag{7}$$

We see that some fact about the world is sufficient to show that the logical theory $\mathcal{L}$ is incorrect. Regardless of what $\mathcal{L}$ says, (6) shows (determines) that S_2 does not follow logically (or even materially) from S_1. For a logical theory to be correct, it has to take the world into account. (For S_2 to be a consequence of S_1, the world has to cooperate.)[23] Now, assume both conditions $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are satisfied in the world, but $\mathfrak{C}_1$ being satisfied does not guarantee, in any modally significant way, that $\mathfrak{C}_2$ is satisfied:

$$(\textit{Level of World}) \qquad \langle \mathfrak{C}_1, \mathfrak{C}_2, \ \textbf{not}\ [\mathfrak{C}_1 \Rightarrow \mathfrak{C}_2] \rangle \tag{8}$$

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[21] Notation: "$\vDash$ is a symbol for the general (or unspecified) relation of *consequence*, understood in terms of truth (i.e., as a semantic relation), and the subscript "L" stands for "logical", so that "$X \vDash_L Y$" means "Y is a *logical* consequence of X".

[22] The subscripts "M" and "N" stand for "material" and "nomic" respectively.

[23] Someone might say that this is a matter not of *the world*, but of the totality of physically possible worlds. Let me clarify. We can think of the laws grounding various types of consequence in two ways: as governing *the world* or as governing some totality of *possible worlds*. In the case of physical laws and nomic consequence, we can think of a physical law as a fact that *materially* holds in *the totality of physically possible worlds*, or we can think of it as a fact that holds with a certain *significant modal force* in *the world* (the one and only world there is). These two ways of thinking are exchangeable. But in this paper I prefer to use the *second* way of thinking. For that reason, I also speak of the grounding of logical consequence in *the world* rather than in a certain totality of possible worlds.

Then, although S_2 is a material consequence of S_1, it is neither a nomic or a logical consequence of S_1:

$$(\textit{Level of Logic}) \qquad \textbf{Not:} \quad S_1 \vDash_{N,L} S_2. \tag{9}$$

Once again, a fact about the world is sufficient to show that $\mathcal{L}$ is incorrect. Regardless of what $\mathcal{L}$ says, (8) shows (determines) that S_2 is not a logical consequence of S_1. In the absence of a *strong* connection (of some appropriate kind) between the situations required for S_1 and S_2 to be true, the claim that S_1 logically implies S_2 is unwarranted. It appears that for S_2 to be a logical or even a nomic consequence of S_1, there has to be a *law* that connects $\mathfrak{C}_1$ to $\mathfrak{C}_2$—a connection with a significant modal force between $\mathfrak{C}_1$ and $\mathfrak{C}_2$.

Finally, suppose that the world being as S_1 requires carries a nomic guarantee for its being as S_2 requires, but not a *stronger guarantee*—in particular, not a guarantee as strong as that required by logical consequence. Figuratively:

$$(\textit{Level of World}) \qquad \langle \mathfrak{C}_1, \mathfrak{C}_2, \ \textbf{not}\ [\mathfrak{C}_1 \boldsymbol{\Rightarrow} \mathfrak{C}_2] \rangle \tag{10}$$

where "$\boldsymbol{\Rightarrow}$" stands for a relation of an especially strong modal force between situations. As before, this fact about the world is sufficient to show that $\mathcal{L}$ is incorrect. It is sufficient to challenge $\mathcal{L}$'s claim that S_2 is a *logical* consequence of S_1.

We have seen that there are a number of ways in which the world can challenge, and even establish the incorrectness of, a proposed logical theory. A correct theory of logical consequence must respect the connections (or lack thereof) between certain conditions on the world: the conditions that have to be satisfied by the world for the premise-sentence to be true and the conditions that have to be satisfied by the world for the conclusion-sentence to be true. We may say that *the world limits the options open for logical theories*.

These considerations show that logic is *constrained* by the world, i.e., the world has at least a *negative* impact on logic. And this in turn shows that a proposed logical theory (our own logical theory included), requires a grounding in the world. I.e, we need to justify the claim that our logical theory is not undermined by the world. More generally, we need to show that it is possible to construct a logical theory that satisfies its designated role without being undermined by situations like those delineated in (6), (8), and (10) above. It is important to note that the conflicts between claims of logical consequence and the world that occur in these cases are *not accidental* in the following sense: They are due (i) to something that is *central* to, or *inherent* in, *logical consequence*—it's being a relation of transmission of *truth*, and (ii) to something that is *central* to, or *inherent* in, *truth*—its having to do with the way the *world* is.

Having shown that logic is *constrained* by the world (possibly along with some other things—e.g., the mind), our next task is to show that it is also *enabled* by the world. In particular, we need to show that by examining the relation between logic and the world we can arrive at a *positive* explanation of *why* certain claims of *logical consequence* are *true*, and that this explanation involves the world in a significant way. In the best case we will show that the world might *bring it about* that (certain things, like language and meaning being given) some sentences *follow logically* from others.

Let us begin, once again, with consequence in general, starting with *material* consequence. Suppose that in the world both $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are the case. This suffices to guarantee that S_1 materially implies S_2 (and vice versa). I.e., the world can justify claims of material consequence. Indeed, the world can bring it about that some sentences are material consequences of others.

Next, let us assume that $\mathfrak{C}_1$ necessitates $\mathfrak{C}_2$ with the force of a law of nature; i.e., the world is governed by a law that positively connects $\mathfrak{C}_1$ and $\mathfrak{C}_2$ with the modal force of natural laws. Then, the existence of such a law will sanction not just the material transmission of truth from S_1 to S_2, but also the nomic transmission of truth from S_1 to S_2. For example, if Newton's third law holds in the world, the world gives rise to (guarantees) the nomic consequence (2). We see that *the world can also justify claims of nomic consequence* (bring it about that some sentences are nomic consequences of others).

Finally, suppose the world is governed by a law that positively connects $\mathfrak{C}_1$ and $\mathfrak{C}_2$ with a stronger modal force than that of laws of nature, a force as strong as that commonly associated with logic, or as required for logic to play its designated role as a theory of an especially powerful type of inference that occurs (or might occur) in any field of knowledge. Then, in principle, such a law will substantiate consequence claims with the force appropriate for logic. For example, suppose the world is governed by a highly necessary and universal law (a law whose counterfactual force exceeds that of physical laws) like:

$$\text{Non-empty } \mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) \Rightarrow \text{Non-empty } \mathcal{A} \cup \mathcal{B}, \tag{11}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are any 1st-level[24] properties and $\cup$, $\cap$ are the operations of join and meet on any (extensional) properties in the world, including highly counterfactual properties (i.e., properties that are possible yet unactualized, in an especially broad sense of possibility). Then this law is sufficient to substantiate the claim:

$$(\exists x)(Ax \vee (Bx \;\;\&\;\; Cx)) \vDash_{\mathrm{L}} (\exists x)(Ax \vee Bx). \tag{12}$$

[24]I use "level" in this paper for an informal typology of objects, properties, and related things. Individuals (objects that in a given context are taken as atomic) are of level 0. Properties, relations, functions, and operators whose highest arguments are of level n are of level $n + 1$. Sometimes I extend the level terminology to phenomena, theories, etc. When I talk about logics and theories I also use the term "order" as analogous to "level".

I.e., this law is sufficiently strong to give rise to a consequence that we usually characterize as logical and has in any case the kind of force (guarantee of transmission of truth) that is required for an especially strong, universal theory of inference—a theory that plays the role that logic is designed to play in our system of knowledge.[25]

We can see that, and how, the existence of certain *laws or regularities in the world* has the power to establish certain claims of logical consequence, or at least claims that accord with the designated role of logic noted above. That is to say, by considering the world we can start to provide a *positive* and not just a negative grounding for logic. Theoretically, logic is grounded in reality through (i) its inherent connection with truth, (ii) the inherent connection between truth and reality, and (iii) the inherent relevance of (ii) to (i). Furthermore, logic is grounded in certain *laws* governing the world, laws which possess an especially strong modal force and whose nature we will shortly begin to investigate.

Two notes: (a) In arriving at this conclusion we have reasoned in accordance with the laws of logic, or at least tried to, and this violates the requirements of foundationalism. But this type of reasoning is sanctioned by foundational holism. To study logic we find a standpoint within Neurath boat that allows us a sufficiently broad view of the boat itself, so we can figure out some of its general features and reason about them.

(b) The above conclusion does not deny that things other than the world—language, meaning, mind, ideas, concepts, etc.—play a role in logical consequence or the preservation of truth in general. But it points out that consequence relations must, ultimately, be judged by reference to the world—that the world plays an important, if not the only role, in logical consequence.

Someone, however, might argue that our analysis is compatible with the view that logic is grounded only in the mind, and not at all in the world. Someone might say that we cannot rule out the possibility that nature endowed us with a built-in logical apparatus (or our language has such a built-in logical apparatus) that happens to agree with the world; in this case, we would not need the world to either sanction or constrain our logic. Put in the form of a question, someone might ask: Could it not be the case that while the world is capable of grounding logic, in fact this job is done by the mind alone, with the world playing no role?[26] The answer to this question is, I believe, negative. Whether we have a built-in logical system that works properly on its own or we have to build one for ourselves by consulting the

[25] Here and in (13) and (14) below I use examples from classical logic. But the discussion can be adjusted to accommodate non-classical examples as well. In fact, the view espoused here takes it as an open question whether the "classical" theory of logic is the "right" theory of logic. This was hinted at in the introductory pages of this paper and will become clearer as we proceed.

[26] I would like to thank Pen Maddy for this question.

world (among other things), the veridicality of such a system significantly depends on the world. Were the world different in relevant ways, were it governed by different pertinent laws, the logic generated by our mind would fail in it. Even if a benevolent Nature or God saw to it that if reality changed, our mind would change with it, it would still be reality (and not just our mind or Nature or God) that would ground our logic. It is only if God, Nature, or our mind, could, and would, change the world so it always obeyed our logic, that the world would not play a central role in the grounding of logic. And even then it would, since after all it would be the ability to regulate *the world* that was responsible for the adequacy of logic.

This concludes the first step in our foundational account of logic: Logic is grounded in reality through its inherent connection with truth. In particular, logic is grounded in certain modally strong connections that hold in the world.

Our next task is to figure out what are the facets of reality that logic is grounded in. In preparation for this task let us revisit our last example. In virtue of what is (12)—i.e., the claim of logical consequence expressed by (12)—true? (12) is true since the transmission of truth from its premise to its conclusion is guaranteed (indeed, strongly guaranteed) by the law delineated in (11). What kind of law is this?

