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THE HORN THEORY OF BOOLE'S PARTIAL ALGEBRAS

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Abstract. This paper augments Hailperin's substantial efforts (1976/1986) to place Boole's algebra of logic on a solid footing. Namely Horn sentences are used to give a modern formulation of the principle that Boole adopted in 1854 as *the* foundation for his algebra of logic—we call this principle *The Rule of* 0 *and* 1.

There are two key components to Boole's algebra of logic: (a) his *translations* between verbal propositions about classes[1] on the one hand and algebraic equations on the other; and (b) his *algebra of classes*. This paper concerns the second component, Boole's algebra of classes.[2] Boole said that, with one exception, the laws of his algebra of classes agreed with the laws of the common algebra (of numbers)—the one exception was the idempotent law for variables.[3]

The challenge for the reader to believe that the algebra of numbers could be harnessed to capture the full scope of correct reasoning about classes was compounded by Boole's choice of semantics, namely partial algebras of classes (the binary operations $+$ and $-$ were partial operations). Boole carried out equational arguments for these partial algebras as though they were total algebras, quite unperturbed by the glaring fact that some of the terms used in the equations might be uninterpretable in the partial algebras.

Another obstacle for the reader was the foundation that Boole chose for his algebra of classes. After discussing rather ordinary looking laws and rules of inference, he suddenly placed a remarkable principle before his readers, one that we call *The Rule of* 0 *and* 1. It stated that the axioms, laws and processes for his algebra of classes would be the same as those for an algebra of numbers where the variables were restricted to the values 0 and 1. Boole then said this would be the one and only foundation principle for his algebra of classes.

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[1] The word 'set' is now used where Boole used the word 'class'.

[2] As Hailperin [6] noted, Boole's algebra of classes is *not* what is now called Boolean algebra. For a leisurely introduction to Boole's algebra of logic, see [4].

[3] Boole used the word 'symbol' instead of 'variable'.

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Using the Rule of 0 and 1, Boole proceeded to easily prove his four main theorems: the Expansion Theorem, the Reduction Theorem, the Elimination Theorem and the Solution Theorem. Armed with these theorems, Boole transformed the process of correct reasoning about classes into a routine mechanical algebraic procedure, his *General Method*, a procedure to find consequences for any given collection of (equational) premisses. (This method played an important role in Boole's development of probability theory in [3].)

For over a century following Boole's publications, the scholarly discussions about the meaning and validity of Boole's algebra of classes were mostly dismissive, uninformative, opaque, or just incorrect. A breakthrough occurred in 1976, when Hailperin [6] said that the laws for the common algebra used by Boole were just the familiar laws for commutative rings of characteristic zero, with unity and without nilpotent elements, plus idempotent variables. To justify Boole's equational logic for partial algebras, Hailperin noted that certain rings (that he called *algebras of signed multisets*) satisfied the above laws, and each of Boole's partial algebras could be realized as the partial algebra of idempotent elements of such a ring. Hailperin proved Boole's theorems within this framework; consequently the world finally knew that Boole's algebra of logic indeed offered a viable algebra of classes.

Although he resuscitated much of Boole's algebra of classes, Hailperin was not able to resurrect Boole's Rule of 0 and 1 as the foundational principle. In [6, 2nd ed., p. 76] he noted some of the difficulties encountered with understanding this principle, including the following:

> ... *one interpretation has only two entities 0 and 1, and an adequate Algebra of Logic can't do with only two classes.*

This was a misunderstanding by Hailperin—the restriction of the variables to the values 0 and 1 was not meant to be applied to the algebra of classes, only to the algebra of numbers. Hailperin was also concerned about the scope of Boole's Rule of 0 and 1. In §3.1 we present R01, a precise formulation (using Horn sentences) of Boole's fundamental principle, and show that Boole's laws, rules of inference and main theorems easily follow from it.

