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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Steve Awodey (Managing Editor), John Burgess, Mark Colyvan, Anuj Dawar, Noam Greenberg, Rahim Moosa, Ernest Schimmerling, Alex Simpson, Kai Wehmeier, and Matthias Wille. Authors and publishers are requested to send, for review, copies of books to *ASL, Box 742, Vassar College, 124 Raymond Avenue, Poughkeepsie, NY 12604, USA.*

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References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

LEON HORSTEN. ***The Tarskian Turn: deflationism and axiomatic truth.*** MIT Press, Cambridge, Massachusetts, 2011, xii + 157 pp.

Without exaggerating, deflationism is the most popular philosophical conception of truth at the moment. The philosophical literature on the topic is full of variations and inhomogeneous. And, despite its popularity, deflationism has some drawbacks. One of the main concerns, however is its lack of clarity and an adequate explication of the commitments of deflationary theories is desirable. Axiomatic theories of truth in contrast are as clear and explicit as one could hope for. Since Friedman and Sheard's seminal paper *An Axiomatic Approach to Self-Referential Truth* (1987) the investigation of axiomatic theories of truth has become a field of growing importance. Horsten's book is state of the art in both areas—deflationism and axiomatic theories of truth—and attempts to fruitfully combine them.

In the first three chapters of his book Horsten prepares the ground for his investigation. The first chapter is introductory. The second chapter contains a motivation for investigating axiomatic theories of truth. Horsten sets his undertaking apart from more traditional and substantial conceptions such as correspondence as well as from semantic versions such as Kripke's. There he also introduces the driving criteria for his choice of axiomatic theories: soundness and strength. The third chapter prepares the reader with background information about the syntactical base theory, Peano arithmetic.

The fourth chapter contains a discussion of the most obvious deflationist theory of truth: disquotational truth. Horsten clearly evaluates the disquotational account in different

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versions and convincingly argues that although it is sound it is truth-theoretically unsatisfactory as it does not cover lots of the things one might expect. In the fifth chapter Horsten engages in the difficult task of explicating deflationism. Horsten takes deflationist truth to be a logico-linguistic notion, i.e., a notion that is in many respects a logical notion but it's dependency on a theory of syntax makes it also lingustic.

A Tarskian theory of truth is introduced and discussed in chapter six. Horsten explains why a Tarskian theory is truth-theoretically stronger than disquotational truth. In the case of Tarskian truth this truth-theoretic strengthening goes hand in hand with an increase in arithmetic strength, as the Tarskian theory can prove the consistency of Peano arithmetic. The disquotational theory in contrast is conservative over Peano arithmetic.

The question whether deflationism is committed to a theory of truth which is conservative over its base theory is taken up in chapter seven. Horsten argues that the nonconservativeness is a *phenomenon* that has to be appreciated. Moreover he turns it into a desideratum to have an arithmetically strong axiomatic theory of truth, as long as it is philosophically sound.

Following the philosophical motivation of ever stronger theories without thereby contradicting its truth-theoretic soundness, Horsten presents in the following two chapters versions of type-free axiomatic theories of increasing acceptability. Chapter eight is concerned with the Friedman–Sheard theory FS. FS is an attractive theory because it is highly compositional and also from a proof-theoretic perspective it is strong. But it is doubtful whether FS is sound. For one thing, it is ω-inconsistent. For another, it is not stably true but only nearly stably true.

The Kripke–Feferman theories based on Kripke's semantic account are the topic of the ninth chapter. Regarding the power game KF is the most attractive theory. Moreover, KF is also arithmetically sound. But Horsten argues, following suggestions by Reinhardt, that KF is not philosophically sound because it cannot be closed under the rules of necessitation and co-necessitation. To circumvent this asymmetry Reinhardt suggested to formulate a theory IKF, by taking the inner logic of KF. But this theory seems to withstand a natural axiomatization. This is reason enough for Horsten to try another approach by giving up classical logic and formulating KF in a partial logic known as strong Kleene. The resulting theory of truth, which he calls PKF, and which he favors is the final theory he discusses. By giving up classical logic, Horsten's PKF is a revisionary theory of truth in the form of a paracomplete approach. The main difference is a restriction of the rule of conditionalization to truth-determinate sentences.

