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FIFTY YEARS OF THE SPECTRUM PROBLEM: SURVEY AND NEW RESULTS

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Abstract. In 1952, Heinrich Scholz published a question in The Journal of Symbolic Logic asking for a characterization of spectra, i.e., sets of natural numbers that are the cardinalities of finite models of first order sentences. Günter Asser in turn asked whether the complement of a spectrum is always a spectrum. These innocent questions turned out to be seminal for the development of finite model theory and descriptive complexity. In this paper we survey developments over the last 50-odd years pertaining to the spectrum problem. Our presentation follows conceptual developments rather than the chronological order. Originally a number theoretic problem, it has been approached by means of recursion theory, resource bounded complexity theory, classification by complexity of the defining sentences, and finally by means of structural graph theory. Although Scholz' question was answered in various ways, Asser's question remains open.

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SGANARELLE: Ah! Monsieur, c'est un **spectre**:
je le reconnais au marcher.
DOM JUAN: Spectre, fantôme, ou diable,
je veux voir ce que c'est.
J.B. Poquelin, dit Molière, Dom Juan, Acte V, scène V

§1. Preamble. At the Annual Symposium of the European Association of Computer Science Logic, CSL'05, held in Oxford in 2005, Arnaud Durand, Etienne Grandjean and Malika More organized a special workshop dedicated to the spectrum problem. The organizers and speakers then decided to use the occasion to expand the survey talk given by J. A. Makowsky into the present survey paper, rather than publish the talks.

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§2. The Emergence of the spectrum problem.

2.1. Scholz's problem. In 1952, H. Scholz published an innocent question in The Journal of Symbolic Logic [100]:

> **1**. An unsolved problem in symbolic logic. Let *IK* [Identitätskalkül] be the first order predicate calculus with identity. In *IK* one can formalize an axiom system *BP* [Boole'sche Postulate] for Boolean algebras with

only one binary relation variable. Let θ be the conjunction of the axioms of *BP*. Then θ is satisfiable in a finite domain of m elements if and only if there is an $n > 0$ such that $m = 2^n$.

From this results the following problem. Let H be an expression of *IK*. We call the set of natural numbers, for which H is satisfiable, the *spectrum of* H. Let M be an arbitrary set of natural numbers. We look for a sufficient and necessary condition that ensures that there exists an H, such that M is the spectrum of H.
(*Received September* 19, 1951. *English translation*).

This question inaugurated a new column of *Problems* to be published in The Journal of Symbolic Logic and edited by L. Henkin. Other questions published in the same issue were authored by G. Kreisel and L. Henkin. They deal with a question about interpretations of non-finitist proofs dealing with recursive ordinals and the no-counter-example interpretation (Kreisel), the provability of formulas asserting the provability or independence of provability assertions (Henkin), and the question whether the ordering principle is equivalent to the axiom of choice (Henkin). All in all 9 problems were published, the last in 1956.

The context in which Scholz's question was formulated is given by the various completeness and incompleteness results for First Order Logic that were the main concern of logicians of the period. An easy consequence of Gödel's classical completeness theorem of 1929 states that validity of first order sentences in all (finite and infinite) structures is recursively enumerable, whereas Church's and Turing's classical theorems state that it is not recursive. In contrast to this, the following was shown in 1950 by B. Trakhtenbrot.

THEOREM 2.1 (Trakhtenbrot 1950, [107]). *Validity of first order sentences in all finite structures (f-validity) is not recursively enumerable, and hence satisfiability of first order sentences in some finite structure (f-satisfiability) is not decidable, although it is recursively enumerable.*

Thus, H. Scholz really asked whether one could prove anything meaningful about f-satisfiability besides its undecidability.

2.2. Basic facts and questions. In our notation to be used throughout the paper, H. Scholz introduced the following:

Let τ be a vocabulary, i.e., set of relation and function symbols. Let ϕ be a sentence in some logic with equality over a vocabulary τ. Unless otherwise stated the logic will be first order logic $\mathrm{FO}(\tau)$. Sometimes we shall also discuss second order logic, or a fragment thereof.

DEFINITION 2.2. The *spectrum* $\mathrm{SPEC}(\phi)$ of ϕ is the set of finite cardinalities (viewed as a subset of $\mathbb{N}$), in which ϕ has a model.

We denote by SPEC the set of spectra of first-order sentences, i.e.,

$$\mathrm{SPEC} = \{\mathrm{SPEC}(\phi) \mid \phi \text{ is a first-order formula}\}.$$

We shall use S, S_i to denote spectra. For the definition of spectra it does not matter whether we use function symbols or not. So, unless otherwise stated, vocabularies will be without function symbols. However, we shall allow function symbols when dealing with sentences of special forms.

Clearly, $\textsc{spec}(\phi) = \emptyset$ if and only if ϕ is not f-satisfiable. By definition of satisfiability 0 is never part of a spectrum. Very often a spectrum is finite, cofinite or even of the form $\mathbb{N}^+ = \mathbb{N} - \{0\}$.

QUESTION 2.3. *Is it decidable whether, for a given ϕ, $\textsc{spec}(\phi) = \mathbb{N}^+$?*

As H. Scholz noted, (the set of) powers of 2 form a spectrum, because they are the cardinalities of finite Boolean algebras. Similarly, powers of primes form a spectrum, because they are the cardinalities of finite fields. For $a, b \in \mathbb{N}^+$ there are many ways to construct a sentence ϕ with $\textsc{spec}(\phi) = a + b\mathbb{N}$, one of which consists in using one unary function symbol. With a moment of reflection, one sees that spectra have the following closure properties.

PROPOSITION 2.4. *Let S_1 and S_2 be spectra.*

(i) *Then $S_1 \cup S_2, S_1 \cap S_2$ are also spectra.*
(ii) *Let $S_1 + S_2 = \{m + n \mid m \in S_1, n \in S_2\}$. Then $S_1 + S_2$ is a spectrum.*
(iii) *Let $S_1 \star S_2 = \{m \cdot n \mid m \in S_1, n \in S_2\}$. Then $S_1 \star S_2$ is a spectrum.*

In the spirit of Question 2.3 we can also ask:

QUESTION 2.5. *Which of the following sets are recursive? The set of sentences ϕ such that*

(i) $\textsc{spec}(\phi)$ *is finite, cofinite.*
(ii) $\textsc{spec}(\phi)$ *is ultimately periodic.*
(iii) $\textsc{spec}(\phi)$ *is, for given $a, b \in \mathbb{N}$ of the form $a + b\mathbb{N}$.*
(iv) $\textsc{spec}(\phi) = S$ *for a given set $S \subseteq \mathbb{N}^+$.*

We shall answer Questions 2.3 and 2.5 in Section 3.4.

2.3. Immediate responses to H. Scholz's problem. The first to publish a paper in response to H. Scholz's problem was G. Asser [6]. A. Robinson's review [97] summarizes it as follows:

> (...) The present paper is concerned with the characterization of all representable sets [=spectra]. A rather intricate necessary and sufficient condition is stated for arithmetical function $X(n)$ to be the characteristic function of a representable set. The condition shows that such a function is elementary in the sense of Kalmar. (...) On the other hand, the author establishes that there exist non-representable sets whose characteristic function is elementary. Examples of representable sets (some of which are by no means obvious) are given without proof and the author suggests that further research in this field is desirable.

Asser also noted that his characterization did not establish whether the complement of a spectrum is a spectrum.

About the same time, A. Mostowski [85] also considered the problem. H. Curry [25] summarizes Mostowski's paper as follows:

> (...) The author proves that for each function $f(n)$ of a class K of functions, which is like the class of primitive functions except that at each step all functions are truncated above at n, there is a formula H that has a model in a set of $n+1$ individuals if and only if $f(n) = 0$. From this he deduces positive solutions to Scholz's problem in a number of special cases.

It is usually considered that A. Mostowski really proved

THEOREM 2.6. *All sets of natural numbers, whose characteristic functions are in the second level of the Grzegorzcyk Hierarchy $\mathcal{E}^2$, are first order spectra.*

The detailed definitions and contents of this theorem will be discussed in Section 4.

In the last 50 years a steady stream of papers appeared dealing with spectra of first order and higher order logics. The problem seems not too important at first sight. However, some of these papers had considerable impact on what is now called Finite Model Theory and Descriptive Complexity Theory.

OPEN QUESTION 1 (Scholz's Problem). *Characterize the sets of natural numbers that are first order spectra.*

Scholz's Problem, as stated, is rather vague. He asks for a characterization of a family of subsets of the natural numbers without specifying, what kind of an answer he had in mind. The answer could be in terms of number theory, recursion theory, it could be algebraic, or in terms of something still to be developed. We shall see in the sequel many solutions to Scholz's Problem, but we consider it still open, because further answers are still possible.

The same question can be asked for any logic, in particular second order logic SO, or fragments thereof, like monadic second order logic MSO, fixed point logic, etc., as discussed in [31, 75].

OPEN QUESTION 2 (Asser's Problem). *Is the complement of a first order spectrum a first order spectrum*?

Here the answer should be yes or no.

The corresponding problem for SO has a trivial solution. Let $\phi \in \mathrm{SO}(\tau)$ with vocabulary $\tau = \{R_1, \ldots, R_k\}$. An integer n is in $\mathrm{SPEC}(\phi)$ iff

$$n \models \exists R_1 \exists R_2 \ldots \exists R_k \phi.$$

Then, the complement of $\mathrm{SPEC}(\phi)$ is easily seen to be the spectrum of the SO sentence $\neg(\exists R_1 \ldots \exists R_k \phi)$. In passing, note that every SO-spectrum is a spectrum of a SO sentence over the empty vocabulary (note that equality is allowed).

However, for fixed fragments of SO, Asser's Problem remains open. In particular

OPEN QUESTION 3. *Is the complement of a spectrum of an* MSO-*sentence again a spectrum of an* MSO-*sentence*?

2.4. Approaches and themes. In this survey we shall describe the various solutions and attempts to solve Scholz's and Asser's problems, and the developments these attempts triggered (see also Börger's earlier survey on this problem [10]). We shall emphasize more the various ways the questions were approached, and focus less on the historical order of the papers. There are several discernible themes described below.

Recursion theory. The early authors H. Asser and A. Mostowski approached the question in the language of the theory of recursive functions, i.e., they looked for characterization of spectra in terms of recursion schemes, or hierarchies of recursive functions. Most prominently in terms of Kalmar's elementary functions, the Grzegorczyk hierarchy, and hierarchies of arithmetical predicates, in particular the *rudimentary relations*. This line of thought culminates in 1962 in the thesis of J. Bennett [7].[1] Although G. Asser already characterized first order spectra in such terms, his characterization was not considered satisfactory even by himself, because it did not use standard terms and was not useful in proving that a given set of integers is (or not) a spectrum. We shall discuss the recursion theoretic approach in detail in Section 4.

Complexity theory. In the 1970s, D. Rödding and H. Schwichtenberg of the Münster school [98] gave a sufficient but not necessary condition: any set of integers recognizable by a deterministic linear space-bounded Turing machine is a first-order spectrum. (This is also a consequence of results from the 1960s by Bennett and Ritchie [7, 95], obtained before the emergence of complexity theory.) Further, Rödding and Schwichtenberg showed that sets of integers recognizable using larger space bounds are higher order spectra. C. Christen developed this line further [16, 17], independently obtaining a number of the following results.

At the same time the spectrum problem gained renewed interest in the USA. In 1972 A. Selman and N. Jones found an exact solution to Scholz's original question [70]: a set of integers is a first order spectrum if and only if it is recognizable by a non-deterministic Turing machine in time $O(2^{c \cdot n})$.

This result was independently also obtained by R. Fagin in his 1973 thesis [35], which contains an abundance of further results. Most importantly, R. Fagin studies *generalized spectra*, which are the projective classes of Tarski, restricted to finite structures, and really laid the foundations for Finite Model Theory and Descriptive Complexity, as can be seen in the monographs [67, 31, 75]. We shall discuss the complexity theoretic approach in detail in Section 5.

Images and preimages of spectra. It is easily seen that if S is a first order spectrum and p is a polynomial with positive coefficients, then $p(S) =$

[1]It seems that some of Bennett's unpublished results were rediscovered independently in China in the late 1980s by Shaokui Mo [82]. We shall discuss his work in the appendix.

$\{p(m) \mid m \in S\}$ is also a spectrum. In J. Bennett's thesis it is essentially proved that there is a first order spectrum S and an integer k such that $\{n \mid 2^{n^k} \in S\}$ is not a spectrum. It is natural to ask what happens to a spectrum under images and preimages of number theoretic functions. The general line of this type of results states that certain images or preimages of spectra of specific forms of sentences are or are not spectra of other specific forms of sentences.

Spectra of syntactically restricted sentences. Already in a paper by L. Löwenheim from 1915 [78] it is noted that, what later will be called the spectrum of a sentence in monadic second order logic (MSO) with unary relation symbols only, is finite or cofinite. The set of even numbers is the spectrum of an MSO sentence with one binary relation symbol, and it is ultimately periodic. Further, *every* ultimately periodic set of positive integers is a spectrum of a first order MSO sentence with one unary function symbol. Over the last fifty years various papers were written relating restrictions on the use of relation and function symbols, or other syntactic restrictions, to special forms of spectra. R. Fagin, in his thesis, poses the following problem

Open Question 4 (Fagin's Problem for binary relations). *Is every first order spectrum the spectrum of a first order sentence with one binary relation symbol*?

The question is even open, if restricted to any fixed vocabulary that contains at least one binary relation symbol, or two unary function symbols.

Much of this line of research is motivated by attempts to solve Fagin's problem.

Transfer theorems. Another way of studying spectra is given by the following result, again from Fagin's thesis: If S is a spectrum of a purely relational sentence where all the predicate symbols have arity bounded by k, then $S^k = \{m^k \mid m \in S\}$ is a spectrum of a sentence with one binary relation symbol only, or even a spectrum on simple graphs. One can view this an approach combining the study of images and preimages of spectra with either syntactically or semantically restricted spectra. Over the years quite a few results along this line were published. We shall discuss the last three approaches under the common theme of restrictions on vocabularies in detail in Section 6.

Spectra of semantically restricted classes. R. Fagin shows that Asser's problem has a positive answer if and only if it has a positive answer when restricted to the class of simple graphs. Similarly, in order to understand Fagin's problem better, one could consider restricted graph classes K, and study first order spectra restricted to graphs in K. One may think of graphs of bounded degree, planar graphs, trees, graphs of tree-width at most k, etc.

Open Question 5 (Fagin's Problem for simple graphs). *Is every first order spectrum the spectrum of a first order sentence over simple graphs*?

OPEN QUESTION 6. *Is every first order spectrum the spectrum of a first order sentence over planar graphs*?