Under one common usage of the term "formal", we may say that this law is a *formal* law, a law that concerns the behavior of such features of and operations on objects (in a broad sense) as non-emptiness, union, and intersection.

Similar laws can explain the truth of other claims of logical consequence. Consider

$$\text{Pa} \ \ \& \sim \text{Pa} \vDash_{\text{L}} \text{S} \tag{13}$$

(which superficially presents a different case from (12)). In virtue of what is (13) true (for any sentence S)? In virtue of a formal law that rules out the possibility that "Pa & $\sim$ Pa" is true by ruling out the possibility that an object a is in the intersection of a property P and its complement in any universe. Since this possibility is ruled out (with a very strong modal force), it is guaranteed (with the same modal force) that "Pa & $\sim$ Pa" is false, hence that the claim of consequence made by (13) is correct.

Failures of logical consequence are explained in a similar way. E.g., the claim

$$(\exists x)(Ax \ \ \& \ \ (Bx \vee Cx)) \vDash_{\text{L}} (\exists x)(Ax \ \ \& \ \ Bx) \tag{14}$$

is false, since it is not supported by an appropriate *formal law*.[27]

[27]How is the failure of

$$\text{Some fake roses are fragrant} \vDash_{\text{L}} \text{Some roses are fragrant} \tag{15}$$

explained? The formal portion of the content of "fake rose" is non-rose, therefore, it is the absence of a formal law connecting the non-emptiness of [Complement(*Rose*) $\cap$ *Fragrant*] to

We may thus say that *logical consequence is grounded in formal laws governing reality*. When a sentence σ stands in the relation of logical consequence to a set of sentences Σ, this can be explained by a certain formal or structural connection between the situation described by Σ and that described by σ, or alternatively, by a certain formal connection between the properties attributed to objects by Σ and those attributed to them by σ. In other words, σ is a logical consequence of Σ iff the *formal skeleton* of the situation delineated by Σ is related to the *formal skeleton* of the situation delineated by σ by a law that guarantees that if the former situation holds so does the latter. Using a more neutral terminology we can say that a given logical consequence is grounded in a universal law connecting formal elements in the truth conditions of its premises and conclusion.[28]

To offer a more systematic account of the formal nature of logic, we have to turn to logical constants and the nature of logicality. But first let us briefly address the question of how logical consequence works in non-veridical contexts according to our account. Take the context of *commands*. Does the command

Answer all the questions in the exam! (16)

logically imply the command

Answer question #1! ? (17)

And if it does, what is this logical implication grounded in according to our account?

Our answer is that this implication *is* logically valid, and it is grounded in the same formal law as its veridical correlates (e.g., "John answered all the questions in the exam; therefore he answered question #1"). Speaking in functional terms—i.e., in terms of how one would design an all-purpose language and a logical system for it—we can explain this within the parameters of our account as follows: In any context whatsoever, veridical or non-veridical, we have to parse our subject-matter in some way. This parsing is based both on the special nature of our subject-matter and on our cognitive choices. In non-veridical contexts we have more options (we have greater freedom, we are less constrained) in choosing a parsing than in veridical contexts.[29] And therefore it stands to reason that we give priority to veridical contexts in choosing a parsing for our entire language. Following

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the non-emptiness of [*Rose* $\cap$ *Fragrant*] that is responsible for the falsehood of (15). (I would like to thank Graham Priest for this question.)

[28]Someone might complain that we are simply reading off the account of logical consequence from the model-theoretic semantics of logic, but this would be to put the cart in front of the horse. Our account *explains why* the model-theoretic semantics of logic is *sound* and *how* it *works*, and this it builds into a more comprehensive philosophical foundation for logic.

[29]This is because in veridical contexts we have to take into account considerations of truth, which are highly constraining, but in non-veridical contexts we are free of such considerations.

this guideline, we construe (16) and (17) as having the same logical structure and the same logical parameters (logical constants), governed by the same formal laws, as their veridical correlates. Consequently, the inference from (16) to (17) comes out logically valid, and its validity is grounded in the same formal laws as those grounding its veridical counterparts. That is to say, our method of inference for veridical contexts can be extended to other contexts, in which case, our explanation of logical consequence in veridical contexts is extendable to non-veridical contexts as well.

2. Logical constants & the nature of logicality. It is often observed that logical consequence depends on the logical constants of the sentences involved rather than on their non-logical constants. (As far as the latter are concerned, only the pattern of repetition—sameness and difference—plays a role.) Thus, the relation of logical consequence is not affected by uniform changes in the non-logical constants, but is affected by uniform changes in the logical constants. Our next step in the foundational process is to answer the question: Which terms of our language should we build into our logical system as logical constants? To avoid an arbitrary answer to this question (an answer based on taste, or on "gut feelings", or on something else that is theoretically irrelevant) I will consider this question, too, in *functional* terms. Specifically, I will view it as requiring us to *figure out* which choices of logical constants would enable logic to perform its function or designated role. Focusing on logical consequence, I will reformulate the question as follows: Which choice(s) of logical constants produce consequences that guarantee the transmission of *truth* from premises to conclusion with an *especially strong modal force* (when plugged into a semantic-syntactic logical system)? Our investigation in the last section suggests that logical consequence is grounded in *formal* laws, and this, in turn suggests that the role of logical constants is to designate the relevant parameters of these laws. I.e., if logical consequence is grounded in laws connecting the *formal* skeleton of the claims made by the premises of a given valid consequence to the formal skeleton of the claim made by its conclusion, then the role of logical constants is to designate relevant parameters of these *formal* structures (*formal* skeletons).

That is to say, choosing terms that denote *formal* properties (relations, functions) as logical constants is a sound way of constructing a logical system that satisfies the "job description" of logic. Such a choice is exemplified by (12), whose logical constants are "$\exists$", "$\vee$", and " & ": "$\exists$" denotes the 2nd-level *formal* property of non-emptiness, and "$\vee$" and " & " denote (in the type of open sentence in which they occur in (12)) the *formal* operations of union and intersection, respectively. Accordingly, the formal content of the premise claim is that the union of one 1st-level property with the intersection of a second and a third 1st-level properties is not empty, and the formal content of the conclusion claim is that the union of the first and second properties is not empty. (12) is true because the transmission of truth from premise to conclusion is guaranteed by a law—(11)—governing the formal parameters of these claims, a law that by virtue of its formality, has an especially strong modal force.

Our next task in the grounding of logic is to give a general, informative, and precise characterization of formality for properties and operators.[30] So far we have presented a few examples of formal parameters (operators) and laws, but this is not sufficient for a theoretical foundation. In turning to a theoretical characterization of formality, however, we face a dilemma: On the one hand, we aim at a general and informative characterization that does not commit us to a special theory of formal structure, and in particular not to a theory embroiled in current controversies. On the other hand, we aim at a precise account, one that will identify the scope of formality, explain its nature in exact terms, and have precise and informative results. But this aim requires the use of sophisticated mathematical resources, i.e., of choosing one mathematical theory to work with over others, a choice that in some logical and philosophical circles will be viewed as highly controversial.

To resolve this dilemma, we will proceed in three steps: In step 1 we will prepare the ground for a general characterization of formality based on several pertinent observations (concerning the world, its formal features, the task of logic, and how to structure a system so it can perform logic's task). In step 2 we will offer a general, non-technical characterization of formality (cum logicality) that fits in with our observations in Step 1, and trace some of its philosophical roots. And in step 3 we will use the resources of a specific mathematical theory—classical set theory, or more specifically, ZFC—to formulate precise criteria of formality and logicality, applicable to systems of the kind delineated in Step 1 and based on the general principles delineated in Step 2.[31] This will enable those who reject ZFC, or even classical (bivalent) logic, to understand our formality principle in non-mathematical terms and to use our precise formulation as an example that can be reworked using a different background theory.

STEP 1. Starting with a few general observations about the world, we note that its objects have, in addition to physical properties, also properties that might be characterized as formal or structural. And whereas the former—properly physical, geological, psychological, and other properties—hold in

[30]Operators are functions, including functions representing properties, relations, and possibly other types of objects. For example, an operator representing a (classical) property is a function that assigns to a given object a fixed value X (usually the True) if it has this property and a fixed value Y (usually the False) otherwise.

[31]The third step provides another example of constructive circularity, or a legitimate use of circularity in foundational projects. The "additional" things that mitigate the circularity include those introduced in steps 1 and 2, which are independent of the mathematics used in 3.

limited "regions" of the world, the latter hold in all its regions: Individuals of all kinds are *self-identical*, properties and relations of all kinds have cardinality properties (they hold of a certain *number* of objects), objects of all kinds stand in *intersections* of properties, relations in all areas might be *transitive*, and so on. Given the universal applicability of the formal[32] and the special modal force of formal laws, it is extremely useful to have a method of inference based on it. Such a method would be systematized by a "logical" system that would use formal features as its active ingredients, represented by special linguistic expressions—logical constants. Due to their special nature, formal features of objects are governed by laws that are (i) universal and (ii) have an especially strong modal force; as such, they can be used as a basis for an especially strong and broadly applicable method of inference. We may implement this idea by building a syntactic-semantic system in which:

(α) Logical constants designate formal properties (or operators)[33] and logical forms designate formal structures of objects (formal skeletons of situations), actual and counterfactual; these designations are held fixed through all operations of the system.

(β) Rules of proof encode laws governing the formal properties (operators) designated by the logical constants.

(γ) Models represent formal possibilities, i.e., all the ways the world could have been, given the laws governing the formal properties designated by the logical constants. As a result, preservation of truth in all models has a strong modal force, i.e., the strong modal force of these formal laws.

(δ) Accordingly, the system has two methods for determining whether a putative consequence is logically valid: a narrower, yet effective, proof method, and a broader, non-effective, model-theoretic method. Proof-theoretically, a consequence is valid iff its conclusion is derivable from its premises by the specified rules; model-theoretically, it is valid iff it preserves truth in all models. Due to the way the system is constructed, the two methods do not conflict (the proof method is sound), though they may not fully coincide (the proof method may not be complete).

STEP 2. Here we set out to explain, in informative yet general terms, what the formal is, i.e., what its distinctive characteristics are. Our suggestion is that logical operators are formal in the sense that they distinguish only the *pattern* delineated by their arguments, and that this can be explained by saying that they do not distinguish between arguments that differ with respect to their underlying individuals. I.e., *if we replace any argument of a formal operator by any other which is its image under some* 1–1 *replacement*

[32]By "the formal" I understand in this paper formal properties and operators, as well as, more generally, formal features, and the laws governing them.

