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NON-STANDARD LATTICES AND O-MINIMAL GROUPS

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Abstract. We describe a recent program from the study of definable groups in certain o-minimal structures. A central notion of this program is that of a (geometric) *lattice*. We propose a definition of a lattice in an arbitrary first-order structure. We then use it to describe, uniformly, various structure theorems for o-minimal groups, each time recovering a lattice that captures some significant invariant of the group at hand. The analysis first goes through a local level, where a pertinent notion of pregeometry and generic elements is each time introduced.

§1. Introduction. The study of groups definable in models of a first-order theory has been a core subject in model theory spanning at least a period of thirty years. On the one hand, definable groups are present whenever non-trivial phenomena occur in the models of a theory and their study has played a prominent role in Shelah's classification theory. On the other hand, a large variety of classical groups turn out to be definable in certain structures and their study via model-theoretic methods has given rich applications to other areas of mathematics. For example, an algebraic group is definable in an algebraically closed field and a compact real Lie group is definable in some o-minimal expansion of the real field.

It is the second kind of examples which we seek to embark on here. O-minimal structures provide a rich, yet tame, model-theoretic setting where definable sets enjoy many of the nice topological properties that hold for semi-algebraic sets. For example, a topological notion of *dimension* can be defined for every definable set. It is often said that o-minimality is the correct formalization of Grothendieck's 'topologie modérée' [8].

Groups definable in an o-minimal structure, in their turn, henceforth called 'o-minimal groups', strikingly resemble real Lie groups. The starting point for the study of o-minimal groups was Pillay's theorem in [40] that every such group admits a definable manifold topology that makes it into a topological group. Since then, an increasing number of theorems have reinforced the

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resemblance of o-minimal groups with real Lie groups, culminating in the solution of Pillay's Conjecture (PC) in recent years. (PC) says, in its simplified form, that every definably connected, definably compact o-minimal group G admits a surjective homomorphism onto a real Lie group, whose dimension (as a Lie group) is equal to the o-minimal dimension of G. The reader is referred to [33] for a detailed account on the history of o-minimal groups.

Depending on what kind of ambient o-minimal structure we are dealing with, different methods for analyzing definable groups have been developed. The main dichotomy has been whether the structure expands a real closed field or not. Let us recall some definitions. An *o-minimal* structure $\mathcal{M} = \langle M, <, \dots \rangle$ is a structure with a dense linear order $<$ such that every definable subset of M is a finite union of open intervals and points. The standard setting where definable groups have been studied is that of an o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ of an ordered group (although, newly in [24], the assumption of the ambient group structure is being removed.) It follows from [37] and [10] that we can have exactly the following cases:

A. $\mathcal{M} = \langle M, <, +, \cdot, 0, 1, \dots \rangle$ expands an ordered field (which, by o-minimality, will have to be a real closed field).
B. $\mathcal{M}$ does not expand a real closed field. Here we can further split into two subcases:
   B1. $\mathcal{M}$ contains a definable real closed field whose domain is a bounded interval $I \subseteq M$.
   B2. No real closed field is definable.

We refer to case (A) as the 'field case', to (B1) as the 'semi-bounded case' and to (B2) as the 'linear case'. The typical example of a linear structure is that of an ordered vector space $\mathcal{M} = \langle M, <, +, 0, \{d\}_{d\in D}\rangle$ over an ordered division ring D. An important example of a semi-bounded structure is the expansion $\mathcal{B}$ of the real ordered vector space $\mathbb{R}_{vect} = \langle \mathbb{R}, <, +, 0, \{d\}_{d\in\mathbb{R}}\rangle$ by all bounded semi-algebraic sets. Every bounded interval in $\mathcal{B}$ admits the structure of a definable real closed field. For example, the field structure on $(-1, 1)$ induced from $\mathbb{R}$ via the semi-algebraic bijection $x \mapsto \frac{x}{\sqrt{1+x^2}}$ is definable in $\mathcal{B}$. By [41, 31, 35], $\mathcal{B}$ is the unique structure that lies strictly between $\mathbb{R}_{vect}$ and the real field. The situation becomes more subtle when $\mathcal{M}$ is non-archimedean, where only some intervals admit a definable real closed field. The relevant definitions are given in Section 3.2 below.

When $\mathcal{M}$ expands a real closed field, a rich machinery from *o-minimal algebraic topology* is at our disposal. For example, the triangulation theorem is known to hold in this case, giving rise to (co)homology theory [16]. This machinery has been successfully used to describe the torsion points of o-minimal groups [14], a key point in the solution of (PC).

On the contrary, if $\mathcal{M}$ does not expand a field, the above machinery is not always present. We illustrate, for example, how the triangulation theorem fails in the linear case.

EXAMPLE 1.1. Let $\mathcal{M} = \langle M, <, +, 0 \rangle$ be a non-archimedean ordered divisible abelian group. This is naturally a vector space over the field of rationals. Consider the parallelogram $S = [0, a] \times [0, b]$, where b is infinitely smaller than a; that is, for every $n \in \mathbb{N}$, $nb < a$. It is easy to see that S cannot be written as a finite union of (closures of) triangles. The reason is that the definable lines in $\mathcal{M}$ can only have rational slopes, and therefore any triangle that can be placed inside the parallelogram will have sides of length infinitely smaller than a.

A substitute for o-minimal cohomology in arbitrary o-minimal structures was recently proposed in [15, 13]. However, a serious consequence of the failure of the triangulation theorem was illustrated in [17] by an example of a semi-linear group which cannot be definably, homeomorphically embedded in the affine space.

Despite the lack of machinery from algebraic topology, a straightforward analysis of definable groups in case (B) has been proposed. This analysis differs from the previous approaches for studying o-minimal groups in that any algebraic topological facts about definable groups *follow* from the analysis rather than being used in it. The analysis consists of two steps:

Step I. Study the group behavior locally, around suitable generic elements.
Step II. Extend the local analysis to a global one.

The structure theorems for definable groups that are obtained from this approach have one aspect in common: each time a *lattice* is recovered which captures some significant invariant of the group at hand.

In this communication, we aim to introduce the notion of a lattice for an arbitrary first-order structure and exemplify its role in the study of o-minimal groups. Through various theorems and conjectures, we suggest that the above program has the potential to find further applications in other o-minimal settings, as well as in the general study of definable groups in model theory.

Structure of this paper. In Section 2, we introduce the notion of a lattice and discuss briefly some of its properties. In Section 3, we use lattices to describe structure theorems for o-minimal groups and conjecture a similar result in a broader context. In Section 4, we deal with the aforementioned Step I: the local analysis. In Section 5, we motivate the study of lattices from a different viewpoint.

§2. Lattices.

We first recall the standard definition of a lattice in $\mathbb{R}^n$ and then proceed to generalize it. Even though our definition makes sense in an arbitrary first-order structure, our main intention is to employ lattices within the o-minimal setting.

DEFINITION 2.1. A (geometric) *lattice L in $\mathbb{R}^n$ of rank m* $(\leq n)$ is a subgroup of $\langle \mathbb{R}^n, + \rangle$ which satisfies any of the following equivalent statements:

(1) L is discrete and spans an m-dimensional subspace of $\mathbb{R}^n$ (over $\mathbb{R}$).
(2) L is discrete and is generated by m $\mathbb{Z}$-independent elements of $\mathbb{R}^n$.
(3) L is generated by m $\mathbb{R}$-independent elements of $\mathbb{R}^n$.
(4) L is generated by m elements of $\mathbb{R}^n$ and the quotient group $\mathbb{R}^n/L$ is a connected Lie group (equipped with the quotient topology).

For example, $\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$ of rank n and $\mathbb{R}^n/\mathbb{Z}^n$ is the well-known n-torus.