For restrictions to graph classes of bounded tree-width, the answer is negative. The reason for this is that spectra of graphs of bounded tree-width are ultimately periodic. In fact, this holds for a much wider class of spectra. E. Fischer and J. A. Makowsky, [41], have analyzed under what conditions MSO-spectra are ultimately periodic. We shall discuss their results in detail in Section 8. This line of thought has not been extensively explored; it may well be a fruitful avenue for studying spectra in the future.

In the sequel of this survey we shall summarize what is known about spectra within these themes. Various solutions to Scholz's Problem were offered in the literature, varying with the tastes of the times, but there may still be more to come. Asser's and Fagin's Problems are still open. Both problems are intimately related to our understanding of definability hierarchies in Descriptive Complexity Theory. They may well serve as benchmarks of our understanding.

§3. Understanding Spectra: counting functions and number theory. In this section we formulate various ways to test our understanding of spectra. It will turn out that there are still many questions we do not know how to answer.

3.1. Representation of spectra and counting functions. Spectra are sets of positive natural numbers. These sets can be represented in various ways. We shall use the following:

DEFINITION 3.1. Let $M \subseteq \mathbb{N}^+$, and let $m_1, m_2, \dots$ an enumeration of M ordered by the size of its elements.

(i) $\chi_M(n)$ is the characteristic function of M, i.e.,

$$\chi_M(n) = \begin{cases} 1 & \text{if } n \in M, \\ 0 & \text{else.} \end{cases}$$

(ii) $\eta_M(n)$ is the enumeration function of M, i.e.,

$$\eta_M(n) = \begin{cases} m_n & \text{if it exists,} \\ 0 & \text{else.} \end{cases}$$

(iii) $\gamma_M(n)$ is the counting function of M, i.e., $\gamma_M(n)$ is the number of elements in M that are strictly smaller than n.

(iv) A *gap* of M is a pair of integers g_1, g_2 such that $g_1, g_2 \in M$ but for each n with $g_1 < n < g_2$ we have that $n \notin M$. Now let $\delta_M(n)$ be the length of the nth gap of M. Clearly, $\delta_M(n) = \eta_M(n+1) - \eta_M(n)$.

(v) If M is the spectrum of a sentence ϕ, we write χ_ϕ instead of $\chi_{\text{SPEC}(\phi)}$. Similarly for η_ϕ, γ_ϕ and δ_ϕ.

Obvious questions are of the following type:

OPEN QUESTION 7. *Which strictly increasing sequences of positive integers, are enumerating functions of spectra? For instance, how fast can they grow?*

OPEN QUESTION 8. *If M is a spectrum how can $\delta_M(n)$ behave?*

Coding runs of Turing machines one can easily obtain the following.

PROPOSITION 3.2. *For every recursive monotonically increasing function f there is a first order formula ϕ such that $\delta_\phi(n) \geq f(n)$.*

Various other partial answers to these questions will appear throughout our narrative.

3.2. Prime numbers. An obvious question is whether the primes form a spectrum. If one gets more ambitious one can ask for special sets of primes such as Fermat primes (of the form $2^{2^n} + 1$), Mersenne primes (of the form $2^p - 1$ with p a prime), or the set of primes p such that $p + 2$ is also a prime (twin primes). Even if we do not know whether such a set is finite, which is the case for twin primes, it may still be possible to prove that it is a spectrum. The answer to all these question is yes, because all these sets are easily proved to be rudimentary, see Section 4.

In the sense of the above definitions we have χ_{primes} is the charactersitic function of the set of primes, $\eta_{primes}(n) = p_n$, and $\gamma_{primes}(n)$ is the counting function of the primes, usually denoted by $\pi(n)$. $\delta_{primes}(n)$ is usually denoted by d_n. All these functions related to primes are subject to intensive study in the literature, see, e.g., [93]. As we have said that the primes form a first order spectrum, all the features of these functions observed on primes do occur on spectra.

For instance, $\pi(n)$ is approximated by the integral logarithm $li(n)$, and it was shown by J. E. Littlewood in 1914, cf. [93], that $\pi(n) - li(n)$ changes sign infinitely many often. For logical aspects of Littlewood's theorem, see [73].

Let us define

$$\pi^+ = \{n \mid \pi(n) - li(n) > 0\},$$
$$\pi^- = \{n \mid \pi(n) - li(n) \leq 0\}.$$

A less obvious question concerning spectra and primes is

OPEN QUESTION 9. *Are the sets π^+ and π^- spectra?*

3.3. Density functions. Many combinatorial functions are defined by linear or polynomial recurrence relations. Among them we have the powers of 2, factorials, the Fibonacci numbers, Bernoulli numbers, Lucas numbers, Stirling numbers and many more, cf. [47].

QUESTION 3.3. *Are the sets of values of these combinatorial functions first order spectra?*

The answer is yes in all of these cases. We shall sketch a proof in the appendix. that is based on the existence of such recurrence relations.

In addition, these functions also allow combinatorial interpretations as counting functions: The powers of 2 count subsets, the factorials count linear orderings, the Stirling numbers are related to counting equivalence relations. We shall see below that in these three examples the combinatorial definitions allow us to give alternative proofs that these sets of numbers are first order spectra.

The spectrum of a sentence ϕ witnesses the existence of models of ϕ of corresponding cardinalities. Instead, one could also ask for the number of ways the set $\{0, 1, \ldots, n-1\} = [n]$ can be made into a model of ϕ. Alternatively one could count models up to isomorphisms or up to some other equivalence relation.

Combinatorial counting functions come in different flavours;

Definition 3.4. Let $\mathcal{C}$ be a class of finite τ-structures. With $\mathcal{C}$ we associate the following counting functions:

(i) $f_{\mathcal{C}}(n)$ is the number of ways one can interpret the relation symbols of τ on the universe $[n]$ such that the resulting structure is in $\mathcal{C}$. This corresponds to counting labeled structures.

(ii) Let $Str(\tau)(n)$ denote the number of labeled τ-structures of size n. We put

$$prob_{\mathcal{C}}(n) = \frac{f_{\mathcal{C}}(n)}{Str(\tau)(n)}$$

which can be interpreted as the probability that a labeled τ-structure of size n is in $\mathcal{C}$.

(iii) $f_{\mathcal{C}}^{iso}(n)$ is the number of non-isomorphic models in $\mathcal{C}$ of size n.

(iv) For an equivalence relation E on $\mathcal{C}$ we denote by $f_{\mathcal{C}}^{E}(n)$ the number of non-E–equivalent models in $\mathcal{C}$ of size n.

(v) If E is the k-equivalence from Ehrenfeucht–Fraïssé games [32, 42], $f_{\mathcal{C}}^{E}(n)$ is denoted by $N_{\mathcal{C},k}(n)$, and is called an Ash-function, cf. [5] and Section 7.

(vi) If $\mathcal{C}$ consists of all the finite models of a sentence ϕ we write $f_{\phi}(n)$ instead of $f_{\mathcal{C}}(n)$. Similarly for $f_{\phi}^{E}(n)$ and $prob_{\phi}(n)$.

Counting labeled and non-labeled structures has a rich literature, cf. [58, 110]. Note that counting non-labeled non-isomorphic structures is in general much harder than the labeled case. The first connection between counting labeled structures and logic is the celebrated 0–1 Law for first order logic:

Theorem 3.5 (0–1 Laws). *For every first order sentence ϕ over a purely relational vocabulary τ we have*:

(i) (Y. Glebskii, D. Kogan, M. Liogonki and V. Talanov [45]; R. Fagin [35]) $\lim_{n\to\infty} prob_{\phi}(n)$ *always exists and is either* 0 *or* 1. *In other words, either ϕ or $\neg\phi$ is almost surely true.*

(ii) (E. Grandjean [48]) *Furthermore, the set of sentences* ϕ *such that*

$$\lim_{n \to \infty} prob_\phi(n) = 1$$

is decidable, and in fact PSPACE-*complete.*

What we are interested in here, is the relationship of such counting functions to spectra. Our example of powers of 2 shows that

$$2^n = f_\phi(n) = \eta_\psi(n)$$

where ϕ is an always-true first order sentence with one unary relation symbol, and ψ is the conjunction of the axioms of Boolean algebras. Similarly,

$$n! = f_{\phi_{LIN}}(n) = \eta_\psi(n)$$

where ϕ_{LIN} are the axioms of linear orders, and ψ describes the following situation:

(i) E is a ternary relation that is a bijection between the universe (first argument x) and all the linear orderings R on a subdomain P (remaining two arguments y, z).
(ii) First we say that to each x corresponds a linear order R. Then, we state that for $x \neq x'$ the orderings are different. For example by saying that at least two elements compare differently. This implies that E is injective.
(iii) To ensure that we get all the orderings on P we say that for every ordering and every transposition of two elements in this ordering, there is a corresponding ordering.

Hence, the size of the model of ψ is the number of linear orderings on P.

Clearly, if f_ϕ is not strictly increasing, there is no ψ with $f_\phi(n) = \eta_\psi(n)$. For instance, for ϕ which says that some function is a bijection of a part of the universe to its complement, we have

$$f_\phi(n) = \begin{cases} \binom{2m}{m} \cdot m! & \text{if } n = 2m, \\ 0 & \text{else.} \end{cases}$$

Open Question 10. *Let* ϕ *a first order sentence, and* f_ϕ *be the associated labeled counting function that is monotonically increasing. Is there a first order sentence* ψ *such that for all n*

$$f_\phi(n) = \eta_\psi(n).$$

The converse question seems more complicated. For instance, as we have noted before, the primes p_n are of the form η_ψ for some first order ψ, but we are not aware of any labeled counting function that will produce the primes.

R. Fagin [36] calls ϕ *categorical* if $f_\phi^{iso}(n) \leq 1$ for every n. For instance, ϕ_{LIN} is categorical. The counting function up to isomorphisms can be bounded by any finite number m, using disjunctions of different categorical sentences. So it may be less promising to study for which first order sentences ϕ there is a ψ such that $f^{iso}_{\phi}(n) = \eta_{\psi}(n)$, or vice versa.

Surprisingly, C. Ash [5] has found a connection between Asser's Problem and the behaviour of the Ash functions defined in Definition 3.4(v). We shall discuss this in Section 7.

3.4. Sentences with prescribed spectra. In the light of Theorem 3.5 we note that if $prob_{\phi}(n)$ tends to 1 then $spec(\phi)$ is cofinite. Obviously, the converse does not hold, because there are categorical sentences with models in all finite cardinalities.

Trakhtenbrot's Theorem says that it is undecidable whether a spectrum is empty, and Grandjean's Theorem says that it is decidable, whether a sentence is almost always true, i.e., $prob_{\phi}(n)$ tends to 1. As a partial answer to Questions 2.3 and 2.5 we have:

PROPOSITION 3.6. *Let ϕ be a first order sentence. The following are undecidable*:

(i) $\mathrm{SPEC}(\phi)$ *is finite, cofinite.*
(ii) $\mathrm{SPEC}(\phi)$ *is ultimately periodic.*
(iii) $\mathrm{SPEC}(\phi)$ *is, for given $a, b \in \mathbb{N}$ of the form $a + b\mathbb{N}$.*
(iv) $\mathrm{SPEC}(\phi) = S$ *for some set $S \subseteq \mathbb{N}^+$.*

SKETCH OF PROOF. Let $\varphi \in$ FO. We describe the construction of FO-sentences $\psi_1, \psi_2, \psi_3, \psi_4$ and ψ_5 such that $\mathrm{SPEC}(\varphi) = \emptyset$:

- iff $\mathrm{SPEC}(\psi_1)$ is finite.
- iff $\mathrm{SPEC}(\psi_2) = \mathbb{N}^+$.
- iff $\mathrm{SPEC}(\psi_3) = \{f(i) \mid i \in \mathbb{N}^+\}$ for a given function f such that $f(i) \geq i$ for all i and the graph $n = f(i)$ seen as a binary relation is rudimentary (see Section 4.2 for a precise definition).
- iff $\mathrm{SPEC}(\psi_4)$ is cofinite.
- iff $\mathrm{SPEC}(\psi_5)$ is ultimately periodic.

Since the problem of emptiness of spectra is undecidable by Trakhtenbrot's theorem, the announced result follows.

Let 0 and max be two constant symbols, let $\leq$ be a binary predicate symbol and let $+$ and $\times$ be two ternary predicate symbols. Let $Arithm(0, \max, \leq, +, \times)$ denote a first-order sentence axiomatizing the usual arithmetic predicates. Our sentences ψ_i $(i = 1, \dots, 3)$ consist of the conjunction of $Arithm(0, \max, \leq, +, \times)$ with a specific part ψ'_i that we will describe below. We shall use the fact that the *Bit* predicate[2] is definable from $+$ and $\times$ in finite structures, as well as the ternary relation $a = b^c$. For simplicity, we will assume w.l.o.g. that the vocabulary of φ consists of a binary relation R only. Let n and y be new variable symbols. Let $\varphi'(n, y)$ be the formula obtained from φ by replacing every quantification $\forall x$ by $\forall x < n$ and $\exists x$ by $\exists x < n$, and every atomic formula $R(x, x')$ by $Bit(y, x + nx')$. The idea is

[2] $Bit(a, b)$ is true iff the bit of rank b of a is 1.

that a graph R on a set of n elements seen as $\{0,\dots,n-1\}$ is encoded by the number $y < 2^{n^2}$ written in binary with a 1 in position $a + bn$ if and only if $R(a,b)$ holds. Hence for all $n \in \mathbb{N}^+$, we have $\exists y < 2^{n^2}\, \varphi'(n,y)$ if and only if φ has a model with n elements.

