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VAUGHT'S THEOREM ON AXIOMATIZABILITY BY A SCHEME

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Abstract. In his 1967 paper Vaught used an ingenious argument to show that every recursively enumerable first order theory that directly interprets the weak system VS of set theory is axiomatizable by a scheme. In this paper we establish a strengthening of Vaught's theorem by weakening the hypothesis of direct interpretability of VS to direct interpretability of the finitely axiomatized fragment VS_2 of VS. This improvement significantly increases the scope of the original result, since VS is essentially undecidable, but VS_2 has decidable extensions. We also explore the ramifications of our work on finite axiomatizability of schemes in the presence of suitable comprehension principles.

§1. Introduction. Craig's well known trick (introduced in [3]) shows us that every RE theory[1] in predicate logic (of finite signature) has a decidable axiomatization. Inspection of the argument shows that we even get a p-time decidable axiomatization. Craig's trick is widely known, e.g., among philosophers of science. It is less well known that there is a substantial class of theories that can be axiomatized by a scheme.[2] This was shown by Robert Vaught in his [16]. Here is a precise formulation of the result. Consider the following axioms (in the language with $=$ and $\in$):

$\mathsf{vs}_n \vdash \forall x_0, \ldots, x_{n-1}\, \exists y\, \forall u\, (u \in y \leftrightarrow \bigvee_{i<n} u = x_i)$.

We use the convention that the empty disjunction is identified with $\bot$. Thus, vs_0 tells us that $\exists y\, \forall u\, u \notin y$. This means—since no assumption of extensionality is imposed on $\in$—that there are one or more empty sets. For $0 < k \leqslant n$, we have $\mathsf{vs}_n \vdash \mathsf{vs}_k$.

The theory VS_n is axiomatized by $\mathsf{vs}_0, \ldots, \mathsf{vs}_n$, or, more economically, just by vs_0 plus vs_n. Vaught's Set Theory, VS, is the theory axiomatized by all the vs_n.

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[1]We will use 'RE' for *recursively enumerable*. Nowadays this is also known as *c.e.*, i.e., *computably enumerable*.

[2]For a philosophico-historical exploration of the notion of scheme in logic, see [2].

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We say that U *directly interprets* V iff there is an interpretation K of V in U, that is *direct*, i.e., that is not relativized and that translates identity to identity. We call a theory that directly interprets VS *a Vaught theory*. So, a Vaught theory contains a formula Axy that satisfies all axioms of VS when $x \in y$ is replaced by Axy. (We allow extra parameters in A.)

Vaught's theorem tells us that:

All RE Vaught theories are axiomatizable by a scheme.

The following is a list of RE Vaught theories that are not *prima facie* axiomatizable by a scheme.

- The arithmetical consequences of ZF (under the von Neumann interpretation of arithmetic in ZF).
- The purely set-theoretical consequences of the Kelley–Morse theory of classes.
- $\mathrm{I}\Delta_0$. This is the theory of Δ_0-induction. See e.g., [8] or [7]. This theory *is* axiomatizable by a scheme with substitutions of a restricted class, to wit the Δ_0-formulas. The notion of scheme studied in this paper has unrestricted substitutions.
- $\mathsf{EA} + \{\mathsf{con}^n(\mathsf{EA}) \mid n \in \omega\}$, where EA is Elementary Arithmetic,
  aka Elementary Function Arithmetic or EFA,
  aka $I\Delta_0 + \mathsf{Exp}$, and $\mathsf{con}^0(\mathsf{EA}) := \top$,
  $\mathsf{con}^{n+1}(\mathsf{EA}) := \mathsf{con}(\mathsf{EA} + \mathsf{con}^n(\mathsf{EA}))$.
- PRA, or Primitive Recursive Arithmetic, in a variant with finite signature.

It is not difficult to see that VS interprets Tarski, Mostowski and Robinson's theory R, which was introduced in [14]. Hence, by the results of [14], the theory VS and therewith every consistent Vaught theory is essentially undecidable.