We propose below a definition of a lattice in the non-standard setting which we view as a 'definable analogue' of statement (4) of the above definition. The choice of our definition is motivated by our wish to view a lattice also as a topological object, besides an algebraic one.

When we deal with an arbitrary vector space, not necessarily topological, one can find in the literature the following adaptation of statement (3) of the last definition:

Let $\mathcal{V}$ be a vector space over a field K. Then a *lattice in $\mathcal{V}$ of rank m* is a subgroup of $\mathcal{V}$ which is generated by m K-independent elements of $\mathcal{V}$. $(*)$

From our perspective this definition involves the following subtlety. If we consider the reals as a vector space over the rationals, then according to $(*)$ any number of $\mathbb{Q}$-independent elements of $\mathbb{R}$ generates a lattice. In particular, such a lattice need not be discrete. This situation may of course seem natural in the setting of the infinite dimensional vector space $\mathbb{R}$ over the rationals. But it is also exactly what shows that the topological nature of a lattice is not taken into account in Definition $(*)$.

On the other hand, requiring further that a lattice be discrete, as in (1) and (2) above, only imposes a trivial restriction in the non-standard setting. Indeed, in a non-archimedean extension $\mathcal{R}$ of the real field any number of $\mathbb{Z}$-independent elements always generates a discrete subgroup of $\langle R, + \rangle$, by saturation.

We thus consider possibility (4). A direct adaptation of that statement would also be problematic, albeit in a minor way. If $\mathcal{R}$ is a non-archimedean extension of the real ordered group $\langle \mathbb{R}, <, +, 0 \rangle$, the quotient $R/\mathbb{Z}$ is very distant from being a connected Lie group. Yet, we would like to have $\mathbb{Z}$ as a lattice. We remedy this by replacing R by its finite part $Fin(R) = \bigcup_{n \in \mathbb{N}}(-n, n)$; the subgroup of $\mathcal{R}$ consisting of all elements of $\mathcal{R}$ bounded by some natural number. Then the quotient group $Fin(R)/\mathbb{Z}$ is isomorphic to a group *definable* in $\mathcal{R}$, namely the group $\langle [0, 1), \oplus, 0 \rangle$ with operation $x \oplus y = x + y \mod 1$. The latter group, in its turn, projects under the standard part map onto the real 1-torus, illustrating a simple case of (PC), and establishing the connection with Lie groups in this case. We conclude that it is more suitable to speak of lattices in $\bigvee$-*definable groups*, such as $Fin(R)$, rather than lattices in $\mathcal{R}$.

Our plan is to define the notion of a $\bigvee$-definable group and then give our definition of a lattice. Our definition will be *stricter* than Definition $(*)$. Besides the above motivations, our choice will be further supported in the next section where lattices will show up as natural objects in the study of o-minimal groups.

Let $\mathcal{M}$ be any sufficiently saturated first-order structure. By a 'definable' set or function, we mean 'definable in $\mathcal{M}$ possibly with parameters'.

Definition 2.2. [38, 11] A $\bigvee$-*definable group* is a group $\langle \mathcal{U}, \cdot \rangle$ whose universe is a directed union $\mathcal{U} = \bigcup_{i \in I} X_i$ of definable subsets of M^n, for some fixed n, where I is countable, and for every $i, j \in I$, the restriction of the group multiplication to $X_i \times X_j$ is a definable function (by saturation, its image is contained in some X_k).

A map $\phi \colon \mathcal{U} \to \mathcal{H}$ between $\bigvee$-definable groups is called $\bigvee$-*definable* if for every definable $X \subseteq \mathcal{U}$ the restriction $\phi_{\restriction X}$ is definable.

In the case where the ambient structure $\mathcal{M}$ is o-minimal, we define the *dimension* of $\mathcal{U}$ to be the maximum dimension among all X_i's.

Note: the condition that I is countable is often relaxed in the literature to the condition that I be small, relative to the saturation of $\mathcal{M}$. In our context, countability of I is used to guarantee that certain quotients of a $\bigvee$-definable group are $\bigvee$-definable (Fact 2.4 below).

Definition 2.3. [11] A $\bigvee$-definable subset A of $\mathcal{U}$ is called *compatible in* $\mathcal{U}$ if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap A$ is a definable set.

Clearly, the kernel of a $\bigvee$-definable map $\phi \colon \mathcal{U} \to \mathcal{V}$ between two $\bigvee$-definable groups is a compatible normal $\bigvee$-definable subgroup of $\mathcal{U}$. In the o-minimal setting, we also obtain the converse.

Fact 2.4. [11, Theorem 4.2] *If $\mathcal{U}$ is a $\bigvee$-definable group in an o-minimal structure and $\mathcal{H} \subseteq \mathcal{U}$ is a $\bigvee$-definable normal subgroup, then $\mathcal{H}$ is a compatible subgroup of $\mathcal{U}$ if and only if there exists a $\bigvee$-definable surjective homomorphism $\phi \colon \mathcal{U} \to \mathcal{V}$ of $\bigvee$-definable groups whose kernel is $\mathcal{H}$.*

We think of the above $\bigvee$-definable group $\mathcal{V}$ as the quotient of $\mathcal{U}$ by $\mathcal{H}$, and we are interested in the case where this quotient is actually definable.

Definition 2.5. [25, 22] Given a $\bigvee$-definable group $\mathcal{U}$ and a normal subgroup $L \subseteq \mathcal{U}$, the quotient group $\mathcal{U}/L$ is called *definable* if there is a definable group K and a surjective $\bigvee$-definable homomorphism $\phi \colon \mathcal{U} \to K$ whose kernel is L. We write $K = \mathcal{U}/L$.

Here is our definition of a lattice.

DEFINITION 2.6. Let $\mathcal{U}$ be a $\bigvee$-definable group and m an integer. A normal subgroup $L \subseteq \mathcal{U}$ is called a *lattice in $\mathcal{U}$ of rank m* if it is generated by m $\mathbb{Z}$-independent elements, and the quotient $\mathcal{U}/L$ is definable.

Being the kernel of a $\bigvee$-definable homomorphism, a lattice L in $\mathcal{U}$ is a compatible $\bigvee$-definable subgroup of $\mathcal{U}$. Note also that since L is finitely generated, it has small size relative to the saturation of $\mathcal{M}$. Hence, for a definable set X, the condition that $X \cap L$ be definable amounts to $X \cap L$ be finite. It follows that any subgroup of a lattice L will be compatible as well.

As an example, if $\mathcal{R}$ is a non-archimedean extension of the real ordered group $\langle \mathbb{R}, <, +, 0\rangle$, the group $\mathbb{Z}$ is a lattice in $Fin(R)$, according to our earlier discussion.Le

In the rest of this section, we list a number of desirable properties about lattices which have been proved under some (broad enough) assumptions. In the next section we will present the main theorems for o-minimal groups and lattices. The results of this section are independent of those theorems.

First, we recall some standard definitions.

DEFINITION 2.7. Let G be a definable group in an o-minimal structure. By [40], G admits a unique definable manifold topology that makes it a topological group. We simply refer to this topology as the *group topology*. We call G *definably compact* if for every definable map f from an open interval (a, b) into G, the limits of $f(x)$ as x tends to a and to b (with respect to the group topology) exist. We call G *definably connected* if it contains no proper non-empty definable subset which is both closed and open.

A $\bigvee$-definable group $\mathcal{U}$ can also be endowed with a manifold topology that makes it into a topological group. We refer the reader to [2, Theorem 4.8] for details. We call $\mathcal{U}$ *connected* [1] if it contains no $\bigvee$-definable compatible proper non-empty subset which is both closed and open.

DEFINITION 2.8. Given a $\bigvee$-definable group $\mathcal{U}$, let us say that $\mathcal{U}$ *admits a lattice* if there is a lattice in $\mathcal{U}$.