– Let $\psi_1' \equiv \exists m, n, y < \max\,(\max = 3^m \times 2^{n^2} \wedge y < 2^{n^2} \wedge \varphi'(n,y))$.
  It is easy to verify that if $\textsc{spec}(\varphi) = \emptyset$, then $\textsc{spec}(\psi_1)$ is also empty (hence finite), and conversely, if $\textsc{spec}(\varphi) \neq \emptyset$, then $\textsc{spec}(\psi_1)$ contains all the integers of the form $3^m \times 2^{n^2}$ for some $m \in \mathbb{N}^+$ and $n \in \textsc{spec}(\varphi)$, i.e., $\textsc{spec}(\psi_1)$ is infinite.
– Let $\psi_2' \equiv (\forall n < \max\, \max \neq 2^{n^2}) \vee (\exists n < \max\,(\max = 2^{n^2} \wedge \forall y < 2^{n^2}\, \neg\varphi'(n,y)))$.
  If $\textsc{spec}(\varphi) = \emptyset$, then for all n and y, the condition $\exists y < 2^{n^2}\, \varphi'(n,y)$ is false, hence $\textsc{spec}(\psi_2) = \mathbb{N}^+$. Conversely, if $\textsc{spec}(\varphi) \neq \emptyset$, then the integers of the form 2^{n^2} with $n \in \textsc{spec}(\varphi)$ are not in $\textsc{spec}(\psi_2)$.
– Since the binary relation $y = f(x)$ is rudimentary, it is definable from $+$ and $\times$ in finite structures. Let $\psi_3' \equiv \exists i < \max\,(\max = f(i) \wedge ((\forall n < i\; i \neq 2^{n^2}) \vee (\exists n < i\,(i = 2^{n^2} \wedge \forall y < 2^{n^2}\, \neg\varphi'(n,y)))))$.
  The verification that $\textsc{spec}(\varphi) = \emptyset$ if and only if $\textsc{spec}(\psi_3) = \{f(i) \mid i \in \mathbb{N}^+\}$ is similar to the previous case.
– Let $\psi_4' \equiv \forall n < \max\,(2^{n^2} \leq \max \longrightarrow \forall y < 2^{n^2}\, \neg\varphi'(n,y))$.
  If $\textsc{spec}(\varphi) = \emptyset$, then for all n and y, the condition $\exists y < 2^{n^2}\, \varphi'(n,y)$ is false, hence $\textsc{spec}(\psi_4) = \mathbb{N}^+$ (and so is cofinite). Conversely, if $\textsc{spec}(\varphi) \neq \emptyset$, then the integers greater than 2^{n^2} with $n \in \textsc{spec}(\varphi)$ are not in $\textsc{spec}(\psi_2)$, which is not cofinite.
– Let $\psi_5' \equiv \exists n, m < \max\,(\max = 2^{n^2} \times m^2 \wedge \exists y < 2^{n^2}\, \varphi'(n,y) \wedge \forall n' < n\, \forall y' < 2^{n^2}\, \neg\varphi'(n',y')))$.
  If $\textsc{spec}(\varphi) = \emptyset$, then $\textsc{spec}(\psi_5) = \emptyset$ (hence is ultimately periodic). Conversely, observe that $\textsc{spec}(\psi_5) = \{n_0 \times m^2 \mid m \in \mathbb{N}+\}$, where $n_0 = \inf(\textsc{spec}(\varphi))$. Hence $\textsc{spec}(\psi_5)$ is not ultimately periodic. $\dashv$

3.5. Real numbers and spectra. Let $\chi_\phi(n)$ be the characteristic function of the spectrum of a first order sentence ϕ. We can associate with ϕ and $a \in \mathbb{Z}$ the real number $r_\phi = a + \sum_n \chi_\phi(n) 2^{-n}$.

DEFINITION 3.7. A real number is *first order spectral* if it is of the form r_ϕ for some $a \in \mathbb{Z}$ and some first order sentence ϕ.

As we have noted ultimately periodic sets of natural numbers are first order spectra, and correspond to rational numbers. Also every ultimately periodic spectrum can be realized by a formula with one function symbol only. We have

PROPOSITION 3.8. *Every rational number q is first order spectral using a formula with one function symbol only.*

QUESTION 3.9. *Do the the first order spectral reals form a field*?

E. Specker, [104], proved that there is a real x primitive recursive in base 2 such that $3x$, $x + \frac{1}{3}$, x^2 are not primitive recursive in base 2. A modern treatment can be found in [14]. H. Friedman [43] mentioned on an internet discussion site that primitive recursive can be replaced in Specker's Theorem by much lower complexity within the Grzegorczyk Hierarchy, discussed in detail in Section 4. All we need here is that the characteristic functions of spectra are in $\mathcal{E}^3$.

J. Miller kindly provided us, [81], with the more precise statement of Friedman's statement.

THEOREM 3.10 (E. Specker, 1949 and H. Friedman 2003). *There is a real x that is primitive recursive in base 2 (and can even be taken to be in $\mathcal{E}^2$ of the Grzegorczyk Hierarchy), such that $3x$, $x + \frac{1}{3}$, x^2 are not primitive recursive in base 2.*

SKETCH OF PROOF.[3] This can be proved by exploiting the fact that none of these functions $3x, x^2, x + 1/3$ are continuous as functions on binary expansions of reals. They can take a number x that is not a binary rational to one that is.

Let us focus on $3x$. So, for example, if $x = 0.0101010101\ldots$, then $3x = 1$. We can exploit this as follows. Say we have built the binary expansion of x up to position $n - 1$ and it looks like $0.b$ (where b is a finite string) and that we want to diagonalize against the ith primitive recursive function p_i. Compute $p_i(n)$, step by step. As long as it does not converge, keep building x to look like $0.b0010101010101\ldots$. If $p_i(n)$ converges at stage s, then use position $n+2s$ or position $n+2s+1$ to spring the delayed trap. If $p_i(n) = 0$, then let $x = 0.b0010101\ldots01011$. If $p_i(n) = 1$, then let $x = 0.b0010101\ldots0100$. Either way, $p_i(n)$ does not correctly compute the nth bit of $3x$. Note that we spread out the unbounded search, so that each bit is computed by a bounded (primitive recursive) procedure.

In this way we can diagonalize against $3x$, x^2 and $x + 1/3$ being primitive recursive while making x primitive recursive. To make x to be in $\mathcal{E}^2$ one uses the fact that $\mathcal{E}^2$ is the same as computable in linear space [95].

In the argument above, when we are trying to figure out bit t of x, we compute $p_i(n)$ (for some i and n determined earlier in the construction of x) for t steps and if it does not halt we output the default bit (alternating between 0 and 1), so it can be made in linear time. Actually, in the case of x^a2, one has to work a little harder to determine the default bit, but this can definitely be done in linear space (and polynomial time). $\dashv$

We shall see in Section 4, Theorem 4.11, that the characteristic functions of spectra are in $\mathcal{E}^3$, therefore Theorem 3.10 covers all the spectral reals,

[3] Due to J. Miller.

hence the spectral reals do not form a field. More precisely we have the following Corollary:

COROLLARY 3.11. *The spectral reals are not closed under addition nor under multiplication. Furthermore, they are closed under the operation* $1 - x$ *iff the complement of a spectrum is a spectrum.*

We now turn the question of algebraicity and transcendence of spectral reals. Clearly, every first order spectral real is a recursive real in the sense of A. Turing [108]. Using Liouville's Theorem,[4] we can see that many transcendental reals are first order spectral.

We call a real number $r \in (0, 1)$ *automatic* if the n-th bit of its binary expansion can be generated by a finite automaton from the binary representation of n. In 1968 A. Cobham, [18] conjectured that automatic numbers are transcendental. This was proven in 2007 by B. Adamczewski and Y. Bugeaud, [1].

OPEN QUESTION 11. (i) *Are there any irrational algebraic reals that are spectral*?

(ii) *Is every automatic real a spectral real*?

Recursive reals of low complexity have recently gained renewed attention, cf. [101, 102]. However, the analysis of computable reals in binary or b-adic presentation is tricky because of the behaviour of the carry, cf. [15].

§4. Approach I: Recursion theory.

This approach has generated all in all four papers (namely [6] by G. Asser in 1955, [85] by A. Mostowski in 1956, [95] by R. Ritchie, and [82] by S. Mo in 1991) and two Ph.D. dissertations, namely [94] by R. Ritchie in 1960 and [7] by J. Bennett in 1962. These works share the common feature of being hard to obtain for many readers for various reasons: Asser's and Mostowski's papers are difficult to read because they are more than fifty years old and Asser's paper is in German. Bennett's thesis, cited in many papers, is almost equally old, and in addition has remained unpublished. Finally, Mo's paper, though more recent, is in Chinese. This is the reason why we outline in the appendix these references in some detail, including several sketches of proofs in modern language. In the present section, after some background material, we present a synthesis of recursion-theoretic approaches to the spectrum problem.

4.1. Grzegorczyk's Hierarchy.

For a detailed presentation of the material in this subsection, see, e.g., [99]. A. Grzegorczyk's seminal paper [55] about classification of primitive recursive functions was published in 1953, one year after Scholz's question, and two years before Asser's paper. Hence, Grzegorczyk's Hierarchy was not the standard way to consider primitive recursive functions in the mid-1950s. And actually, G. Asser and A. Mostowski deal

[4] Liouville's Theorem states, in simplified form, that a real of the form $r = \sum_n 2^{-f(n)}$ where $f(n) \geq n!$ is transcendental.

with recursion-theoretic aspects of spectra, but not explicitly with Grzegorczyk's classes, though it is the usual framework in which their results are presented. It is only in J. Bennett's thesis in 1962 and especially in S. Mo's paper in 1991 that one finds an explicit study of spectra in terms of Grzegorczyk's classes.

From now on, a function is always intended to be a function from some $\mathbb{N}^k$ to $\mathbb{N}$ (total, unless otherwise specified).

Definition 4.1 (Elementary functions). The class $\mathcal{E}$ of elementary functions is the smallest class of functions containing the zero, successor, projections, addition, multiplication and modified subtraction functions, and which is closed under composition and bounded sum and product (i.e., $f(n, \vec{x}) = \sum_{i=0}^{n} g(i, \vec{x})$ and $f(n, \vec{x}) = \prod_{i=0}^{n} g(i, \vec{x})$, with previously defined g). We denote by $\mathcal{E}_\star$ the elementary relations, i.e., the class of relations whose characteristic functions are elementary.

The class $\mathcal{E}$ was introduced by Kalmár [72] and Csillag [24] in the 1940s, and contains most usual number-theoretic functions. It also corresponds to Grzegorczyk's class $\mathcal{E}^3$, that we define below.

Definition 4.2 (Primitive recursion). Let f, g, h be functions. We say that f is defined from g and h by primitive recursion when it obeys a schema:
$$\begin{cases} f(0, \vec{x}) & = g(\vec{x}), \\ f(n+1, \vec{x}) & = h(n, \vec{x}, f(n, \vec{x})). \end{cases}$$
The class of primitive recursive functions, denoted by $\mathcal{PR}$, is the smallest class of functions containing the zero function, the successor function, the projection functions, and which is closed under composition and primitive recursion.

For instance, elementary functions are primitive recursive. The following binary function Ack, known as Ackermann's function, is provably not primitive recursive, whereas all unary specialized functions $Ack_x : y \mapsto Ack(x, y)$ are primitive recursive:
$$\begin{cases} Ack(0, y) = y + 1, \\ Ack(x+1, 0) = Ack(x, 1), \\ Ack(x+1, y+1) = Ack(x, Ack(x+1, y)). \end{cases}$$

In order to introduce Grzegorczyk's hierarchy, we need a weaker version of primitive recursion, in which the newly defined functions have to be bounded by some previously defined function.

Definition 4.3 (Bounded recursion). Let f, g, h, j be functions. We say that f is defined from g, h and j by bounded recursion when it obeys a schema:
$$\begin{cases} f(0, \vec{x}) & = g(\vec{x}), \\ f(n+1, \vec{x}) & = h(n, \vec{x}, f(n, \vec{x})), \\ f(n, \vec{x}) & \leq j(n, \vec{x}). \end{cases}$$

Let f_n $(n = 0, 1, 2, \dots)$ be the following sequence of primitive recursive functions:

- $f_0(x, y) = y + 1$,
- $f_1(x, y) = x + y$,
- $f_2(x, y) = (x + 1) \cdot (y + 1)$,
- and for $k \geq 0$ $\begin{cases} f_{k+3}(0, y) & = f_{k+2}(y + 1, y + 1), \\ f_{k+3}(x + 1, y) & = f_{k+3}(x, f_{k+3}(x, y)). \end{cases}$

Roughly speaking, the important feature is that the functions f_n are more and more rapidly growing. Several other similar sequences of increasingly growing functions can be used to define Grzegorczyk's classes.

Definition 4.4 (Grzegorczyk's hierarchy). The Grzegorczyk's class $\mathcal{E}^n$ is the smallest class of functions containing the zero function, the projections functions and f_n and which is closed under composition and bounded recursion. The associated classes of relations $\mathcal{E}^n_\star$ are defined as the class of relations on integers with a characteristic function in $\mathcal{E}^n$.

Note that, for sake of simplicity, we use the same notation for a class of relations of various arities (e.g., $\mathcal{E}^3_\star$) and the class of unary relations (i.e., sets) that it contains. Which one is intended will always be clear from the context.

The main features of Grzegorczyk's classes were studied by A. Grzegorczyk in [55] and by R. Ritchie in [95].

Theorem 4.5 (A. Grzegorczyk (1953)).

- *The functional hierarchy is strict for $n \geq 0$, i.e., we have $\mathcal{E}^n \subsetneq \mathcal{E}^{n+1}$.*
- *The relational hierarchy is strict for $n \geq 3$, i.e., we have $\mathcal{E}^n_\star \subsetneq \mathcal{E}^{n+1}_\star$.*
- *For the initial levels of the relational hierarchy, we have $\mathcal{E}^0_\star \subseteq \mathcal{E}^1_\star \subseteq \mathcal{E}^2_\star \subseteq \mathcal{E}^3_\star$.*
- *The Kalmár–Csillag class of elementary functions $\mathcal{E}$ is equal to $\mathcal{E}^3$.*
- *Finally, the full hierarchy corresponds to primitive recursion, i.e., $\mathcal{PR} = \bigcup_{n=0}^{+\infty} \mathcal{E}^n$.*

Theorem 4.6 (R. Ritchie (1963)). *We have $\mathcal{E}^2_\star \neq \mathcal{E}^3_\star$* [95].

Note that the possible separation of the relational classes $\mathcal{E}^0_\star, \mathcal{E}^1_\star, \mathcal{E}^2_\star$ is still an open question.

Open Question 12. *Are the inclusions in*

$$\mathcal{E}^0_\star \subseteq \mathcal{E}^1_\star \subseteq \mathcal{E}^2_\star$$

proper?

An important point is that the functional hierarchy deals with the rate at which functions may grow: intuitively, functions in the low level of the hierarchy grow very slowly, while functions higher up in the hierarchy grow very rapidly. However, this feature does not hold for the relational hierarchy, because characteristic functions do not grow at all (they are 0–1 valued). For instance, the ternary relations $z = x + y$, $z = x \times y$, $z = x^y$ as well as $z = Ack(x, y)$ all belong to $\mathcal{E}^0_\star$, whereas the corresponding functions provably do not lie in $\mathcal{E}^0$.

4.2. Rudimentary relations and strictly rudimentary relations. In addition to primitive recursive classes of relations, we also introduce two new classes of relations with an arithmetical flavour, namely the rudimentary and strictly rudimentary relations. These classes were originally introduced by R. Smullyan [103], and a major reference about rudimentary relations and subclasses is J. Bennett's thesis [7].

Definition 4.7 (Rudimentary relations). Denote by Rud the smallest class of relations over integers containing the graphs of addition and multiplication (seen as ternary relations) and closed under Boolean operations ($\neg, \wedge, \vee$) and bounded quantifications ($\forall x < y \dots$ and $\exists x < y \dots$).