[33]Due to the systematic correlation between properties/relations and operators (noted above) we will freely switch from talk of the former to talk of the latter.

of individuals by individuals of any type (e.g., a 1–1 *replacement of humans by numbers), the formal operator "will not notice"*: it will assign the same truth-value to the two arguments. And the same holds for properties (relations, function): a formal property is satisfied (possessed) by a given object iff it is satisfied (possessed) by all its images under 1–1 replacements of individuals by individuals of any types. One example of a formal operator under this characterization is the 2nd-level operator of *non-emptiness*, i.e., the operator denoted by the existential quantifier of standard 1st-order logic. This operator, whose arguments are 1st-level properties, is formal because it does not distinguish between two 1st-level properties whose extensions can be obtained from each other by a 1–1 replacement of their members. This operator assigns the value True to the former iff it assigns the value True to the latter. (Speaking in terms of properties: the 2nd-level property of *non-emptiness* is satisfied by the former iff it is satisfied by the latter). Another example is the 1st-level *identity* operation (relation). This operator assigns the value True to one pair of individuals iff it assigns the value True to any other pair of individuals obtained from the former by a 1–1 replacement of individuals. A third example is the *intersection* operation—the denotation of " & " in open contexts of the form "$Bx \quad \& \quad Cx$".[34] An individual is in the *intersection* of two properties iff its image is in the *intersection* of the images of these properties under any 1–1 replacement of individuals. In contrast, the operators (relations, functions, properties) denoted by "is taller than", "father of", "is a property of humans" are not formal. All of these are affected by a 1–1 replacement of, say, individual humans by individual numbers: while humans stand in the taller-than relation, numbers do not; while humans have fathers, numbers do not; and while the property of being-married is a property of humans, its numerical image is not a property of humans. Most properties do not distinguish *some* 1–1 replacements of individuals, but only *formal* properties do not distinguish *all* such replacements.[35]

The idea that the formality of logic consists in its abstracting from differences between objects has old roots. We can trace it to, e.g., Kant's statement that general logic "treats of understanding without **any regard** to **difference** in the **objects** to which the understanding may be directed"

[34]Here we are stepping into the domain of mathematics that was to be limited to Step 3. But the use of the mathematical term "intersection" is not essential, serving merely to simplify the presentation of an example. Alternative terms include "overlap", "meet", etc.

[35]In his 2000 dissertation, MacFarlane distingued 3 senses in which logic is commonly said to be formal. Formality 1: Logic is formal "in the sense that it provides constitutive norms for thought as such" (p. ii). Formality 2: Logic is formal "in the sense that it is indifferent to the particular identities of objects" [*ibid.*]. Formality 3: Logic is formal "in the sense that it abstracts entirely from the *semantic content* of thought" [*ibid.*]. The sense of formality proposed here is closest to MacFarlane's formality 2. For a generalized theory of formality which treats the formality of logic as a particular case, see Beck [2011]. For a "conceptual archeology" of "logical formality" see Dutilh Novaes [2011].

[1781/7, A52/B76] or to Frege's statement that "pure logic ... **disregard[s]** the **particular characteristics of objects**" [1879, p. 5]. (It should be noted, however, that the exact meaning of these historical characterizations is open to different interpretations. (See MacFarlane [2000]).)

We have discussed the formality of logic in terms that are naturally applicable to predicate logic. What about sentential logic? Here we do not have individuals and properties but only sentences (or propositions), truth-values, and perhaps states of affairs (delineated by sentences). In what sense is sentential logic formal? The formality of sentential logic, on the present account, is an extension of that of predicate logic to a more abstract (and less powerful) level of thought. The logical operators of sentential logic, like those of predicate logic, are formal in distinguishing only the *pattern* delineated by their arguments, but here this is explained by saying that they do not distinguish between arguments that differ in all but in the truth value / being or not being the case of their atomic constituents (atomic sentences / states of affairs). I.e., if we *replace* any argument of a formal connective/operator by any other which is its image under some 1–1 *replacement* of true/false atomic sentences by any other true/false atomic sentences or of existent/non-existent atomic states-of-affairs by any other existent/non-existent atomic states of affairs, the formal connective/operator *will not notice*: it will assign the same truth-value to the two arguments. We can see the sense in which logical constants (operators) of both sentential and predicate logic are formal by considering *classical negation*. The complementation expressed by classical sentential-negation is the Boolean correlate of the set-theoretic complementation expressed by classical predicative-negation.[36] This is reflected in the equivalence between the sentential and predicative truth-conditions of classical negation, as in: "$\sim$ Pa" is true *iff* "Pa" is false *iff* the referent of "a" is in the complement of the extension of "Px" in a given universe.

STEP 3. We are now ready to present a mathematically precise criterion of formality (for both sentential and predicative operators).[37] The conception of formality delineated in Step 2 can be given a precise representation by two mathematical criteria: the well-known Boolean (truth-functional) criterion of logicality in the case of formal sentential-operators, and the *invariance*

[36] By "sentential-negation" I mean negation as attached to a well-formed formula of sentential logic or a sentence (closed formula) of predicate logic, and by "predicative-netation" I mean negation as attached to an open formula of predicate logic. (Note that "predicative" as used here has nothing to do with "predicative" as used in connection with the logical paradoxes and constrasted with "impredicative".) Below, I extend the terminology of "sentential" and "predicative" to operators as well: sentential/predicative operators are those that (would) correspond to sentential/predicative constants.

[37] In formulating mathematical criteria of formality and logicality I will not relate to infinitistic logics, which are outside the scope of the present paper. It is of interest to note, however, that these criteria can be expanded to infinitistic logics and that McGee [1996] discusses the ideas expressed in this section in terms of such a logic.

under isomorphisms criterion, whose three main originators are Mostowski [1957], Lindström [1966], and Tarski [1986], in the case of formal predicative-operators. The Boolean criterion is so well-known that there is no need to describe it here, but the invariance-under-isomorphisms criterion is not as familiar and as such requires an introduction. I will take the liberty of introducing the latter criterion in a form that naturally fits the present investigation.

In preparation, let us start with a semi-technical matter: Some operators take into account features of their arguments that involve not just their extension but also their *counter-extension*. For example, the quantifier-operator "all" takes into account the size of the counter-extension of its arguments (which must be 0 for "all" to be satisfied) and the quantifier-operator "all but John" takes into account which particular individuals are in the counter-extension of its arguments (they must be limited to John for "all but John" to be satisfied). This fact is directly relevant to the formality test for these operators: this test has to check their behavior under 1–1 replacements of individuals in an entire underlying universe, so their behavior under changes in the counter-extension of their arguments is also checked. I.e., an operator is formal iff it does not distinguish any 1–1 replacements of individuals in an underlying universe. To express this idea, we will coin the notion "argument-structure" and formulate the criterion of formality using this notion.

Before turning to a precise specification of "argument-structure", however, let me clarify my use of "structure" in this paper.

One of the characteristic features of the foundation for logic developed in this paper is its *objectual* nature, i.e., its conception of logic as grounded not just in the mind but also (and significantly so) in the world, and its view of formality as having to do with certain features of objects and the laws governing them. Accordingly, my notions of *structure*, *operator*, *argument*, and *argument-structure* are *objectual*. What this means is that a *structure*, in this paper, is not associated with a language (or a "signature", as this term is used in model theory). A structure is any sequence of objects whose first element is a universe—a non-empty set of individuals—and whose other elements are individuals or constructs of individuals (k-tuples of individuals, sets/properties of individuals, etc.) in the universe. When I want to associate a structure with a language (signature), I will use the term "model". Similarly, my notions of "operator" and "argument (of an operator)" are objectual: an operator/argument does not need to have a name in a given language, though it might have. The notion of "argument-structure", explained below, is also objectual.

Now, given an n-place operator, an argument-structure of this operator is an $n+1$-structure whose last n elements (viewed as an n-tuple) form a (potential) argument of that operator. For example, an argument-structure for an operator corresponding to a 1-place 2nd-level property of 1-place 1st-level properties, e.g., $\forall$ or $\exists$, is a pair $\langle A, B\rangle$, where A is a universe and B is a subset of A. Although the truth-conditions of $\exists$, unlike those of $\forall$, do not take the universe into account, for the sake of uniformity we will formulate both truth-conditions in terms involving a universe (the former vacuously): $\exists_{\mathrm{A}}(\mathrm{B}) = \mathrm{T}$ iff $\mathrm{B} \neq \varnothing$, $\forall_{\mathrm{A}}(\mathrm{B}) = \mathrm{T}$ iff $\mathrm{A} - \mathrm{B} = \varnothing$.

Using the same terminology, we say that two argument-structures are isomorphic iff each is the image of the other under some bijection from the universe of one to that of the other. I.e., $\langle A, \beta_1, \ldots, \beta_m\rangle \cong \langle A', \beta'_1, \ldots, \beta'_n\rangle$ iff $m = n$ and there is some bijection from A to A′ such that for each i, $1 \leqslant i \leqslant n$, β'_i is the image of β_i under this bijection. We say that an operator O is invariant under an isomorphism of the argument-structures $\langle A, \beta_1, \ldots, \beta_m\rangle$ and $\langle A', \beta'_1, \ldots, \beta'_n\rangle$ iff it assigns the same truth value to $\langle \beta_1, \ldots, \beta_m\rangle$ in A as to $\langle \beta'_1, \ldots, \beta'_n\rangle$ in A′ (or $\langle \beta_1, \ldots, \beta_m\rangle$ satisfies it in A iff $\langle \beta'_1, \ldots, \beta'_n\rangle$ satisfies it in A′). We can now formulate the formality criterion for (predicative) operators as follows:

(FORMALITY) *An operator is formal iff it is invariant under all isomorphisms of its argument-structures.*

All the standard logical operators—=, $\exists$, $\forall$, and the logical connectives in open-sentence contexts—satisfy this criterion. (For example, $\forall$ satisfies this criterion because for any of its argument-structures, $\langle A, B\rangle$ and $\langle A', B'\rangle$: if $\langle A, B\rangle \cong \langle A', B'\rangle$, B satisfies $\forall$ in A iff B′ satisfies $\forall$ in A′.) Paradigmatically non-logical operators—i.e., operators (properties) like *red, is of the same color as, is a property of Napoleon*, and *is a relation between red objects*—do not satisfy this criterion.

With this characterization we have provided a precise explanation of the sense in which logic is grounded in laws governing "formal features" or "parameters" of reality. In fact, however, we have done more than that. We have prepared a (partial) basis for a precise characterization of "logicality"—the feature distinguishing logical from non-logical *constants* (or, more generally, *expressions*).