Note that VS_2 is a theory of unordered pairing. This pairing is not assumed to be functional. So, more than one object might code an unordered pair. Moreover, this pairing is not surjective. In fact empty sets fail to be unordered pairs. We call a theory that directly interprets VS_2: *a pair theory*. The main theorem of this paper is that:

All RE pair theories are axiomatizable by a scheme.

There are decidable pair theories. This is an immediate consequence of the results on ordered pairing by Richard Tenney [15], reported in [5], p. 162. A beautiful special case is the decidability of the true theory of the Cantor Pairing Function C with Successor, $\mathsf{Th}(\mathbb{N}, \mathsf{C}, \mathsf{S})$, which is proved in [1].

Ferrante and Rackoff show that pair theories for functional ordered pairing are non-elementary.[3] This means that there is no elementary decision

[3] I think that their result must also work in the non-extensional case, but did not try to prove this. Their result trivially transfers to functional unordered pairing.

algorithm for such theories.[4] See [5]. The existence of decidable pair theories shows that our main theorem is a true improvement on Vaught's result. Ferrante and Rackoff's result shows that *to be a pair theory* is still a substantial restriction on theories.

Vaught provides several examples of RE theories that are not axiomatizable by a scheme. E.g., let X be an RE, non-recursive set. Let $\mathsf{E!}_n$ be the statement, in the language of identity, that there are precisely n elements. Let U be the theory in the language of identity, axiomatized by the $\neg\,\mathsf{E!}_n$, for $n \in X$. If U were axiomatizable by a scheme, then the class of finite models (say, as coded in the natural numbers in some standard way) would be decidable, since it is decidable whether a finite model satisfies a given scheme. Quod non.

Plan of the paper. Consider an RE axiomatized theory U with an axiom set represented by a Σ_1-formula α. We want to create a scheme that says:

All axioms in α are true.

To be able to express this, we need a modicum of arithmetic and coding of syntax. In the presence of only a theory of pairing we cannot get enough arithmetic and coding machinery, but we can compensate for that by creating better and better approximations of the necessary arithmetic and coding machinery using schematic variables. The basic apparatus to realize this idea is developed in Section 3.

We cannot possibly have a full satisfaction predicate, so we work with approximations of the satisfaction predicate. A schematic variable will range over these approximations. To build the approximations of satisfaction predicates, we need more than just approximations of a theory of numbers, we need approximations of a theory of sets (or sequences, or some other appropriate kind of *container*). We develop the approximations of a theory of sets in Section 4. We implement the approximations of satisfaction in Section 5.

We note that we could have completely avoided the use of a theory of numbers by doing our coding directly in set theory. However, it is convenient to work with numbers because there is such a large legacy of work in metamathematics which uses numbers. E.g., in this way we have the notion of Σ_1-formula available without the need to tell a further story.

There is a problem that can best be described as *how can we keep our scheme from saying too much*? As is well known, first-order logic has a problem in capturing the notion of *finite*. How do we prevent the quantifier *all axioms* from ranging also over non-standard axioms? Here, surprisingly, the Liar Paradox comes to the rescue. By self-reference we show that only *standard* approximations of satisfaction are possible. We use the standard numbers

[4]The elementary functions are a hierarchy of functions with at most multi-exponential growth-rate. They are the provably total functions of EA.

provided by these standard approximations to gain the necessary control over the axioms our scheme is talking about. The self-referential argument is our Theorem 5.2.[5]

In Section 6, we put all the ingredients to work to prove the main theorem.

At the end of our paper, we briefly discuss the possibility of compressing schemes to single sentences in the presence of comprehension principles. This is done in Section 7.

Vaught's paper. The present paper is in many respects a re-presentation or re-make of Section 3 of Vaught's beautiful seven page paper [16]. Vaught's Theorem 2 states that any RE Vaught theory is axiomatizable by a scheme. Vaught's theory S is our theory VS, and Vaught's theory T_1 is a theory of sequences and numbers.