In Facts 2.9–2.11 below we assume the following:

(1) $\mathcal{M}$ is a sufficiently saturated o-minimal expansion of an ordered group.
(2) $\mathcal{U}$ is a connected abelian $\bigvee$-definable group which admits a lattice.

FACT 2.9. [22] *If L and L' are lattices in $\mathcal{U}$, then L and L' are isomorphic.*

FACT 2.10. [23] *If L is a lattice in $\mathcal{U}$, then* $\operatorname{rank}(L) \leq \dim(\mathcal{U})$.

FACT 2.11. *If L is a compatible subgroup of $\mathcal{U}$ isomorphic to some $\mathbb{Z}^m$, then L extends to a lattice; that is, there is a lattice L' in $\mathcal{U}$ with $L \subseteq L'$.*

The last fact does not appear in the literature but follows easily from the results we quote in Section 5 below. We include a proof there. For the moment, we remark that the assumption of L being compatible in $\mathcal{U}$ is necessary. Indeed, if L extends to a lattice L', then as remarked earlier, L' is compatible and, thus, so is L.

EXAMPLE 2.12. Let $\mathcal{M} = \langle M, <, +, 0\rangle$ and a, b be as in Example 1.1, $\mathcal{U}$ the subgroup of $\langle M, +\rangle$ generated by $[-a, a]$ and $L = \mathbb{Z}b$. Then $L \cap [-a, a]$ is not definable, so L is not compatible in $\mathcal{U}$. Hence, L does not extend to a lattice. On the other hand, $\mathbb{Z}a$ is a lattice in $\mathcal{U}$.

Here is an example of a $\bigvee$-definable group $\mathcal{U}$ that does not admit any lattice.

EXAMPLE 2.13. Let $\mathcal{M} = \langle M, <, +, 0\rangle$ be as in the previous examples and consider an infinite increasing sequence of elements $0 < a_1 < a_2 < \dots$ such that, for every $n \in \mathbb{N}$, we have $na_i < a_{i+1}$. The subgroup $\mathcal{U} = \bigcup_i(-a_i, a_i)$ of $\langle M, +\rangle$ is a $\bigvee$-definable group which does not admit a lattice. Indeed, for every $b \in (-a_i, a_i)$, the set $\mathbb{Z}b \cap (-a_{i+1}, a_{i+1})$ is not definable.

Observe that in the last example $\mathcal{U}$ is not generated by any definable set. In Conjecture 3 in Section 5 below, we suggest that this is the only obstacle to a $\bigvee$-definable group admitting a lattice.

§3. Structure theorems for o-minimal groups.

This section contains the main theorems about lattices and o-minimal groups. Unless stated otherwise, $\mathcal{M} = \langle M, <, +, 0, \dots\rangle$ denotes a sufficiently saturated o-minimal expansion of an ordered group.

Definition 2.6 suggests that a lattice presupposes a definable group. The converse is also true: given a definable group, we can always recover a lattice.

THEOREM 3.1. [12] *Let G be an abelian, definably connected definable group. Then there is a connected, divisible, torsion-free $\bigvee$-definable group $\mathcal{U}$ and a lattice L in $\mathcal{U}$ such that $G = \mathcal{U}/L$.*

This theorem is very general in nature. The $\bigvee$-definable group $\mathcal{U}$ is the 'universal cover' of G (see [12]) and the lattice L is isomorphic to the fundamental group of G. Observe, however, that no information is given about how $\mathcal{U}$ (or L) are related to the ambient o-minimal structure. Moreover, no information is given about the rank of L, even when G is definably compact. One motivation for the subsequent theorems is to recover some of this information when we work in structures of sort (B1) or (B2) from the Introduction. Let us first consider the linear case.

3.1. Linear case.

THEOREM 3.2. [25, 19] *Assume $\mathcal{M} = \langle M, <, +, 0, \{d\}_{d\in D}\rangle$ is an ordered vector space over an ordered division ring. Let G be a definably connected definable group of dimension n. Then $G = \mathcal{U}/L$, with $\mathcal{U}$ and L as in Theorem* 3.1 *and, moreover*:

1. *$\mathcal{U}$ is a* subgroup *of $\langle M^n, +\rangle$ generated by a definable set.*
2. $\operatorname{rank}(L) = \dim(G/H)$, *where H is a maximal torsion-free definable subgroup of G.*

A few comments are in order. Observe that (1) implies that G is abelian. By [39], we know that every abelian group G, definable in any o-minimal structure, contains a maximal torsion-free definable subgroup H and that G/H is definably compact. We can call $\dim(G/H)$ the *compact dimension* of G. Theorem 3.2, then, says that the lattice we recover in the linear case is a subgroup of $\langle M^n, +\rangle$ which encodes the compact dimension of G. A corollary is that the subgroup $G[k]$ of k-torsion points of G is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^s$, where s is the compact dimension of G (see [19, Corollary 3.12]). In the special case where G is definably compact, we obtain that $G[k]$ is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$, where n is the dimension of G. This was known to be a crucial step in establishing (PC) in various cases, and indeed, together with work from [4], (PC) in the linear case was established in [25].

EXAMPLE 3.3. 1. (Simplest example). Let $\mathcal{M} = \langle M, <, +, 0\rangle$ be an ordered divisible abelian group. Pick any positive $a \in M$ and define:

- the $\bigvee$-definable subgroup $\mathcal{U}_a = \bigcup_{n\in\mathbb{N}}[-na, na]$ of $\langle M, +\rangle$.
- the lattice $L = \mathbb{Z}a$ in $\mathcal{U}_a$.

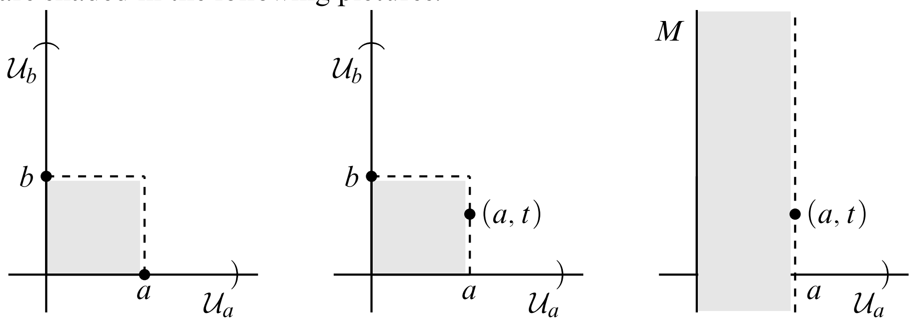
The quotient group $G_a = \mathcal{U}_a/L$ is then definable, definably compact and of dimension $1 = \operatorname{rank}(L)$.

The rest of the examples have their origins in [42] and [39].

2. (Two-dimensional examples). In the same $\mathcal{M}$, we pick two positive elements $a, b \in M$ and define:

- $\mathcal{U} = \mathcal{U}_a \times \mathcal{U}_b$, with notation from Example 1.
- $L_1 = \mathbb{Z}(a, 0) + \mathbb{Z}(0, b)$
- $L_2 = \mathbb{Z}(a, t) + \mathbb{Z}(0, b)$, where $0 < t < b$ and $t \notin \mathbb{Q}a$.

Then $G_1 = \mathcal{U}/L_1$ and $G_2 = \mathcal{U}/L_2$ are both definable, definably compact of dimension $2 = \operatorname{rank}(L_1) = \operatorname{rank}(L_2)$. The group G_1 is the product of G_a and G_b (with notation from Example 1), whereas G_2 *cannot* be written as the product of two 1-dimensional definable subgroups (the reason being that there is no definable line segment connecting the origin to the point (a, t)). The domains of the corresponding definable groups K, from Definition 2.5, are shaded in the following pictures.