Operators, as we recall, can have names, which are linguistic entities. Their names are "constants" of a given language, and the (named) operators themselves are "the objectual correlates" of such *constants*. Given our precise criterion of formality for operators we can formulate a precise criterion of logicality for constants as follows:

(LOGICALITY) *A constant is logical iff*

(i) *it denotes a formal operator, and*

(ii) *it satisfies additional conditions that ensure its proper functioning in a given logical system: i.e., it is a rigid designator, its meaning is exhausted by its extensional denotation, it is semantically-fixed (its denotation is determined outside rather than inside models and is built into the apparatus of models), it is defined over all models, etc.*

Since the issue we are focusing on here is exhausted by (i), however, I will direct the reader to Sher [1991, Chapter 3] for a discussion of (ii), and in what follows I will identify LOGICALITY with (i). I will call the combination of FORMALITY and LOGICALITY "the formality criterion of logicality". I will also refer to it as "the invariance-under-isomorphism criterion" or "ISOM". (To the extent that the current background theory is accepted,) LOGICALITY plays a substantial role in constructing, choosing, and evaluating logical systems. It provides an adequacy condition for logical systems, and it delineates a totality of potential logical constants to chose from.

The formality criterion of logicality for predicative constants, like that for sentential constants (i.e., the familiar Boolean or truth-functional criterion), licenses the acceptance of many logical constants that traditionally were not considered logical. But unlike the logical constants sanctioned by the sentential criterion, those sanctioned by the predicative criterion are not all reducible to the "standard" (standardly accepted) logical constants. Among the irreducibly new logical constants sanctioned by LOGICALITY are infinite cardinality quantifiers ("exactly α", for any infinite cardinal α), "finitely/infinitely/indenumerably many", the 1- and 2-place "most" quantifiers (as in "Most things are B's" and "Most B's are C's", respectively), the "well-ordering" quantifier, and more. Naturally, the question arises whether we should treat LOGICALITY as a *necessary and sufficient* condition for logical constants or only as a *necessary* condition. While most adherents of classical logic accept it as at least a necessary condition, its status as a sufficient condition is more controversial.

Now, objections to LOGICALITY as an adequate (necessary and sufficient) criterion for logical constants can be made on various grounds, some relevant to our project, others not. Among the former would be objections concerning one or another of our main theses, e.g., that logic is grounded in formal laws governing reality, or that formality is tantamount to invariance under isomorphism. The latter are "red herrings" from our perspective.

3. Red herrings and real problems. Some criticisms of LOGICALITY as a necessary and sufficient condition focus on the use of logical constants in natural language. They focus on the existence, in natural languages, of expressions that allegedly satisfy both parts of LOGICALITY yet intuitively seem non-logical. From the point of view of the present foundational project, whose focus is *theoretical*, whose concerns are *epistemic*, and whose interest is in the *construction* of a logical system fulfilling certain theoretical epistemic roles, these criticisms are largely irrelevant.[38]

[38] From other perspectives, of course, they might be relevant. For examples of criticisms in which linguistic intuitions play a significant role, see, e.g., Hanson [1997] and Gómez-Torrente [2002]. For responses see Sher [2001], [2003]. Some issues conncerning part (ii) of LOGICALITY play a role in these responses, but as noted above, they are not needed for the present discussion.

Other criticisms focus on issues that are more pertinent to the present investigation, and these we do have to consider here. A rich source of such criticisms is Feferman [1999], [2010]. Feferman raises three objections to what he referred to as the "Tarski–Sher thesis":

> A. The thesis assimilates logic to mathematics, more specifically to set theory.
> B. The set-theoretical notions involved in explaining [the thesis] are not robust.
> C. No natural explanation is given by it of what constitutes the *same* logical operation over arbitrary basic domains. [1999, p. 37][39]

Concerning (A), Feferman presents it as a matter of "gut feeling":

> [I]t will evidently depend on one's gut feelings about the nature of logic as to whether this [objection] is considered reasonable or not. [*Ibid.*]

He is especially disturbed by the fact that we can express substantive mathematical statements in purely logical vocabulary and that the mathematical version of the thesis saddles logic with substantial ontological commitments:

> [W]e can express the Continuum Hypothesis and many other substantial mathematical propositions as logically determinate statements on the Tarski–Sher thesis.[40] ... [I]nsofar as one or the other version of the thesis requires the existence of set-theoretical entities of a special kind, or at least of their determinate properties, it is evident that we have thereby transcended logic as the arena of universal notions independent of "what there is". [*Ibid.*, p. 38]

My response is:

(a) *Gut feelings*. From the present perspective, which treats the foundational problem of logic as a theoretical problem, gut feelings cannot play a major role in either accepting or rejecting the thesis. The relation between logic and mathematics does require an explanation, but this should be theoretical in nature. I will offer such an explanation in Section 4.

(b) *Continuum Hypothesis* (CH) *and ontological commitment*. First, the view that logic must be devoid of any commitments involving the world is a "purist" view that goes hand in hand with the foundationalist approach to logic but has no place in my foundational-holist approach. Second, as Feferman seems to be aware (last citation), his criticism does not apply to the general version of the Tarski–Sher thesis, at least on my construal—namely, Steps 1 and 2 above. This version of the thesis is not couched in any specific mathematical theory and therefore is not committed to the expressibility

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[39] To maintain a consistent numbering system in the paper I have replaced Feferman's "1", "2", "3", by "A", "B", "C".

[40] One way we can do so is by using the quantifiers "There are $2^{\aleph_0}$" and "There are $\aleph_1$".

of CH or to the existence of set theoretical entities. The expressibility of CH is merely an artifact of choosing a particular mathematical theory as a background theory of formal structure, and so is the commitment to the existence of sets.[41] Someone might also be bothered by the fact that we do not know whether the Continuum hypothesis is true or false, but this is not a problem from our perspective: Lack of knowledge—both temporary and lasting—is a fact of life in all fields of knowledge, and there is no reason why logic should be an exception.

Turning to (B), Feferman admits that the "notion of 'robustness' for set-theoretical concepts is vague", but his underlying idea is "that if logical notions are at all to be explicated set-theoretically, they should have the same meaning independent of the exact extent of the set-theoretical universe" [*ibid.*]. One mathematical condition that can be used to capture this idea is *absoluteness*, where given a set, T, of axioms (in the standard language of set theory), a "formula φ ... is defined to be *absolute with respect to* T if φ is invariant under end-extensions for models of T" [2010, p. 13]. Feferman's motivation for requiring all logical constants to be definable by "robust" notions has to do with the idea that logic should "not encapsulate any problematic set-theoretical content" and that the meaning of logical constants should "not depend on any special set-theoretical assumptions about what exists beyond the most elementary set-constructions" [*Ibid.*, p. 17]. Under the absoluteness condition, constants like "the quantifier 'there exist uncountably many x'", that satisfies LOGICALITY, "would not be logical" [1999, p. 38].

This criticism, too, applies at most to the specialized version of the Tarski–Sher thesis. With respect to this version, my response is that if features of the background vocabulary like absoluteness were shown to be centrally relevant to the foundational problem of logic, it would be reasonable to require a revision of the logicality criterion based on this; but to the best of my knowledge, they have never been shown to be directly pertinent to this problem. Furthermore, Feferman concedes that "the notion of absoluteness is itself relative and is sensitive to a background set theory, hence again to the question of what entities exist" [*ibid.*]. This raises the question why, if non-robust notions should be avoided in one place in a foundation for logic they should not be avoided in others. Why would a non-robust notion like absoluteness be allowed to play a central role in formulating the logicality criterion or even in constraining its formulation?

Speaking more generally about the "problematic" features of standard set theory itself and the "non-robustness" of many of its notions, I think it is important to distinguish between two types of problems: (i) problems concerning the basic ideas of set theory, and (ii) problems arising

[41] Indeed, as we will see in Section 4, it is an open question whether set theory itself, as a theory of the formal, is committed to the existence of formal individuals like sets.

from specific features of the logical framework used to formulate set theory. To a significant extent, the non-robustness of many set-theoretical notions is due to the expressive limitations of standard 1st-order logic which lead to phenomena like the Löwenheim–Skolem–Tarski (LST) phenomenon, non-standard models, and de-stabilization of set-theoretical notions. "Uncountably many", for example, is satisfied by countable sets in non-standard models, and this renders it highly volatile. This problem, however, can in principle be solved by strengthening the expressive power of the logical framework by introducing new logical constants, and therefore they are of a lesser significance for the present investigation. (As a logical constant, one whose meaning is fixed across models, "uncountably many" is more robust than its non-logical correlate, since it does not have countable models.) As in the Neurath-boat metaphor of patching a hole in two stages (first temporarily, then more permanently, using the temporary patch to create tools for producing a more enduring replacement), the process of constructing a criterion of logicality can in principle involve two or more stages. Starting with a set-theoretical background-theory formulated within a logical framework that was selected prior to the introduction of a criterion of logicality and is subject to destablilizing phenomena like LST, we produce LOGICALITY under less than ideal conditions. Then, using LOGICALITY to construct a stronger logical framework for set theory, we strengthen this theory and with it our logicality criterion, redefining the logical constants it licenses in more robust terms.

Moving on to (C), Feferman's criticism is:

> It seems to me there is a sense in which the usual operations of the first-order predicate calculus have the *same meaning* independent of the domain of individuals over which they are applied. This characteristic is *not* captured by invariance under bijections. As McGee puts it "the Tarski–Sher thesis does not require that there be any connections among the ways a logical operation acts on domains of different sizes. Thus, it would permit a logical connective which acts like disjunction when the size of the domain is an even successor cardinal, like conjunction when the size of the domain is an odd successor cardinal, and like a biconditional at limits" [1996, p. 577]. ... I ... believe that if there is to be an explication of the notion of a logical operation in semantical terms, it has to be one which shows how the way an operation behaves when applied over one domain M_0 connects naturally with how it behaves over any other domain M_0'. [1999, pp. 38–39]

My response to this criticism is that in systematizing a theory we are often forced to accept entities that when viewed from *outside the theory* seem strange, lack internal unity, have no rhyme or reason. Such entities make good sense within the theory, however, where their naturalness, internal unity, and raison d'être rest on the principles of that theory. This is something all mathematicians are aware of, and Feferman himself [2000] points to many *legitimate* mathematical objects that are, in his words, "monstrous" or "pathological". Indeed, even in the case of logic, Feferman accepts operators that, intuitively, lack unity of meaning. Consider, for example, a 135-place truth-functional connective that behaves like a conjunction on rows with exactly 2, 101, 103, 104, or 120–130 T's (i.e., rows in which 2, 101, . . . , sentential variables are assigned the value "True"), like disjunction on rows with 3, 4, 5,6, and 70–100 T's, and like some highly irregular connective on all other rows. Does this connective have "the same meaning" in all rows of its truth-table (correlates of models)? Is there a natural connection between the way it behaves in rows with 100 T's, rows with 101 T's and rows with 102 T's? But this logical operator is sanctioned by Feferman, and for a good reason. What makes it "the same operator in all rows" is the criterion of logicality for sentential operators, i.e., the fact that it is a *truth-functional* or a *Boolean* operator.

