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THE HYPERUNIVERSE PROGRAM

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Abstract. The *Hyperuniverse Program* is a new approach to set-theoretic truth which is based on justifiable principles and leads to the resolution of many questions independent from ZFC. The purpose of this paper is to present this program, to illustrate its mathematical content and implications, and to discuss its philosophical assumptions.

§1. Introduction. The purpose of this paper is to discuss and illustrate the *Hyperuniverse Program* (as well as the *Inner Model Hypothesis* (IMH) and its variants as a proposal for realizing it), an approach due to the second author (see [7]) which is inspired by the search for solutions to questions known to be independent from the axiomatic system ZFC.[1]

In recent years, different research programs, motivated by independence phenomena, have been formulated in set theory. The stage for most of them has been set by Gödel's program for new axioms, announced in [9] at a time when the independence of the Continuum Hypothesis from ZFC could only be (correctly) conjectured. [9], and its revised and extended version [10], have played a fundamental role in the debate concerning the foundations of set theory. In defense of the views there expressed, Gödel invoked philosophical considerations on the nature of mathematics, analysis of logico-mathematical concepts, and technical arguments of a purely mathematical character. Similar ingredients can be found in most of the subsequent proposals for overcoming independence results.

Gödel's program is worth a closer look. As a fundamental motivation for the program of extending ZFC through the addition of new axioms, the conviction is expressed in [9] that it is possible to give a final answer

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[1]By independent questions we mean sentences ϕ of the first-order language of ZFC such that ZFC can prove neither ϕ nor $\neg\phi$.

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to the question of the cardinality of the continuum, despite its probable independence from ZFC. This conviction explicitly rests on a Platonistic view of mathematics, according to which set-theoretical concepts and theorems describe some well determined reality, "in which Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality", ([9], p. 181).

When it comes to discussing proposals for new axioms, the point is made in [9] that the candidates for new axioms should be justified, displaying conformity to motivating principles more evident and persuasive than the candidates themselves. The concept of set is called upon for this purpose, where the view is taken that a set is something obtainable from the integers (or some other well-defined object) by iterated application of the operation "set of" ([9], p. 180). A special emphasis is put on the *maximizing* implications of that concept, to the effect that axioms "stating the existence of still further iterations of the operation set of", like "small" large cardinal hypotheses, are regarded as fully legitimate candidates for new set-theoretic axioms.[2] [9], however, does not rule out the possibility that, beyond the concept of set, there may be other motivations that succeed in indicating reasonable strategies for extending ZFC. In fact it is conjectured that "there may exist besides the ordinary axioms [...] other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts" ([9], p. 182). The suggestion is also made, in [10], that some *maximum* property of the system of sets may be devised that is not directly suggested by the concept of set, yet may work as a reasonable new axiom for set theory ("[...] from an axiom in some sense opposite to this one [$V = L$] the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which [...] would state some maximum property of the system of all sets [...]", [10], p. 478).

It is by invoking the criterion of *success* as contributing to a decision about the truth of a candidate for an axiom for set theory that the way is opened in

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[2]By "small large cardinals" are here meant large cardinals whose existence is compatible with the axiom of constructibility $V = L$. Gödel pronounces on one of them, the axiom stating the existence of an inaccessible cardinal, as follows:

> The simplest of these "strong axioms of infinity" asserts the existence of inaccessible numbers (and of numbers inaccessible in the stronger sense) $> \aleph_0$. The latter axiom, roughly speaking, means nothing else but that the totality of sets obtainable by exclusive use of the procedure of formation of sets expressed in the other axioms forms again a set (and therefore a new basis for further applications of these processes). ([9], p. 182)

The axiom stating the existence of a measurable cardinal, as well as its incompatibility with the axiom of constructibility, was known to Gödel as he wrote [10]. However, Gödel apparently did not consider this axiom as implied by the concept of set (see [10], footnote 16).

[9] for bringing considerations of a purely mathematical character into the discussion of proposals for new axioms. The success of an axiom is meant to consist in its fruitfulness in consequences, its "shedding light upon a whole discipline", and its yielding "powerful methods for solving given problems" ([9], p. 183). Mathematical results (facts "not known at Cantor's time") are also invoked in the attempt to explain the prediction that Cantor's conjecture will turn out to be wrong. Thus, the moral of [9] is that in formulating axiom candidates for set theory, one not only is committed to the search for general motivating principles that justify them, but one must also take into account a corpus of already existing and accepted mathematical results, upon which the new axiom(s) should shed light, or at the very least, not irreconcilably contradict.

The approach that we present here shares many features, though not all, of Gödel's program for new axioms. Let us briefly illustrate it. The Hyperuniverse Program is an attempt to clarify which first-order set-theoretic statements (beyond ZFC and its implications) are to be regarded as true in V, by creating a context in which different pictures of the set-theoretic universe can be compared. This context is the *hyperuniverse*, defined as the collection of all countable transitive models of ZFC. The comparison of such models evokes *principles* (principles of *maximality* and *omniscience*, as we will name two of them) that suggest *criteria* for preferring, on justifiable grounds, certain universes of sets over others.[3] Starting from criteria for preferred universes, one applies the principle that first-order statements that hold across all preferred universes (hopefully including solutions to independent questions) also hold in V (an assumption based partly on the downward Löwenheim–Skolem theorem), and adopts these statements as new axioms of set theory.

This being, in a nutshell, the Hyperuniverse Program, one clearly sees that it shares the fundamental aim of Gödel's program of extending ZFC by new set-theoretic axioms resulting from "a more profound understanding of basic concepts underlying logic and mathematics". In fact, within the Hyperuniverse Program one formulates principles and criteria for preferred universes that are suggested by a logico-mathematical analysis of the hyperuniverse. Also, Gödel's suggestion to consider a "*maximum* property of the system of all sets" for extending ZFC is addressed by this program. Indeed *maximality* works well as a principle inspiring criteria for preferred universes. Moreover, in both Gödel's program and the Hyperuniverse Program, one seeks to find solutions to independent questions in a way that may be regarded as ultimate and not revisable, and hence may be regarded as definitive or *true* in V, *the* universe of all sets.