$G_1 = G_a \times G_b$ $\quad$ $G_2 = \frac{\mathcal{U}_a \times \mathcal{U}_b}{\mathbb{Z}(a,t)+\mathbb{Z}(0,b)}$ $\quad$ $G_4 = \frac{\mathcal{U}_a \times M}{\mathbb{Z}(a,t)}$

3. (Non-compact examples). In the same $\mathcal{M}$, we pick a positive $a \in M$, and define:

- $\mathcal{U}' = \mathcal{U}_a \times M$
- $L_3 = \mathbb{Z}(a, 0)$
- $L_4 = \mathbb{Z}(a, t)$, where $0 < t \in M$ and $t \notin \mathbb{Q}a$.

Then $G_3 = \mathcal{U}'/L_3$ and $G_4 = \mathcal{U}'/L_4$ are both definable, each of dimension 2 and compact dimension $1 = \mathrm{rank}(L_3) = \mathrm{rank}(L_4)$. The group G_3 is the product of G_a and M, whereas G_4 cannot be written as the product of two 1-dimensional definable subgroups.

3.2. Semi-bounded case. We now proceed to the semi-bounded case, (B1), from the Introduction. Assume $\mathcal{M} = \langle M, <, +, 0, I \rangle$ is a semi-bounded o-minimal structure, where I is a bounded interval on which there is definable real closed field. Following [36], we call an interval $J \subseteq M$ *short* if it is in definable bijection with I; equivalently, if there is a definable real closed field with domain J. Otherwise, we call J *long*. A definable set $X \subseteq M^n$ is called *short* if it is in definable bijection with a subset of I^n. Given a definable set $X \subseteq M^n$, we define [20] the *long dimension* of X to be the maximum k such that X contains a definable homeomorphic image of J^k, for some long interval J. Clearly, the long dimension of X is at most equal to the dimension of X. It holds that a definable set is short if and only if it has long dimension 0.

A *short* $\bigvee$*-definable group* $\mathcal{U} = \bigcup_{i \in I} X_i$ is a $\bigvee$-definable group such that each X_i is short.

Let Λ be the set of all $\emptyset$-definable partial endomorphisms of $\langle M, <, +, 0 \rangle$. A set which is ($\bigvee$-)definable in the reduct $\mathcal{M}_{lin} = \langle M, <, +, 0, \{\lambda\}_{\lambda \in \Lambda} \rangle$ is called ($\bigvee$-)*semi-linear*.

THEOREM 3.4. [21] *Assume $\mathcal{M}$ is semi-bounded. Let G be an abelian, definably connected, definably compact definable group of long dimension k. Then $G = \mathcal{U}/L$, with $\mathcal{U}$ and L as in Theorem* 3.1 *and, moreover, $\mathcal{U}$ is a group extension of*

- *a $\bigvee$-definable group $\mathcal{K}$ generated by a short definable set, by*
- *a $\bigvee$-definable subgroup $\mathcal{H}$ of $\langle M^k, + \rangle$ generated by a semi-linear set of long dimension k.*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{U} \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] & 0 \\ & & G & & \end{tikzcd}$$

That is, although the universal cover $\mathcal{U}$ of G may not be a subgroup of $\langle M^n, + \rangle$, $n = \dim(G)$, we can recover a subgroup $\mathcal{H}$ of $\mathcal{U}$ which is a subgroup of $\langle M^k, + \rangle$, where k is the long dimension of G and of $\mathcal{H}$. Still, the lattice L

we recover contains no information about the long dimension of G. We can achieve this at a second step.

THEOREM 3.5. [21] *Assume $\mathcal{M}$ and G are as in Theorem 3.4. Then there is a divisible $\bigvee$-definable group $\overline{\mathcal{U}}$, a lattice $\overline{L}$ in $\overline{\mathcal{U}}$ and a finite subgroup $F \subseteq \overline{\mathcal{U}}$, such that $G = \overline{\mathcal{U}}/(\overline{L} \times F)$ and*

(1) *$\overline{\mathcal{U}}$ is a group extension of*
- *a short definably compact* definable *group $\overline{K}$ by*
- *a connected $\bigvee$-definable subgroup $\mathcal{H}$ of $\langle M^k, +\rangle$ generated by a semi-linear set of long dimension k.*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H} \arrow[r] & \overline{\mathcal{U}} \arrow[r] \arrow[d] & \overline{K} \arrow[r] & 0 \\ & & G & & \end{tikzcd}$$

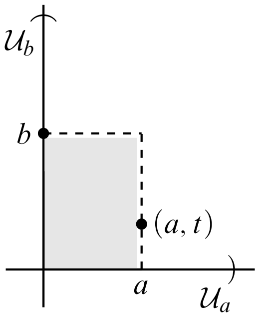
(2) $\mathrm{rank}(\overline{L}) = k$.

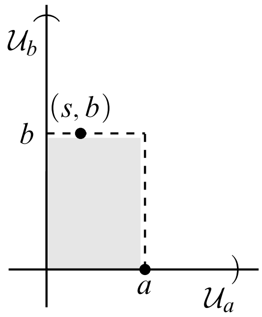
Two remarks are in order. First, observe that G is *almost* a quotient of $\overline{\mathcal{U}}$ by $\overline{L}$; it is not known if one can choose the finite F to be trivial. Second, the universal cover $\mathcal{U}$ of G has now been replaced by $\overline{\mathcal{U}}$, which is just a $\bigvee$-definable cover of G (in the sense of [12]). The improvement of Theorem 3.5 over 3.4 is that we have now recovered a lattice $\overline{L}$ which encodes the long dimension of G. This is the *first interpretation* of the comparison between the two theorems that we give in this paper. Three more interpretations follow below. For the moment, let us give some examples.

EXAMPLE 3.6. Let $\mathcal{M} = \langle M, <, +, 0, R\rangle$ be an expansion of an ordered divisible abelian group by a real closed field R with bounded domain $(0, a) \subseteq M$. In particular, $\mathcal{M}$ is semi-bounded and $(0, a)$ is short. Let also b be any element in M such that $(0, b)$ is long. With the same notation as in Example 3.3, we define:

- the 1-dimensional subgroup $\mathcal{H} = \mathcal{U}_b$ of $\langle M, +\rangle$.
- $\mathcal{U} = \mathcal{U}_a \times \mathcal{U}_b$.
- $L_1 = \mathbb{Z}(a, t) + \mathbb{Z}(0, b)$, where $0 < t < b$ and $t \notin \mathbb{Q}a$.
- $L_2 = \mathbb{Z}(a, 0) + \mathbb{Z}(s, b)$, where $0 < s < a$ and $s \notin \mathbb{Q}b$.

Then the quotient groups $G_1 = \mathcal{U}/L_1$ and $G_2 = \mathcal{U}/L_2$ are definable, definably compact of dimension 2 and of long dimension 1.



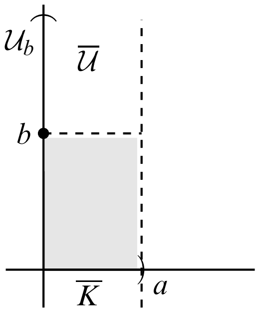


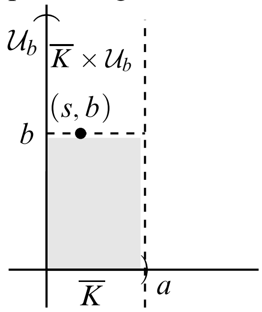
$G_1 = \frac{\mathcal{U}_a \times \mathcal{U}_b}{\mathbb{Z}(a,t)+\mathbb{Z}(0,b)}$ $\qquad$ $G_2 = \frac{\mathcal{U}_a \times \mathcal{U}_b}{\mathbb{Z}(a,0)+\mathbb{Z}(s,b)}$

If we define further

- the short definably compact definable group $\overline{K} = \mathcal{U}_a/\mathbb{Z}a$,
- a *suitable* extension $\overline{\mathcal{U}}$ of $\overline{K}$ by $\mathcal{H}$,
- $\overline{L}_1 = \mathbb{Z}(0,b)$,
- $\overline{L}_2 = \mathbb{Z}(s,b)$

then we obtain $G_1 = \overline{\mathcal{U}}/\overline{L}_1$, $G_2 = (\overline{K} \times \mathcal{U}_b)/\overline{L}_2$, and $\overline{L}_1$, $\overline{L}_2$ have rank 1. Note that G_2 does not contain any definable subgroups of long dimension 1.