I should also note that the *sentential connective* in the citation from Feferman above is *not* a logical connective of *sentential logic* on my version of the Tarski–Sher thesis. This is because my logicality criterion for sentential logic is the usual Boolean or truth-functional criterion, and this criterion does not sanction operators that take into account things other than the truth-value of their arguments; in particular, it does not sanction operators that take into account things like universes of discourse and their features.

As for predicate logic, here sentential connectives can be licensed in two ways: through the logicality criterion for sentential operators and through the logicality criterion for predicative operators. When introduced through the former, they cannot behave like Feferman's connective, for the reason noted above. (When used in open formulas, through the Tarskian definition of truth for predicate logic, this definition determines that they can only coincide with predicative operators that do not take the cardinality of the universe of their argument-structures into account. For example, in contexts like "Bx & Cx" or "Bx & Cy", " & " coincides with the objectual operator intersection-in-all-universes or Cartesian-product-in-all-universes, respectively.) When introduced through the latter criterion, they are introduced as objectual logical operators that turn into sentential logical operators in closed-sentence contexts. Such operators are logical due to their formality, and they receive their internal unity from their characteristic trait of distinguishing *only* formal features of their argument-structures, including their universes. Not all features of the universe of an argument-structure are formal, but its cardinality is, and therefore logical operators are in principle sensitive to this feature. The predicative correlate of Feferman's logical connective might, at first blush, seem "monstrous", but in fact it is theoretically sound.

While Feferman's specific objections are unwarranted, the question still arises whether, assuming part (ii) of LOGICALITY is satisfied, formality is a necessary-and-sufficient condition for logical constants or only a necessary condition. My considered view is this: If the question is "Which logical constants should we include in the logical system we are working with?", then this question has multiple dimensions, and at different times we should make different decisions, depending on what dimensions are most important to us and what our goals are at those times. The logicality criterion by itself does not suffice to determine our choice and other, e.g., pragmatic considerations, might play a role, leading us to limit the logical constants licensed by LOGICALITY to, say, the standard logical constants. But if the question is "Which choice of logical constants will give rise to a logical system whose consequences transmit truth from premises to conclusion with an especially strong modal force in all fields of knowledge?", then, I believe, the answer is that any choice of logical constants satisfying LOGICALITY will do. From this perspective our criterion demarcates a maximalist conception of logicality under a unified theme, formality, a conception of a family of logical systems, each of which satisfies logic's designated role in a partial manner. And with respect to this conception, our criterion sets a necessary-and-sufficient condition on logicality. One important credential of this criterion is its playing a *significant* role in a *substantive*, *unified* foundation of logic (rather than being an ad hoc criterion or a criterion integrated into a piecemeal or trivial account of logic).

It should be noted that using the formality criterion to expand logic, and in particular 1st-order logic, has advantages that go beyond the purview of this paper. For example, ISOM (the invariance-under-isomorphism criterion, or the formality criterion of logicality—see Section 2 above) has proven to yield extremely fruitful and interesting results, involving "generalized" quantifiers, in mathematics and linguistics. Generalized quantifiers are quantifiers like "most" and "indenumerably many" that like the standard 1st-order quantifiers, $\forall$ and $\exists$, represent formal operators (properties, relations, functions) of level 2 and are used in conjunction with 1st-order (individual) variables. Working within the framework of generalized logic has led to important results in model-theory and abstract logic, including Lindström's [1974] seminal characterization of standard 1st-order logic, proofs that standard 1st-order logic is not the strongest complete logic (Keisler [1970] and others), work on generalized quantifiers in finite and infinite models (see Barwise and Feferman [1985] and Väänänen [1997], [2004]). It has also led to important results in linguistic semantics (for recent work and references see Peters and Westerståhl [2006]).

Our next step in working out the grounding of logic concerns the reality of the formal, the discipline by which it is studied, and the joint grounding of logic and mathematics in it.

4. A structuralist foundation for logic & its connection to mathematics. If our theory is right, logic is grounded in formal laws governing the behavior of objects (properties, relations, functions, states of affairs or situations), actual and formally possible. An adequate logical theory based on these principles requires the resources of a background theory of formal structure. This theory will determine the totality of formally-possible structures of objects (basis for structures/models), the totality of formal features of objects (basis for logical constants), and the universal laws governing formal features and formally-possible structures of objects (basis for logical laws and ground of logical consequences). Traditional philosophy, with its foundationalist mindset and its resulting ban on circularity of any kind, does not allow logic to be connected to other theories in this way, but foundational holism does. The question arises: Which theory in our system of knowledge does, or at least can, play the role of a background theory of formal structure for logic?

Before we can answer this question, however, we need to consider a more basic question. In order for any theory to study the laws governing formal features of objects (actual and potential), objects (including actual objects) must have formal features. Our first question, therefore, concerns the *reality of the formal*. Some philosophers, especially extreme nominalists, contest the reality of the formal, and this issue has been widely debated in the philosophy-of-mathematics literature. I will not be able to go into the diverse views expressed on this topic here; instead, I will offer one basic, rather common-sensical, argument for the reality of the formal. One way in which my task is easier than that of other philosophers who might attempt to defend the reality of the formal (in the objectual sense) is that I am not required to defend the existence of *formal individuals*. This follows from a corollary of our characterization of formality (in both its general and mathematical formulations):

There are no formal individuals. (FI)

To see why this is so, let us try to apply FORMALITY to individuals. First, we see that FORMALITY does not apply to individuals directly. Since individuals have no arguments, they cannot be differentiated according to what features of their arguments they take (do not take) into account. So, the formality criterion does not sanction the formality of any individuals. Second, when we use this criterion to check the formality of individuals indirectly, by checking whether the *identity* of individuals is invariant under isomorphisms,[42] the criterion gives a negative result: Given any individual c and a structure $\langle A, c\rangle$, there is a structure $\langle A', c'\rangle$ such that $\langle A', c'\rangle \cong \langle A, c\rangle$ yet $c' \neq c$.[43]

[42] In the sense that if a structure with a given individual is isomorphic to another structure, then the *same* individual appears in both.

[43] (i) Another way to view this indirect test is as testing whether the 1-place 1st-level property "is identical to c", where c is a fixed individual, is formal. Clearly, it is not.

(ii) Note that (FI) follows from our non-technical characterization of formality as well.

It follows that to ground logic in the formal (in our sense), one has to establish the reality of *formal features* but not the existence of *formal individuals*. This, as we will see below, positions us in an interesting situation vis-à-vis nominalists: someone who is a nominalist with respect to individuals can accept our foundational account of logic as grounded in the formal features of reality or in the laws governing these features.

So far we have assumed that reality has formal features. But does it have such features? To defend the reality of formal features, assume they are *not* real. What does this assumption mean? Given our understanding of formality (Section 2 above), it means that objects in the world are neither identical to themselves nor different from any other objects, that collections of objects have no size, that properties of objects do not form unions and intersections, that relations of objects exhibit no formal patterns (e.g., no relations are reflexive, symmetric, or transitive), and so on. These claims, however, are quite unreasonable. Take the students in my recent graduate seminar. It is hard to deny that the students and I are real, that the students are each *identical* to him-/her-self and *different* from me, that the students and I form a collection of individuals of a definite *cardinality*, that the properties of being a student and being a professor have a *union* and an *intersection*, that the students stand in the *reflexive*, *symmetric*, and *non-transitive* relation "x takes a common seminar with y", and so on.[44] So if my students and I are real, and if we have the 1st-level properties and relations mentioned above, and if these properties and relations have the 2nd-level properties mentioned above, then objects in the world do have formal features, and such features are real.

But if formal features are real, then they, like other features of objects, potentially exhibit regularities and are governed by laws. And there are good reasons for believing that features like identity, cardinality, intersection, union, reflexivity, transitivity, symmetry, and so on, are not irregular or lawless. So, even if you start as a nominalist with respect to individuals, it will be hard for you to deny that the individuals you sanction have formal features (self-identity), that their properties and relations have formal qualities (cardinality) and stand in formal configurations (union), that these formal qualities and configurations exhibit certain regularities and are governed by certain laws (laws of identity, cardinality, union), and so on. And it is such formal laws that the theory of the formal studies.

Which theory, then, studies the formal? The most natural answer is: Mathematics. Some, of course, might reject this answer. They might say that mathematics is purely conventional, or that it is too general and too

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(iii) As far as the mathematical criterion of formality is concerned, the result does not depend on its specific articulation in this paper. (See Lindström [1966] and Tarski [1986].)

[44] Here, again, it is possible to express this point without commitment to controversial entities.

abstract to engage with the world, or that only *applied* mathematics has anything to do with the world (or something else of a similar kind).

First I should say that the issue is not whether all of mathematics is engaged in studying the formal, or even whether some mathematical theory is *exclusively* engaged in this study. Mathematics is a broad and diverse discipline, with multiple goals and interests. The question is whether one of the significant things that mathematics does is to provide a theory of the formal (in our sense). A negative answer to this question would make little sense. It would be quite strange if things in the world and their properties had formal features, say if properties of objects had cardinality features and these were governed by certain laws, yet mathematicians exclusively studied other, "unreal" cardinalities, governed by laws that are completely different from those of real cardinalities.

Next, let us turn to the last two objections mentioned above. (Since mathematical conventionalism, like logical conventionalism, is widely thought to be inadequate, there is no need to address it here.) Once we acknowledge the existence of a formal layer of reality and are aware of the strong modal force and great generality of its laws, we see that mathematics is not too general or too abstract to engage with its laws, and that "pure" and not only applied mathematics must be concerned with them. This is because accounting for universal and highly necessary laws in a *precise manner* and in *full generality* requires a highly general and abstract theory—something on the order of "pure" mathematics in the case of formal laws. For example, to state in *complete generality* the laws of finite cardinalities we need something on the order of an infinite set. And once an infinite set is introduced, to state in *complete generality* the law governing the cardinality relation between sets and power-sets (or the relation between properties and power-properties it represents), we need something as general and as abstract as the full-scale *Cantor's theorem*.

Now, given the existence of mathematical theories that study such laws, it is reasonable to think that, whether originally built, or intentionally pursued, for this purpose, they are in a position to serve as background theories of formal structure for logic.