[3]The formulation of criteria for preferred universes is not an easy task. In particular the possibility of conflicting desiderata to be imposed on preferred universes of sets cannot be excluded at the outset. Essential to the Hyperuniverse Program is thus the effort to combine the desired criteria into a coherent synthesis; this will be explained in detail below.

It must be explicitly said, however, that in formulating the Hyperuniverse Program, Platonism is nowhere invoked, either with regard to V or to the hyperuniverse. To the contrary, some of its characteristic features clearly express an anti-Platonistic attitude, which makes the program radically different from Gödel's. No well-determined reality is called upon within the Hyperuniverse Program in arguing for the legitimacy of the search for solutions to independent questions. Rather, one considers that in spite of the abundance of independence results obtained in set theory, there are no a priori grounds against the goal of finding ultimate answers to questions like CH. This shifts the burden of proof onto those who claim that there are such grounds.[4] Moreover, in formulating the Hyperuniverse Program the expression "*true* in V" is not used to reflect an ontological state of affairs concerning the universe of all sets as a reality to which existence can be ascribed independently of set-theoretic practice. Instead "*true* in V" is meant as a *façon de parler* that only conveys information about set-theorists' epistemic attitudes, as a description of the status that certain statements have or are expected to have in set-theorists' eyes. Sentences "true in V" are meant to be sentences that are or should be regarded by set-theorists as definitive, i.e., ultimate and not revisable. Within the Hyperuniverse Program two sorts of statements qualify for this status. The first are those set-theoretic statements that, due to the role that they play in the practice of set theory and, more generally, of mathematics, should not be contradicted by any further candidate for a set-theoretic statement that may be regarded as ultimate and not revisable. Let us call these statements "*de facto*" set-theoretic truths. The axioms of ZFC and the consistency of ZFC + large cardinal axioms are examples of such truths. But secondly, within the Hyperuniverse Program, one is ready to regard as true in V statements that, beyond not contradicting *de facto* set-theoretic truth, obey a condition for truth explicitly established at the outset. Let us call these "*de jure*" set-theoretic truths. The condition which they obey is that they are sentences that hold in all preferred universes of the hyperuniverse. The latter, in turn, is not meant as an independent, well-determined reality, but as a mathematical construct, produced along with the developments of set theory and of the program. Hence, within the Hyperuniverse Program, Platonism is invoked neither with regard to V nor with regard to the hyperuniverse. In fact, as intended by the program, formulating *de jure* set-theoretic truths is an autonomously regulated process. No "external" constraint is imposed while engaged in it, such as an already existing reality to which one must be faithful. Instead, in searching for *de jure* set-theoretic truths one is only expected to follow justifiable procedures. It cannot be excluded at the outset that at some time the need will arise to

[4]This assumption seems to lie behind Shelah's considerations in [18], [19] as well as the *multiverse view* advocated by Hamkins in [11]. A critical appraisal of the former is given in [1].

modify the procedures adopted in order to integrate them with other, equally reasonable procedures.[5]

In short, formulating *de jure* set-theoretic truth, which lies at the core of the Hyperuniverse Program, may be understood as the *active* response of a non-Platonistically minded mathematician, who believes that it makes sense to search for new truths in V beyond *de facto* set-theoretic truths. This contrasts with any form of skepticism concerning such a search, be it motivated by the assumption that such a search is hopeless or by the confidence, possibly grounded on Platonism, that whatever the well-determined features of V are, they will somehow manifest themselves without any effort of our own. Equivalently, one may characterize the Hyperuniverse Program as a *dynamic* approach to set-theoretic truth, *free* from external constraints (although internally regulated), in contrast to any *static* Platonistic view that truth concerning sets is *restricted* to a fixed state of affairs to which one must be "faithful".

The stance of the advocate of the Hyperuniverse Program towards existing set-theoretic developments is both complex and surprising. Of course the latter explicitly enter into the program insofar as one aims at obtaining preferred universes that, beyond conforming to certain criteria and not contradicting existing *de facto* set-theoretic truth, are successful in deciding independent questions. Moreover the techniques needed to establish the existence of preferred universes are provided by existing developments in set theory or by new developments inspired by the program which extend existing developments. There is another reason, however, for the Hyperuniverse Program to explicitly call upon set-theoretic developments, albeit in a negative way. When declaring the intention of extending ZFC so as to settle independent questions, one also requires that one be as *unbiased* as possible as to the way such questions should be settled and as to which principles and criteria for preferred universes one should formulate. In particular, the latter must not be chosen at the outset so as to be apt for settling questions independent of ZFC, or for meeting the needs of some particular area of existing set-theoretic practice. Nor should specific mathematical hypotheses be invoked in formulating such principles and criteria (e.g., large cardinal or forcing axioms). The rationale behind unbiasedness is twofold. On the one hand, one wants to be as cautious as possible as to what set-theoretic developments beyond ZFC belong to the realm of *de facto* set-theoretic truth. Endorsing this attitude means doing justice to the fact that disparate views have been advanced within the set-theoretic community on this matter.[6] Secondly, one aims at formulating principles and criteria starting from

[5] An example of this will be given in Section 3, where the criterion of *power set maximality* is modified so as to be compatible with the criterion of *ordinal maximality*.

[6] See, e.g., [22] and [19] for different positions on the status of large cardinal axioms or the axiom of determinacy $AD_{L[\mathbb{R}]}$ in contemporary set theory.

an analysis of the hyperuniverse which focus exclusively on its most general features. As a result, the principles chosen and the criteria derived from them are expected to yield a justified selection of preferred universes on the sole basis of one's acquaintance with the most basic aspects of set theory.