$G_1 = \frac{\overline{\mathcal{U}}}{\overline{L}_1}$ $\qquad$ $G_2 = \frac{\overline{K} \times \mathcal{U}_b}{\overline{L}_2}$

We observe that the groups G_1 and G_2 in this example are actually semilinear groups. However, we can easily equip the domain of $\overline{K}$ (and, hence, also that of $\mathcal{U}_a$) with a different group structure (such as the multiplicative group structure of R), and obtain groups G_1 and G_2 which are not semilinear.

Here is a *second interpretation* of Theorem 3.5 versus 3.4. On the one hand, we know from the last example that a definable group G cannot always be an extension of a short definable group by a long definable group, since it may not contain any definable long subgroup in the first place. On the other hand, Theorem 3.4 says that the universal cover $\mathcal{U}$ of G is *always* an extension of a short $\bigvee$-definable group by a $\bigvee$-semilinear group $\mathcal{H}$. What

Theorem 3.5 advances is that this extension can actually be achieved 'closer' to G, in terms that the new cover $\overline{\mathcal{U}}$ is now an extension of a short *definable* group by $\mathcal{H}$.

Before looking into the proofs of the above theorems, we conclude this section with a conjecture about definable groups beyond the o-minimal setting.

3.3. Beyond the o-minimal setting. It is common among model-theorists to seek extensions of the o-minimal framework that preserve nice behavior. Examples are:

(a) dense pairs [7],
(b) expansions of the real field by a multiplicative subgroup with the Mann Property [9],
(c) structures with o-minimal open core [32], [6].

These structures are not purely topological and tameness does not extend to all definable sets in the structure, but it does extend to all *open* definable sets. Our goal of studying definable groups in this setting is to analyze them in terms of their topological, 'o-minimal part', and their part that corresponds to the extra structure.

We fix ourselves at the general framework of an expansion of a real closed field by a 'small' predicate. These expansions include examples (a) and (b) above and, under certain conditions, they have o-minimal open core (see [5]). The notion of smallness appears in various places in the literature. We adopt here the definition from [7] and [5].

DEFINITION 3.7. Let $\mathcal{M} = \langle M, <, \dots \rangle$ be any o-minimal structure and $P \subseteq M$. We call P *large in* $\mathcal{M}$ if there is some m and a definable function $f\colon M^m \to M$ such that $f(P^m)$ contains an open interval in M. We call P *small in* $\mathcal{M}$ if it is not large.

So now let $\langle \mathcal{M}, P \rangle$ be an expansion of a real closed field $\mathcal{M}$ by a small (in $\mathcal{M}$) predicate P. For example, let $\mathcal{M}$ be the real field and P:

(A) the field of real algebraic numbers (yielding a dense pair), or
(B) $2^{\mathbb{Z}}$ or $2^{\mathbb{Z}}3^{\mathbb{Z}}$ or $2^{\mathbb{Q}}$ (multiplicative subgroups with the Mann Property).

Given a definable set $X \subseteq M^n$, we define the *large dimension* of X to be the maximum k such that there is an injective function $g\colon M^k \to M^n$ definable in $\mathcal{M}$, and a large $J \subseteq M$ with $g(J^k) \subseteq X$. We call X *small* if it has large dimension 0. As can easily be verified, this definition agrees with Definition 3.7 for $X \subseteq M$.

Together with A. Günaydin, we conjecture the following.

CONJECTURE 1. *Let $\langle \mathcal{M}, P \rangle$ be a sufficiently saturated expansion of a real closed field $\mathcal{M}$ by a small predicate P, such that $\langle \mathcal{M}, P \rangle$ has o-minimal open core. Let G be an abelian definable group which is 'connected' and has large dimension k. Then there is a divisible $\bigvee$-definable group $\overline{\mathcal{U}}$ and a lattice L in $\overline{\mathcal{U}}$ such that $G = \overline{\mathcal{U}}/L$, and*

(1) *$\overline{\mathcal{U}}$ is a group extension of*
- *a small definable group $\overline{K}$ by*
- *a $\bigvee$-semialgebraic group $\mathcal{H}$ of dimension k generated by a semialgebraic set.*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H} \arrow[r] & \overline{\mathcal{U}} \arrow[r] \arrow[d] & \overline{K} \arrow[r] & 0 \\ & & G & & \end{tikzcd}$$

(2) $\operatorname{rank}(L) = k$.

A few comments are in order:

(a) We can again think of $\overline{\mathcal{U}}$ as a $\bigvee$-definable 'cover' of our group G, although the theory of covers has not yet been developed in this context.

(b) There is a missing notion of 'connectedness' to be defined for G. We envision that this will be one of the following two: either (1) G is connected if its topological closure (in the topology of $\mathcal{M}$) is definably connected (as a set definable in the o-minimal, open core of $\langle \mathcal{M}, P \rangle$), or (2) G is connected if it does not contain any definable subgroups of small (in the above sense) index.

(c) It is known by [26] that the pairs $\langle \mathcal{M}, P \rangle$ in examples (A) and (B) above have NIP. The notion of 'finitely satisfiable generics' is a generalization of definable compactness in the NIP context [27]. So it may be natural to impose further on G the condition that it has finitely satisfiable generics, strengthening also the conclusion that $\overline{K}$ has finitely satisfiable generics.

(d) Finally, the lattice L encodes the large dimension of G, which can be interpreted as 'how semi-algebraic' G is.

We are now in a position to offer our *third interpretation* of Theorem 3.5 over 3.4. We believe that Theorem 3.5 is more natural, in the following sense. Conjecture 1 suggests that if we are to apply the proposed program to some general context of expansions of well-behaved structures by a predicate, then we have to deal with the fact that 'small' groups (with respect to the predicate) may not necessarily have universal covers. This restriction leaves us with the possibility to express *some* cover of the group in question as an extension of a small group by some $\bigvee$-definable group in $\mathcal{M}$. It is desirable then to have a result of this form in the semi-bounded case, as well.

Since each of the above structure theorems claims that a cover of our group G relates to the ambient structure, it should be clear that at a local level a tighter connection must exist between G and the ambient structure. This observation makes up Step I of our proposed analysis in the Introduction. As we are about to see, the study of local behavior of G in the above settings presents a remarkable uniformity.

§**4. Local analysis.** It is a general thrust of o-minimality that, given a definable set X, around its generic elements the definable objects are best-behaved. For example, if X and $f\colon X \to M$ are definable over A, then every element of X, generic over A, is contained in an open neighborhood inside X on which f is continuous. The key point of our analysis is to obtain, each time, a nice local behavior *relative to* the particular nature of the ambient structure. To this end, we employ each time a relative notion of genericity. We assume the reader is familiar with the basic notions of pregeometries and generic elements, as presented, for example, in [30].