PKF is therefore different from all the theories considered before and this complicates an evaluation. Although Horsten presents the main results regarding PKF, at this point it would have been helpful to present more details. Horsten also discusses some objections, especially McGee's methodological objection against changing the logic. Here Horsten illustrates his justification by comparing the truth case to set theory. He claims that whereas in set theory there is a conceptual motivation for restricting the comprehension scheme, there is none in the case of truth motivating a restriction of the Tarski-biconditionals. But his argument is at this point not totally convincing. On the one hand, Kripke's concept of groundedness or the revision theoretic concept of stability could be interpreted as such a conceptual motivation for a restriction. On the other hand, it may well be that some motivation could be given, even if those two examples are not accepted.

The tenth chapter is devoted to a new reading of deflationism: inferential deflationism. Horsten intertwines the philosophical story behind PKF with inferentialism. The idea is that the meaning of truth is given by the rules and axioms similar to the conception 'that the meaning of the classical logical connectives is determined by their role in valid reasoning'. In PKF the *essential* inferential character of truth is highlighted by the fact that it contains rules of inference, which cannot be replaced by axioms. Moreover Horsten argues that the strategy of remaining silent with respect to reflection upon the theory leads to a revenge free account. The strategy of silence is to be applied whenever asked about the semantic status of a paradoxical sentence, such as the liar.

In his book Leon Horsten accomplishes something very important. He bridges the gap between formal investigations on truth and philosophical considerations. In the past few years one could easily have the impression that this gap is there for a reason and that there is no way to fruitfully combine the two. But Horsten shows that this is not the case. It is possible to draw upon technical results to gain insight into philosophical questions and to clarify deflationism. But not only that: his inferential deflationism is an innovative and tempting reading of deflationism. This new reading of deflationism draws upon similarities of fields far apart and witnesses for Horsten's ability to combine different philosophical areas like axiomatic theories of truth and inferentialism.

Horsten manages to present the difficult technical material of the field of axiomatic theories of truth in a very accessible way without getting lost in the details and, at the same time, without oversimplifying the issues at hand. The book is therefore very well suited for a course on axiomatic theories of truth, as well as for a course on deflationism itself. Besides that it is a valuable contribution to the contemporary literature. With its fluent style, Horsten's book is a must read for anybody interested in truth.

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ENRIQUE CASANOVAS. ***Simple theories and hyperimaginaries.*** Lecture Notes in Logic, vol. 39. ASL and Cambridge University Press, Cambridge, 2011, 184 pp.

Simple theories were first defined in Shelah's paper, *Simple unstable theories*, ***AML***, (1980). In it he embarked on a general study of theories in which the notion of dividing had local character—a desirable property of an abstract independence relation. This paper was not well understood and in fact was cited only once by anyone other than Shelah (Keisler, *Measures and forking*, ***APAL***, (1987)) before the mid-1990's. The situation changed dramatically in about 1995 with Byunghan Kim's thesis and related papers (see Kim and Pillay, *Simple theories*, ***APAL***, (1997)) in which dividing in simple theories was shown to be symmetric and in fact, to be a very well-behaved abstract independence relation satisfying among other things the independence theorem. This ushered in a golden era in the study of simple theories and much of the foundational work in the subject was done in the five years immediately afterwards. The role of Lascar strong types and Lascar group were clarified and the apparent necessity of hyperimaginaries, equivalence classes of type-definable equivalence relations, was realized. One of the high points of that period was the proof by Buechler, Pillay and Wagner (*Supersimple theories*, ***JAMS***, (2000)) that for supersimple theories, hyperimaginaries were eliminable, i.e., they were interdefinable with sets of imaginary elements. At the end of this period, the two most important foundational questions left were whether all simple theories eliminated hyperimaginaries and whether the stable forking conjecture holds (essentially if all dividing in simple theories is the result of dividing by a stable formula). Both of these problems remain open today.