A surprise is that, unbiasedness notwithstanding, the Hyperuniverse Program leads to results that strongly affect our understanding of the corpus of already existing set-theoretic developments. This is the case, e.g., if one adopts the *Inner Model Hypothesis* (IMH), as formulated in [7], as a criterion for preferred universes, providing a suitable description of what it means for a countable transitive model of ZFC to be maximal (fixing the ordinals). This hypothesis settles many questions independent of ZFC, but also has implications of a revisionary character with respect to what is sometimes assumed without question by the set theory community: although the IMH is compatible with the internal consistency of very large cardinals (i.e., their existence in inner models), it contradicts their existence in the universe V as a whole. This may be regarded as disruptive, providing evidence *contra* rather than *pro* the hypothesis. By taking it seriously, however, one may nonetheless come to the unexpected conclusion that the IMH does not contradict the practice of set theory after all, as it is the existence of large cardinals in inner models, and not in V, that has gained the status of an ultimate, unrevisable assumption in set theory, one which we are constrained not to contradict in proposing new axioms. In other words, one recognizes the internal consistency of large cardinals, as opposed to their actual existence in the universe, as a *de facto* set-theoretic truth. An analagous phenomenon regards *projective determinacy* (PD): the IMH contradicts PD but is consistent with the determinacy of sets of reals which are ordinal-definable without real parameters. Thus the IMH violates the *principle of uniformity*, which asserts that natural projective statements relativise to real parameters, and one recognizes ordinal-definable determinacy without real parameters, as oppposed to PD, as a *de facto* set-theoretic truth. This discussion of the effects of the IMH on existing set-theoretic developments also applies to other criteria for preferred universes that arise in the Hyperuniverse Program.

The plan of this paper is as follows. In Section 2 we describe the hyperuniverse and consider its relation to V. In Section 3 we introduce criteria for preferred universes based upon principles of *maximality* and *omniscience*. The current state of the *Hyperuniverse Program* is summarized in Section 4, while the final appendix is devoted to a broader discussion of *maximality* as well as to the roles of large cardinals and projective determinacy in set-theoretic practice.

§2. The hyperuniverse. In contemporary set theory many methods are available for creating new universes, i.e., models of ZFC, starting from given ones: set-forcing, class-forcing, hyperclass-forcing (i.e., forcing whose conditions are classes),[7] and model-theoretic techniques.[8] As a result, a multitude of different universes are available to set-theorists. This abundance of ZFC models has recently led to the introduction of the *multiverse* as a new set-theoretic notion, and to related discussions about whether the multiverse may represent the proper starting point for addressing questions concerning truth in set theory. Depending on one's view as to which ZFC models should enter into it, quite different pictures of the multiverse have been suggested in the literature. Diverging views have been expressed as well as to how the multiverse may work as a proper framework for pronouncing on matters of set-theoretic truth. In this section we will review existing alternative proposals concerning the multiverse and present the *hyperuniverse* as an optimal realization of the multiverse concept.

Both Woodin and the second author have used the term *multiverse* for collections of universes obtained from one or more initial models of ZFC via some method for manipulating them. In particular, in [23] Woodin starts from countable transitive models M of ZFC, and takes the multiverse around M to be the collection generated by closing under set-generic extensions and set-generic ground models (this is what Woodin calls the (*set-*)*generic multiverse* generated from M).[9] Also V is regarded by Woodin so that a (set-)generic multiverse may be generated from it. To this purpose one considers (set-)generic extensions as Boolean valued models, i.e., models having the form $V^{\mathbb{B}}$, where $\mathbb{B}$ is a complete Boolean algebra. In contrast to this work, where Woodin *de facto* regards the notions "generic-extension" and "set-generic-extension" as synonymous, earlier work of the second author of this paper led to the introduction of the *class-generic multiverse around* L, obtained by closing L under class-forcing and class-generic ground models, as well as inner models of class-generic extensions that are not necessarily themselves class-generic (see [5]).[10] The *set-generic multiverse* and the *class-generic multiverse* are quite different: the former preserves large cardinals notions and does not lead beyond set forcing, whereas the latter can destroy large cardinals and leads to models that are not directly obtainable by class

[7]See [6].

[8]See, e.g., [14]. It is by manipulating universes of sets via these methods that mutually exclusive truth values can be assigned to set-theoretic sentences, thus proving them to be independent from ZFC.

[9]The restriction to countable transitive models of ZFC is due to the fact that the existence of forcing extensions for such models can be proved in ZFC.

[10]Woodin explicitly rejects the possibility of considering a multiverse built on class forcing: "there is no reasonable candidate for the definition of an expanded version of the set-generic multiverse which allows *class-forcing* extensions and yet which preserves the existence of large cardinals across the multiverse" ([23], p. 107). There are difficulties with Woodin's position. Class-forcing and hyperclass-forcing have the same status as set-forcing within set theory. Moreover, by restricting to set-forcing and assuming the existence of large cardinals, one artificially avoids a real difficulty of any unbiased search for new set-theoretic axioms, that of dealing with justifiable principles that may destroy large cardinal axioms.

forcing. Also Hamkins has recently formulated a view of the multiverse, apparently dissociating from both Woodin and the second author. What in [11] is referred to as the multiverse is, in fact, not a collection of ZFC models that can be generated from initial universes by closing under specified procedures. Rather, the multiverse is described as the multitude consisting of all set-theoretic universes that have been constructed so far and may be produced in the future, possibly including non well-founded models and models of systems other than ZFC. The result is a heterogeneous open-ended plurality, of which no overall unified description can be given.