§1. Boole's partial algebras. Although there is much in Boole's writings on logic that one finds lacking in clarity, he was perfectly clear as to the algebraic structures he was working with. Given a universe U, let $\mathbf{S}(U)$ be the partial algebra $(\mathrm{Su}(U), +, \cdot, -, 0, 1)$ whose domain is $\mathrm{Su}(U)$, the collection of subsets of U, and whose operations and constants are given by:

$$\begin{aligned} A + B &:= A \cup B \quad \text{provided } A \cap B = \varnothing;\ \text{otherwise undefined,} \\ A \cdot B &:= A \cap B, \\ A - B &:= A \smallsetminus B \quad \text{provided } B \subseteq A;\ \text{otherwise undefined,} \\ 1 &:= U, \\ 0 &:= \varnothing. \end{aligned}$$

Let $\mathfrak{S}$ be the class of Boole's partial algebras, namely the algebras $\mathbf{S}(U)$ where $U \neq \varnothing$. These partial algebras are clearly not our Boolean algebras.[4]

Boole's partial algebras will be considered as partial algebraic structures using the language of rings—rings are normally presented with '$-$' treated as a unary operation called the additive inverse, and subtraction is defined by $x - y := x + (-y)$. However one can just as easily view rings with '$-$' being subtraction, and define the additive inverse by $-x := 0 - x$; this requires only a modest change in the defining equational axioms. From now on we assume all first-order terms and formulas are for this language.

For terms $s(\mathbf{x})$, $t(\mathbf{x}, \mathbf{y})$, it will always be assumed that $\mathbf{x}$ is the list of variables $x_1, \dots, x_m$ and $\mathbf{y}$ is the list of variables $y_1, \dots, y_n$. The notion of when a partial algebra $\mathbf{S}(U)$ satisfies an equation $s(\mathbf{x}) = t(\mathbf{x})$ is complicated by the fact that the term functions defined by these terms on $\mathbf{S}(U)$, namely $s^{\mathbf{S}(U)}$ and $t^{\mathbf{S}(U)}$, can be partial functions. However for certain terms this difficulty disappears.

DEFINITION 1.1. A term $t(\mathbf{x})$ is $\mathfrak{S}$-*total* if for every nonempty U, $t^{\mathbf{S}(U)}$ is a total function on the domain of $\mathbf{S}(U)$. A first-order formula is $\mathfrak{S}$-total if every term in the formula is $\mathfrak{S}$-total.

If a term $t(\mathbf{x})$ is $\mathfrak{S}$-total, clearly every subterm is $\mathfrak{S}$-total.

Boole gave many examples demonstrating his General Method, to go from premisses to conclusions in the ordinary verbal logic of classes. On the algebraic side the premisses and conclusions were always equations $s(\mathbf{x}) = t(\mathbf{x})$. From the examples and various comments by Boole, it is clear that these equations, properly constructed, would be $\mathfrak{S}$-total.

§2. The class $\mathfrak{R}$ of rings. Let $\mathfrak{R}$ be the collection of commutative rings $\mathbf{R} = (R, +, \cdot, -, 0, 1)$ of characteristic 0 with unity and no nilpotent elements, and let $\Sigma_{\mathfrak{R}}$ be the obvious set of first-order axioms for this class. Direct powers $\mathbf{Z}^U$, where $U \neq \varnothing$, of the ring of integers $\mathbf{Z}$ are in this class.