The book under review, Simple theories and hyperimaginaries by E. Casanovas, takes stock of these foundational issues and sheds light on them a decade later. Much of the book looks at the modern state of the technical tools that were brought to bear on these foundational issues; examples are included or referenced to highlight technical issues and not to make the reader more familiar with simple theories or their applications. This book is not for the uninitiated. Having said this, the book plays an important role in consolidating much of the material that has developed in the past 10 years and which has, in many cases moved on beyond simple theories.

The introductory material is standard and has appeared in a number of places but I have always liked the author's approach (expounded in *Simplicity simplified* ***Revisita Colombiana Matemàtica***, (2007)) and which is followed here. This has always seemed to be the quickest line of obtaining all the basic results about dividing in simple theories. This material takes up the first ten chapters which sounds like a lot but the chapters are refreshingly short and ten chapters is only slighlty more than sixty pages. There is attention to the local theory, developing the theory formula-by-formula, and concentration on the stable part of the theory—the author never mentions the stable forking conjecture directly but he presents all the necessary tools that exist to take a crack at it. Although much of what is presented is couched in terms of simple theories, in many cases the results are general and could be seen as quite useful to someone who wanted to look at stable domination and the role of invariant global types. Chapter 12 is a joy to read for anyone who works on abstract independence relations. It is concise, based on work of Adler, but it gives short proofs of many results which have been floating around and lays the ground work for much work to come.

The one problem I felt in this early part of the book was Chapter 11 where a discussion of canonical bases occurs before the introduction of hyperimaginaries. This makes some of the work in this chapter artificial (the assumption of stationarity in the simple case, for instance) and means that results have to be reproven after the introduction of hyperimaginaries. Still, this chapter reads quite smoothly as an introduction to canonical bases in stable theories.

The real concentration on simple theories starts in Chapter 13 with the introduction of supersimple theories. Hyperimaginaries are introduced and now the issues of dividing, forking and canonical bases in simple theories are revisited with this new tool. The justification for this comes in a rather short and elegant reworking of the proof that supersimple theories eliminate hyperimaginaries. I believe much of the effort put into these latter chapters will pay off for those who come afterwards and want to either work on the problem of elimination of hyperimaginaries in simple theories or on generalizations.

There is the inevitable comparison to the book, ***Simple theories*** by F. Wagner. This book was published in 2000 at the end of the era I described above. It played a valuable role in getting much of the folklore out into the open and concentrated on several topics not touched upon in the book under review, most noticeably definable groups in simple theories. I think that the revisiting of these foundational topics after some time has passed has allowed Casanovas to make judicious choices about what to include. As an introduction to simple theories with an eye toward directions that the subject is taking now, this book by Casanovas is a fine addition to the literature.

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Jouko Väänänen. ***Models and games.*** Cambridge Studies in Advanced Mathematics, vol. 132. Cambridge University Press, Cambridge, 2011, xii + 367 pp.

This book is about games used in model theory. The games in question are two-player games of perfect information, played by 'player **I**' and 'player **II**'. There are three main kinds: back-and-forth 'Ehrenfeucht–Fraïssé' games to compare two model-theoretic structures, semantic 'Hintikka' games to evaluate logical sentences in structures, and model existence games to construct models of sets of sentences. In their basic forms, there are finitely many or ω rounds, while the most general games come with a partial order in which player **I** must play an ascending sequence, the game ending when he runs out of elements. This flexibility allows the games to handle not only first-order logic but various infinitary logics ($L_{\infty\omega}$, $L_{\kappa\lambda}$) and, using enhanced games, first-order logic augmented with generalized quantifiers as well.