Having identified the basic relation between logic and mathematics, i.e., that logic is grounded in the formal and the formal is studied by mathematics, we can turn to Feferman's claim that our criterion of logicality is tantamount to "assimilating logic to mathematics". An examination of both our general characterization and its precise version, LOGICALITY, shows that if by "assimilating" Feferman means "identifying", this claim is inaccurate: on our account *logic and mathematics stand in a systematic and fruitful* relationship to each other but are *not identical to each other*. They differ in at least two significant things: (i) subject-matter, and (ii) formality of their objects. The first difference is straightforward: Although logic is concerned with the world, it approaches it through language. Its direct subject-matter is linguistic (– sentences, inferences, theories), while the direct subject-matter of mathematical is objectual (– objects and structures of objects). The second difference is more subtle: One significant result of the invariance criterion of logicality (noted, e.g., in Tarski [1986]) is that classical mathematical notions are logical when construed as higher-level notions yet non-logical when construed as lower-level notions. In particular, mathematical individuals and many of their 1st-level mathematical properties do not satisfy the formality part of this criterion, but their higher-level correlates do. Thus, as individual-cardinals, 2 and $\aleph_0$ are not formal (logical), but as quantifier-cardinals (2nd-level entities) they are; as a relation between individuals, the membership relation (ϵ) is not formal, but as a relation between lower-level and higher-level entities (as e.g., in "*a* IS A MEMBER OF *B*", where *a* is a 0-level object and *B* is a 1st-level object") it is. Tarski concluded that it is an *arbitrary* matter whether we regard mathematics as logic, but in my view he was mistaken. There is a systematic division of labor between logic and mathematics, and the difference between mathematical individuals and their higher-level correlates with respect to formality is part of it: Mathematics studies the formal (largely) through mathematical individuals and their properties (which are strictly speaking not formal), while logic uses formal operators (which, for the most part, are higher level correlates of lower level mathematical objects) to develop a method for valid reasoning and inference.

Now, one may ask why in studying the formal, mathematics commonly uses *lower-level* (or 1st-order) theories. Why, if the formal (largely) resides on the level of properties of properties, does mathematics study it on the level of individuals and their properties? For example, why does mathematics study cardinalities, which are 2nd-level properties, by means of 1st-order theories, like Peano Arithmetic and ZFC, which construe them as individuals? And can such theories provide accurate knowledge of the formal?

The answer to the last question is positive: although 1st-order mathematical theories cannot not provide accurate knowledge of the formal directly, they can do so indirectly. As for the "why" question, the key to answering it lies in the observation that theories, qua theories, are creations of the mind, and the more intricate the mind is, the more likely (and capable) it is to devise indirect yet fruitful and correct accounts of reality. Thinking in functional terms, it is easy to see what kind of advantage such an indirect study might have. Suppose we, humans, work better with systems of individuals than with systems of higher-level objects, i.e., for one reason or another, we are more competent in discovering regularities and systematizing them when we work with lower-level concepts. Then, it would be advantageous for us to study the formal on that level. This we could do by, say, constructing a 1st-level model of reality (in the everyday sense of "model"), or of those parts/aspects of reality we wish to study. For example, we would create a 1st-order arithmetic theory that would study the (higher-level) laws of cardinality through their 1st-level correlates and in this way would provide an indirect account of these laws. 1st-order arithmetic (if correct) would then be connected to reality in a systematic manner, but its connection to reality would be *indirect*. True 1st-order mathematical theories of higher-level phenomena would be true of reality *less directly* than true 1st-order theories of 1st-level phenomena, but they would be true of it *just as much*.

Figuratively, we can represent the difference between direct and indirect connection to reality in terms of direct and indirect, or simple and composite, reference (using numerical superscripts to distinguish types of linguistic/ontological elements, and different kinds of arrows to distinguishes different reference relations and constituents of such relations):[45]

Simple Reference

Language:	Individual Constants^0	Predicates^1
	$\downarrow$	$\downarrow$
World:	Individuals^0	Properties^1

Composite Reference

Language:	Individual Constants^0	Predicates^1
	$\downarrow$	$\downarrow$
Posit:	Individuals^0	Properties^1
	$\Downarrow$	$\Downarrow$
World:	Properties^2	Properties^3

One can discern certain similarities between our conception of mathematical truth and that of some fictionalists (e.g., Field's [1989]). In particular, both view mathematical individuals as fictions (of some kind). But there are very significant differences as well: On the fictionalist account reality has no genuinely formal features, while on the present account it does; for the fictionalist, 1st-order arithmetic theorems are false, whereas on the present account they are true; according to the fictionalist, applied arithmetic theorems are conservative extensions of physical truths, but on the present account they are applications of formal truths. We may say that if you know how to connect them to reality, the laws of 1st-order mathematics do not lie . . .

[45] Hodes [1984] is a precursor of this account. I would like to thank Stewart Shapiro for pointing this out to me.

But if mathematics studies the formal indirectly, through 1st-order theories, the question arises: Where, exactly, does the formal enter into (1st-order) mathematical theories? My answer is: Through structures. 1st-order mathematical theories study the formal by studying mathematical structures. Numerical individuals are not formal, but numerical structures, i.e., structures of numerical individuals, are. The same holds for set-theoretical structures: a set as an individual is not formal, but a set-theoretic structure is. The mark of formality for mathematical structures is the same as that for logical operators: *invariance under isomorphisms*. A mathematical structure preserves its *mathematical identity* under isomorphisms. Both in the case of mathematical structures and in the case of formal operators we can say that identity is identity-up-to-isomorphism. Two isomorphic systems of the natural numbers are identical (as natural-numbers systems), regardless of whether their universes consist of the same individuals. In this sense, mathematical systems, like logical operators, do not distinguish the identity of individuals. Mathematics studies the formal through an ontology of structures whose individuals represent formal features of objects through their role in structures, and the laws governing these structures are the mathematical representations of the laws governing formal features of objects.

It is readily seen that the formality of mathematics on our account bears certain significant similarities to its *structurality* in the sense of structuralist philosophies of mathematics. This similarity is reflected in, e.g., the centrality of *isomorphism* for both. Its centrality for mathematical structuralism comes into view in the following citation from Shapiro:

> No matter how it is to be articulated, structuralism depends on a notion of two systems that exemplify the "same" structure. That is its point . . . . [W]e need to articulate a relation among systems that amounts to "have the same structure".
>
> There are several relations that will do for this . . . . The first is isomorphism, a common (and respectable) mathematical notion. Two systems are isomorphic if there is a one-one correspondence from the objects and relations of one to the objects and relations of the other that preserve the relations. . . . Informally, it is sometimes said that isomorphism "preserves structure". [1997, pp. 90–91]

And even those who do not identify isomorphism with structure-identity regard it as very central to structuralism. Resnik [1997], for example, regards mathematical structures as representing *patterns* of objects having properties and standing in relations, and isomorphism he regards as representing congruence of structures, where "congruence is the strongest" of all "equivalence relationships which occur between patterns". [*ibid.*, p. 209][46] The close connection between the formal and the structural is yet another aspect of the

[46]Inverted order of parts of a sentence.

link between logic and mathematics on our account. Mathematical individuals represent (2nd-level) formal properties through their role in structures, and the laws governing them in these structures represent the formal laws grounding logic.

The great foundational systems of the late 19th- and early 20th-century drew a close connection between logic and mathematics. Searching for a certain (solid) foundation for mathematics, logicism sought to ground mathematics in logic, intuitionism sought to ground both mathematics and logic in mental construction, and (proof-theoretic) formalism sought to ground both in syntax. Two ways in which the present proposal diverges from these traditional approaches are (i) in shifting its focus to a foundation for logic, and (ii) in replacing the traditional foundationalist methodology by a new, holistic (yet still foundational) methodology. But rather than breaking away from the old logic-mathematics connection, it draws us back to it, albeit with a new understanding.

Methodologically, a joint account of logic and mathematics has a clear advantage over disjoint accounts: it reduces two philosophical mysteries—the nature of logic and the nature of mathematics—to one, and it saddles us with one foundational task instead of two. A joint account can take several forms; three of these are: (i) *logicism*—a reduction of mathematics to logic, (ii) *mathematism*—a reduction of logic to mathematics, and (iii) *third element*—a grounding of both logic and mathematics in a third element, which in our case is the structural or the formal. (We can call this "logico-mathematical structuralism".) Briefly, we can compare these three options as follows:

Logicism: Although logicism has the advantages of a distinguished ancestry, familiarity, rich body of literature, and important attempts at renovation, it is highly problematic from the point of view of a foundation for *logic*. Logicism uses logic to explain and ground mathematics, but it leaves logic itself unexplained and ungrounded. Some philosophers tried to pair the logicist foundation of mathematics with a conventionalist foundation for logic, but logical conventionalism, as we have noted above, is highly problematic. So far, to the best of my knowledge, there is no adequate foundation for logic within logicism.

Mathematism: Mathematism has the same methodological advantage as logicism (though without the distinguished ancestry and the rich body of literature), and it may have better prospects for a joint foundation due to the existence of several substantive accounts of mathematics that do not put the main burden of explanation on logic (e.g., mathematical Platonism, mathematical naturalism, and mathematical structuralism). Still, I do not know of any well-developed (let alone successful) attempt at a joint mathematist foundation. Of the non-logicists accounts of mathematics, structuralism appears to me the most promising, but I prefer to construe it as falling under the "third element" category.

Logico-mathematical Structuralism: The third option is to form a common foundation for logic and mathematics based on a third thing that grounds both in interrelated yet different ways. The present proposal falls under this category, grounding mathematics and logic in the same thing, the *formal* or the *structural* (in the objectual sense). The structurality of logic and mathematics, on this account, consists in their discerning only formal *patterns*: formal patterns of objects in mathematics, and formal patterns of linguistic expressions in logic. The latter ground logical truth and inference, and are themselves grounded (in a composite manner) in laws governing the former.

In addition to the link between formality and structurality, there are other points of similarity between the present "formalist" account of logic and structuralist accounts of mathematics. Three of these are: (i) tendency toward realism (e.g., Resnik [1997] and Shapiro [1997]), (ii) rejection of foundationalism and endorsement of holism (e.g., Resnik and Shapiro), (iii) attribution of strong modal force to mathematical/logical laws (e.g., Hellman [1989]). However, the idea of extending mathematical structuralism to logic, or of having a common structuralist foundation for both logic and mathematics, has not been thoroughly examined by mathematical structuralists. For this reason, and because of the considerable variability of views among mathematical structuralists,[47] let us put the connection between "logical formalism" (our approach) and mathematical structuralism aside at this point and proceed with the interrelation between logic and mathematics independently of it.