In spite of its very restricted definition, the class Rud is surprisingly robust (e.g., it has several equivalent definitions in the fields of computational complexity, recursion theory, formal languages etc.) and large. For instance, the following formula defines the set of prime numbers:

$$x > 1 \wedge \forall y < x\, \forall z < x\, \neg(x = y.z).$$

More complex formulas prove that the ternary relation $z = x^y$ is rudimentary (Bennett [7]), as well as the graph $z = Ack(x, y)$ of Ackermann's function (Calude [11]), which is not primitive recursive (as a function), or the four-ary relation $x^y \equiv z\ [mod\ t]$ (Hesse, Allender, Barrington [62]). Actually, we are not aware of a natural number theoretic relation that is provably not rudimentary.

The following is easy to see.

Proposition 4.8. Rud $\subseteq \mathcal{E}^0_\star \subseteq \mathcal{E}^1_\star \subseteq \mathcal{E}^2_\star$.

However, the equality remains an open question (and would imply Rud $= \mathcal{E}^2_\star$ as well, since the closure of $\mathcal{E}^0_\star$ under polynomial substitution is $\mathcal{E}^2_\star$ whereas Rud is closed under polynomial substitution).

Open Question 13. *Are the inclusions in* Rud $\subseteq \mathcal{E}^0_\star \subseteq \mathcal{E}^1_\star \subseteq \mathcal{E}^2_\star$ *proper*?

It remains to introduce the strictly rudimentary relations. Let us consider the dyadic representation of integers, i.e., $n \in \mathbb{N}$ is represented by a word in $\{1, 2\}^*$. Compared to binary notation, dyadic notation avoids the problem of leading 0s and yields a bijection between integers and words. When integers are seen as words, it is natural to consider subword quantifications instead of ordinary bounded quantification. We say that $w = w_1 \dots w_k$ is a subword of $v = v_1 \dots v_p$ and we denote $w \upharpoonright v$ when there exists $1 \leq i \leq p$ such that $w_1 = v_i, \dots, w_k = v_{i+k-1}$. Of course, if $x \upharpoonright y$, then $x \leq y$.

DEFINITION 4.9 (Strictly rudimentary relations). Denote by SRUD the smallest class of relations over integers containing the graphs of dyadic concatenation (seen as a ternary relation) and closed under Boolean operations ($\neg$, $\wedge$, $\vee$) and subword quantifications ($\forall x \restriction y \dots$ and $\exists x \restriction y \dots$).

There are only a few natural examples of strictly rudimentary relations, e.g., x begins (or ends or is a part of) y (as dyadic words), $x = y$, the dyadic representation of x is a single symbol, the dyadic representation of x contains only one type of symbol. On the other hand, several relations are provably not strictly rudimentary such as $x \leq y$, $x = y + 1$, x and y have the same dyadic length and the dyadic representation of x is of the form $1^n 2^n$ for some n (V. Nepomnjascii 1978, see [88]).

Note that rudimentary relations were originally (and equivalently) defined by Smullyan [103] using dyadic concatenation as a basis relation instead of addition and multiplication. Clearly, we have SRUD $\subsetneq$ RUD.

4.3. Recursive and arithmetical characterizations of spectra. In the 1950s and 1960s, following the tastes of their time, logicians tried to characterize SPEC by recursion and arithmetic. Typically, they wished to obtain the characteristic functions of spectra as the 0-1-valued functions in a class defined by closure of a certain set of simple functions under certain operators (such as composition or various recursion schemas). From this point of view, their results are not totally satisfactory because the results are either partial, or somehow cumbersome or unnatural. However, these studies show that the class of spectra is very broad, and that most classical arithmetical sets are spectra.

The class SPEC is contained in Grzegorczyk's hierarchy (by G. Asser in [6] and A. Mostowski in [85]), from which we can deduce that all rudimentary sets are spectra.

THEOREM 4.10 (G. Asser (1955)). SPEC $\subsetneq \mathcal{E}^3_\star$.

Asser's theorem is based on a rather complicated and artificial arithmetical characterization of spectra (see the appendix). In particular, Asser's construction is of no help in proving that a particular set is (or not) spectrum.

Though he actually uses a slightly different class (see the appendix), the following result is usually attributed to Mostowski:

THEOREM 4.11 (A. Mostowski (1956)). $\mathcal{E}^2_\star \subseteq$ SPEC.

Note that equality in Mostowski's theorem remains an open question.

OPEN QUESTION 14. *Is the inclusion in $\mathcal{E}^2_\star \subseteq$ SPEC proper?*

The following corollary is not stated by A. Mostowski, but can be found in J. Bennett's thesis. It is worth noting because one of the most fruitful ways in proving that various arithmetical sets are spectra is to prove that they are actually rudimentary.

COROLLARY 4.12. RUD $\subseteq$ SPEC.

For instance, any set defined by a linear or polynomial recurrence condition, such as the Fibonacci numbers (i.e., those numbers appearing in the sequence defined by $u_0 = u_1 = 1$ and $u_{n+2} = u_n + u_{n+1}$), is rudimentary (see [34]). From Corollary 4.12, we deduce that such sets are spectra, as announced in Section 3. Similarly, using the fact that the set of prime numbers is rudimentary and that exponentiation has a rudimentary graph, one proves that the sets of Fermat primes (of the form $2^{2^n} + 1$), Mersenne primes (of the form $2^p - 1$ with p a prime), or twin primes (p prime such that $p + 2$ is also a prime) are rudimentary (hence also spectra).

Note that the question of whether the inclusion in Corollary 4.12 is proper is still open.

OPEN QUESTION 15. *Do we have* RUD = SPEC?

This problem is further investigated in Subsubsection 6.3.3.

An arithmetic characterization of SPEC in terms of strictly rudimentary relations is also given, among many other results (see the appendix), by J. Bennett in his thesis.

THEOREM 4.13 (J. Bennett (1962)). *A set $S \subseteq \mathbb{N}$ is in* SPEC *iff it can be defined by a formula of the form $\exists y \leq 2^{x^j} R(x, y)$ for some $j \geq 1$, where R is in* SRUD. *i.e.,*

$$S = \{x \in \mathbb{N} \mid \exists y \leq 2^{x^j} R(x, y)\}$$

for some $R \in$ SRUD *and $j \geq 1$.*

J. Bennett also characterizes spectra of higher order logics and shows that the union of spectra of various orders equals the class of elementary relations $\mathcal{E}^3_\star$.

The characterization of spectra stated in Theorem 4.13 is rather simple and elegant. However, once again, it is not really useful in proving that a given set is a spectrum, now because SRUD is very restrictive. A somehow similar characterization of SPEC using RUD instead of SRUD would have been more powerful—but, one gets in this way second-order spectra.

Finally, let us note a late paper on the recursion-theoretic aspect of spectra, namely [82], due to the Chinese logician Mo Shaokui in 1991 (see the appendix). The solution to Scholz's problem proposed there is of the same type as Bennett's characterization. However, the only bibliographic references in Mo's paper are Scholz's question [100] and Grzegorczyk's paper [55], so that it can be considered completely independent from all other contributions about spectra. The appendix summarizes this paper's results.

§5. **Approach II: Complexity theory.** The spectrum problem, formulated in the early 1950s, predates complexity theory since the notions of time or space bounded Turing machines first emerged in the 1960s (see [61, 74]). However, the first results about complexity of spectra appeared very soon (see Subsection 5.3), and computational complexity characterizations of spectra were found, in at least three independent early contexts (see Subsection 5.4). Later on, several refinements and developments of these seminal results have been published (see Subsections 5.7 and 5.8).

Turing machines and other standard models of computation operate on words, not on numbers. Let $L \subseteq \Sigma^*$ be a language, i.e., a set of finite words over a fixed finite alphabet Σ. Without loss of generality we assume $\Sigma = \{0, 1\}$, and that input words have no leading zeros.

The archetypical task, given a language L, is to study the complexity of deciding membership in L of a word x as a function of the length $|x|$, i.e., asymptotic growth rate of the time, space or other computational resources needed to decide whether $x \in L$.

5.1. Complexity and spectra. In this section, for a fixed sentence ϕ, the set of natural numbers $\textsc{spec}(\phi)$ is seen as the set of positive instances of a decision problem (given a number n, is there a model of ϕ with n elements?).

When dealing with computational complexity, we convert spectra (sets of natural numbers) into languages (over alphabet $\{0, 1\}$). The spectrum problem can thus be rephrased as: What is the computational complexity of the decision problems for spectra?

Complexity classes. Denote by $\mathrm{NTIME}(f(n))$ (resp. $\mathrm{DTIME}(f(n))$) the class of binary languages accepted in time $O(f(n))$ by some non-deterministic (resp. deterministic) Turing machine, where n is the length of the input. Similarly, let us denote by $\mathrm{DSPACE}(f(n))$ the class of languages accepted in space $O(f(n))$ by some deterministic Turing machine. Some well-known complexity classes that concern us here are:

$$\mathrm{L} = \mathrm{DSPACE}(\log n) \subseteq \mathrm{NL} = \mathrm{NSPACE}(\log n),$$
$$\mathrm{LINSPACE} = \mathrm{DSPACE}(n) \subseteq \mathrm{NLINSPACE} = \mathrm{NSPACE}(n),$$
$$\mathrm{P} = \bigcup_{c \geq 1} \mathrm{DTIME}(n^c) \subseteq \mathrm{NP} = \bigcup_{c \geq 1} \mathrm{NTIME}(n^c),$$
$$\mathrm{E} = \bigcup_{c \geq 1} \mathrm{DTIME}(2^{c \cdot n}) \subseteq \mathrm{NE} = \bigcup_{c \geq 1} \mathrm{NTIME}(2^{c \cdot n}).$$

Finally, if C denotes a complexity class, we denote its complement class, i.e., the class of binary languages L such that $\Sigma^* - L \in \mathrm{C}$, by coC.

Of course, the perennial open questions are:

Open Question 16. (i) *Are any of the inclusions* $\mathrm{L} \subseteq \mathrm{NL}$, $\mathrm{LINSPACE} \subseteq \mathrm{NLINSPACE}$, $\mathrm{P} \subseteq \mathrm{NP}$ *and* $\mathrm{E} \subseteq \mathrm{NE}$ *proper*?

(ii) *Do any of the equalities* $\mathrm{NP} = \mathrm{coNP}$ *and* $\mathrm{NE} = \mathrm{coNE}$ *hold*?

Surprisingly, the following was shown independently by N. Immermann and R. Szelepcźenyi in 1982, cf. [67]:

THEOREM 5.1 (Immermann, Szelepcźenyi 1982).

$$\mathrm{NL} = \mathrm{coNL} \quad \textit{and} \quad \mathrm{NLINSPACE} = \mathrm{coNLINSPACE}.$$

In Section 6 we shall also make use of the polynomial time hierarchy PH and its linear analogue LTH. The class RUD lies between L and LINSPACE, and must be different from one of them.

PROPOSITION 5.2.

(i) (Nepomnjascii 1970, [87]) $\mathrm{L} \subseteq \mathrm{RUD}$.
(ii) (Wrathall 1978, [112]) $\mathrm{RUD} = \mathrm{LTH}$.
(iii) (Myhill 1960, [86]) $\mathrm{LTH} \subseteq \mathrm{LINSPACE}$.

OPEN QUESTION 17. *Which of the inclusions*

$$\mathrm{L} \subseteq \mathrm{RUD} = \mathrm{LTH} \subseteq \mathrm{LINSPACE}$$

are proper?

Number representations by binary or unary words. It is natural to use binary notation for natural numbers. An alternative without leading zeros is Smullyan's *dyadic* notation [103]. The shortest binary length and dyadic length of the natural number n are very close to $\lceil \log_2 n \rceil$, whereas its unary length is of course n, and we have $n = 2^{\log_2 n}$. Consequently, the same (mathematical) computation that is performed by some Turing machine in time, e.g., $O(2^{c \cdot |n|})$ when $|n|$ is the binary length of the natural number input, is also performed by a slightly different Turing machine in time $O(n^c)$ when n is the (unary length of the) natural number input.

Unary notation (also called *tally* notation, in which the number n is represented by the word $1 \dots 1$ composed of n ones) also has its fans, for reasons explained in the description of Fagin's work. Most results in this section may be stated in either notation, but for sake of simplicity, and unless explicitly stated otherwise, we use binary notation.

Recall that SPEC denotes the set of spectra of first-order sentences, i.e.,

$$\mathrm{SPEC} = \{\mathrm{spec}(\phi) \mid \phi \text{ is a first-order sentence}\}.$$

5.2. Spectra, formal languages, and complexity theory. Formal language theory was much studied in the early 1960s, cf. [59], in particular the Chomsky hierarchy. While the regular and context-free language classes were well-understood, several questions remained open for larger classes. We need here the following:

THEOREM 5.3.

(i) (Ritchie 1963, [95]) $\mathcal{E}^2_\star = \mathrm{LINSPACE}$.
(ii) (Kuroda 1964, [74]) *A language L is* context sensitive *iff* $L \in \mathrm{NLINSPACE}$.

For our discussion one should remember that at that time it was then (as now) unknown whether LINSPACE = NLINSPACE and also unknown whether NLINSPACE was closed under complementation. The latter was only resolved positively more than 20 years later, see Theorem 5.1.

These open questions showed a tantalizing similarity to Scholz' and Asser's questions. If we identify characteristic functions with sets, then Bennett's 1962 thesis combined with Asser, Mostowski and Ritchie's results, yield

$$\text{LINSPACE} \subseteq \text{Spec} \subseteq \mathcal{E}^3_\star \text{ and } \text{LINSPACE} \subseteq \text{NLINSPACE} \subseteq \mathcal{E}^3_\star.$$

This led to a conjecture $\text{Spec} \stackrel{?}{=} \text{NLINSPACE}$, that spectra might be coextensive to the context sensitive languages. The analogy fails, though: more than n "bits of storage" seem to be needed to store an n-element model $\mathcal{M}$ of a sentence ϕ.

5.3. An early paper. One of the first papers explicitly relating spectra to bounded resource machine models of computation is [98] (in German), due to Rödding and Schwichtenberg from Münster in 1972. This transition from recursion theory to complexity theory had been prepared ten years before by Bennett and Ritchie, and Rödding and Schwichtenberg made a step further. The model of computation they use is not Turing machines, but register machines. As Bennett does, Rödding and Schwichtenberg not only consider spectra of first-order sentences, but also higher order spectra, namely spectra of sentences using i-th order variables. Let us denote by $\text{HO}-\text{SPEC}_i$ the class of spectra of sentences using i-th order variables. Let us define the following sequence of functions: let $exp_0(n) = n$, and $exp_{i+1}(n) = 2^{exp_i(n)}$. Along with other results in the field of recursion theory, Rödding and Schwichtenberg prove the following theorem.

THEOREM 5.4 (Rödding and Schwichtenberg 1972 [98]). *For all $i \in \mathbb{N}$, we have* $\text{DSPACE}(exp_i(n)) \subseteq \text{HO}-\text{SPEC}_{i+1}$.

In particular, taking $i = 0$, first-order spectra are thereby shown to contain $\text{DSPACE}(n)$. Let us finally remark that Rödding and Schwichtenberg did not consider non-deterministic complexity classes.