The book offers a fine introduction to modern work on games, and climbs quite rapidly to considerable heights. There are ten chapters. Chapter 1 is a one-page introduction highlighting the 'Strategic Balance of Logic', a 'red thread' by which the three kinds of game are linked. Chapter 2 is a short primer on necessary background including cardinals and ordinals, with 34 exercises. Chapter 3 introduces games, strategies, closed games and determinacy. Chapter 4 concerns semantic and back-and-forth games on graphs. First-order logic for graphs is introduced, and the n-round back-and-forth game shown to characterise equivalence for sentences of quantifier depth n. Structures for arbitrary vocabularies are developed in Chapter 5, with examples including unary relations, equivalence relations, linear orders, boolean algebras, and trees. Back-and-forth games are considered, and associated sets of partial isomorphisms. First-order logic for arbitrary structures is introduced in Chapter 6, which proves the Löwenheim–Skolem theorem using cub sets, an unusual approach. Semantic and model existence games are also brought in here, and compactness, omitting types, interpolation and Beth definability theorems are proved for countable and uncountable vocabularies.

The book then moves to less well trodden territory. Chapter 7 considers 'dynamic' back-and-forth games EFD_α for an ordinal α, in which player **I** must choose in successive rounds smaller and smaller ordinals $< \alpha$; the game stops when he cannot continue. Infinitary logic ($L_{\infty\omega}$) and a suitable semantic game are introduced, and Scott watersheds and sentences. Definability of well-orderings is considered extensively. Chapter 8 is about model theory of infinitary logics. It considers the model existence game, Craig interpolation, large models based on indiscernibles, and homogeneous models, again with an interest in definability of well-orderings. There is a final, attractively-written section on game logic—short, elegant proofs—and a little on Henkin quantifiers. (Given the author's recent work, it is surprising that the book has rather little on independence-friendly logics.) Chapter 9 covers $L_{\infty\lambda}$, and new back-and-forth games in which players can choose λ elements from structures in a single round. Scott watersheds are reconsidered here. A strong section on posets leads to transfinite dynamic back-and-forth games in which player **I**'s ordinal α (above) is replaced by an arbitrary poset. The chapter ends with topology of uncountable models. A detailed history is followed by 59 exercises. Finally, Chapter 10 is about generalized quantifiers. The writing in the early part is strikingly clear and should be accessible to readers dipping in at this point. Strategies are mostly treated informally, whereas in earlier chapters they are often formalized as families of functions, and there are many lucid diagrams. After some material on monotone and other types of quantifiers, the author focuses on the largeness quantifier $\exists^{\geq\omega_1}x$ and the cofinality quantifier stating that a formula $\varphi(x, y)$ defines a linear order of cofinality ω. Games are used to obtain an omitting types theorem, leading to completeness and compactness.

Every chapter has many exercises—sometimes more than 100. They are so numerous that they must be regarded as a core part of the text. They are ordered by topic, not difficulty. This is at first alarming because their difficulty varies widely. Several require prior knowledge—e.g., of forcing—but they can be skipped by students lacking the necessary background. Some are phrased in novel, informal ways, and many are most stimulating.

The early chapters and parts of Chapter 10 could be managed by masters students in mathematics (or possibly computer science), but I thought the book as a whole most suited to professionals or fairly strong doctoral students or postdocs in mathematical logic. There are several reasons for this. Although it is largely self-contained as to definitions, the book would be easier going for readers who have worked with basic logic and set theory. Most of the material lies within mathematical logic, and the examples are mostly from mathematics: analysis, algebra, set theory. The proofs are often quite terse and tough, needing some sophistication, and there is rather little introductory text. There are also some mistakes. As well as typos, concepts are sometimes used before being defined: e.g., reducts, the game

EFD_α for $\alpha = 1, 2$, and fragments (used in §8.4 but defined only in §8.5). Some theorems and proofs have omitted or extraneous countability or normal form hypotheses (e.g., 6.18, 10.60, 10.78), and others have inaccuracies (e.g., 4.15 misassociates number of free variables with quantifier rank, 6.19 does not define the set X_n). With one possible exception, all the mistakes I found could be fixed by strong students and they would benefit from doing so.