Nor is there consensus among set-theorists as to whether and how the multiverse should be regarded as a context for determining questions of truth. Hamkins' proposal of a heterogeneous open-ended multiverse, for instance, is accompanied by the twofold invitation to abandon the "dream solution template for the CH", according to which the truth value of CH must be decided by some new axiom for set theory, and to consider the question whether CH holds or not as *already* definitively settled, as the result of our knowledge of the different truth values it may assume in the different universes of the multiverse. In [23], instead, the *set-generic multiverse* is introduced in order to scrutinize the *set-generic multiverse conception of truth*. According to the latter, a sentence formulated in the language of set theory is true if it holds absolutely in the multiverse generated by V, i.e., if it holds in each universe belonging to that multiverse. Were one to adopt the set-generic multiverse conception of truth, one should declare that a sentence like CH lacks truth value. This is not Woodin's conclusion, however. In fact he argues that the set-generic multiverse conception of truth is untenable because it violates principles that he regards as essential for any notion of truth for the set-theoretic universe (see [23]).

However note that, despite Woodin's and Hamkins' different mathematical understanding of the multiverse, and their diverging positions as to the status of sentences independent of ZFC, there is a point at which their views of the multiverse are more similar than may appear at the outset. In considering whether by invoking the multiverse one may introduce a suitable notion of truth for set-theoretic sentences, both Woodin and Hamkins tacitly start from the assumption that one should regard the multiverse as an ultimate plurality of ZFC models that cannot be transcended, i.e., reduced to something that is more fundamental. As a result, they are both led to candidates for a notion of set-theoretic truth that are highly incomplete, allowing sentences of set theory which are neither true nor false. This assumption, shared by Woodin and Hamkins, is worth emphasizing as it is clearly rejected by the Hyperuniverse Program (see Desideratum 2 below), which we now present as a distinctive approach to set-theoretic truth making use of the multiverse concept.

The *Hyperuniverse Program* may be understood as an attempt to arrive at new *de jure* set-theoretic truths by starting from a picture of the multiverse that faithfully summarizes the full plethora of results obtainable in contemporary set theory. That one focuses on well-founded models of ZFC when using this approach just amounts to expressing the twofold conviction that the axioms of ZFC are *de facto* set-theoretic truths and that it is only the well-founded models of this theory that provide plausible pictures of the set-theoretic universe. The Hyperuniverse Program thus begins by asserting that the multiverse should satisfy a maximality and a well-definedness criterion that only the collection of all countable transitive models of ZFC can meet. More precisely:

DESIDERATUM 1. *The multiverse should be as rich as possible but it should not be an ill-defined or open-ended multiplicity.*

In stating this, one has two aims. First, one is motivated by the fact that the methods for creating well-founded universes existing in contemporary set theory go well beyond set-forcing or class-forcing (hence the multiverse should include more than set- or class-generic extensions and ground models). Since the *hyperuniverse*, the collection of all countable transitive models of ZFC, is closed under all possible universe-creation methods, one is led to identifying the multiverse with it. Second, requiring in Desideratum 1 that the multiverse be given a precise mathematical formulation enables one to put it to work for the aim of enriching the realm of set-theoretic truth. This is done in the Hyperuniverse Program by formulating justifiable preferences for certain members of the hyperuniverse over others, thereby obtaining a selection of preferred universes. The requirement that the multiverse be well-defined is a necessary condition for this selection process to be possible, which would not be the case were the multiverse ill-defined or open-ended.

DESIDERATUM 2. *The hyperuniverse is not an ultimate plurality. One can express preferences for certain members of it according to criteria based on justified principles.*

Another key point in the Hyperuniverse Program is that first-order properties which are true across preferred universes of the hyperuniverse are true in V.

DESIDERATUM 3. *Any first-order property of V is reflected into a countable transitive model of ZFC which is a preferred member of the hyperuniverse.*

An important consequence of Desideratum 3 is that, while the criteria for preferred universes formulated within the Hyperuniverse Program may be non first-order (indeed the criteria that we will introduce in Section 3 are not—they quantify over the entire hyperuniverse), nonetheless in the Hyperuniverse Program one arrives at first-order axioms for set theory, these being the first-order truths shared by the preferred universes.

In justifying Desideratum 3 one may invoke the downward Löwenheim–Skolem theorem, which, however, *per se* only implies that there must be members of the hyperuniverse into which V first-order reflects. That these may be chosen as preferred elements of the hyperuniverse is an assumption that is made *ad hoc* within the Hyperuniverse Program, by arguing that it expresses a reasonable procedure for enlarging the realm of set-theoretic truth. There is no need in the Hyperuniverse Program to show that this strategy is the "right" one for arriving at new truths of set theory. In fact no Platonistic assumption underlies the program, no commitment to a view of V as a well-determined reality existing independently of mathematical practice to which one should be faithful when extending set-theoretic knowledge. As a result, within the Hyperuniverse Program no a priori distinction is drawn between right and wrong strategies for arriving at new set-theoretic truths. Instead one aims to formulate and justify procedures for finding new set-theoretic statements that one wishes to regard as ultimate and definitive. The reasonableness of the suggested procedure is the sole ground for the claim that the statements arrived at deserve to be regarded as true in V. Desiderata 2 and 3 amount to a proposal for a strategy toward finding new set-theoretic truths (a proposal which, in its full form, has to include explicit criteria for preferred elements of the hyperuniverse; we consider this in Section 3). How is one to argue for the reasonableness of this strategy?

Consider the aim of the Hyperuniverse Program. One wishes to master the wide variety of different pictures of V with which one is confronted in contemporary set theory, and which is faithfully represented by the hyperuniverse. Due to the downward Löwenheim–Skolem theorem, members of the hyperuniverse are candidates for conveying first-order information about V. Being confronted with a bewildering number of different options is a situation which we are familiar with not only in contemporary set theory. A behavior which we naturally adopt in such a situation is the following: we analyze what the possibilities are, choose among them those that under justified criteria look better than others (hence could be privileged on a priori grounds), and decide in favour of these. This is exactly what one does in the Hyperuniverse Program. In one's search for new truths of V one starts from the hyperuniverse, which most faithfully reflects the possible pictures of the set-theoretic universe. As one is not content with the hyperuniverse as an ultimate, non-transcendable context, one is led to the program described by desiderata 2 and 3, which amounts to singling out members of the hyperuniverse that possess optimal meta-mathematical properties (i.e., those which obey the criteria for preferred universes), so as to decide in favour of them for the purpose of enriching the realm of truth in V. The strategy of the Hyperuniverse Program is thus perfectly reasonable in light of its aims.