5.4. First-order spectra and non-deterministic exponential time. Scholz's original question from [100] was finally answered after twenty years, when Jones and Selman related first-order spectra to non-deterministic time bounded Turing machines. The result was first published in a conference version [70] in 1972, and the journal version [71] appeared in 1974.

THEOREM 5.5 (Jones and Selman 1972 [70]). $\text{Spec} = \text{NE}$.

This leads to a complexity theory counterpart of Asser's question:

COROLLARY 5.6. $\text{Spec} = \text{coSpec}$ *if and only if* $\text{NE} = \text{coNE}$.

They note that this does not answer Asser's question, but it shows the link with a wide range of "closure under complement" questions in complexity theory. Presently, we know that many of them are very difficult questions.

PROOF IDEAS. To see that $\text{Spec} \subseteq \text{NE}$, consider $\text{spec}(\phi) \in \text{Spec}$. Since the sentence ϕ is fixed, satisfaction $\mathcal{M} \models \phi$ can be decided in time that is at most polynomial in the size of model $\mathcal{M}$, where the polynomial's degree depends on the quantifier nesting depth in ϕ. A simple guess-and-verify algorithm is: given number x, non-deterministically guess an x-element model $\mathcal{M}$, then decide whether $\mathcal{M} \models \phi$ is true. Time and space $2^{c \cdot n}$ suffice to store an x-element model and check $\mathcal{M} \models \phi$, where constant c is independent of x and n is the length of x's binary notation. Thus the algorithm works in non-deterministic exponential time (as a function of input length n).

To show Spec $\supseteq$ NE, let Z be a nondeterministic time-bounded Turing machine that runs in time $2^{c \cdot n}$ on inputs of length n. Here the input x is a word of length n. Think of x as a binary-coded natural number. Computation C can be coded as a word $C = \mathit{config}_0 \mathit{config}_1 \dots \mathit{config}_{2^{c \cdot n}}$ where config_0 contains the Turing machine's input, and each config_t encodes the tape contents and control point at its t-th computational step.

Now we have to find a first-order sentence ϕ such that:

(i) For every input-accepting Z computation, $\mathcal{M} \models \phi$ for some model $\mathcal{M}$ of cardinality x.
(ii) If Z has no computation that accepts its input x, then ϕ has no model of cardinality x.

Each config_t length is at most $2^{c \cdot n}$, so C has length bound $2^{c' \cdot n} = x^{c'}$ for c' independent of n. A model $\mathcal{M}$ of cardinality x contains, for each k-ary predicate symbol P of sentence ϕ, a relation $\overline{P} \subseteq \{0, 1, \dots, x-1\}^k$. Thus a model "has enough bits" $x^k = 2^{k \log x} = 2^{O(|x|)}$ to encode all the symbols of computation C.

The remaining task is to actually construct ϕ so it has a model $\mathcal{M}$ of cardinality x if and only if Z has a well-formed computation C that accepts input x. In effect, the task is to use predicate logic to check that C is well-formed and accepts x. The technical details are omitted from this survey paper; several approaches may be seen in [71, 35, 16, 67]. $\dashv$

5.5. Relationship to the question $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$**.** Let $\mathrm{UN} = \{L \mid L \subseteq 1^*\}$ be the set of *tally languages* (each is a set of unary words over the one-letter alphabet $\{1\}$) and let $\mathrm{NP}_1 = \mathrm{NP} \cap \mathrm{UN}$. Since there is a natural identification between NP_1 and NE, we can deduce that if P = NP, then NP = coNP and NE = coNE, i.e., the complement of a spectrum is a spectrum. Of course, it also holds that if there is a spectrum whose complement is not a spectrum, i.e., if NE $\neq$ coNE, then NP $\neq$ coNP and P $\neq$ NP. The converse implication remains open.

5.6. Independent solutions to Scholz's problem. The characterization of spectra by non-deterministic complexity classes was independently found also by Christen on the one hand and Fagin on the other hand during their PhD studies.

Claude Christen's thesis[5] [16] (1974, ETH Zürich, E. Specker) remains unpublished, and only a small part was published in German [17]. Christen discovered his results independently, and only in the late stage of his work was his attention drawn to Bennett's work [7] and the paper of Jones and Selman [70]. It turned out that most of his independently found results were already in print or published by Fagin after completion of Christen's thesis.

Ronald Fagin's thesis (1973, UC Berkeley, R. Vaught) is a treasury of results introducing *projective classes of finite structures*, which he called *generalized spectra* (see Subsection 5.7) that had wide impact on what is now called descriptive complexity and finite model theory. Most of our knowledge about spectra until about 1985 and, to some extent, far beyond that, is contained in the published papers (see [36, 38, 37]) emanating from Fagin's thesis [35]. In this survey, these papers are pervasive. We begin by reviewing what is said in [36] about the consequences of the complexity characterization of spectra *per se*.

Recall that $\mathrm{E} = \bigcup_{c \geq 1} \mathrm{DTIME}(2^{c \cdot n}) \subseteq \mathrm{NE}$ and let us examine the problem of closure under complementation. Since $\mathrm{E} = \mathrm{coE}$, it is clear that if a first-order spectrum is in E, then its complement is also a first-order spectrum. Of course, the question $\mathrm{E} \stackrel{?}{=} \mathrm{NE}$ is still open. Fagin notes that E contains the spectra of categorical sentences, i.e., sentences that have at most one model for every cardinality. Thus, one obtains a model theoretic question closely related to Asser's question.

Open Question 18. *Is every spectrum the spectrum of a categorical sentence*?

Besides reviewing many natural sets of numbers that are spectra, Fagin also proves by a complexity argument the existence of a spectrum S such that $\{n \mid 2^n \in S\}$ is *not* a spectrum (see also [65] for a recent proof by diagonalization).

5.7. Generalized first-order spectra and NP: Fagin's result. This section concerns what was called *generalized first-order spectra* by Fagin in his 1974 paper [36], and is nowadays more usually referred to as (classes of finite structures definable in) *existential second-order logic*. Our main goal is to clarify the differences and the connections with ordinary first-order spectra.

In this subsection, we are no longer interested in the *size* of the finite models of some given sentence, but in the models themselves. Hence, let σ and τ be two disjoint vocabularies, and let ϕ be a first-order $\sigma \cup \tau$-sentence. The *generalized spectrum* of ϕ is the class of finite τ-structures that can be expanded to finite models of ϕ. In other words, it is the class of finite models of the existential second-order sentence $\exists\sigma\phi$ with vocabulary τ. The vocabulary τ is usually referred to as the *built-in* vocabulary, whereas σ is

[5]Claude Christen, born 1943, joined the faculty of CS at the University of Montreal in 1976 and died there, a full professor, prematurely, April 10, 1994.

often called the *extra* vocabulary. Note that generalized spectra are finite counterparts to Tarski's *projective classes* (see [105]). Fagin's theorem states the equivalence between generalized spectra and classes of finite structures accepted in NP.

THEOREM 5.7 (Fagin 1974, [36]). *Let τ be a non-empty vocabulary. A class of finite τ-structures K is a generalized spectrum iff $K \in$ NP.*

If the built-in vocabulary τ is empty, then the τ-structures are merely sets. From a computational point of view, it is natural to see such empty structures as unary representations of natural numbers. From a logical point of view, one obtains ordinary spectra. Hence, Fagin rephrases Jones and Selman's complexity characterization of first-order spectra as follows:

PROPOSITION 5.8. *A set S, if regarded as a set of unary words, is a first-order spectrum if and only if $S \in \mathrm{NP}_1$.*

Concerning the complement problem for generalized spectra, in view of Fagin's theorem it is not surprising that the general case remains open. However, the following is known. If σ consists of unary predicates only, it is called unary. It has been proved in several occasions that unary generalized spectra are not closed under complement (see Fagin 1975, [37], Hajek 1975, [57], Ajtai and Fagin 1990, [2]). For instance, it is shown in [37] by a game argument that the set of connected simple graphs is not a unary generalized spectrum. In contrast, it is easy to design a monadic existential second-order sentence defining the class of non-connected simple graphs.

Since our survey deals with spectra and not with descriptive complexity as a whole, we will not say any more on this subject. However, let us note that *descriptive complexity* [67] emerged as a specific field of research out of Fagin's paper about generalized spectra.

5.8. Further results and refinements. During the late 1970s and the 1990s, several results were published that generalize Jones and Selman's result to higher order spectra on the one hand, and that refine this result in order to obtain correspondences between certain complexity classes and the spectra of certain types of sentences.

In 1977, Lovász and Gács showed that there are generalized first order spectra such that their complement cannot be expressed with a smaller number of variables [77]. To do this they introduced first order reductions, which became a very important tool in finite model theory and descriptive complexity. In fact they were the first to show the existence of decision problems that are **NP**-complete with respect to first order reductions.

First order reductions were used in (un)decidability results early on, [106], and more explicitly in [92]. For systematic surveys, see [19, 80]. In the context of generalized spectra they were rediscovered independently also by Immerman in [66], Vardi in [109] and Dahlhaus in [26]. First order reductions are of very low complexity, essentially they are uniform $\mathbf{AC}^0$

transductions. The first use of low complexity reduction techniques seems to be Jones [69] who termed them log-rudimentary. Allender and Gore [3] showed them equivalent to uniform $\mathbf{AC}^0$ reductions.

Open Question 19. *Is there a universal* (*complete*) *spectrum* S_0 *and a suitable notion of reduction such that every spectrum* S *is reducible to* S_0?

Note that this question has two flavors, one where we look at spectra in terms of sets of natural numbers, and one where we look at sets of (cardinalities of) finite models and their defining sentences. First order reductions may be appropriate in the latter case.

In 1982, Lynch [79] relates the computation time needed to decide a property on set of integers to the maximal arity of symbols required in the sentence to define this property. Below, we refine the definition of classes of spectra in order to take into account some specific syntactic restrictions on sentences.

Definition 5.9. For $d \in \mathbb{N}^*$, define the following classes:

(i) SPEC_d (resp. F-SPEC_d) is the class of spectra of first-order sentences over arbitrary predicate (resp. predicate and function) symbols of arity at most d.
(ii) $\text{SPEC}(k\forall)$ (resp. $\text{SPEC}_d(k\forall)$) is the class of spectra of prenex first-order sentences involving at most k variables that are all universally quantified (resp. and involving predicate symbols of arity at most d). The classes $\text{F-SPEC}(k\forall)$ and $\text{F-SPEC}_d(k\forall)$ are analogously defined.
(iii) Finally, let $\text{SPEC}_d(+)$ be the class of spectra of first-order sentences over the language containing one ternary relation interpreted as the addition relation over finite segments of $\mathbb{N}$ and predicate symbols of arity at most d.

Some inclusions between these classes are easy to obtain: more resources (in terms of arity or number of variables) means more expressive power. Hence, for example, for all $i, j \in \mathbb{N}$ such that $i < j$:

$$\text{SPEC}_i \subseteq \text{SPEC}_j,\ \text{F-SPEC}_i \subseteq \text{F-SPEC}_j \text{ and } \text{SPEC}_i \subseteq \text{F-SPEC}_i.$$

The following result is a first relationship between computational time complexity and "syntactic" complexity.

Proposition 5.10 (Lynch 1982 [79]). *For all* $d \geq 1$,

$$\text{NTIME}(2^{d\cdot n}) \subseteq \text{SPEC}_d(+).$$

The question of proper containment remains open. this question refines the complexity characterization of first-order spectra and has many further developments that we present in Section 6. From a technical point of view, note that Lynch works with so-called "word-models". Namely, a binary word w with length n is seen as a structure with universe $\{0, \ldots, n-1\}$, equipped with some arithmetic (e.g., successor predicate or addition predicate) and with a unary predicate that indicates the positions of the 1 digits in w. The methods developed in this paper are re-used later on by several authors.

OPEN QUESTION 20. *Is the inclusion* $\mathrm{NTIME}(2^{d \cdot n}) \subseteq \mathrm{SPEC}_d(+)$ *proper* (*for* $d \geq 1$)?

Finally, Lynch explains that, from an attentive reading of Fagin's proof, one can only deduce that if some language L is in $\mathrm{NTIME}(2^{d \cdot n})$, then L is in SPEC_{2d} i.e., is a the spectrum of a first-order sentence involving predicates of arity at most $2d$. Even though Lynch's result is not an exact characterization, but only an inclusion, it has had much influence on other researchers.

In a series of papers published between 1983 and 1990 (see [48, 50, 49, 51, 52, 54]), Grandjean proposes two fruitful ideas. The first one is to use RAM machines as a natural model of computation for general logical structures instead of Turing machines, which best suit languages (or word structures). The second idea is to remark that the time complexity seems closely related to the syntactical form of the sentences (and more specifically in this case with the number of universally quantified variables). Let $\mathrm{NRAM}(f(n))$ be the class of binary languages accepted in time $O(f(n))$ by a non-deterministic RAM (with successor), where n is the length of the input.

THEOREM 5.11 (Grandjean 1983 [50, 51, 52, 54]). *For all* $d \geq 1$, *we have* $\mathrm{NRAM}(2^{d \cdot n}) = \text{F-SPEC}(d\forall) = \text{F-SPEC}_d(d\forall)$.

The case $d = 1$ i.e., the case of sentences with one universally quantified variable, is more involved: it requires encoding of arithmetic predicates such as linear order or addition that appear intrinsically in the characterization of computation by sentences with one variable. This theme is developed in the papers [52, 54]). In passing, this implies that the presence of the addition relation is not mandatory in Proposition 5.10 provided (unary) functions are allowed in the language.

An interesting corollary of the latter characterization is that when the number of (universally quantified) variables is fixed, restricting the language to contain only function or relation symbols of arity bounded by d does not weaken the expressive power of sentences, and defines the same class of spectra. In other words, the following holds.

COROLLARY 5.12 (Grandjean 1983 [50, 51, 52, 54]). *For all* $d \geq 1$, *it holds that* $\text{F-SPEC}(d\forall) = \text{F-SPEC}_d(d\forall)$.

The original proof of this result relies on complexity arguments using the characterization of $\text{F-SPEC}(d\forall)$. The following is a purely logical proof.[6]

PROOF OF COROLLARY 5.12. For simplicity of notation, we give the proof in the case $d = 1$. Let $\varphi \equiv \forall t \Psi$ where Ψ is quantifier-free and whose vocabulary is composed of function symbols of various arities. Let $\mathsf{Term}(\Psi)$

[6] We thank Étienne Grandjean for kindly giving us this proof.

be the set of terms and subterms of Ψ. The first idea is to associate with each element τ of $\mathsf{Term}(\Psi)$ a new unary function f_τ and to rewrite the formula in the new vocabulary. The definition of f_τ is as follows:

(i) if $\tau = t$ or τ is a constant symbol, then $f_\tau(t) = \tau$,

(ii) if $\tau = f(\tau_1(t), \dots, \tau_k(t))$ for some function symbol f of arity k, then $f_\tau(t) = f(f_{\tau_1}(t), \dots, f_{\tau_k}(t))$.