The exception is unfortunately a major result crowning the end of the book: completeness for the generalized quantifier $Qxy\varphi(x, y)$, expressing that $\varphi(x, y)$ defines a linear order of countable cofinality (Theorem 10.91, due originally to Shelah (1975)). In outline, given a consistent theory T, the proof starts with a 'weak' model $M_0 \models T$ of uncountable regular cardinality $\kappa \geq |T|$ in which all linear orders definable with parameters have cofinality κ. It then constructs an ω-chain $M_0 \prec M_1 \prec \cdots$ of models of cardinality κ, with union M_ω. For orders required to have countable cofinality, this is achieved in M_ω by extending them infinitely often in the chain. The idea is for the others to be given cofinality κ in M_ω by 'keeping them fixed' once they appear, using an omitting-types argument that requires them to already have cofinality κ. But as the chain proceeds, new linear orders with new parameters arise. Their cofinalities are unknown, so omitting-types may not apply to them, and even if it did, their cofinalities in M_ω would also be unknown. The book's author has suggested two possible fixes. First, departing some way from the text, add that $\kappa \geq 2^\omega$, make the elementary chain of length κ^+, and arrange that in the union, all definable linear orders required to have countable cofinality have cofinality κ^+, while the others have cofinality at most κ. Now make all cofinalities uncountable, using an ultrapower, and finally decrease cofinality κ^+ to ω using an ω-chain of elementary substructures. Alternatively, to allow the proof of Lemma 10.89 to work, strengthen Lemma 10.88 by additionally establishing that for every $\psi(x, y)$ with parameters in M defining a linear order of cofinality κ in $(\mathcal{M}, Q)$, M is cofinal in the order defined by ψ on $(\mathcal{M}', Q')$. The author will give more details on his web-page.

To me, these issues do not detract substantially from the book. The reader simply needs to be professional enough to be stimulated rather than frustrated by such things. Such a reader has much to gain from the book. The choice of material is interesting and wide ranging. Although orientational text is sparse, the text has a pleasant, rather taciturn tone, and short, helpful comments are sprinkled throughout, giving it the feel of seminar notes and allowing the character of the author to come through strongly. Usefully, the bibliography items have page numbers pointing to the main text, and the many diagrams are quite enjoyable, neatly summarising complex ideas.

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Alasdair Urquhart. *Henry M. Sheffer and notational relativity*. ***History and Philosophy of Logic***, vol. 33 (2012), pp. 33–47.

Henry Maurice Sheffer's name is an inherent part of the history of logic. Almost everybody who has come into closer contact with logic is familiar with the ambitious aim of practicing logic with the smallest possible set of primitives and axioms. The introduction of the "Sheffer Stroke" to the system of ***Principia Mathematica*** in 1913—replacing the primitive symbols for adjunction and negation—marks an important contribution to modern formal logic in the early 20th century. No less a figure than Bertrand Russell supported Sheffer throughout his lifetime. But apart from Sheffer's most basic propositional connective, most of us know almost nothing about his further scientific work.

Whereas already in 2000 Michael Scanlan published an excellent outline of Sheffer's biography (*The known and unknown H. M. Sheffer*, in ***Transactions of the Charles S. Peirce Society***, vol. XXXVI/2, pp. 193–224), it is Alasdair Urquhart who has written a supplementary article on his few further works in logic, most of them only abstracts. Apart from his motivation, to bring Sheffer's almost unknown ambitious logical project into light, Urquhart's interest is dedicated to the question why Sheffer published so little (cf. pp. 42f.). After outlining some historiographical aspects on Sheffer's famous primitive operator (pp. 34f.), Urquhart discusses his logical work in three steps: first, Sheffer's research program in logic up to 1921 (pp. 35ff.)—the year of birth of his unpublished and only privately circulated monograph ***The general theory of notational relativity***; second, the monograph itself (pp. 37ff.) as a "key document in understanding Sheffer's research in logic" (p. 35); and third, his later publications up to 1947 (pp. 41ff.). From 1910 on Sheffer attempted to found a general theory of systems through determining all possible axiomatic formulations of a given system by its intrinsic properties, "that is to say, those [properties, M.W.] that are independent of the particular primitive notions and axioms employed" (p. 35). For this early period Urquhart diagnoses several unfulfilled announcements by Sheffer as well as the problem of realizability of such a far-reaching goal: "from our current point of view, [it is, M.W.] probably impossible in general" (p. 35). While interpreting Sheffer's "idiosyncratic notation" and "clumsy terminology" (p. 37) in concepts of permutation groups, Urquhart describes the content of his monograph ***The general theory of notational relativity*** in a lucid manner. Since Sheffer's small monograph is still unpublished, for the time being Urquhart's reconstruction seems to be the only source to get an understanding of Sheffer's concepts and results on isotropy, stratigrafy, permutivity, and their applications. Because of a series of grand but unproven claims, the fragmentary character of the sections of the monograph, the unsatisfactory methodology and some other aspects, Urquhart raises doubts whether the manuscript would have ever been accepted for publication by a journal (cf. p. 41).