Let us emphasize that there is *per se* no guarantee that the criteria which we list below will lead to new axioms that both resolve independent questions and are compatible with *de facto* set-theoretic truth. I.e., by following them one is not sure at the outset that one will succeed in enlarging the realm of truth in V beyond the sentences that are already accepted as definitive in set theory. This is the result of the unbiased nature of the criteria for preferred universes being used. However, it turns out that by selecting universes according to our suggested criteria, one indeed obtains solutions to independent questions without conflicting with existing definitive truths of set theory. That this *de facto* happens may be invoked as a relevant a posteriori argument (an argument from *success*) for the reasonableness of the strategy suggested by the Hyperuniverse Program.

§3. Criteria for preferred universes. Which universes are preferred in the Hyperuniverse Program?

In Section 1 we made the point that by subscribing to the Hyperuniverse Program one is expected to conform to principles and criteria for preferred universes that arise from an *unbiased* look at the hyperuniverse, so as to obtain a selection of universes that is justifiable. The program thus excludes the possibility that needs arising from specific areas of set-theoretic or mathematical practice play a role in formulating criteria for preferred universes. Therefore statements to the effect that one should prefer universes in which principles hold that resolve the difficulties arising in a specific area of set theory or mathematics are not candidates for such criteria. Let us give some examples of such non-criteria.

a. The generalised continuum hypothesis (GCH), which is very effective in resolving a wide range of questions in set theory;[11]
b. $V = L$, a theory which yields a powerful infinitary combinatorics which can be used to resolve even more problems in set theory than GCH;
c. Projective Determinacy (PD), which yields an attractive theory of projective sets of reals;
d. Forcing axioms (such as MA, BPFA, BMM), which, like $V = L$, have great combinatorial strength.[12]

Criteria of this kind reflect the interests of specific groups of set-theorists or mathematicians. As a result, there may be as many different such criteria as there are areas of set theory or mathematics. Moreover, as interests in set theory change, so may these criteria. Thus at the outset, no selection of universes can be made according to them that can presume to be universally recognized as legitimate within the set-theoretic community as a whole. Is there a better way to select preferred universes?

The positive answer to this question given by the Hyperuniverse Program is that by focusing instead only on the most general features of the hyperuniverse and formulating principles based upon them, one is capable of suggesting (and justifying) criteria for preferred universes. This is based on the obvious fact that the hyperuniverse consists of ZFC models that may be

[11] See, e.g., [19] on the advantages of assuming GCH as an axiom.

[12] One may add to the list Woodin's axiomatic proposals and conjectures, introduced in [22], based on Ω-logic. The latter is a logic which can be proved (under the assumption of the existence of a proper class of Woodin cardinals) to be unaffected by set-forcing. But as discussed earlier, one cannot justify an exclusive focus on set-forcing in suggesting new axioms and conjectures.

mutually related (some universes may be, e.g., forcing extensions, ground models, or rank initial segments of others), and one may justifiably choose elements of the hyperuniverse that are "preferable" in terms of this comparison. These are explicitly identified with the universes that, with respect to those to which they are related, satisfy principles such as *maximality* or *omniscience*.

Before considering how an element of the hyperuniverse may succeed in being maximal, let us mention a danger of selecting universes according to principles and criteria derived from an unbiased look at the hyperuniverse. In doing so one may be led to the adoption of first-order statements which contradict *de facto* set-theoretic truth. Let us give an example. One may wish to make a selection of preferred universes based on a principle of *minimality*. One's criterion would therefore be that preferred universes should be as small as possible. This criterion may lead to the choice of just one universe, the *minimal model* of ZFC, which would have as an implication that the statement that set models of ZFC do not exist expresses a property of V. This is however in obvious conflict with set-theoretic practice, i.e., the existence of set models of ZFC does belong to the realm of *de facto* set-theoretic truth. The same applies to a weaker criterion inspired by a minimality principle, according to which one should prefer universes that satisfy the axiom of constructibility, $V = L$. Although the axiom of constructibility does allow for the existence of set models of ZFC (and more), it does not allow for the existence of inner models of ZFC with measurable cardinals. This too stands in conflict with set-theoretic practice, i.e., the existence of such models belongs to the realm of *de facto* set-theoretic truth (the point will be further discussed in the Appendix).

We turn now to the principle of *maximality*. A first point to make about maximality is that one cannot have "structural maximality" within the hyperuniverse, in the sense that a preferred universe should contain all possible ordinals or real numbers. For there is no tallest countable transitive model of ZFC and over any such model new reals can be added to obtain another such model. What principle of *maximality* may be then imposed on elements of the hyperuniverse?

(*Logical*) *Maximality*: let v be a variable that ranges over the elements of the hyperuniverse. v is (*logically*) *maximal* if all set-theoretic statements with certain parameters which hold externally, i.e., in some universe containing v as a "subuniverse", also hold internally, i.e., in some "subuniverse" of v.

Depending on what one takes as parameters and what one takes for the concept of "subuniverse", different criteria for maximal universes arise from (and are justified in light of) this principle. Here are two examples.