One obtains a new sentence Ψ' instead of Ψ by replacing each term $\tau \in \mathsf{Term}(\Psi)$ by $f_\tau(t)$ in conjunction with the definition of each function symbol f_τ. Let us explain the transformation on an example.

Let Ψ be the following very simple sentence with f of arity 2 and g of arity 1:

$$E(\underbrace{f(\overbrace{f(t, \underbrace{g(t)}_{\tau_1})}^{\tau_2}, t)}_{\tau_3}, t).$$

Then, Ψ' corresponds to:

$$\begin{aligned}E(f_{\tau_3}(t), f_t(t)) \wedge f_{\tau_3}(t) &= f(f_{\tau_2}(t), f_t(t)) \wedge f_{\tau_2}(t) \\ &= f(f_t(t), f_{\tau_1}(t)) \wedge f_{\tau_1}(t) = g(f_t(t)) \wedge f_t(t) = t.\end{aligned}$$

It is easily seen that the only non unary symbols (here f) appear (if they do at all) only as an outermost symbol in an atomic formula. Let $f(\overline{\sigma_1}(t)), \dots, f(\overline{\sigma_h}(t))$ be the list of terms in Ψ' involving f. The idea is now to replace in Ψ' each $f(\overline{\sigma_i}(t))$ by some new term $F_i(t)$ where F_i is of arity one (let's call this new sentence Ψ'') and to write down the relations between each pair $F_i(t)$ and $F_j(t)$ for $i, j \leq h$. This provides a new sentence $\varphi' = \forall t\, \forall t'\, (\Psi'' \wedge \Delta)$ where

$$\Delta \equiv \bigwedge_{i,j \leq h} (\overline{\sigma}_i(t) = \overline{\sigma}_j(t') \rightarrow F_i(t) = F_j(t')).$$

The above method shows, when the number of variables is $d = 1$ how to replace h-ary functions by unary functions. However, in order to control the definition of the $F_i s$, one additional quantified variable has been introduced. To get rid of this additional variable one can proceed as follows. First, remark that, for each $i \leq h$, the number of tuples $(\overline{\sigma}_i(t), i, t)$ is linear in the size of the domain D of the finite structure under consideration. Then, one maylexicographically order such tuples in such a way that two tuples $(\overline{\sigma}_i(t), i, t)$ and $(\overline{\sigma}_{i'}(t'), i', t')$ having the same first coordinate also verify $F_i(t) = F_{i'}(t')$. On a suitable vocabulary, only one variable is necessary to express such a constraint. This is described in detail below.

The vocabulary is enriched with a binary predicate $<$ interpreted as a linear order on the domain D, and h unary functions $\overline{N}_j$ for $j \leq h$. Let Δ'

be the following sentence.

$$
\begin{aligned}
&\Delta' \equiv\\
&\quad \forall(i,t)\,\exists(j,x)\,\overline{N}(j,x) = (\overline{\sigma}_i(t), i, t) \wedge\\
&\quad \forall(j,x) \neq (h,\max)\,\exists(j',x')\,(j',x') = (j,x)+1 \wedge \overline{N}(j,x) < \overline{N}(j',x') \wedge\\
&\quad \forall(j,x) \neq (h,\max)\,\exists(j',x')\,\exists(i,t)\,\exists(i',t')\\
&\qquad\quad (j',x') = (j,x)+1 \wedge \overline{N}(j,x) = (\overline{\sigma}_i(t), i, t) \wedge\\
&\qquad\qquad\qquad\qquad\qquad\quad \overline{N}(j',x') = (\overline{\sigma}_{i'}(t'), i', t') \wedge\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\overline{\sigma}_i(t) = \overline{\sigma}_{i'}(t') \to F_i(t) = F_{i'}(t')).
\end{aligned}
$$

where $\forall(i,t)$ stands for $\bigwedge_{1\leq i\leq h} \forall t$ and $\exists(j,x)$ for $\bigvee_{1\leq i\leq h} \exists x$ and $N(j,x)$ stands for $N_j(x)$. Similarly, $(j,x)+1$ represents the successor of pair (j,x) in the lexicographic ordering of pairs (j,x), $j \in \{1,\dots,h\}$ and $x \in D$. The above sentence expresses the fact that the function $\overline{N}$ (in fact the union of functions $\overline{N}_j$, $j \leq h$) is an increasing bijection from the set $\{1,\dots,h\} \times D$ to the set $\{(\overline{\sigma}_i(t), i, t) \mid i \leq h, t \in D\}$. This sentence plays the same role as the sentence Δ with one difference: tuples $(\overline{\sigma}_i(t), i, t)$ with the same first component $\overline{\sigma}_i(t)$ are contiguous in the numbering $\overline{N}$. Using a result of Grandjean [54], one can replace the linear ordering $<$ by additional unary functions. ⊣

To be complete, one should also mention the earlier (and weaker) 1975 result obtained by Pudlák [91] by purely logical argument.

Proposition 5.13 (Pudlák 75 [91]). F-SPEC$(d\forall) \subseteq$ F-SPEC$_d(2d\forall)$ *for all* $d \geq 1$.

In the next section, we will examine more closely the expressive power of spectra on restricted vocabularies. The results of this section show that a tight connection exists between nondeterministic complexity classes and classes of spectra defined by limiting the number of universally quantified variables in sentences. A natural question is whether such a connection exists when the language itself is limited. In particular

Open Question 21. *Is there a characterization as a complexity class of the classes* F-SPEC$_d$ *for all* $d \geq 1$?

This question has also some connections with problems addressed in Section 6.4.

§6. Approach III: Restricted vocabularies.

6.1. Spectra for monadic predicates. The perhaps simplest way to restrict vocabularies is by limiting the arity of the symbols. In that direction, the smallest restricted class of spectra that can be studied is that of sentences involving only relation symbols of arity one (so-called monadic in the literature). In this case, the following can be proved:

Proposition 6.1 (Löwenheim 1915 [78], Fagin 1975 [37]). *Let τ be a vocabulary consisting of unary relation symbols only and $\phi \in \mathrm{MSO}(\tau)$. Then the spectrum of ϕ is finite or co-finite.*

Proof (hints). Use quantifier elimination or Ehrenfeucht–Fraïssé games. ⊣

Remark that the even numbers are a spectrum of the following sentence with one unary function:

$$\forall x\, f(x) \neq x \wedge f^2(x) = x.$$

Hence, the most trivial extension of monadic relational vocabulary already provides a spectrum that is neither finite nor co-finite. A natural question is whether the converse of Proposition 6.1 is true or not. The following observation can be made by noting that one can express the cardinality of a finite domain set by an existential first-order formula.

Observation 6.2. Every finite or co-finite set $X \subseteq \mathbb{N}$ is a first-order spectrum for a sentence with equality only (i.e., no relation or function symbols).

This contrasts with the fact that every SO-spectrum is also an SO-spectrum over equality only. Further, the following result holds.

Proposition 6.3. *If τ consists of a finite (possibly empty) set of unary relation symbols, the* $\mathrm{MSO}(\tau)$*-spectra are exactly all finite and cofinite subsets of* $\mathbb{N}$.

6.2. Spectra for one unary function. As remarked above, one unary function is enough to define nontrivial spectra. It turns out, however, that a complete characterization of spectra for one unary function (with additional unary relations) is possible.

Definition 6.4. A set $X \subseteq \mathbb{N}$ is *ultimately periodic* if there are $a, p \in \mathbb{N}$ such that for each $n \geq a$ we have that $n \in X$ iff $n + p \in X$.

The set of even numbers is ultimately periodic with $a = p = 2$. Again, one may observe

Observation 6.5. Every ultimately periodic set $X \subseteq \mathbb{N}$ is a first order spectrum for a sentence with one unary function and equality only (this is true even if the function is restricted to be a permutation).

Surprisingly, ultimately periodic sets are precisely the spectra of sentences with one unary function [29, 56].

Theorem 6.6 (Durand, Fagin, Loescher 1997, Gurevich, Shelah 2003). *Let ϕ be a sentence of* $\mathrm{MSO}(\tau)$ *where τ consists of only*

- *finitely many unary relation symbols*; *equality*; *and*
- *one unary function symbol.*

Then $\mathrm{SPEC}(\phi)$ *is ultimately periodic.*

PROOF (HINT). The proof of [29] uses Ehrenfeucht–Fraïssé game argument and is restricted to the EMSO(τ) case. The generalization of [56] is an application of the Feferman–Vaught–Shelah decomposition method. $\dashv$

There exist alternative ways to characterize ultimately periodic sets. Among others, they can also be seen as sets of integers definable in Presburger Arithmetic. Also, since ultimately periodic sets are closed under complementation, one has:

COROLLARY 6.7. *Spectra involving a single unary function symbol are closed under complement.*

6.3. Beyond one unary function; and transfer theorems. There exist several ways to extend a vocabulary with one unary function: one may choose to add one (or several) new unary function(s) or, in the opposite direction, one may consider vocabularies with only one unrestricted binary relation symbol. It turns out that, up to what will be called "transfer theorems" in the following, both kinds of extension lead to very expressive formalisms.

Before going further, notations about classes of spectra need to be refined to take into account the number of symbols of distinct arities. Again, we will distinguish in the following *whether function symbols are allowed or not*. We write F-SPEC with various indices when function symbols are allowed, and SPEC when function symbols are not allowed.

DEFINITION 6.8. A set S of integers is in $\text{F-SPEC}_i^{\alpha,\beta}$) if there exists a first-order sentence ϕ such that $S = \text{SPEC}(\phi)$ and the vocabulary of φ contains only

- α function symbols of arity i and
- at most β function symbols of arity less than i, or relation symbols of arity less than or equal to i.

Similarly, a set of integers is in $\text{SPEC}_i^{\alpha,\beta}$ if it can be defined by a first order formula over the language of α relation symbols of arity i and β relation symbols of arity less than i.

When the number of symbols of arity less than i is not restricted, the respective classes of spectra are denoted by $\text{SPEC}_i^{\beta,\omega}$ and $\text{F-SPEC}_i^{\beta,\omega}$. For example, the class of first order spectra over one unary function and an arbitrary number of monadic relation and constant symbols (studied in Section 6.2) is denoted $\text{F-SPEC}_1^{1,\omega}$. Similarly SPEC_i^{α} and F-SPEC_i^{α} are abbreviations for $\text{SPEC}_i^{\alpha,0}$ and $\text{F-SPEC}_i^{\alpha,0}$ Finally, in this setting SPEC_i abbreviates $\text{SPEC}_i^{\omega,\omega}$ (the same holds for F-SPEC_i).

Let us examine the relationships among these different classes of spectra. Recall that for all $i, j \in \mathbb{N}$ such that $i < j$:

$$\text{SPEC}_i \subseteq \text{SPEC}_j,\ \text{F-SPEC}_i \subseteq \text{F-SPEC}_j \text{ and } \text{SPEC}_i \subseteq \text{F-SPEC}_i.$$

The following inclusions are also easy to prove. For all $i, \beta \in \mathbb{N}$,

$$\text{SPEC}_i^{\alpha,\omega} \subseteq \text{SPEC}_i^{\alpha+1} \text{ and } \text{F-SPEC}_i^{\alpha,\omega} \subseteq \text{F-SPEC}_i^{\alpha+1}.$$

The relative expressive power, in terms of definable spectra, of formulas with i-ary functions and formulas with $i+1$-ary relations can be compared more precisely. In [30], it is shown that a spectrum of a first-order formula involving any number of unary function symbols is also the spectrum of a formula using only one binary relation. This can be generalized to any arity.

PROPOSITION 6.9. [30, 28] *For all integers $i \geq 1$, $\text{F-SPEC}_i \subseteq \text{SPEC}_{i+1}^1$*

Thus the following chain of inclusions holds. (None of the reverse inclusions are known to be true.)

$$\text{SPEC}_i \subseteq \text{F-SPEC}_i^1 \subseteq \text{F-SPEC}_i \subseteq \text{SPEC}_{i+1}^1 \subseteq \text{SPEC}_{i+1}$$

A natural way to study the expressive power of relational languages with respect to spectra is to reduce them through functional (here polynomial) transformation.

DEFINITION 6.10. If $f\colon \mathbb{N} \to \mathbb{N}$ and S is a set of integers, then $f(S) = \{f(n)\colon n \in S\}$ and $S^i = \{n^i\colon n \in S\}$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of spectra and $f\colon \mathbb{N} \to \mathbb{N}$ be a function. A natural question is then the following:

Let S be a spectrum in $\mathcal{C}_1$ and f be a $\mathbb{N} \to \mathbb{N}$ function, is $f(S)$ a spectrum in $\mathcal{C}_2$?

In [38], Fagin showed an interesting equivalence between spectra defined by different relational languages. Such results have since been called transfer theorems. The proof can be seen as an extension of the well-known interpretation method of Rabin [92] with an additional constraint that describes how the domain size of the structure needs to change.

THEOREM 6.11 (Fagin 1975 [38]). *For every $i \geq 1$,*

$$S \in \text{SPEC}_{2i} \iff S^i \in \text{SPEC}_2.$$

Since relational spectra of arity one are finite or cofinite the "transfer" theorem above cannot be extended to SPEC_1. Not too surprisingly, if function symbols are allowed, a similar and more uniform equivalence can be proved.

PROPOSITION 6.12. *For every $i \geq 1$, $S \in \text{F-SPEC}_i \iff S^i \in \text{F-SPEC}_1$.*

The expressive power of unary functions is well described by the following result that shows that any spectrum can be polynomially transformed into a spectrum involving unary functions only.

THEOREM 6.13 (Durand, Ranaivoson 1996 [30]). *Let $P(X) \in \mathbb{Q}[X]$ of degree $m \geq i$ and with a strictly positive dominating coefficient. Then,*

$$S \in \text{F-SPEC}_i \Rightarrow \lceil P(S) \rceil = \{\lceil P(n) \rceil,\ n \in S\} \in \text{F-SPEC}_1.$$

This last result raises the following natural question: what are the smallest languages that capture the essence of the spectrum problem?

6.3.1. *Spectra for two unary functions and more.* If function symbols are allowed, it turns out that two unary functions are "enough":

THEOREM 6.14 (Durand, Fagin and Loescher, 1998, [29]). *Given k and a spectrum S in* F-SPEC_1^k. *Then* $kS = \{kn : n \in S\} \in \text{F-SPEC}_1^2$.

Combined with Proposition 6.12 or with Theorem 6.13, this implies that there is a first order spectrum over two unary function symbols that, written in unary, is **NP**-complete.

The above result is, in some sense, optimal. One function is not enough to represent all spectra (even after polynomial transformation) as shown by the following result.