Unfortunately, such shortcomings are not only features of the unpublished manuscript, since Sheffer's work "shows the marks of insufficient mathematical technique" (p. 42). Urquhart explains these deficiencies with respect to Sheffer's self-imposed isolation from the scientific community. There seems to be no evidence that he was familiar with the mainstream of logic of his days and that was "almost a guarantee of failure" (p. 42). Despite his shortcomings in research, "Sheffer inspired considerable loyalty and affection among his students" (p. 43) and his most formative influence on modern times (apart from the "Sheffer stroke") is due to the work of Cooper Harold Langford, who saw his own successful general program of research as having been initiated by Sheffer's largely failed one (cf. pp. 44ff.).

Urquhart's paper is a valuable contribution to the history of modern formal logic. It sheds light on the logical work of Henry M. Sheffer and elucidates the background, why it was neglected for decades.

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GEORGE BARMPALIAS, ANDREW E. M. LEWIS, and KENG MENG NG. *The importance of* Π^0_1 *classes in effective randomness*. ***The Journal of Symbolic Logic***, vol. 75 (2010), pp. 387–400.

GEORGE BARMPALIAS, ANDREW E. M. LEWIS, and FRANK STEPHAN. Π^0_1 *classes, LR degrees and Turing degrees*. ***Annals of Pure and Applied Logic***, vol. 156 (2008), pp. 21–38.

ANTONIN KUČERA. *Measure,* Π^0_1 *classes and complete extensions of PA*. ***Recursion Theory Week*** (***Oberwofach, 1984***). Lecture Notes in Mathematics, vol. 1141. Springer, Berlin, 1985, pp. 245–259.

FRANK STEPHAN. *Martin-Löf randomness and PA complete sets*. ***Logic Colloquium '02.*** Lecture Notes in Logic, vol. 27, Association for Symbolic Logic, La Jolla, CA, 2006, pp. 342–348.

The importance of Π^0_1 classes in effective randomness can be traced back to the seminal paper of Kučera and has continued through the recent powerful work of Barmpalias, Lewis, Ng and Stephan.

Effectively closed sets, or Π^0_1 classes, have long been a central theme in computability theory. Effectively closed sets are important because they can represent the set of solutions to effectively posed problems in many areas of mathematics. Most research has been focused on Π^0_1 subsets of the Cantor space $\{0,1\}^{\mathbb{N}}$ of infinite sequences $X = (X(0), X(1), \dots)$ of 0's and 1's, and the closely related space $[0,1]$ of real numbers. Elements of $\{0,1\}^{\mathbb{N}}$ are often said to be real numbers. A key representation example from mathematical logic is the set of complete consistent extensions of an axiomatizable theory (such as Peano Arithmetic). An example from combinatorics is the set of k-colorings of a computable graph. An example from analysis is the set of zeroes of a computable continuous function. The classic 1972 papers of Jockusch and Soare, *Π^0_1 classes and degrees of theories*, [***Transactions of the American Mathematical Society***, vol. 173 (1972), pp. 33–56] and *Degrees of members of Π^0_1 classes* [***Pacific Journal of Mathematics***, vol. 40 (1972), pp. 605–616], are landmarks in the field and introduce the notion of forcing with Π^0_1 classes, used to prove the classic low basis theorem, that every Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$ has a member of low degree.