- *Criterion of ordinal* (*or vertical*) *maximality*: this criterion appeals to maximality with respect to the ordinals, where models have fixed the power-set operation. Let us define a universe w to be a *lengthening* of v if v is a (proper) rank initial segment of w. v is *ordinal maximal* iff it has a lengthening w such that for all first-order formulas φ and subsets A of v belonging to w, if $\varphi(A)$ holds in w then $\varphi(A \cap v_\alpha)$ holds in v_β for some pair of ordinals $\alpha < \beta$ in v (where v_α denotes the collection of sets in v of rank less than α).

- *Criterion of power set (or horizontal) maximality*: this criterion appeals to maximality with respect to power set, where models have fixed ordinals. If a parameter-free sentence holds in some outer model of v (i.e., in some universe w containing v with the same ordinals as v), then it holds in some inner model of v (i.e., in some universe v_0 contained in v with the same ordinals as v).

Ordinal (or vertical) maximality has a long history in set theory. It is also known as a higher-order *reflection principle*, and has been shown to imply (and to justify) the existence of "small" large cardinals (i.e., large cardinal notions consistent with $V = L$ such as inaccessibles, weak compacts, ω-Erdős cardinals, ...).[13] Power set maximality, instead, has been only recently formulated. In fact it is equivalent to the IMH, which formally speaking states that by passing to an outer model of v, *internal consistency* remains unchanged, i.e., the set of parameter-free sentences which hold in some inner model of v is not increased. Assessing the compatibility of power set maximality with *de facto* set-theoretic truths is no trivial matter. For the IMH refutes the existence of inaccessible cardinals as well as projective determinacy (PD) (see [7]). These implications have forced a re-examination of the roles of large cardinals and determinacy in set-theoretic practice. As a result one sees that power set maximality may be compatible with *de facto* set-theoretic truths after all. For, if one accepts that the role of large cardinals in set theory is correctly described by saying that their existence in inner models, and not their existence in V, is a *de facto* set-theoretic truth, and that the importance of PD is captured by its parameter-free version, then the compatibility of power set maximality with set-theoretic practice is restored: the IMH is in fact consistent both with inner models of very large cardinals and with parameter-free PD (indeed with OD-determinacy without real parameters).[14] We will return to the

[13] Stronger forms of reflection lead to much larger cardinals. These are the principles in which the parameter A is allowed to be a more complex object, such as a hyperclass (class of classes), hyperhyperclass (class of hyperclasses) .... Carrying this out in the natural way leads quickly to inconsistency as Koellner has pointed out (see [15]). Carrying this out using the concept of embedding restores consistency and via work of Magidor (see [17] or [13], Theorem 23.6) leads to an equivalence with the very large supercompact cardinals. However, it is not clear how to justify embedding reflection principles as unbiased or even natural principles of ordinal maximality, due to the arbitrary nature of the embeddings involved (the relationship between A and its "reflected version" is given by an embedding with no uniqueness properties).

[14] In particular the IMH is consistent with the regularity of all parameter-free definable projective sets of reals. Allowing arbitrary real parameters makes a big difference and converts a principle compatible with the IMH to one which is not.

role of both large cardinal axioms and PD within set theory in the Appendix.

What conclusion can be drawn as to justified criteria for preferred universes? We have so far formulated two candidate criteria: ordinal maximality and power set maximality. The ideal situation would be to combine them into a single consistent criterion, i.e., a criterion that is satisfied by at least one element of the hyperuniverse. This is not trivial, since power set maximality and ordinal maximality contradict each other. One is thus led to the following conjecture:

SYNTHESIS CONJECTURE. Let *power set maximality** (IMH*) be *power set maximality* (IMH) restricted to ordinal maximal universes (i.e., the statement that if a sentence holds in an ordinal maximal outer model of v then it holds in an inner model of v). Then the conjunction of *power set maximality** (IMH*) and ordinal maximality is consistent. I.e., there are universes which simultaneously satisfy both criteria.

A proof of the *Synthesis conjecture* is within reach, as it only demands the existing method for proving the consistency of the IMH (see [8]) together with a careful understanding of how Jensen coding can be done in the presence of small large cardinal properties. Via the Hyperuniverse Program, the *Synthesis Conjecture* is effective in yielding new (first-order) set-theoretic axioms, including solutions to independent questions. As universes which witness the Synthesis Conjecture (i.e., which are ordinal maximal and satisfy IMH*) are preferred universes, first-order properties shared by all such universes are true in V and may be adopted as new axioms. Examples of such statements are the following (see [7], [8], [1]):

1. There are *small* large cardinals and inner models with *measurable* cardinals of arbitrary Mitchell order.
2. For some real R, $R^{\#}$ does not exist and so Jensen covering holds with respect to $L[R]$, the constructible universe relativised to R. As a result:
3. There are no measurable cardinals, the *singular cardinal hypothesis* is true, the continuum is not real-valued measurable, *projective determinacy* (PD) is false, the *proper forcing axiom* is false and there are non-Borel analytic sets which are not Borel isomorphic.

The *Continuum Hypothesis* remains undecided, even assuming that there exists a universe that obeys the *Synthesis Conjecture*. One needs a stronger version of power set maximality than the Inner Model Hypothesis to settle CH, i.e., the hypothesis for formulas with globally absolute parameters.[15] A consistency proof for the resulting *Strong Inner Model Hypothesis* (SIMH) is however still lacking.

Omniscience and a Grander Synthesis? Yet another source of criteria for preferred universes is the principle of *omniscience* A universe is *omniscient*

[15] See [8].

if it is able to describe what can be true in alternative universes. A precise criterion based upon this principle is the following.

CRITERION OF OMNISCIENCE. Let Φ be the set of sentences with arbitrary parameters from v which can hold in some outer model of v. Then Φ is first-order definable in v.

This kind of statement first appeared in unpublished work of Mack Stanley, where he shows that there are omniscient universes (in our terminology), assuming a bit less than the consistency of a measurable cardinal (stationary-many Ramsey cardinals, roughly speaking). One may be tempted to regard omniscience as a form of power set maximality; this is unlikely, however, for whereas power-set maximality does not allow any parameters, the principle of omniscience allows arbitrary set parameters.