DEFINITION 2.1. Given a first-order formula $\varphi(\mathbf{x})$, let $\varphi(\mathbf{x})^\star$ be the formula obtained by relativizing the quantifiers of $\varphi(\mathbf{x})$ to idempotent elements, that is, in $\varphi(\mathbf{x})$ replace each occurrence of a subterm of the form $\big[(\forall x)\psi\big]$ by $\big[(\forall x)\big(x^2 = x \rightarrow \psi\big)\big]$; and each occurrence of a subterm of the form

[4]Our notion of Boolean algebra, with the operations of union, intersection and complement, is due to W. S. Jevons in 1864 [7]—he had been a student of De Morgan, and found Boole's algebra of logic decidedly unsatisfactory. In 1863 Jevons wrote to Boole recommending replacing Boole's $+$ with union. Boole flatly rejected the possibility, and eventually broke off the correspondence. Accepting the law $x + x = x$ would have destroyed the possibility of using common algebra since common algebra plus $x + x = x$ implies $x = 0$. The ironic naming of the algebras that Boole had rejected as 'Boolean algebras' was evidently due to Sheffer [10] in 1913, depriving us of the natural name for the partial algebras that Boole actually used.

$\big[(\exists x)\psi\big]$ by $\big[(\exists x)\big((x^2 = x) \wedge \psi\big)\big]$. The replacements start with subterms of quantifier depth 1, then quantifier depth 2, etc.

LEMMA 2.2. *For any set $U \neq \varnothing$:*

(a) *The map which sends each subset A of U to its characteristic function χ_A is an embedding of $\mathbf{S}(U)$ into $\mathbf{Z}^U$.*

(b) *For φ a $\mathfrak{S}$-total first-order sentence, $\mathbf{S}(U) \models \varphi$ iff $\mathbf{Z}^U \models \varphi^\star$.*

PROOF. Item (a) is obviously true, and (b) follows from an easy induction using the definition of $\models$. ⊣

A *basic Horn formula* is a disjunction of atomic and negated atomic formulas, with at most one atomic formula. A *Horn formula* is a first-order formula in prenex form with the matrix, the quantifier-free part of the formula, being a conjunction of basic Horn formulas. As is well-known, Horn sentences are preserved by direct products.

It is easy to see that if $\varphi(\mathbf{x})$ is a Horn formula then $\varphi(\mathbf{x})^\star$ is logically equivalent to a Horn formula with the same quantifier prefix. For example, if $\omega(x, y, z)$ is a conjunction of basic Horn formulas, then

$$\big[(\forall x)(\exists y)(\forall z)\omega(x, y, z)\big]^\star$$

is logically equivalent to

$$(\forall x)(\exists y)(\forall z)\Big[(x^2 = x) \rightarrow \Big((y^2 = y) \wedge \big((z^2 = z) \rightarrow \omega(x, y, z)\big)\Big)\Big],$$

and the matrix of this sentence is equivalent to a conjunction of basic Horn formulas.

From [8] we know that every $\mathbf{R} \in \mathfrak{R}$ is isomorphic to the ring of global sections of its Pierce sheaf under the mapping $\pi : \mathbf{R} \rightarrow \prod_{x \in X} \mathbf{R}/Rx$, where $X = X(\mathbf{R})$ is the Stone space of the Boolean algebra of idempotents of the ring $\mathbf{R}$; furthermore the stalks $\mathbf{R}/Rx$ are directly indecomposable members of $\mathfrak{R}$, so the idempotents of each $\mathbf{R}/Rx$ are 0, 1, and they generate a subring of the stalk isomorphic to $\mathbf{Z}$. Then a simple argument shows that the idempotents of $\mathbf{R}$ generate a subring of $\mathbf{R}$ isomorphic to $\mathbf{C}(X, \mathbf{Z})$, the ring of continuous functions from X to $\mathbf{Z}$, where $\mathbf{Z}$ has the discrete topology.

Now we are ready to state the main theorem of this paper. The notation $\Sigma_{\mathfrak{R}} \vdash \varphi$ means there is a derivation of φ from $\Sigma_{\mathfrak{R}}$ in first-order logic.