The set of complete consistent extensions of Peano Arithmetic is denoted by *PA* and may be viewed as the Π^0_1 class of sets which separate two disjoint c.e. sets, that is, $X \in PA$ if and only if X contains all theorems of Peano Arithmetic and does not contain any negations of those theorems. A Turing degree is called *PA* if it is the degree of a complete extension of Peano Arithmetic. The *PA* degrees play an important role in the study of Π^0_1 classes. In particular, it was shown by Scott and Solovay that a degree **a** is *PA* if and only if the degrees below **a** form a basis for the Π^0_1 classes. Thus the class *PA* is Π^0_1 complete with respect to the Muchnik degrees of difficulty, where a class P is said to be Muchnik reducible to a class Q if, for any $Y \in P$, there exists $X \in Q$ such that $Y \leq_T X$. It should be mentioned that Scott sets are precisely the sets of degrees coded in a nonstandard model of arithmetic.

Effective or algorithmic randomness arises from considering the randomness of a single real. There are several approaches to this notion. One approach is based on Kolmogorov's idea of compressibility of strings. Here we fix a universal prefix-free machine M and define the complexity $K(\tau)$ of a string τ to be the length $|\sigma|$ of the shortest string σ such that $M(\sigma) = \tau$ (we say that σ is a *description* of τ). A real X is called Chaitin–Levin random if its initial segments have high complexity, that is, if there is a constant c such that $K(X \upharpoonright n) \geq n - c$ for all n. This notion is well-defined since it is invariant under the choice of a universal prefix-free machine.

Another approach to randomness is via effective null sets. Martin-Löf originated in *The definition of random sequences* [***Information and Control***, vol. 9 (1966), pp. 602–619] the notion that a real X is random if it does not belong to any effectively null set. Here a set E is effectively null if $E = \cap_j E_j$, where $\langle E_j \rangle$ is a uniformly computable sequence of c.e. open sets such that $\mu(E_j) < 2^{-j-1}$. For example, let $E_j = \{X : (\forall n \leq j) X(n) = 0\}$, so that $\cap_j E_j = \{0^\omega\}$. Schnorr proved the surprising result that this definition of Martin-Löf randomness coincides exactly with the notion of Chaitin–Levin randomness defined above. Thus Martin-Löf randomness is a robust notion. A degree **a** is called Martin-Löf if it is the degree of some Martin-Löf random real.

Since a Π^0_1 class is exactly the complement of a c.e. open set, we observe that a real X is *not* Martin-Löf random if, for any effective sequence $\langle Q_n \rangle$ of Π^0_1 classes such that $\mu(Q_n) > 1 - 2^{-n}$, X belongs to one of the sets Q_n. But there is more to say. Martin-Löf showed that there is a *universal* Martin-Löf test $\langle E_j \rangle$, meaning that X is Martin-Löf random if and only if $X \notin \bigcap_n E_n$. Kučera (1985) proved this result and observed that for the complementary sequence $\langle P_n \rangle$ of Π^0_1 classes, the set of Martin-Löf random reals is exactly the union $\bigcup_n P_n$. In particular, each P_n is a Π^0_1 class of positive measure containing only Martin-Löf random reals.

Kučera used Π^0_1 classes of positive measure to obtain results about the degrees of Martin-Löf random sets. He showed that any Π^0_1 class of positive measure contains reals of every Martin-Löf degree. He showed that every degree above $\mathbf{0}'$ is a Martin-Löf degree but the Martin-Löf degrees do not form an upper cone. He also showed that any Martin-Löf random set is bi-effectively immune and hence is fixed-point-free but not every fixed-point-free degree is Martin-Löf random.