THEOREM 6.15 (Loescher 1997, [76]). *The set* $\{n^2 : n \in \mathbb{N}\}$ *belongs to the class* $\text{F-SPEC}_1^2 \backslash \text{F-SPEC}_1^1$, *hence the inclusion* $\text{F-SPEC}_1^1 \subset \text{F-SPEC}_1^2$ *is proper.*

6.3.2. *Spectra for one binary relation symbol.* On relational languages, an easy consequence of Theorem 6.11 is that for every spectrum S, there exists $i \in \mathbb{N}$, such that $S^i \in \text{SPEC}_2^1$. Then, up to polynomial transformation, any spectrum is a spectrum of such a sentence.

But, a little more can be proved. Let $BIN \subseteq \text{SPEC}_2^1$ be the set of spectra of symmetric, irreflexive relations (simple graphs). The whole complexity of the spectrum problem is already contained in the apparently weaker question of whether BIN is closed under complement.

THEOREM 6.16 (Fagin 1974, [36]). *BIN is closed under complement iff the complement of every first order spectrum is also a spectrum. Moreover, the complement of every first order spectrum is also a spectrum iff for all $X \in BIN$ the complement* $\mathbb{N} - X \in \text{SPEC}_2^1$.

OPEN QUESTION 22 (Fagin 1974, [36]). *Is every first order spectrum in* SPEC_2^1?

6.3.3. *Rudimentary sets and spectra of restricted vocabularies.* The relation between rudimentary sets and spectra have been investigated. It was first observed that the set of primes is in F-SPEC_1 (Grandjean 1988 [53]). More generally, it holds that:

$$\text{RUD} \subseteq \text{F-SPEC}_1 \text{ (Olive 1996 [89]).}$$

Due to the closure of RUD under polynomial and inverse polynomial transformation [60] and to the existence of the above described transfer results for spectra, it is clear that, not only $\text{RUD} \subseteq \text{F-SPEC}_1^2$ but also:

$$\text{RUD} = \text{SPEC} \text{ if and only if } \text{RUD} = \text{F-SPEC}_1^2.$$

However, in view of the following surprising result, there are evidences that none of these equalities hold.

THEOREM 6.17 (Woods, 1981, [111]). *If* RUD $=$ SPEC *then* NE $=$ coNE *and* NP $\neq$ coNP.

Since the proof is not easily available and given in a different framework in Woods's thesis, we sketch it below.

PROOF. Recall that LTH is the linear time analog of the polynomial hierarchy PH and that Celia Wrathall proved in [112] that LTH $=$ RUD.

The following facts are easy to prove (the last item follows from the preceeding ones).

(i) LTH $\subseteq$ DSPACE(n) $\subseteq$ NE.
(ii) If LTH $=$ DSPACE(n) then LTH collapses to some level k.
(iii) If LTH collapses to some level k then PH collapses to some level k'.
(iv) LTH $=$ DSPACE(n) implies PH $=$ PSPACE.

If RUD $=$ SPEC then, since SPEC $=$ NE and RUD is closed under complement, it holds that NE $=$ coNE.

Let us examine the other consequence. If RUD $=$ SPEC then RUD $=$ LTH $=$ DSPACE(n) $=$ NE $=$ SPEC. But then, by item (iv), PH $=$ PSPACE. Suppose now that, NP $=$ coNP. In this case, the polynomial hierarchy collapses to NP i.e., NP $=$ PH. Then, since NP $\subseteq$ NE one obtains PH $\subseteq$ NE and then, PSPACE $=$ PH $\subseteq$ NE $=$ DSPACE(n), which contradicts the well-known result DSPACE(n) $\subsetneq$ PSPACE. ⊣

Hence, although the equality RUD $=$ F-SPEC$_1^2$ might seem plausible at first sight, its consequences make it probably hard to prove.

6.4. The unary and the arity hierarchies. The results of the preceding section show that for any spectrum S, there is a polynomial P such that $P(S)$ is the spectrum of a first order sentence with two unary functions. This underlines the expressive power of this latter class of spectra *up to polynomial transformation*. However, it is often an open problem whether *equality* holds between particular classes of spectra, for example, classes defined by restriction on the number or the arity of the predicates in the languages. Taking a very particular case, it is not even known whether three unary functions "say" more than two as far as spectra are concerned. This leads to the following open problems about spectra of unary functions.

OPEN QUESTION 23 (The unary hierarchy). *Is the following hierarchy proper*:

$$\text{F-SPEC}_1^1 \subset \text{F-SPEC}_1^2 \subseteq \text{F-SPEC}_1^3 \subseteq \cdots \subseteq \text{F-SPEC}_1^k \subseteq \cdots ?$$

OPEN QUESTION 24. *Is* RUD $=$ F-SPEC$_1^2$?

OPEN QUESTION 25. *Is* SPEC$_2^1$ $=$ F-SPEC$_1$ *or even* SPEC$_2^1$ $=$ F-SPEC$_1^2$?

Both positive or negative answers to these questions would have nontrivial consequences. For example, proving RUD $\neq$ F-SPEC$_1^2$ would separate the classes RUD and SPEC. In the opposite, we already know that RUD $=$ F-SPEC$_1^2$ implies NP $\neq$ coNP and NE $=$ coNE.

Proving that $\text{F-SPEC}_1^k \subsetneq \text{F-SPEC}_1^{k+1}$ for some integer k would also separate SPEC from RUD. Similarly, a collapse of the unary hierarchy to some level k would have strong consequences. It is easily seen that, testing whether a number n (as input i.e., in unary) is in the spectrum of a first order sentence over k unary functions can be decided by a deterministic polynomial time RAM algorithm that uses $k \cdot n$ additional nondeterministic steps. Since $\text{NRAM}(2^n) \subseteq \text{F-SPEC}_1$ from Theorem 5.11, the inclusion $\text{F-SPEC}_1 \subseteq \text{F-SPEC}_1^k$ would imply immediately the following "trade-off" result on nondeterministic RAM computations (see [29]): for any arbitrary constant c, any nondeterministic RAM that, given a number n as input, runs in time $c \cdot n$ can be simulated by a RAM that runs in $k \cdot n$ nondeterministic steps and in a polynomial number of deterministic steps. For c greater than k (which is fixed) such a result is rather unexpected and would strongly modify our understanding of the relationships between determinism and nondeterminism.

Another natural question concerns the relative expressive power of first order sentences defined by restriction on the arity of the symbols involved in the language. It is open whether any spectrum is the spectrum of a sentence over one binary relation only. The question may be refined as follows (See Fagin [38, 39]).

OPEN QUESTION 26 (The arity hierarchy). *Is the following hierarchy proper*:

$$\text{SPEC}_1 \subseteq \text{SPEC}_2 \subseteq \text{SPEC}_3 \subseteq \cdots \subseteq \text{SPEC}_k \subseteq \cdots ?$$

The same question could be asked for spectra over i*-ary functions.*

Although the above problem is still open, Fagin proved the following partial result.

THEOREM 6.18 (Fagin 1975 [38]). *If* $\text{SPEC}_k = \text{SPEC}_{k+1}$ *for some integer* k, *then the arity hierarchy collapses and* $\text{SPEC}_k = \text{SPEC}_m$ *for every* $m \geq k$.

6.4.1. *Collapse of hierarchies and closure under functions.* Let $\mathcal{C}$ be a class of spectra and f be a $\mathbb{N} \to \mathbb{N}$ function. Class $\mathcal{C}$ is *closed* under f if for any spectrum S in $\mathcal{C}$, $f(S)$ is in $\mathcal{C}$. In the spirit of Theorems 6.11 and 6.13 and Proposition 6.12 one can relate the collapse of hierarchies to the possible closure of class of spectra under some functions. In [64], such problems are studied and several related results are given.

THEOREM 6.19.

(i) (Hunter, [64]) *The arity hierarchy collapses to* SPEC_2^1 *if and only if* SPEC_2^1 *is closed under function* $f\colon n \mapsto \lceil \sqrt{n}\, \rceil$.

(ii) *The unary hierarchy collapses to* F-SPEC_1^2 *if and only if* F-SPEC_1^2 *is closed under function* $f\colon n \mapsto \lceil \frac{n}{2} \rceil$.

In [64], a result similar to (ii) is proved about the number of binary predicates instead of the number of unary functions. It is not hard to see that his result extends to the case of (ii).

6.5. Higher order spectra. Lynch generalizes Proposition 5.10 to higher order spectra in [79]. It is shown that the polynomial time hierarchy corresponds to second-order logic. In other words, let $\mathbf{PH}_1 = \mathbf{PH} \cap \mathrm{UN}$. Then a set $X \subseteq \mathbb{N}$ is a second order spectrum if and only if $\{1^n : n \in X\} \in \mathbf{PH}_1$.

Let SO_k be the class of second-order formulas where all the second order variables are of arity at most k. Let $2^{X^k} = \{2^{n^k} \in \mathbb{N} : n \in X\}$ for $X \subseteq \mathbb{N}$. The following precise characterization can be obtained.

Theorem 6.20 (More and Olive 1997, [83]). *A set X is a spectrum of a sentence in* SO_k *iff* 2^{X^k} *is rudimentary.*

6.6. Spectra of finite variable logic $\mathbf{FO}^k$. We denote by FO^k first order logic with only k distinct variables (bound or free), and by *Spec* FO^k the set of spectra of sentences of FO^k. FO^k has been studied extensively in finite model theory, [90, 46], but somehow the spectrum problem was not adressed for FO^k in the literature.[7] One gets easily the following.

Proposition 6.21. *The spectrum of a sentence in* FO^1 *is finite or cofinite.*

Let M be a set of natural numbers. Recall that a *gap* of M is a pair of integers g_1, g_2 such that $g_1, g_2 \in M$ but for each n with $g_1 < n < g_2$ we have that $n \notin M$. Proposition 3.2 can be extended as follows. By coding structures that model iterated powersets one gets immediately the following.

Proposition 6.22. *For every k there is a $\phi \in FO^4$ such that the spectrum* $\mathrm{SPEC}(\phi)$ *contains gaps of size $exp_k(n)$.*

Four variables are used here to express extensionality of the membership relation, and closure under unions with singletons. Working a bit harder one can prove the following.[8]

Theorem 6.23 (Grohe). (i) *For every Turing machine T there is a sentence $\phi_T \in$* FO^3 *such that*

$$\mathrm{SPEC}(\phi_T) = \{t^2 \mid T \textit{ has an accepting run of length } t\}.$$

(ii) *For every recursive function f there is a sentence ϕ_f in* FO^3 *such that the gaps in the spectrum of ϕ_f grow faster than f.*

The key idea is to encode Turing machines on grids, hence the t^2 in the statement of the Theorem.

In contrast to the above, it follows from the proof that FO^2 has the finite model property and hence is decidable, [46], that the gaps are bounded.

Theorem 6.24. *For every $\phi \in FO^2$ the spectrum* $\mathrm{SPEC}(\phi)$ *has gaps of size at most $2^{poly(n)}$.*

[7] While the third author was lecturing in Chennai in January 2009 on the spectrum problem, Dr. S. P. Suresh asked about it. The results below are the fruit of discussions with Amaldev Manuel and Martin Grohe.

[8] M. Grohe, personal communication

COROLLARY 6.25. *The following inclusions are proper*:

$$\mathit{Spec}\,\mathrm{FO}^1 \subset \mathit{Spec}\,\mathrm{FO}^2 \subset \mathit{Spec}\,\mathrm{FO}^3$$

OPEN QUESTION 27 (Finite Variable Hierarchy). *Does the hierarchy* $\mathit{Spec}\,\mathrm{FO}^k$ *collapse at level* 3?

§7. **The Ash conjectures.** The notion of k-equivalence of structures was introduced by Fraïssé [42] in 1954 and the game presentation is due to Ehrenfeucht [32] in 1961.

DEFINITION 7.1. Let σ be a vocabulary, let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures and let k be an integer. The two structures $\mathcal{A}$ and $\mathcal{B}$ are k-equivalent if and only if they satisfy the same σ-sentences with quantifier depth $\leq k$.

For a detailed presentation of k-equivalence, we refer the reader to [31, 63]. For our purpose, the two most important features of the above notion are the following. For each finite vocabulary σ and for each quantifier depth k, the number of k-equivalence classes of σ-structures is finite and we denote it by $M_{\sigma,k}$. For each finite vocabulary σ, for each quantifier depth k and for each k-equivalence class of σ-structures $\mathcal{C}$, there exists a σ-sentence $\Psi_{\mathcal{C}}$ of quantifier depth k such that, for all σ-structure $\mathcal{A}$, we have $\mathcal{A} \in \mathcal{C}$ if and only if $\mathcal{A} \models \Psi_{\mathcal{C}}$.

In 1994, Ash [5] introduces a counting function relative to the k-equivalence classes:

DEFINITION 7.2. Let σ be a finite relational vocabulary, and let k be a positive integer. Ash's function $N_{\sigma,k}$ counts, for each positive integer n, the number of k-equivalence classes of σ-structures of size n.

This function is obviously bounded by the total number of classes, $M_{\sigma,k}$, and Ash's conjecture deals with its asymptotic behavior.

OPEN QUESTION 28 (Ash's constant conjecture). *Is it true that for any finite relational vocabulary* σ *and any positive integer* k, *the Ash function* $N_{\sigma,k}$ *is eventually constant*?

A weaker version of his conjecture is also proposed:

OPEN QUESTION 29 (Ash's periodic conjecture). *Is it true that for any finite relational vocabulary* σ *and any positive integer* k, *the Ash function* $N_{\sigma,k}$ *is eventually periodic*?

Ash shows by a very neat proof that both conjectures imply the spectrum conjecture (i.e., SPEC = coSPEC). Let us present the idea of the proof. Fix σ and k and assume for simplicity that $N_{\sigma,k}(n) = a$ for all n. Take a quantifier depth k first-order σ-sentence ϕ. We exhibit a quantifier depth k first-order σ'-sentence θ_a such that $\mathrm{SPEC}(\theta_a) = \mathbb{N}^+ \setminus \mathrm{SPEC}(\phi)$, where σ' consists of a disjoint copies of σ. Let $\mathcal{C}_1, \ldots, \mathcal{C}_{M_{\sigma,k}}$ be the classes of k-equivalence of σ-structures. All the structures in a given class $\mathcal{C}_i$ agree on ϕ, i.e., either

$\forall \mathcal{A} \in \mathcal{C}_i\ \mathcal{A} \models \phi$ or $\forall \mathcal{A} \in \mathcal{C}_i\ \mathcal{A} \models \neg\phi$, because ϕ has quantifier depth k. Form the set X_a consisting of all sets of a distinct $\mathcal{C}_i$s such that $\forall \mathcal{A} \in \mathcal{C}_i\ \mathcal{A} \models \neg\phi$. Take $\theta_a \equiv \bigvee_{Y \in X_a} \bigwedge_{\mathcal{C} \in Y} \Psi'_{\mathcal{C}}$, where $\Psi'_{\mathcal{C}}$ characterizes $\mathcal{C}$ and is written using a distinct copy of σ. The sentence θ_a has quantifier depth k and $n \in \textsc{spec}(\theta_a)$ if and only if there are (at least) a distinct k-equivalence classes containing a structure $\mathcal{A}$ with size n and $\mathcal{A} \models \neg\phi$. Since the total number of k-equivalence classes containing a structure $\mathcal{A}$ with size n is exactly a, there is no class left for a model of ϕ, and the anounced result follows.