THEOREM 2.3. *Let φ be a Horn sentence.*

(a) $$\Sigma_{\mathfrak{R}} \vdash \varphi^\star \quad \textit{iff} \quad \mathbf{Z} \models \varphi^\star.$$

(b) *If φ is $\mathfrak{S}$-total then*

$$\mathfrak{S} \models \varphi \quad \textit{iff} \quad \mathbf{Z} \models \varphi^\star.$$

PROOF. For (a), the direction $\Rightarrow$ is trivial since $\mathbf{Z} \models \Sigma_{\mathfrak{R}}$, so suppose $\mathbf{Z} \models \varphi^\star$, and let $\mathbf{R} \in \mathfrak{R}$. Since $\varphi^\star$ is logically equivalent to a Horn sentence, it follows from Theorem 3.1 of [5] that $\mathbf{C}(X, \mathbf{Z}) \models \varphi^\star$, where $X = X(\mathbf{R})$. Since the subring of $\mathbf{R}$ generated by its idempotents is isomorphic to $\mathbf{C}(X, \mathbf{Z})$, it follows that $\mathbf{R} \models \varphi^\star$. Thus $\mathfrak{R} \models \varphi^\star$. By Gödel's completeness theorem for first-order logic, $\Sigma_{\mathfrak{R}} \vdash \varphi^\star$.

For (b) we have

$$\begin{aligned} \mathfrak{S} \models \varphi \;&\Leftrightarrow\; \mathbf{S}(U) \models \varphi && \text{for } U \neq \varnothing \\ &\Leftrightarrow\; \mathbf{Z}^U \models \varphi^\star && \text{for } U \neq \varnothing, \text{ by Lemma 2.2(b)} \\ &\Leftrightarrow\; \mathbf{Z} \models \varphi^\star && \text{since } \varphi^\star \text{ is equivalent to a Horn sentence.} \quad \dashv \end{aligned}$$

§**3. Boole's algebra of classes.** In this section we briefly review Boole's laws and rules of inference, present his Rule of 0 and 1, and then state R01, our version of this principle. After this we note that Boole's laws and rules of inference, as well as his main theorems, readily follow from R01.[5]

3.1. Laws, rules of inference, and the rule of 0 and 1. In his 1847 publication [1] Boole believed that three simple (equational) laws plus the familiar rules of 'equals plus equals give equals', etc., provided a solid foundation for his algebra of logic. In his book [3] of 1854, the number of laws increased to seven; however he no longer claimed that the list of laws was adequate.

After noting that his algebra of logic had all but one of its laws in common with the laws of numbers, Boole used the one law that differentiates them, the idempotent law, to segue on page 37 of [3] to a surprise announcement of a completely new foundation for his algebra of logic, a principle that we call the *Rule of* 0 *and* 1. Here is Boole's justification and statement of this foundational principle:

> 15. *However, it is not with the symbols of quantity generally that it is of any importance, except as a matter of speculation, to trace such affinities. We have seen* (II. 9) *that the symbols of Logic are subject to the special law,* $x^2 = x$. *Now of the symbols of Number there are but two, viz.* 0 *and* 1, *which are subject to the same formal law. We know that* $0^2 = 0$, *and that* $1^2 = 1$; *and the equation* $x^2 = x$, *considered as algebraic, has no other roots than* 0 *and* 1. *Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity admitting only of the values* 0 *and* 1. *Let us conceive, then, of an Algebra in which the symbols* x, y, z, *etc. admit indifferently of the values* 0 *and* 1, *and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established.*

[5] In his analysis of Boole's work, Hailperin [6] justified Boole's results using a two-sorted version of $\Sigma_{\mathfrak{R}}$, with one sort for the idempotent elements, the other for any elements. The semantics were given by algebras of signed multisets; these are isomorphic to our rings $\mathbf{C}(X, \mathbf{Z})$.

Our version of Boole's Rule of 0 and 1 is simply:

> R01: *The laws, rules of inference, and theorems for Boole's algebra of classes are precisely those expressed by members of the class* $\mathsf{Th}^{\star}_{\mathsf{Horn}}(\mathbf{Z})$ *of Horn sentences* φ *for which* $\mathbf{Z} \models \varphi^{\star}$.