Synthesizing ordinal maximality with omniscience should not be difficult, using indiscernibles for the Dodd–Jensen core model in the presence of Ramsey cardinals. An intriguing open question is how to achieve a grander synthesis with power set maximality. The obvious approach, asserting power set maximality for omniscient and ordinal maximal universes, appears to be inconsistent. It is nevertheless reasonable to conjecture that some such grand synthesis is possible, but its formulation will be subtle and the mathematics required to verify consistency may be challenging.

§4. Conclusions. The Hyperuniverse Program presented in this paper is a new approach to set-theoretic truth, aimed at enlarging the realm of true-in-V statements beyond ZFC. To this purpose the program develops a justifiable strategy, and regards the intrinsic reasonableness of this strategy as a guarantee for the truth of the results obtained. More precisely, one introduces the hyperuniverse as the most suitable realization of the multiverse concept and puts it to work for the purpose of comparing different pictures of the set-theoretic universe (countable transitive models of ZFC) in light of criteria for preferring some universes over others. First-order properties shared by all preferred universes are taken to be true in V. By invoking the criteria of ordinal (vertical) maximality and power set (horizontal) maximality, a suitable realization of the program is obtained. By postulating the existence of an element of the hyperuniverse that satisfies the natural synthesis of these criteria (i.e., the Synthesis Conjecture), one arrives at statements which are true in V yet independent from ZFC. These statements contradict the existence of very large cardinals but are consistent with their existence in inner models, and they contradict projective determinacy but are consistent with determinacy for sets of reals which are ordinal-definable without real parameters. This leads to a reassessment of the roles of large cardinals and determinacy in set theory.

It is worth observing that, although the realization of the Hyperuniverse Program presented in this paper fails to resolve many interesting

ZFC-independent questions and raises issues that call for further investigation (starting with the consistency of the Synthesis Conjecture), this by no means undermines the overall validity and mathematical fruitfulness of the program. Quite the contrary, the research outcomes obtained and the questions entailed by the developments inspired by the Hyperuniverse Program attest to its mathematical potential and speak of its promise for the future, as further principles (such as *omniscience*) that motivate criteria for preferred universes are analyzed and discovered, and a synthesis is sought for them in conjunction with maximality.

§5. Appendix: the hyperuniverse program, maximality, large cardinals and PD. This Appendix is devoted to a closer examination of the relation of the *Hyperuniverse Program* to alternative proposals for extending set-theoretic truth (beyond ZFC and other *de facto* true set-theoretic statements), in particular to *large cardinals* and *Projective Determinacy* (PD) as candidates for axioms of set theory.

Gödel's Program for new axioms, sketched in Section 1, includes the recommendation to consider some *maximum* property of the system of all sets for the purpose of extending ZFC. Since *maximality* is used in the Hyperuniverse Program as a motivating principle for criteria for preferred universes, we advocated above that this program meets Gödel's recommendation. Of course, while claiming this, we are aware of the fact that the considerations concerning *maximality* developed within the Hyperuniverse Program are of a different nature than those invoked in alternative proposals for new set-theoretic axioms. This applies, in particular, to proposals to the effect that ZFC should be enlarged through the addition of suitable large cardinal hypotheses, as these being faithful to our expectations concerning the *maximum* character of the universe of all sets. Consider the following quotation by H. Wang ([21], p. 553):

> We believe that the collection of all ordinals is very 'long' and each power set (of an infinite) set is very 'thick'. Hence any axioms to such effect are in accordance with our intuitive concept.

By giving, as Wang does, the length of the ordinals and the thickness of power sets as examples of *maximum* properties of the system of all sets, one *ipso facto* starts from the assumption that by "*maximum* properties of the system of all sets" one means ontological features of V related to what "exists" within it. In making this assumption, one may either intend V as an independently existing well-determined reality (this seems to be the choice of Gödel in [9]), or (at least partly) as a well-determined epistemic notion, a mental representation of the universe which we are naturally led to by our intuitions concerning sets (Wang seems to see V in [21] in this way, appealing to the iterative concept of set). In either case one can only realize the idea that the system of sets displays *maximum* properties by making existential assertions concerning V. In fact, arguments aimed at showing that the existence of large cardinals witnesses both the length of the ordinals and the thickness of power set, and are therefore faithful to the assumption that the universe is *maximal*, have been repeatedly given in the literature.[16] Also *forcing axioms* have been advocated as "natural" axioms for set theory, given their "maximizing" existential implications (see [2]).

In the Hyperuniverse Program, V is nowhere invoked as an independently existing well-determined reality. Nor is it called upon as a determined picture of the universe forced upon us by our intuitions concerning sets. Instead, V is intended as a meta-mathematical outcome. In saying this, we are not thinking of the final result of a concluding process. What we have in mind is an *ideal* condition, which one can only better and better approximate. In the Hyperuniverse Program, V denotes the structure that satisfies whatever set-theoretic statements deserve to be regarded as true (either as a *de facto* or as a *de jure* set-theoretic truth). I.e., the content of V, far from being understood in terms of a reality which is *in se* determined and which we should be faithful to when doing set theory, is meant to be a product of our own, progressively developing along with the advances of set theory, the development of the program with the resulting enrichment of the realm of set-theoretic truth.