Kučera also considered the *PA* degrees and their relation to the Martin-Löf degrees. Since the sets of Martin-Löf degree have measure one, whereas the sets of *PA* degree have measure 0, not every *PA* degree is Martin-Löf random. Kučera constructed *PA* sets which do not have Martin-Löf degree. On the other hand, the *PA* degrees coincide with the Martin-Löf degrees for those degrees above $\mathbf{0}'$. Stephan proved in the 2006 paper that any Martin-Löf random set which is not above $\mathbf{0}'$ is not *PA* and in fact does not compute any *PA* degree. Finally, Kučera used Π^0_1 classes of random reals to show that every *PA* degree $\mathbf{a}$ has the cupping property, where $\mathbf{a}$ has the cupping property if for every degree $\mathbf{c}$ above $\mathbf{a}$ there exists $\mathbf{b}$ such that $\mathbf{c} = \mathbf{a} \vee \mathbf{b}$.

Barmpalias, Lewis and Ng (2010) revealed a further striking connection between the *PA* degrees and the Martin-Löf degrees by proving that every *PA* degree is the join of two Martin-Löf degrees. The techniques used are based on properties of Π^0_1 classes and in particular Π^0_1 classes of positive measure.

Effective randomness is in part a continuation of the study of the computability of sets and functions. The notion of computable functions and computably enumerable sets led naturally to that of relative computability and then to the ever so fruitful development of the the Turing degrees of computability. So also the notion of randomness has led in recent years to the study of relative randomness and degrees of randomness. The two definitions above of algorithmic randomness are easily relativized to define the notion of A being random relative to B, or B-random. Then we say that $A \leq_{LR} B$ if every B-random set is A-random. A is *low for randomness* if $A \leq_{LR} 0$, that is if every A-random set is random. Several other notions of lowness have been studied in this area. In particular, a real A is said to be K-trivial if the complexity of the initial segments have complexity as low as possible, that is, there is a constant c such that for all n, $K(A \upharpoonright n) \leq K(0^n) + c$. These two notions and two other notions of lowness for randomness, are shown to be equivalent in papers by Nies in *Lowness properties and randomness* [***Advances in Mathematics***, vol. 197, pp. 274–305] and by Hirschfeldt, Nies and Stephan in *Using random sets as oracles* [***Journal of the London Mathematical Society***, vol. 75 (2007), pp. 610–622].

Every K-trivial set is low in the Turing degrees and the K-trivial sets induce a Σ^0_3 ideal in the ω-c.e. degrees. Kučera and Slaman use Π^0_1 classes in *Low upper bounds of ideals* [***The Journal of Symbolic Logic***, vol. 74 (2009), pp. 517–534] to construct a low Turing upper bound for the K-trivial degrees. Barmpalias, Lewis and Stephan (2008) constructed a Π^0_1 class consisting of GL_1 reals $\leq_{LR} \mathbf{0}'$ which contains no K-trivial reals. This contrasts with the classic low basis theorem for the Turing degrees.

Barmpalias, Lewis and Ng proved a number of results relating Π^0_1 classes with no K-trivial members and the *LR* degrees which are parallel to classic results of Jockusch and Soare about Π^0_1 classes with no computable members and the Turing degrees. For example, A, B are said to be a *minimal pair* in a degree structure if they have greatest lower bound $\mathbf{0}$. Jockusch and Soare showed that any Π^0_1 class with no computable members contains a minimal pair A and B in the Turing degrees. Barmpalias, Lewis and Ng gave a proof that every Π^0_1 class with no K-trivial member has a minimal pair in the *LR* degrees and uses this to show that there is a minimal pair below $\mathbf{0}'$ in the *LR* degrees. Other parallel results include the following. For any sequence $\langle C_i \rangle$ of sets which are not K-trivial, P has a member B such that $C_i \not\leq_{LR} B$ for all i. The LR degree spectrum of a Π^0_1 class with no K-trivial members contains an antichain of size $2^{\aleph_0}$. The LR upper closure of a Π^0_1 class with no K-trivial members is meager. For any Π^0_1 class P with no K-trivial members, there is a nonempty Π^0_1 class Q with no K-trivial members such that for no $X \in P$ and $Y \in Q$ is $X \leq_{LR} Y$. Finally, there is a perfect Π^0_1 class such that any two distinct members are LR-incomparable.

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