By Theorem 2.3(a),

$$\mathsf{Th}^{\star}_{\mathsf{Horn}}(\mathbf{Z}) = \{\varphi \in \text{Horn sentences}\colon \Sigma_{\mathfrak{R}} \vdash \varphi^{\star}\},$$

and from Theorem 2.3(b),

$$\varphi \in \mathsf{Th}^{\star}_{\mathsf{Horn}}(\mathbf{Z}) \text{ and } \varphi \text{ is } \mathfrak{S}\text{-total} \Rightarrow \mathfrak{S} \models \varphi.$$

Let $\widehat{\Sigma}_{\mathfrak{R}}$ be the laws of $\Sigma_{\mathfrak{R}}$ stated for terms, not just variables. For example, the commutative law for multiplication would be given by the identities $(\forall \mathbf{x})\big(s(\mathbf{x}) \cdot t(\mathbf{x}) = t(\mathbf{x}) \cdot s(\mathbf{x})\big)$. The laws and rules of inference that Boole used for his algebra of classes are easily seen to be available using R01, for example:

(a) $\mathsf{Th}_{\forall\mathsf{Horn}}(\mathbf{Z}) \subseteq \mathsf{Th}^{\star}_{\mathsf{Horn}}(\mathbf{Z})$, where $\mathsf{Th}_{\forall\mathsf{Horn}}(\mathbf{Z})$ is the universal Horn theory of $\mathbf{Z}$.[6] Thus $\widehat{\Sigma}_{\mathfrak{R}} \subseteq \mathsf{Th}^{\star}_{\mathsf{Horn}}(\mathbf{Z})$; and the familiar rules of inference, namely *equals* $\Box$ *equals gives equals*, where $\Box \in \{+, \cdot, -\}$, are expressed by universal Horn sentences in $\mathsf{Th}^{\star}_{\mathsf{Horn}}(\mathbf{Z})$.

(b) $(\forall x)(x^2 = x) \in \mathsf{Th}^{\star}_{\mathsf{Horn}}(\mathbf{Z})$, that is, the *idempotent variables* law is in $\mathsf{Th}^{\star}_{\mathsf{Horn}}(\mathbf{Z})$.[7]

R01 avoids the pitfalls that Boole [3, p. 36] and Hailperin [6, 2nd ed., p. 77] noted, such as the cancellation law

$$\varphi := (\forall x)(\forall y)(\forall z)\big(x \cdot y = x \cdot z \rightarrow x = 0 \ \vee \ y = z\big);$$

also $\varphi := (\forall x)(x = 0 \vee x = 1)$. Neither of these $\mathfrak{S}$-total sentences φ is true for $\mathfrak{S}$, but $\mathbf{Z} \models \varphi^{\star}$ holds for both. The sentences φ in these two examples are *not* equivalent to Horn sentences.

In §3.3 Boole's Theorems are presented as sentences logically equivalent to Horn sentences in $\mathsf{Th}^{\star}_{\mathsf{Horn}}$. But first we need to discuss constituents.

3.2. Constituents. Boole introduced constituents [3, pp. 75–78] to provide a basis for expanding terms. In the next definition, for economy of space, $1 - x_i$ is abbreviated to x_i'; also let $\widetilde{m} = \{1, \dots, m\}$, etc.

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[6] We do not have $\mathsf{Th}_{\forall\exists\mathsf{Horn}}(\mathbf{Z}) \subseteq \mathsf{Th}^{\star}_{\mathsf{Horn}}(\mathbf{Z})$ since $\varphi := (\forall x)(\exists y)\big(x + x = y\big)$ holds in $\mathbf{Z}$, but $\varphi \notin \mathsf{Th}^{\star}_{\mathsf{Horn}}(\mathbf{Z})$.