The interplay between logic and mathematics, on the present account, is a continuous process, integral to both disciplines: mathematics provides logic with a background theory of formal structure, logic provides mathematics with an inferential framework for the development of theories (of formal structure and possibly other things). Functionally, we can describe this process as proceeding in stages: Starting with a rudimentary logic-mathematics that studies some very basic formal operations like complementation, union, intersection, inclusion, and so on, we create the resources for the development of a simple logical system (something on the order of sentential or syllogistic logic). This logic helps us to develop a more sophisticated mathematics. Then, being motivated by methodological problems arising in this mathematics (e.g., problems of axiomatization) and using some of its resources (e.g., set-theoretical resources), we develop a more powerful and systematic logical system (on the order of axiomatized 1st-order logic with standard logical constants). Next, using this system as a framework for mathematics, we develop rigorous axiomatizations of mathematical theories (like arithmetic and Euclidean geometry) as well as rigorous general theories of formal structure (like axiomatic set theory). Using this sophisticated

[47]Which importantly include, in addition to those mentioned above, Parsons [2008], Chihara [1990], Reck. [2003], and others.

theory, we can proceed to develop a systematic definition of logical consequence (like the Tarskian or model-theoretic definition) and a systematic criterion of logicality (like the invariance-under-isomorphism criterion), and based on these we can create an expanded 1st-order logical framework, say, something like "generalized" 1st-order logic (Mostowski [1957], Lindström [1966], Keisler [1970], and others). This expanded logic might enable us in the future to develop a more sophisticated mathematics, and so on.

We are nearing the end of our foundational outline. We have explained in some detail our thesis that logic is grounded in reality and we have delineated an account of the relation between logic and mathematics concordant with it. Let us finish with a brief discussion of three pertinent issues: the *normativity* of logic, the *traits* of logic, and *error* and *revision* in logic.

Normativity. The source of the *normativity* of logic, on the present account, is its *truth*. There is a sense in which truth is not just a property of statements/theories but also a cognitive *value*, parallel to moral value in the practical domain. Following Williams [2002], we will call this value "truthfulness"; here, we are interested primarily in *cognitive* truthfulness. Cognitive truthfulness is a central value in the intersection of ethics and epistemology, and every discipline that upholds this value is a normative discipline. Since logic, like most other disciplines, aims at truth (something that was reflected, in the present foundational study, in our emphasis on its veridicality), it is a normative discipline.

The view that truth is a central source of epistemic normativity and that all veridical disciplines are, therefore, epistemically normative can be traced to Frege:

> Any law asserting what is, can be conceived as prescribing that one ought to think in conformity with it, and is thus in that sense a law of thought. This holds for laws of geometry and physics no less than for laws of logic. (Frege [1893, p. 12]).

MacFarlane elucidates this view as follows:

> On Frege's view, ... it is a feature of *all* descriptive laws [that they are normative]. ... [C]onsider the "game" of thinking about the physical world (not just grasping thoughts, but evaluating them and deciding which to endorse). ... "[M]oves" in this game—judgments—can be assessed as correct or incorrect. Judgments about the physical world are correct to the extent that their contents match the physical facts. Thus, although the laws of physics are descriptive laws—they tell us about (some of) these physical facts—they have prescriptive consequences for anyone engaged in the "game" of thinking about the physical world: such a thinker *ought not* make judgments that are incompatible with them. Indeed, insofar as one's activity is to count as making judgments

> about the physical world at all, it must be assessable for correctness in light of the laws of physics. In this sense, the laws of physics provide *constitutive* norms for the activity of thinking about the physical world. (MacFarlane [2002, pp. 36–37]).

In the case of logic, its laws concern a special type of consequence (truth, consistency), one that occurs in all fields of knowledge, and therefore its normative force encompasses the activities of inferring, asserting, and theorizing in all fields. In Frege's words:

> From the laws of [logic][48] there follow prescriptions about asserting, thinking, judging, inferring [in general]. (Frege [1918, p. 1])

A distinctive aspect of Frege's (and our) account of cognitive normativity is the connection between the descriptive and the prescriptive. A clear explanation is given, once again, by MacFarlane:

> Frege ... says that logic, like ethics, can be called a "normative science" [1979, p. 128]. For although logical laws are [descriptive and] not prescriptive in their content, they *imply* prescriptions .... For example, [they imply that] one ought not believe both a proposition and its negation. Logical laws, then, have a dual aspect: they are descriptive in their content but imply norms for thinking. (MacFarlane [2002, p. 36])

Now, although the source of logic's normativity, on our account as on Frege's, is the same as that of other disciplines, this does not mean that logic's normativity is the same as theirs in other respects as well. We have already noted that the normativity of logic has a broader scope than that of physics. We have also seen that its normativity is grounded in a different type of truth than that of physics, i.e., in *formal* truth. Finally, we can see that logic's normativity is in a certain sense stronger, deeper, and more transparent than that of other disciplines. The transparency of logic's normativity is due to logic's subject-matter: logic deals with assertions, theories, and inferences directly, rather than indirectly, through their objects, and in so doing it carries its normativity on its sleeve, so to speak. As for the strength of logic's normativity, this belongs to a cluster of traits that have traditionally been associated with logic and are best discussed as a group.

Traits. Logic has been traditionally characterized as formal, highly general, topic neutral, basic, modally strong, highly normative, a-priori, highly certain, and analytic. As a foundational holist and a believer in the need to ground logic not just in the mind (or language) but also in reality, I reject the characterization of logic as analytic. But aside from this and a modification of its characterization as apriori, the present account affirms all the traditional traits of logic.

[48] Frege says "From the laws of truth", but for him the laws of logic *are* the laws of truth.

Formality, on the present account, is logic's key trait, and all its other traditional traits are closely connected to it. The connection between formality (in our sense) and *generality* was pointed out by Tarski [1986]. The higher the degree of invariance of a given notion, the more general it is; and the larger the class of transformations under which a given notion is invariant, the higher its degree of invariance. It follows from these principles that since logical notions have a higher degree of invariance than physical, biological, psychological, and many other notions, they are more general.[49] Turning to *topic neutrality*: The formality of logic, i.e., the fact that it has a higher degree of invariance than that of other disciplines, ensures that it abstracts from (is indifferent to, does not notice) their distinctive subject-matters. As a result, logic is applicable to other disciplines regardless of their "topic": i.e., logic is topic neutral. If it works in one science, it works in all. The formality, i.e., strong invariance, of logic also means that logic does not distinguish between objects and situations that are physically (nomically) possible and those that are physically impossible yet formally possible, while physics does. As a result, logical laws hold in a broader space of possibilities than physical laws: logical laws hold in physically impossible yet formally possible states of affairs while physical laws do not. This means that logic has an *especially strong modal force*.

Furthermore, since the scope of formal—hence logical—laws properly includes that of nomic laws, the natural and social sciences have to take into account, and indeed obey, the laws of logic, but not the other way around. In this sense logic is more basic than these disciplines. Now, this basicness is mirrored in logic's *strong normative force*. It follows from the fact that the natural and social sciences are subject to the authority of logic but not vice versa that logic has a stronger normative force. Turning, next to the *certainty* of logic, let us note first that logic is highly certain in a particular sense. It is not that people are unlikely to make errors in applying logical laws, but that logical laws are less likely to be refuted by scientific discoveries than other laws. This does not mean that logic is immune to discoveries altogether (recall Russell's discovery of a paradox in Frege's logic), or that it is completely immune to discoveries in other disciplines (interconnectedness of fields of knowledge). But it means that logic is more shielded from new results than other sciences. Once again, this is connected to its formality or strong invariance. Formal, hence logical, operators are indifferent to most aspects of reality; as a result, investigations concerning those aspects are relatively unlikely to give rise to a new theory of formal structure, one that undermines our current logic.

Concerning the traditional characterization of logic as *apriori*, this is often explained by its strong modal force and/or its analyticity. I share the

[49]For the sake of brevity, I compare logic's traits only to those of the physical sciences.

view that logic has a strong modal force, but not the view that this requires absolute apriority. The category of absolute apriori makes sense within a foundationalist framework, but not within a holistic framework. Traditional apriorism requires *absolute* independence from empirical considerations, but foundational holism allows only *relative* independence (although it does allow *considerable* independence). On the foundational holistic account developed in this paper, logic is *largely* immune to empirical considerations, but not *completely* immune. This, too, is connected to its formality, in much the same way as its considerable certainty (noted above). The features of objects that logic takes into account are too abstract to be directly investigated by empirical methods; therefore, there is priority to reason over sense-perception in acquiring logical (formal) knowledge. But being *largely* based on reason is not being *exclusively* based on reason. Logic is therefore *quasi*- rather than *absolutely* apriori.

Error and revision in logic. We have discussed the traditional traits of logic and explained its strong normativity. Our view of logic as *quasi*-apriori and *less than absolutely* certain, however, requires us to address another issue: *error* and *revision in logic*. As a preliminary let me note that viewing logic as a veridical discipline is not the same thing as viewing it as immune to error. Logical theories, like any other theories, can be mistaken (contain errors), but this does not disqualify them from being veridical. A theory is veridical if it (i) aims at truth (where truth requires a systematic connection with reality), (ii) uses truth as a central standard for its judgments, and (iii) provides substantial tools for checking the truth of its judgments. And logic, on our account, satisfies all these requirements. Some might think that logic is immune to error as a result of its strong modal force. They might reason that since logical truths are necessary—i.e., necessarily true—they *cannot* be false. But the view that theories whose claims have a strong modal force are infallible is simply wrong. The formal necessity of the logical laws does not imply that logic is infallible any more than the nomic necessity of the physical laws implies that physics is infallible. Newtonian and Einsteinian laws do not differ in their modal status, but they do (according to contemporary physics) differ in their truth. We can be wrong about what the laws of nature are without being wrong about the modal status of such laws. And the same holds for logic.

What are some of the possible sources of error in logic? One potential source is its background theory of formal structure. If the laws governing formal configurations of properties and situations are different from what our current background formal theory says they are, there might be errors in our logic. Such errors might justify revision. Another potential source of error is the choice of logical constants. If we select "is taller than" or "is a property of humans" as logical constants, we will mistake material consequences for logical ones, and if we de-select the existential and universal quantifiers as logical constants, we will mistake logical consequences for material ones. A third source of error might lie in the construction of our system. If we construct models as ranging over physically-possible structures of objects (instead of formally-possible structures of objects), we will mistake physical laws for logical laws. Errors of all these kinds would provide a sound reason for revision in logic.