Linear case. [25] Here we use generic elements with respect to the usual pregeometry coming from the algebraic closure operator $acl\colon \mathcal{P}(M) \to \mathcal{P}(M)$, defined as:

$$acl(A) = \{a \in M : \text{there are } \bar{b} \subseteq A \text{ and a formula } \phi(x, \bar{y}), \text{ such that } \phi(\mathcal{M}, \bar{b}) \text{ is finite and } \mathcal{M} \models \phi(a, \bar{b})\}.$$

This pregeometry gives rise to generic elements, as well as to a notion of dimension for definable sets that coincides with their topological dimension. We prove:

PROPOSITION 4.1. *Assume $\mathcal{M}$ is as in Theorem* 3.2, *and that $G = \langle G, \oplus \rangle$ is a definable group of dimension n with $G \subseteq M^n$. Then every generic element a in G* (*over the parameters that define G*) *is contained in an open neighborhood $V_a \subseteq G$ such that for every $x, y \in V_a$,*

$$x \ominus a \oplus y = x - a + y. \tag{1}$$

(*'$\oplus$' and '$\ominus$' denote the group and inverse operations of G.*)

That is, the local sum around a in the sense of G *coincides* with the one in the sense of $\langle M^n, + \rangle$. This yields a local isomorphism between G and $\langle M^n, + \rangle$ in the strongest possible way.

Semi-bounded case. [20] Here we introduce a new notion of a pregeometry, based on the *short closure operator shcl*: $\mathcal{P}(M) \to \mathcal{P}(M)$, defined as:

$$shcl(A) = \{a \in M : \text{there are } \bar{b} \subseteq A \text{ and a formula } \phi(x, \bar{y}), \text{ such that } \phi(\mathcal{M}, \bar{b}) \text{ is a short interval and } \mathcal{M} \models \phi(a, \bar{b})\}.$$

A generic element with respect to the *shcl*-pregeometry is called *long-generic*. In [20] it is proved that the corresponding notion of dimension coincides with the long dimension defined earlier. We prove:

PROPOSITION 4.2. *Assume $\mathcal{M}$ is as in Theorem* 3.5, *and that $G = \langle G, \oplus \rangle$ is a definable group of long dimension k. Then every long-generic element a in G is contained in a definable subset $V_a \subseteq G$ of long dimension equal to k such* *that for every $x, y \in V_a$,*

$$x \ominus a \oplus y = x - a + y.$$

In particular, on V_a, G is locally isomorphic to $\langle M^k, +\rangle$.

Expansion by a small predicate. Here we use the *small closure operator* $smcl\colon \mathcal{P}(M) \to \mathcal{P}(M)$, defined in [5] as:

$$smcl(A) = \{a \in M : \text{there are } \bar{b} \subseteq A \text{ and a formula } \phi(x, \bar{y}), \text{ such that } \phi(\mathcal{M}, \bar{b}) \text{ is a small set and } \mathcal{M} \vDash \phi(a, \bar{b})\}.$$

It is shown in [5] that, under certain conditions, *smcl* defines a pregeometry. Together with A. Günaydin, we conjecture that if $\langle \mathcal{M}, P\rangle$ has o-minimal open core, then *smcl* is a pregeometry. A generic element with respect to the *smcl*-pregeometry is called *large-generic*. We conjecture that the corresponding notion of dimension coincides with the large dimension defined earlier. Moreover, we conjecture:

Conjecture 2. *Assume $\langle \mathcal{M}, P\rangle$ and $G = \langle G, \oplus\rangle$ are as in Conjecture* 1, *and that $G \subseteq M^n$. Then every large-generic element a in G is contained in a definable subset $V_a \subseteq G$ of large dimension k, and there is a semi-algebraic function $f\colon M^{2n} \to M^n$, such that for every $x, y \in V_a$,*

$$x \ominus a \oplus y = f(x, y).$$

That is, on V_a the group G should 'behave' like a semi-algebraic group. Of course, here the set V_a contains a but it is not a 'local neighborhood' in the usual topological sense.

No ambient group assumption. Very recently, in [24], a similar notion of a pregeometry was introduced in a different context; namely, in the study of groups definable in an arbitrary o-minimal structure. In this context, roughly speaking, an interval is called *group-short* if it can be endowed with the structure of a definable group interval. We define:

$$gcl(A) = \{a \in M : \text{there are } \bar{b} \subseteq A \text{ and a formula } \phi(x, \bar{y}), \text{ such that } \phi(\mathcal{M}, \bar{b}) \text{ is a group-short interval and } \mathcal{M} \vDash \phi(a, \bar{b})\}.$$

The corresponding notion of generic elements is then employed to prove that every definable group is in definable bijection with a subset of a product of group intervals. For details and precise definitions, the reader is referred to [24].

4.1. From local to global. We will not be very extensive here. The methods involved in each case appear to be different. One aspect in common, however, is that one needs a structure theorem for definable sets first. In the general o-minimal setting it is well-known that every definable set is a finite union of cells. In the linear case, the union can be refined into *linear* cells [25]. In the semi-bounded setting, a structure theorem is proved which expresses every definable set as a finite union of *cones* [34], [10], [20]. In certain expansions of a real closed field $\mathcal{M}$ by a small predicate P, we have at our disposal a structure theorem saying that every definable set is a boolean combination of existentially definable formulas of the form

$$\exists y\, P(y) \wedge \varphi(\bar{x}, y),$$

where $\varphi(\bar{x}, \bar{y})$ is a quantifier-free formula from the language of $\mathcal{M}$. We conjecture with A. Günaydin that every definable set is a finite *union* of sets of the form

$$\exists y\, P(y) \wedge \varphi_1(x_1, y) \wedge \cdots \wedge \varphi_n(x_n, y),$$

where each $\varphi_i(x_i, b)$ defines a large subset of M.

One of the uses of the structure theorems of sets is to extend the nice local behavior around generic elements to a sufficiently large subset X of G. Such a set X was used in each of the proofs of Theorems 3.2 and 3.4 in order to generate the corresponding group $\mathcal{U}$.

§5. $\bigvee$-definable groups and lattices. The study of lattices becomes important also from a different point of view. It is clear that in order to derive Theorem 3.5 from Theorem 3.4 one needs to prove, among others, that $\mathcal{K}$ admits a $\bigvee$-definable homomorphism $\phi\colon \mathcal{K} \to \overline{K}$ onto a *definable* group $\overline{K}$. Equivalently, $\mathcal{K}$ admits a lattice. This brings up an important conjecture which is the theme of this last section.

Let $\mathcal{M}$ be a sufficiently saturated o-minimal expansion of an ordered group. By a *bounded* size we mean a size less than the saturation of M.

CONJECTURE 3. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is generated by a definable set. Then $\mathcal{U}$ admits a lattice.*

As we saw in Example 2.13 the assumption that $\mathcal{U}$ is generated by a definable set is necessary.

This conjecture was posed in [22] and was proved to be equivalent to a number of different statements. Given $\mathcal{U}$ as above, we say that $\mathcal{U}^{00}$ *exists* if there is a smallest type-definable subgroup of bounded index in $\mathcal{U}$, which we denote by $\mathcal{U}^{00}$. A subset $X \subseteq \mathcal{U}$ is called *generic* if boundedly many translates of it cover $\mathcal{U}$.

THEOREM 5.1. [22] *If $\mathcal{U}$ satisfies the assumptions of Conjecture* 3, *then the following are equivalent.*

1. *$\mathcal{U}$ admits a lattice.*
2. *$\mathcal{U}$ contains a definable generic set.*
3. *$\mathcal{U}^{00}$ exists, it is torsion-free, and $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times \mathbb{T}^r$, where $\mathbb{T}^r$ is the real r-torus.*

COROLLARY 5.2. *The group $\mathcal{K}$ from Theorem* 3.4 *admits a lattice.*

PROOF. Since $\mathcal{U}$ in Theorem 3.4 is a cover of a definable group G, it admits a lattice. Hence it must contain a definable generic set X. Now the image of X under the map $: \mathcal{U} \to \mathcal{K}$ is a definable generic subset of $\mathcal{K}$. Hence $\mathcal{K}$ admits a lattice. $\dashv$

We now come to the *fourth interpretation* of Theorem 3.5 over 3.4 which also concerns the most important application to the study of o-minimal groups. Recall from the introduction that (PC) says that every definably connected, definably compact o-minimal group G admits a surjective homomorphism π onto a real Lie group, whose dimension (as a Lie group) is equal to the o-minimal dimension of G. The Compact Domination Conjecture (CDC) says that for every definable $X \subseteq G$, if $\dim(X) < \dim(G)$, then the Haar measure of $\pi(X)$ is 0.