[7] It is interesting to note that R01 picks up on a distinction that Boole glossed over in his presentation of his laws, namely that one is not permitted to do a substitution in the idempotent law to obtain an equation $t(\mathbf{x})^2 = t(\mathbf{x})$ without further justification. However for his other laws, one could carry out arbitrary substitutions, for example, to obtain $s(\mathbf{x}) \cdot t(\mathbf{x}) = t(\mathbf{x}) \cdot s(\mathbf{x})$ from $xy = yx$, his commutative law for multiplication.

DEFINITION 3.1. Given a list of variables $\mathbf{x}$, the 2^m **x-constituents** are

$$\begin{gathered} x_1 x_2 \dots x_m \\ x_1' x_2 \dots x_m \\ \vdots \\ x_1' x_2' \dots x_m'. \end{gathered}$$

A useful notation for referring to them is as follows. For $\sigma \in 2^{\widetilde{m}}$, that is, for σ a mapping from $\{1, \dots, m\}$ to $\{0, 1\}$, let

$$\mathcal{C}_\sigma(\mathbf{x}) := \mathcal{C}_{\sigma_1}(x_1) \dots \mathcal{C}_{\sigma_m}(x_m),$$

where $\mathcal{C}_1(x_j) := x_j$ and $\mathcal{C}_0(x_j) := x_j'$.

Thus, for example, with $m = 5$ and $\sigma = 01101$, $\mathcal{C}_\sigma(\mathbf{x}) = x_1' x_2 x_3 x_4' x_5$.
In the following, whenever an operator is simply indexed by σ, etc, for example $\sum_\sigma$ or $\bigwedge_\sigma$, it is understood that the index runs over the full range of possibilities for the list of variables being considered. It is easily verified that $\mathbf{Z}$ satisfies $\varphi^\star$ for φ any of the following where $\sigma, \tau \in 2^{\widetilde{m}}$:

(a) $(\forall \mathbf{x})\big(C_\sigma(\mathbf{x})^2 = C_\sigma(\mathbf{x})\big)$,
(b) $(\forall \mathbf{x})\big(C_\sigma(\mathbf{x}) \cdot C_\tau(\mathbf{x}) = 0\big)$ if $\sigma \neq \tau$,
(c) $(\forall \mathbf{x})\big(1 = \sum_\sigma C_\sigma(\mathbf{x})\big)$.

Given a term $t(\mathbf{x}, \mathbf{y})$ and $\sigma \in 2^{\widetilde{m}}$, $t(\sigma, \mathbf{y})$ is the term obtained by replacing each x_i by σ_i.

3.3. Boole's theorems. The following gives Boole's four main theorems as consequences of R01.

THEOREM 3.2. *If φ is any of the following four kinds of sentences, then φ is logically equivalent to a Horn sentence $\psi \in \mathsf{Th}^\star_{\mathsf{Horn}}(\mathbf{Z})$:*

(a) (*Development/Expansion Theorem*) (Boole [3, pp. 72–76])

$$(\forall \mathbf{x}, \mathbf{y})\Big(t(\mathbf{x}, \mathbf{y}) = \sum_\sigma t(\sigma, \mathbf{y}) \cdot C_\sigma(\mathbf{x})\Big).$$

(b) (*Reduction Theorem*) (Boole [3, p. 121])

$$(\forall \mathbf{x})\Big(\bigwedge_i t_i(\mathbf{x}) = 0 \leftrightarrow \sum_i t_i(\mathbf{x})^2 = 0\Big).$$

(c) (*Elimination Theorem*) (Boole [3, p. 101])

$$(\forall \mathbf{y})\Big((\exists \mathbf{x})(t(\mathbf{x}, \mathbf{y}) = 0) \leftrightarrow \prod_\sigma t(\sigma, \mathbf{y}) = 0\Big).$$