Other revisions in logic could be pragmatic: suppose no veridical considerations favor one logical theory over another, but pragmatic or methodological considerations do; then, as holists, we can judiciously use such considerations to motivate revision. What about experience? What role, if any, can experience play in discovering error and initiating revision in logic? Although abstract theoretical considerations play a greater role in revision in logic than empirical considerations, we cannot rule out the possibility that empirical discoveries of "a very fundamental nature" might significantly impact logic. Something along those lines was suggested by Tarski in a letter to Morton White:

> Axioms of logic are of so general a nature that they are rarely affected by ... experiences in special domains. However, ... I can imagine that certain new experiences of a very fundamental nature may make us inclined to change just some axioms of logic. And certain new developments in quantum mechanics seem clearly to indicate this possibility. (Tarski [1987, pp. 31–32])

One has to be careful in attributing to an author a view informally delineated in a personal letter; so I will speak only for myself. Due to the strong invariance of the formal, formal laws are not directly discoverable by empirical methods. But the possibility that some combination of experiential and theoretical considerations might suggest the superiority of some theory of formal structure over another, and through this, of one logic over another, cannot be ruled out. Due to the special nature of logic, theoretical considerations will always carry more weight than experiential considerations, but we allow the possibility that problems/discoveries in physics might *point beyond themselves*, so to speak, to something more abstract. In particular, we allow the possibility that a problem in physics points to a problem in one of physics' background theories, including its formal or logical background theory. Finally, we should not forget that failed instantiations, including failed empirical instantiations, can pose a challenge (albeit a defeasible challenge) to abstract laws.

It is important to emphasize, however, that allowing experience a limited role in the revision of logic does not, by itself, render logic contingent (in the way the natural and social sciences are). Nor does it interfere with its strong invariance. We can replace our logical theory by another with an equally strong modal force and equally invariant notions, whether we do so based on purely theoretical or partly empirical considerations.

Needless to say, the revisability of logic extends to the philosophy of logic, including our own foundational account. One specific feature that has been subject to proposals for revision is our invariance criterion of logicality. Feferman [1999], for example, called for the replacement of invariance-under-*isomorphisms* by invariance-under-*homomorphisms*; Bonnay [2008] and others offered other alternatives.[50] These proposals require, of course, serious consideration, but the philosophical bar our investigation set on such revisions is quite high. In particular, not every criterion of logicality is equally conducive to a unified, substantive, theoretical foundation for logic.

This concludes our outline of a foundation for logic. I hope the attempt in this paper to devise a foundational methodology and develop a substantive (albeit incomplete) foundation for logic based on it will encourage others to engage in this field of investigation and expand it beyond mathematical logic.

REFERENCES

J. Barwise and S. Feferman (editors) [1985], ***Model-theoretic logics***, Springer-Verlag.

A. Beck [2011], ***A general theory of formality***, Dissertation, University of California at San Diego.

G. Birkhoff and J. von Neumann [1936], *The logic of quantum mechanics*, ***Annals of Mathematics***, vol. 37, pp. 823–843.

L. BonJour [1985], ***The structure of empirical knowledge***, Harvard University Press, Cambridge.

D. Bonnay [2008], *Logicality and invariance*, this Bulletin, vol. 14, pp. 29–68.

C. Chihara [1990], ***Constructibility and mathematical existence***, Oxford University Press.

C. Dutilh Novaes [2011], *The different ways in which logic is (said to be) formal*, ***History and Philosophy of Logic***, vol. 32, pp. 303–332.

J. Etchemendy [1990], ***The concept of logical consequence***, Harvard University Press, Cambridge.

S. Feferman [1999], *Logic, logics, and logicism*, ***Notre Dame Journal of Formal Logic***, vol. 40, pp. 31–54.

S. Feferman [2000], *Mathematical intuition vs. mathematical monsters*, ***Synthese***, vol. 125, pp. 317–322.

S. Feferman [2010], *Set-theoretical invariance criteria for logicality*, ***Notre Dame Journal of Formal Logic***, vol. 51, pp. 3–20.

H. Field [1989], ***Realism, mathematics and modality***, Basil Blackwell, Oxford.

G. Frege [1879], *Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought*, ***From Frege to Gödel: A source book in mathematical logic, 1879–1931*** (J. von Heijenoort, editor), Harvard University Press, Cambridge, 1967, pp. 1–82.

G. Frege [1893], ***The basic laws of arithmetic: Exposition of the system***, The University of California Press, Berkeley, 1964, translated by M. Furth.

G. Frege [1918], *Thoughts*, ***Logical investigations*** (P. T. Geach, editor), Basil Blackwell, Oxford, 1977, pp. 1–30.

[50] I should perhaps note that during a recent workshop on logical constants (ESSLLI 2011, Slovenia) Feferman said he no longer wished to defend his alternative proposal although he still abided by his objections to isomorphism.

G. Frege [1979], ***Posthumous writings: Gottlob Frege***, (H. Hermes, F. Kambartel, and F. Kaulbach, editors), Chicago, translated by P. Long and R. White.

C. Glymour [1980], ***Theory and evidence***, Princeton University Press.

M. Gómez-Torrente [2002], *The problem of logical constants*, this Bulletin, vol. 8, pp. 1–37.

A. Gupta and N. Belnap [1993], ***The revision theory of truth***, MIT Press, Cambridge.

S. Haack [1993], ***Evidence and inquiry***, Blackwell, Oxford.

R. Hanna [2006], ***Rationality and logic***, MIT Press, Cambridge.

W. H. Hanson [1997], *The concept of logical consequence*, ***Philosophical Review***, vol. 106, pp. 365–409.

G. Hellman [1989], ***Mathematics without numbers: Towards a modal-structural interpretation***, Oxford University Press.

H. Hodes [1984], *Logicism and the ontological commitments of arithmetic*, ***Journal of Philosophy***, vol. 81, pp. 123–149.

I. Kant [1781/7], ***Critique of pure reason***, Macmillan, London, translated by N. Kemp Smith, 1929.

H. J. Keisler [1970], *Logic with the quantifier 'there exist uncountably many'*, ***Annals of Mathematical Logic***, vol. 1, pp. 1–93.

J. Kvanvig [2007], *Coherentist theories of epistemic justification*, internet.

K. Lehrer [1974], ***Knowledge***, Oxford University Press.

K. Lehrer [1990], ***Theory of knowledge***, Westview, Boulder.

P. Lindström [1966], *First order predicate logic with generalized quantifiers*, ***Theoria***, vol. 32, pp. 186–195.

P. Lindström [1974], *On characterizing elementary logic*, ***Logical theory and semantic analysis*** (S. Stenlund, editor), D. Reidel, Dordrecht, pp. 129–146.

J. MacFarlane [2000], ***What does it mean to say that logic is formal***, PhD Dissertation, Pittsburgh.

J. MacFarlane [2002], *Frege, Kant, and the logic in logicism*, ***The Philosophical Review***, vol. 111, pp. 25–65.

P. Maddy [2007], ***Second philosophy: A naturalistic method***, Oxford University Press.

V. McGee [1996], *Logical operations*, ***Journal of Philosophical Logic***, vol. 25, pp. 567–580.

A. Mostowski [1957], *On a generalization of quantifiers*, ***Fundamenta Mathematicae***, vol. 44, pp. 12–36.

O. Neurath [1921], *Anti-Spengler*, ***Empiricism and sociology*** (R. S. Cohen and M. Neurath, editors), Reidel, Dordrecht, 1973, pp. 158–213.

O. Neurath [1932], *Protocol statements*, ***Philosophical papers*** (R. S. Cohen and M. Neurath, editors), Reidel, 1973, Dordrecht, pp. 91–99.

C. Parsons [2008], ***Mathematical thought and its objects***, Cambridge University Press.

S. Peters and D. Westerståhl [2006], ***Quantifiers in language and logic***, Oxford University Press.

W. V. Quine [1951], *Two dogmas of empiricism*, ***From a logical point of view***, Harvard University Press, Cambridge, 2nd ed., pp. 20–46.

W. V. Quine [1970/86], ***Philosophy of logic***, 2nd ed., Harvard University Press, Cambridge.

J. Rawls [1971], ***A theory of justice***, Harvard University Press.

E. H. Reck. [2003], *Structures and structuralism in contemporary philosophy of mathematics*, ***Synthese***, vol. 137, pp. 369–419.

M. D. Resnik [1997], ***Mathematics as a science of patterns***, Oxford University Press.

B. Russell [1938], ***The principles of mathematics***, 2nd ed., Norton, New York, introduction to the second edition.

S. Shapiro [1991], ***Foundations without foundationalism: A case for second-order logic***, Oxford University Press.

S. Shapiro [1997], ***Philosophy of mathematics: Structure and ontology***, Oxford University Press.

G. Sher [1991], ***The bounds of logic: A generalized viewpoint***, MIT Press, Cambridge.

G. Sher [2001], *The formal-structural view of logical consequence*, ***Philosophical Review***, vol. 110, pp. 241–261.

G. Sher [2003], *A characterization of logical constants is possible*, ***Theoria***, vol. 18, pp. 189–197.

G. Sher [forthcoming], *Forms of correspondence*: *The intricate route from thought to reality*, ***Alethic pluralism: Current debates*** (C. D. Wright and N. J. Pedersen, editors), Oxford University Press.

A. Tarski [1933], *The concept of truth in formalized languages*, In Corcoran Tarski [1983], translated by J. H. Woodger, pp. 152–278.

A. Tarski [1936], *The establishment of scientific semantics*, In Corcoran Tarski [1983], translated by J. H. Woodger, pp. 401–408.

A. Tarski [1983], *Logic, semantics, metamathematics*, (J. Corcoran, editor), Hackett, Indianapolis, 2nd ed., translated by J. H. Woodger.

A. Tarski [1986], *What are logical notions*?, ***History and Philosophy of Logic***, vol. 7, pp. 143–154.

A. Tarski [1987], *A philosophical letter of Alfred Tarski. presented by M. White*, ***Journal of Philosophy***, vol. 84, pp. 28–32.

J. Väänänen [1997], *Generalized quantifiers*, ***Bulletin of the European Association for Theoretical Computer Science***, vol. 62, pp. 115–136.

J. Väänänen [2004], *Barwise*: *Abstract model theory and generalized quantifiers*, this Bulletin, vol. 10, pp. 37–53.

B. Williams [2002], ***Truth and truthfulness: An essay in genealogy***, Princeton University Press.

L. Wittgenstein [1921], ***Tractatus logico-philosophicus***, Routledge & Kegan Paul, 1961, translated by Pears and McGuinness.

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