In particular, in the Hyperuniverse Program V plays the role of an outcome that one can only approach by starting from the hyperuniverse as the most suitable instantiation of the multiverse notion. The fact that within the Hyperuniverse Program one endorses a multiverse perspective is explained as by Woodin, who says that "the refinements of Cohen's method of *forcing* in the decades since its initial discovery and the resulting plethora of problems shown to be unsolvable, have in a practical sense almost compelled one to adopt" a multiverse position in contemporary set theory ([23], p. 103). Consider that, from a multiverse perspective, one works not with a unique "system of all sets" but with many different ones, and deals with them as meta-mathematical constructions, as models. From a multiverse perspective one is thus naturally led to understand the expression "maximal properties of the system of all sets" in terms of meta-mathematical features revealed by comparing set-theoretic models. This is what is done in the Hyperuniverse Program. Criteria like vertical and horizontal maximality are the rigorous expression of what it means for an element of the hyperuniverse, i.e., a countable transitive model of ZFC, to display "maximum properties". To put it in other terms, no need is seen in the Hyperuniverse Program to make existential assertions concerning V in order to be faithful to the idea that the system of all sets must be maximal; in particular no need is seen to assume the existence of large cardinals in the universe. Vice-versa, implications like those

[16]See [16] for an extensive review of arguments given by set-theorists as to the faithfulness of large cardinals to maximality.

of the *Synthesis Conjecture* concerning large cardinals are not seen as contradicting one's maximality expectations concerning the "system of all sets".

As discussed earlier, in the Hyperuniverse Program the non-existence of very large cardinals (above a measurable) in V is not only supposed to be compatible with maximality expectations concerning models of ZFC, it is also viewed as compatible with *de facto* set-theoretic truth. This follows from a cautious examination of the role played by large cardinal assumptions in contemporary set theory, leading to the view that, although large cardinals arise in set theory in a number of ways, their importance derives from their existence in inner models. Indeed, when proving that the consistency strengths of large cardinal extensions of ZFC fall into a well-ordered hierarchy one need only consider large cardinal existence in inner models. This is also the case for *consistency upper and lower bound results*, the most important use of large cardinals in set theory. For upper bound results one starts with a model M of ZFC which contains large cardinals and then via forcing produces an outer model $M[G]$ in which some important statement holds. Notice that in the resulting model, large cardinals may fail to exist; they only exist in an inner model, namely the original M. And of course we do not have to assume that the initial M is the full universe V, it is sufficient for it to be any inner model with large cardinals. In lower bound results, one starts with a model M satisfying a statement of interest and then constructs an inner model with a large cardinal; this is the Dodd–Jensen *core model program*; see [12]. As Steel points out, "we know of no way to compare the consistency strengths of PFA and the existence of a total extension of Lebesgue measure except to relate each to the large cardinal hierarchy" ([20], footnote 22, p. 427). By invoking this fact he adds: "the large cardinal hierarchy is essential". However once again, in proving the consistency results which make large cardinals "essential", one only assumes their existence in inner models.[17] A similar argument applies to the *inner model program*, whose aim is to show that if large cardinals exist in V then they also exist in well-behaved inner models; this is equivalent to the program of showing that if large cardinals exist in an inner model then they also exist in an even smaller, well-behaved inner model.

A possible objection to the above is that one uses large cardinals in V, rather than in inner models, to prove forms of the *axiom of determinacy*, such as PD, determinacy for all projective sets of reals. There are two common reasons given for asserting that PD is "true". One reason is based

[17]Similar views as to the role of large cardinals in set theory are expressed by Shelah. See [19]. An opposing view, expressed in [23], is that the only basis for believing in the consistency of large cardinal axioms is believing in their truth in V. One can object, however, that Woodin's argument is based on a false analogy between large infinities and large finite sets. It is true that the existence of large finite sets is implied by their consistency; this is simply because V_ω has no proper inner models and therefore the existence of large finite sets is the same as their existence in inner models. This is obviously not the case with large infinities.

on *extrapolation*: Since Borel and analytic sets are well-behaved (in the sense that they are Lebesgue measurable and have the Baire and perfect set properties) and PD extends this to all projective sets, then PD must be "true". But there are clear rebuttals to this argument. Consider, for instance, Lévy–Shoenfield absoluteness, the absoluteness of Σ^1_2 statements with respect to arbitrary outer models. This is provable in ZFC even if one allows arbitrary real parameters. Extrapolation then naturally leads to Σ^1_n absoluteness with arbitrary real parameters. But even Σ^1_3 absoluteness with arbitrary real parameters is provably false. With arbitrary real parameters a consistent principle can only be obtained by artificially taking "outer model" to mean "set-generic outer model". As soon as one relaxes this to class-generic outer models, the principle becomes inconsistent.

So, if one is so easily led to inconsistency when extrapolating from Σ^1_2 to Σ^1_3 absoluteness, how can one justify the extrapolation from Σ^1_1 measurability to projective measurability? More reasonable would be the extrapolation *without parameters*. Indeed, parameter-free Σ^1_3 absoluteness, unlike the version with arbitrary real parameters, is consistent with (and indeed follows from) the IMH. Thus a natural conclusion with regard to projective statements is the following: The principle of *uniformity*, which asserts that properties that hold for parameter-free projective sets also hold for arbitrary projective sets is false. Thus the regularity of projective sets is a reasonable extrapolation from the regularity of Borel and analytic sets *provided one does not allow parameters*. Indeed, parameter-free PD (or even ordinal-definable determinacy without real parameters) and the existence of inner models with very large cardinals are consistent with the IMH (and very likely with a witness to the *Synthesis Conjecture*), but PD with parameters and the existence of inner models with very large cardinals containing an arbitrarily given real are not.

A second reason for asserting the "truth" of PD is that it "settles all natural questions about HC (the set of hereditarily countable sets)". This assertion is based on the fact that assuming large cardinals, you cannot change the first-order theory of HC by set-forcing and this theory is in some sense described by PD. But this ignores the fact the theory of HC can change, even at the least possible level (Σ^1_3) if one allows other ways of enlarging the universe, even ways which preserve the existence of very large cardinals. And there are simple examples of such statements (such as the existence of models of very large cardinals with a small amount of "iterability"). The conclusions reached by the Hyperuniverse Program through the use of maximality principles also yield strong conclusions about the theory of HC (conflicting with PD), but without any need to refer to "set-forcing".

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