(d) (*Solution Theorem*) (Boole [3, pp. 86–93]; Schröder [9, Vol. I, p. 447])[8] *Let $t^\star(\boldsymbol{x}, y)$ be the sum of the $\boldsymbol{x}, y$-constituents whose coefficient in the expansion of $t(\boldsymbol{x}, y)$ is non-zero.*

$$(\forall \mathbf{x}, y)\Big(\big[t(\mathbf{x}, y) = 0\big] \leftrightarrow \big[t(\mathbf{x}, 1)\cdot t(\mathbf{x}, 0) = 0\big] \wedge (\exists z)\Big[y = t^\star(\mathbf{x}, 0) + z\cdot\big(1 - t^\star(\mathbf{x}, 1) - t^\star(\mathbf{x}, 0)\big)\Big]\Big).$$

Proof. A sentence φ in any of the above forms is clearly equivalent to an $\forall\exists$ Horn sentence ψ, with $\mathbf{Z} \models \psi^\star$ iff $\mathbf{Z} \models \varphi^\star$. Showing $\mathbf{Z} \models \varphi^\star$ is routine since the quantified variables range over $\{0, 1\}$. For example, to prove (c) it suffices to show

$$\bigwedge_\tau \Big(\bigvee_\sigma (t(\sigma, \tau) = 0) \leftrightarrow \prod_\sigma t(\sigma, \tau) = 0\Big)$$

holds in $\mathbf{Z}$, where σ ranges over $2^{\widetilde{m}}$ and τ ranges over $2^{\widetilde{n}}$. This is clearly true in $\mathbf{Z}$ since at least one of a finite collection of elements of $\mathbf{Z}$ is zero iff their product is zero.

The most complicated result is (d); to prove this it suffices to show that

$$\bigwedge_{\sigma,\tau}\Big(\big[t(\sigma, \tau) = 0\big] \leftrightarrow \big[t(\sigma, 1)\cdot t(\sigma, 0) = 0\big] \wedge \bigvee_\rho \Big[\tau = t^\star(\sigma, 0) + \rho\cdot\big(1 - t^\star(\sigma, 1) - t^\star(\sigma, 0)\big)\Big]\Big)$$

holds in $\mathbf{Z}$, where σ ranges over $2^{\widetilde{m}}$ and τ, ρ range over 2, that is, over $\{0, 1\}$. By putting in values for ρ and τ, this reduces to showing

$$\bigwedge_\sigma\Big(\big[t(\sigma, 1) = 0\big] \leftrightarrow \big[t(\sigma, 1)\cdot t(\sigma, 0) = 0\big] \wedge \Big[1 = t^\star(\sigma, 0) \vee 1 = 1 - t^\star(\sigma, 1)\Big]\Big) \wedge$$
$$\bigwedge_\sigma\Big(\big[t(\sigma, 0) = 0\big] \leftrightarrow \big[t(\sigma, 1)\cdot t(\sigma, 0) = 0\big] \wedge \Big[0 = t^\star(\sigma, 0) \vee 0 = 1 - t^\star(\sigma, 1)\Big]\Big)$$

holds in $\mathbf{Z}$, where σ ranges over $2^{\widetilde{m}}$. This is easy to verify. ⊣

[8]The Solution Theorem we are using is based on a simple version due to Schröder for modern Boolean algebra; it is easily adapted to Boole's setting. Hailperin's proofs in [6, 2nd ed.], of these theorems span some 25 pages, after 135 pages of preliminaries.

Boole's General Method uses each of these four theorems like rewrite rules, from left to right, transforming a term into its expansion, collapsing several equations into one, eliminating variables, and solving for a variable (with side conditions on the other variables). His General Method often led to expressions that he wanted to simplify; for this he turned to ad hoc algebraic manipulations based on $\Sigma_{\mathfrak{R}}$. Thus both $\Sigma_{\mathfrak{R}}$ and Boole's main theorems play important roles in his General Method; R01 readily justifies applications of the General Method.

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