These ideas of Ash's have not been exploited afterwards, and his paper has remained isolated until recently. In 2006, Chateau and More [13] published a second paper related to Ash's counting functions. For all $i \in \mathbb{N}^+$, note $N^{-1}_{\sigma,k}(i) = \{n \in \mathbb{N}^+ \mid N_{\sigma,k}(n) = i\}$, the inverse image of the positive integer i under the function $N_{\sigma,k}$. Both Ash's conjectures can be rephrased in terms of the sets $N^{-1}_{\sigma,k}(i)$ and are subsumed under the following condition:

Open Question 30 (Ultra-weak Ash conjecture). *Is it true that for any finite relational vocabulary σ, for any positive integer k and for all $i \in \mathbb{N}^+$, the set $N^{-1}_{\sigma,k}(i)$ is a spectrum*?

This conjecture is related to the spectrum problem as follows:

Theorem 7.3. *Let σ be a finite relational vocabulary, and let k be a positive integer. For all $i \in \mathbb{N}^+$, the set $N^{-1}_{\sigma,k}(i)$ is a spectrum if and only if for every σ-sentence φ of quantifier depth $\leq k$, the set $\mathbb{N}^+ \setminus \textsc{spec}(\varphi)$ is a spectrum.*

All in all, the spectrum conjecture is true if and only if the ultra-weak Ash conjecture is true.

Note that, in some cases, one gets interesting additional information about complements of spectra. E.g., if the Ash function $N_{\sigma,k}$ is eventually constant, then for every σ-sentence φ of quantifier depth $\leq k$, the set $\mathbb{N}^+ \setminus \textsc{spec}(\varphi)$ is the spectrum of a sentence of the same quantifier depth as φ, over a vocabulary with the same arities as σ.

In order to make some progress, particular cases of Ash conjectures may be considered, i.e., by restricting the the sets of pairs (σ, k). In his original paper, Ash [5] already did so. For instance, he shows that if σ is a unary vocabulary, then for all $k \geq 1$, Ash's function $N_{\sigma,k}$ is eventually constant. In [13], it is also proved that for all finite relational vocabulary σ, Ash's function $N_{\sigma,2}$ is eventually constant. However, these results are of very limited interest because unary vocabularies or quantifier depth two (see [84]) only allow definition of finite and cofinite spectra.

In other cases, solving restricted versions of Ash conjectures happens to be as difficult as solving the full conjectures. Let $\mathcal{G}$ be a vocabulary restricted to a single binary relation symbol (i.e., the language of graphs). As already said, Fagin [38] shows that, up to a polynomial padding, every spectrum is a spectrum in $\textsc{spec}^1_2$. Since spectra are closed under inverse images of polynomials, one obtains:

PROPOSITION 7.4. *If for all $k \geq 1$ and for all $i \in \mathbb{N}^+$, the set $N^{-1}_{\mathcal{G},k}(i)$ is a spectrum, then the (full) spectrum conjecture holds.*

It is less known that, as an easy corollary of a result of Grandjean in [54], up to a polynomial padding, every spectrum is the spectrum of a quantifier depth 3 sentence, using an unbounded number of binary relations. Once again, it follows:

PROPOSITION 7.5. *If for all binary vocabulary σ and for all $i \in \mathbb{N}^+$, the set $N^{-1}_{\sigma,3}(i)$ is a spectrum, then the spectrum conjecture holds.*

Since there is no known padding result that uses a finite set of pairs (σ, k), these results are presently the best possible. In order to make further progress in solving particular cases of Ash conjectures, Chateau and More studied the problem for restricted classes of structures.

DEFINITION 7.6. Let σ be a finite relational vocabulary, and let $\mathcal{C}$ be a class of finite σ-structures. For any positive integer k, the Ash function for the class $\mathcal{C}$, denoted by $N_{\mathcal{C},k}$, counts the number $(\leq M_{\sigma,k})$ of non k-equivalent structures in $\mathcal{C}$ of size n for all $n \geq 1$.

Let $\mathcal{B}$ be the set of finite Boolean algebras. Clearly, we have: $N^{-1}_{\mathcal{B},k}(1) = \{2^\alpha \in \mathbb{N} \mid \alpha \geq 2\}$, $N^{-1}_{\mathcal{B},k}(0) = \mathbb{N}^+ \setminus \{2^\alpha \in \mathbb{N} \mid \alpha \geq 2\}$ and $N^{-1}_{\mathcal{B},k}(i) = \emptyset$ for $i \notin \{0, 1\}$. From this example, it follows that we cannot expect Ash's constant or periodic conjectures to hold for classes of structures. A natural question arises: "Which functions can be Ash's function for some class of structures?". Let $g\colon \mathbb{N}^+ \mapsto \mathbb{N}$ be a function bounded by some constant M_g and computable in E. Then, it is proved that there exist a vocabulary σ, a σ-sentence φ and a quantifier depth K such that $g = N_{Mod_f(\varphi),K}$, where $Mod_f(\varphi)$ denotes the class of finite models of φ.

Now, let us turn to the ultra-weak Ash conjecture for classes of structures. Consider the class $Mod_f(\Psi)$ of finite models of a first-order sentence Ψ.

OPEN QUESTION 31 (Ultra-weak Ash conjecture for classes of structures). *Is it true that for any finite relational vocabulary σ, for any first-order σ-sentence Ψ, for any positive integer k and for all $i \in \mathbb{N}$, the set $N^{-1}_{Mod_f(\Psi),k}(i)$ is a spectrum*?

Not all classes of structures are interesting in this context. In particular it is natural to require that $Mod_f(\Psi)$ contains at least one structure of size n, for all positive integer n.

THEOREM 7.7. *Let σ be a finite relational vocabulary, let Ψ be a first-order σ-sentence such that $Mod_f(\Psi)$ contains at least one structure of size n for all positive integers n, and let k be a positive integer. For all $i \in \mathbb{N}^+$, the set $N^{-1}_{Mod_f(\Psi),k}(i)$ is a spectrum if and only if for every σ-sentence φ of quantifier depth $\leq k$ that implies Ψ (i.e., that every model of φ is also a model of Ψ), the set $\mathbb{N}^+ \setminus \textsc{spec}(\varphi)$ is a spectrum.*

In some particular cases, we obtain more information about complements of spectra. For instance, if Ψ expresses that all binary relations in σ are functional (which can be done using a quantifier depth 3 sentence), and if $k \geq 3$, then all binary relations in the sentence defining $\mathbb{N}^+ \setminus \text{SPEC}(\varphi)$ are functional too.

Let us turn to classes of structures for which the study of Ash's functions is as difficult as the general case. More precisely, let us consider the following classes of structures: let $\mathcal{SG}$ be the class of simple graphs; let $2\mathcal{E}$ be the class of finite structures whose vocabulary is composed of two binary relations interpreted as equivalence relations, and let $2\mathcal{F}$ be the class of finite structures whose vocabulary is composed of two binary relations interpreted as functions.

Proposition 7.8. *Let $\mathcal{C} \in \{\mathcal{SG}, 2\mathcal{E}, 2\mathcal{F}\}$. If for all positive integer k and for all $i \in \mathbb{N}^+$, the set $N_{\mathcal{C},k}^{-1}(i)$ is a spectrum, then the spectrum conjecture holds.*

Once again, this is a consequence of various padding results. Fagin proves in [38] that simple graphs allow a polynomial padding for all spectra. This also holds for two unary functions, as proved by Durand, Fagin and Loescher in [29]. The last result concerning two equivalence relations is proved in [12, 33].

§8. Approach IV: Structures of bounded width.

8.1. From restricted vocabularies to bounded tree-width. In this section we look again at spectra of sentences with one unary function and a fixed set of unary predicates. The finite structures which have only one unary function consist of disjoint unions of components of the form given in Figure 1. They are directed forests in which the roots are joined by a directed cycle.

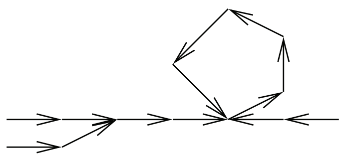

Figure 1. Models of one unary function.

The unary predicates are just colours attached to the nodes. The similarity of labeled graphs to labeled trees can be measured by the notion of *tree-width*, and in fact, these structures have tree width at most 2. Inspired by Theorem 6.6, E. Fischer and J. A. Makowsky [41] generalized Theorem 6.6 by replacing the restriction on the vocabulary by a purely model theoretic condition involving the width of a relational structure. In this section we discuss their results.

8.1.1. *Tree-width.* In the 1980s the notion of tree-width of a graph became a central focus of research in graph theory through the monumental work of Robertson and Seymour on graph minor closed classes of graphs, and its algorithmic consequences [96]. The literature is very rich, but good references and orientation may be found in [27, 8, 9]. Tree-width is a parameter that measures to what extent a graph is similar to a tree. Additional unary predicates do not affect the tree-width. Tree-width of directed graphs is defined as the tree-width of the underlying undirected graph.[9]

DEFINITION 8.1 (Tree-width). A k-tree decomposition of a graph $G = (V, E)$ is a pair $(\{X_i \mid i \in I\}, T = (I, F))$ with $\{X_i \mid i \in I\}$ a family of subsets of V, one for each node of T, and T a tree such that

(i) $\bigcup_{i \in I} X_i = V$.
(ii) for all edges $(v, w) \in E$ there exists an $i \in I$ with $v \in X_i$ and $w \in X_i$.
(iii) for all $i, j, k \in I$: if j is on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$. In other words, the subset $\{t \mid v \in X_t\}$ is connected for all v.
(iv) for all $i \in I$, $|X_i| \leq k + 1$.

A graph G is of *tree-width at most* k if there exists a k-tree decomposition of G. A class of graphs K is a $TW(k)$-class iff all its members have tree width at most k.

Given a graph G and $k \in \mathbb{N}$ there are polynomial time, even linear time, algorithms, which determine whether G has tree-width k, and if the answer is yes, produce a tree decomposition, cf. [9]. However, if k is part of the input, the problem is NP-complete [4].

Trees have tree-width 1. The clique K_n has tree-width $n - 1$. Furthermore, for fixed k, the class of finite graphs of tree-width at most k, denoted by $TW(k)$, is MSO-definable. The tree-width of a class of graphs can often be bounded as a function of other parameters. Planar graphs of radius r have tree-width at most $3r$, chordal graphs with maximal clique of size c have tree-width at most $c - 1$. Typical graph classes of unbounded tree-width are the class of planar graphs, in particular the class of all planar grids $G_{m,n}$, and the class of all regular graphs of degree $r, r \geq 3$.

Tree-width for labeled graphs can be generalized to arbitrary relational structures in a straightforward way. Clause (ii) in the above definition is replaced by

(ii-rel): For each r-ary relation R, if $\bar{v} \in R$, there exists an $i \in I$ with $\bar{v} \in X_i^r$.

This was first used in [40].

It is now natural to ask whether Theorem 6.6 can be generalized to arbitrary vocabularies, provided one restricts the spectrum to structures of fixed

[9] In [68] a different definition is given, which attempts to capture the specific situation of directed graphs. But the original definition is the one which is used when dealing with hypergraphs and general relational structures.

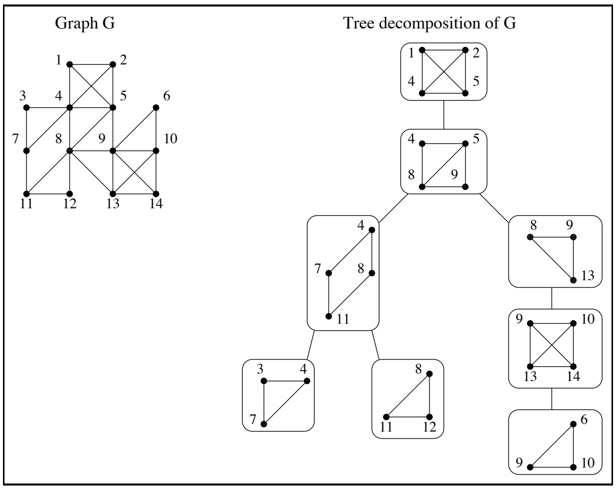

FIGURE 2. A graph and one of its tree-decompositions.

tree-width k. Indeed, E. Fischer and J. Makowsky [41] have established the following:

THEOREM 8.2 (E. Fischer and J. A. Makowsky (2004)). *Let ϕ be an* MSO *sentence and $k \in \mathbb{N}$. Assume that all the models of ϕ are in $TW(k)$. Then $spec(\phi)$ is ultimately periodic.*

8.2. Extending the logic. First one observes that the logic MSO can be extended by modular counting quantifiers $C_{k,m}$, where $C_{k,m}x\ \phi(x)$ is interpreted as "there are, modulo m, exactly k elements satisfying $\phi(x)$". We denote the extension of MSO obtained by adding, for all $k, m \in \mathbb{N}$ the quantifiers $C_{k,m}$, by CMSO.

Typical graph theoretic concepts expressible in FO are the presence or absence (up to isomorphism) of a fixed (induced) subgraph H, and fixed lower or upper bounds on the degree of the vertices (hence also r-regularity). Typical graph theoretic concepts expressible in MSO but not in FO are connectivity, k-connectivity, reachability, k-colorability (of the vertices), and the presence or absence of a fixed (topological) minor. The latter includes planarity, and more generally, graphs of a fixed genus g. Typical graph theoretic concepts expressible in CMSO but not in MSO are the existence of an eulerian circuit (path), the size of a connected component being a multiple of k, and the number of connected components is a multiple of k. All the non-definability statements above can be proved using Ehrenfeucht–Fraïssé games. The definability statements are straightforward.

Second, one observes that very little of the definition of tree-width is used in the proof of Theorem 8.2. The techniques used in the proof can be adapted to other notions of width of relational structures, such as *clique-width*, which was introduced first in [22] and studied more systematically in [23], and to *patch-width*, introduced in [41].

8.3. Ingredients of the proof of Theorem 8.2. To prove Theorem 8.2 and its generalizations, the authors use three tools:

(i) A generalization of the Feferman–Vaught Theorem for k-sums of labeled graphs to the logic CMSO, due to B. Courcelle, [20], and further refined by J. A. Makowsky in [80].
(ii) A reduction of the problem to spectra of labeled trees by a technique first used by B. Courcelle in [21] in his study of graph grammars.
(iii) An adaptation of the Pumping Lemma for labeled trees, cf. [44].

The proof of Theorem 8.2 is quite flexible. Its proof can be adapted to stronger logics and other notions of width of relational structures. However, the details are rather technical, and the reader is referred to [41].

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