THEOREM 5.3. [21] (CDC) *holds in any o-minimal expansion $\mathcal{M}$ of an ordered group.*

PROOF. (CDC) was only known in the linear [18] and field [29, 28] cases. For the semi-bounded case, one can use Theorem 3.4 to reduce the problem to these cases. Namely, first reduce the problem to the universal cover $\mathcal{U}$ of G and then to the $\bigvee$-definable groups $\mathcal{H}$ and $\mathcal{K}$. We know that $\mathcal{H}$ is the universal cover of a semi-linear group, and thus we obtain (CDC) for it. But *unless* we also know that $\mathcal{K}$ is a cover of a *definable* group from the field case, we cannot (at least, we were not able to) conclude (CDC) for it. The improvement to Theorem 3.5 handles this step. $\dashv$

A positive answer to Conjecture 3 would have far-reaching consequences.

THEOREM 5.4. [22] *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which admits a lattice. Then $\mathcal{U}$ is divisible.*

The conclusion that $\mathcal{U}$ is divisible is a rather desirable property. For example, very recently in [3] it was proved that for such a group $\mathcal{U}$, the subgroup of k-torsion points is finite. Note that in the definable case, we already know by Strzebonski that a connected abelian definable subgroup is divisible.

Conjecture 3 is broadly open. We introduce below the notion of $\bigvee$-dimension for $\mathcal{U}$, which may assist us in doing some inductive arguments. The $\bigvee$-dimension intends to count how 'non-definable' $\mathcal{U}$ is (recall the notion of 'compatible' from Section 2):

DEFINITION 5.5. The $\bigvee$*-dimension* of $\mathcal{U}$, denoted by $\mathrm{vdim}(\mathcal{U})$, is the maximum k such that $\mathcal{U}$ contains a compatible subgroup isomorphic to $\mathbb{Z}^k$, if such k exists, and ∞, otherwise.

In [23], we reduce Conjecture 3 to some properties of $\bigvee$-dimension. Namely, Conjecture 3 is true if and only if, for every $\mathcal{U}$ that satisfies the assumptions of the conjecture, the following hold:

(1) If $\mathcal{U}$ is not definable, then $\mathrm{vdim}(\mathcal{U}) > 0$.
(2) $\mathrm{vdim}(\mathcal{U}) \leq \dim(\mathcal{U})$. (In particular, $\mathrm{vdim}(\mathcal{U})$ is finite.)

A positive answer is also obtained if $\mathcal{U}$ is a $\bigvee$-definable subgroup of some cartesian power of $\langle M, +\rangle$.

In the recent [3], property (2) was established. So Conjecture 3 now reduces to property (1). Moreover, a result in [3] establishes (1) assuming that $\mathcal{U}$ satisfies a suitable 'convexity' condition.

As far as divisibility of $\mathcal{U}$ is concerned, we do not know if the assumption that $\mathcal{U}$ is generated by a definable set is necessary. The following conjecture has been asked in the past by several authors (such as Edmundo in [11]).

CONJECTURE 4. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group. Then $\mathcal{U}$ is divisible.*

We conclude this paper with the proof of Fact 2.11 mentioned earlier.

FACT 5.6. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which admits a lattice. If L is a compatible subgroup of $\mathcal{U}$ isomorphic to some $\mathbb{Z}^m$, then L extends to a lattice; that is, there is a lattice L' in $\mathcal{U}$ with $L \subseteq L'$.*

PROOF. Let $\pi_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times \mathbb{T}^r$ be given by Theorem 5.1. We first claim that the image of L under $\pi_{\mathcal{U}}$ is a discrete subgroup of $\mathbb{R}^k \times \mathbb{T}^r$. Indeed, by [22, Lemma 3.3], for every compact neighborhood $W \subseteq \mathcal{U}/\mathcal{U}^{00}$ of 0, there is a definable set $Z \subseteq \mathcal{U}$ such that $\pi_{\mathcal{U}}^{-1}(W) \subseteq Z$. Since L is compatible in $\mathcal{U}$, $Z \cap L$ is finite and hence $W \cap \pi_{\mathcal{U}}(L)$ must be finite. It follows that $\pi_{\mathcal{U}}(L)$ is discrete.

We observe, moreover, that $\mathcal{U}^{00} \cap L = \{0\}$. Indeed, pick any definable $Z \subseteq \mathcal{U}$ such that $\mathcal{U}^{00} \subseteq Z$. Then $\mathcal{U}^{00} \cap L \subseteq Z \cap L$ is finite, as above. But $\mathcal{U}^{00}$ is torsion-free, so $\mathcal{U}^{00} \cap L = \{0\}$.

It follows that $\pi_{\mathcal{U}}$ restricted to L is injective, and hence $\pi_{\mathcal{U}}(L) \simeq \mathbb{Z}^m$. Let $f\colon R^k \times \mathbb{T}^r \to R^k$ denote the projection onto the first k coordinates. Since $\mathbb{T}^r$ is compact, we obtain $\ker(f) \cap \pi_{\mathcal{U}}(L) = \{0\}$ and hence also $f(\pi_{\mathcal{U}}(L)) \simeq \mathbb{Z}^m$. By a classical result a discrete subgroup of $\mathbb{R}^k$ must be generated by $\leq k$ elements, and therefore $m \leq k$.

Denote $B = f(\pi_{\mathcal{U}}(L))$. We can extend B to a lattice B' of $\mathbb{R}^k$ of rank k

$$B' = B + \mathbb{Z}b_{m+1} + \cdots + \mathbb{Z}b_k,$$

where $b_{m+1}, \ldots, b_k$ are some elements of $\mathbb{R}^k$ which, together with the generators of B, are all $\mathbb{R}$-independent. Now pick any elements $a_{m+1}, \ldots, a_k \in \mathcal{U}$ such that $f(\pi_{\mathcal{U}}(a_i)) = b_i$. We claim that the subgroup L' of $\mathcal{U}$ defined by

$$L' = L + \mathbb{Z}a_{m+1} + \cdots + \mathbb{Z}a_k$$

is a lattice in $\mathcal{U}$.

We need to show that $\mathcal{U}/L'$ is definable. By [22, Lemma 2.1], it suffices to find a definable $Z \subseteq \mathcal{U}$ such that $Z + L' = \mathcal{U}$ and $Z \cap L'$ is finite.

Fix some compact set $H \subseteq \mathbb{R}^k \times \mathbb{T}^r$, such that $H + B' = \mathbb{R}^k \times \mathbb{T}^r$. By [22, Lemma 3.3] there is a definable set $Z \subseteq \mathcal{U}$ which contains $\pi_{\mathcal{U}}^{-1}(H)$. But then:

$$Z + L' \supseteq \pi_{\mathcal{U}}^{-1}(H) + \pi_{\mathcal{U}}^{-1}(B) = \pi_{\mathcal{U}}^{-1}(H + B) = \pi_{\mathcal{U}}^{-1}(\mathbb{R}^k \times \mathbb{T}^r) = \mathcal{U}.$$

To check that $Z \cap L'$ is finite, we observe that, since $b_{m+1}, \ldots, b_k$ together with the generators of B are all $\mathbb{Z}$-independent, $\pi_{\mathcal{U}}$ restricted to L' is injective. Hence $Z \cap L'$ has as many elements as $H \cap B$, that is, finitely many. $\dashv$

REFERENCES

[1] E. BARO and M. EDMUNDO, *Corrigendum to: "Locally definable groups in o-minimal structures" by Edmundo*, ***Journal of Algebra***, vol. 320 (2008), no. 7, pp. 3079–3080.

[2] E. BARO and M. OTERO, *Locally definable homotopy*, ***Annals of Pure and Applied Logic***, vol. 161 (2010), no. 4, pp. 488–503.

[3] A. BERARDUCCI, M. EDMUNDO, and M. MAMINO, *Discrete subgroups of locally definable groups*, preprint, 2012.

[4] A. BERARDUCCI, M. OTERO, Y. PETERZIL, and A. PILLAY, *A descending chain condition for groups definable in o-minimal structures*, ***Annals of Pure and Applied Logic***, vol. 134 (2005), pp. 303–313.

[5] A. BERENSTEIN, C. EALY, and A. GÜNAYDIN, *Thorn independence in the field of real numbers with a small multiplicative group*, ***Annals of Pure and Applied Logic***, vol. 150 (2007), pp. 1–18.

[6] A. DOLICH, D. MARKER, and C. STEINHORN, *Structures having o-minimal open core*, ***Proceedings of the American Mathematical Society***, vol. 139 (2011), pp. 319–330.

[7] L. VAN DEN DRIES, *Dense pairs of o-minimal structures*, ***Fundamenta Mathematicae***, vol. 157 (1998), no. 1, pp. 61–78.

[8] ———, ***Tame topology and o-minimal structures***, Cambridge University Press, Cambridge, 1998.

[9] L. VAN DEN DRIES and A. GÜNAYDIN, *The fields of real and complex numbers with a small multiplicative group*, ***Proceedings of the London Mathematical Society***, vol. 93 (2006), pp. 43–81.

[10] M. EDMUNDO, *Structure theorems for o-minimal expansions of groups*, ***Annals of Pure and Applied Logic***, vol. 102 (2000), pp. 159–181.

[11] ———, *Locally definable groups in o-minimal structures*, ***Journal of Algebra***, vol. 301 (2006), no. 1, pp. 194–223.

[12] M. EDMUNDO and P. ELEFTHERIOU, *The universal covering homomorphism in o-minimal expansions of groups*, ***Mathematical Logic Quarterly***, vol. 53 (2007), pp. 571–582.

[13] M. EDMUNDO, G. JONES, and N. PEATFIELD, *Sheaf cohomology in o-minimal structures*, ***Journal of Mathematical Logic***, vol. 6 (2006), no. 2, pp. 163–179.

[14] M. EDMUNDO and M. OTERO, *Definably compact abelian groups*, ***Journal of Mathematical Logic***, vol. 4 (2004), pp. 163–180.

[15] M. EDMUNDO and N. PEATFIELD, *O-minimal Cech cohomology*, ***Quarterly Journal of Mathematics***, vol. 59 (2008), no. 2, pp. 213–220.

[16] M. EDMUNDO and A. WOERHEIDE, *Comparison theorems for o-minimal singular (co)homology*, ***Transactions of the American Mathematical Society***, vol. 360 (2008), pp. 4889–4912.

[17] P. ELEFTHERIOU, *A semi-linear group which is not affine*, ***Annals of Pure and Applied Logic***, vol. 156 (2008), pp. 287–289.

[18] ———, *Compact domination for groups definable in linear o-minimal structures*, ***Archive for Mathematical Logic***, vol. 48 (2009), no. 7, pp. 607–623.

[19] ———, *Groups definable in linear o-minimal structures: the non-compact case*, ***The Journal of Symbolic Logic***, vol. 75 (2010), pp. 208–220.

[20] ———, *Local analysis for semi-bounded groups*, ***Fundamenta Mathematicae***, vol. 216 (2012), pp. 223–258.

[21] P. Eleftheriou and Y. Peterzil, *Definable groups as homomorphic images of semilinear and field-definable groups*, ***Selecta Mathematica***, vol. 18 (2012), no. 4, pp. 905–940.

[22] ———, *Definable quotients of locally definable groups*, ***Selecta Mathematica***, vol. 18 (2012), no. 4, pp. 885–903.

[23] ———, *Lattices in locally definable subgroups of* $\langle R^n, +\rangle$, ***Notre Dame Journal of Formal Logic***, to appear.

[24] P. Eleftheriou, Y. Peterzil, and J. Ramakrishnan, *Interpretable groups are definable*, preprint, 2011.

[25] P. Eleftheriou and S. Starchenko, *Groups definable in ordered vector spaces over ordered division rings*, ***The Journal of Symbolic Logic***, vol. 72 (2007), pp. 1108–1140.

[26] A. Günaydin and P. Hieronymi, *Dependent pairs*, ***The Journal of Symbolic Logic***, vol. 76 (2011), pp. 377–390.

[27] E. Hrushovski, Y. Peterzil, and A. Pillay, *Groups, measures, and the NIP*, ***Journal of American Mathematical Society***, vol. 21 (2008), pp. 563–596.

[28] ———, *On central extensions and definably compact groups in o-minimal structures*, ***Journal of Algebra***, vol. 327 (2011), pp. 71–106.

[29] E. Hrushovski and A. Pillay, *On NIP and invariant measures*, ***Journal of the European Mathematical Society***, vol. 13 (2011), no. 4, pp. 1005–1061.

[30] D. Macpherson, *Notes on o-minimality and variations*, ***Model theory, algebra, and geometry*** (D. Haskell, A. Pillay, and C. Steinhorn, editors), Mathematical Sciences Research Institute Publications 39, Cambridge University Press, 2000, pp. 97–130.

[31] D. Marker, Y. Peterzil, and A. Pillay, *Additive reducts of real closed fields*, ***The Journal of Symbolic Logic***, vol. 57 (1992), pp. 109–117.

[32] C. Miller and P. Speissegger, *Expansions of the real line by open sets: o-minimality and open cores*, ***Fundamenta Mathematicae***, vol. 162 (1999), no. 3, pp. 193–208.

[33] M. Otero, *A survey on groups definable in o-minimal structures*, ***Model theory with applications to algebra and analysis, Vol. 2*** (Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, editors), London Mathematical Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, pp. 177–206.

[34] Y. Peterzil, *A structure theorem for semibounded sets in the reals*, ***The Journal of Symbolic Logic***, vol. 57 (1992), pp. 779–794.

[35] ———, *Reducts of some structures over the reals*, ***The Journal of Symbolic Logic***, vol. 58 (1993), pp. 955–966.

[36] ———, *Returning to semi-bounded sets*, ***The Journal of Symbolic Logic***, vol. 74 (2009), pp. 597–617.

[37] Y. Peterzil and S. Starchenko, *A trichotomy theorem for o-minimal structures*, ***Proceedings of the London Mathematical Society***, vol. 77 (1998), no. 3, pp. 481–523.

[38] ———, *Definable homomorphisms of abelian groups in o-minimal structures*, ***Annals of Pure and Applied Logic***, vol. 101 (2000), no. 1, pp. 1–27.

[39] Y. Peterzil and C. Steinhorn, *Definable compactness and definable subgroups of o-minimal groups*, ***Journal of the London Mathematical Society***, vol. 69 (1999), no. 2, pp. 769–786.

[40] A. Pillay, *On groups and fields definable in o-minimal structures*, ***Journal of Pure and Applied Algebra***, vol. 53 (1988), pp. 239–255.

[41] A. Pillay, P. Scowcroft, and C. Steinhorn, *Between groups and rings*, ***Rocky Mountain Journal of Mathematics***, vol. 19 (1989), no. 3, pp. 871–885.

[42] A. Strzebonski, *Euler charateristic in semialgebraic and other o-minimal groups*, ***Journal of Pure and Applied Algebra***, vol. 96 (1994), pp. 173–201.

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