Vaught writes the following in the beginning of Section 3 about the improvement his Theorem 2 makes compared to his Theorem 1.

> Theorem 2 improves on Theorem 1 in two ways. For one thing, S appears to be somewhat weaker than T_1 (as well as more elegant). More significant, however, is the fact that Theorem 2 states that T is outright axiomatizable by a schema, rather than by a scheme plus the axioms of S.
>
> Nevertheless, there do not appear to be any very interesting theories whose schematic axiomatizability is obtainable from Theorem 2 but not from Theorem 1. Consequently, we shall give a brief sketch of the proof.

It seems to me that the observation that Vaught's theorem extends to all theories that directly interpret a weak theory of pairing does extend the scope of the theorem sufficiently substantially to justify a more detailed presentation.

Our proof follows Vaught's rather closely. Thus, it can be viewed for a large part as a careful exposition of Vaught's work. Here is the main difference. Vaught uses a fixed coding of numbers and sets. He just varies the satisfaction predicates. To get an axiomatization by a finite scheme, he employs an extra idea to get a schematic axiomatization of his set theory VS. We, on the other hand, employ the fact that the theory of pairing *locally directly interprets* the theory VS. We work with a varying interpretation of larger and larger parts of VS, and, thus, with a varying interpretation of numbers and sets.

Note that the schematic axiomatizability of VS also follows from our result.

§2. Preliminaries. In this paper, we consider RE theories of finite signature. These theories will be one-sorted in most of the paper. The exception is Section 7, where we study theories with a second sort of classes. We will assume that our theories are *officially* relational. However, we will often pretend there are function symbols present. As is well known, the function symbols can be eliminated by an efficient translation.

[5]Variants of the same trick were used, after Vaught's paper, in other contexts, e.g., by Plisko, in his work on the predicate logic of realizability. See [11] and [12].

Interpretations in this paper will be multi-dimensional relative interpretations with parameters. An interpretation K of U in V is based on a translation τ of the language of U to the language of V, such that, for all U-sentences A, we have $U \vdash A$ implies $V \vdash A^\tau$. See e.g., [17] or [19] or [18] for careful definitions.[6] An interpretation is *direct* iff it is unrelativized and does translate identity into identity. Note that direct interpretations are automatically one-dimensional. A theory V *locally interprets* a theory U iff V interprets every finite subtheory of U. We employ the following notations:

- $V \rhd U$ for: V interprets U.
- $U \equiv V$ for: U is mutually interpretable with V, i.e., $U \rhd V$ and $V \rhd U$.

Further determinations of the kind of interpretability intended will be added as subscript. E.g., $V \rhd_{\mathsf{dir}} U$ will mean: V directly interprets U.

A scheme $\mathcal{S}$ for a theory U is obtained as follows. We extend the signature of U, say Σ_U, with a finite number of predicate symbols to a new signature Σ_U^+. We assume that the new symbols are marked in Σ_U^+. A scheme is a sentence in the extended language. Let's say that an *s-translation* is a translation of the language given by Σ_U^+ into the language of U that is constant on the non-marked or old vocabulary; the predicate symbols of the new vocabulary are replaced by formulas of the old vocabulary. We do not have domain restriction in s-translations. The scheme $\mathcal{S}$ is valid in U if, for all s-translations τ, $U \vdash \mathcal{S}^\tau$. Our formulations allow two readings: do we allow parameters or not in the translations τ. I will assume that the default reading of scheme is *scheme with parameters*. This means that the s-translations in the definition of validity range over translations with parameters. These parameters are unrestricted, i.e., they range over the full domain of U, if we do not allow parameters, we will speak about a *closed scheme*.

We will use $[\mathcal{S}]$ for the set of all all $\mathcal{S}^\tau$, where τ is an s-translation (to the given target signature). The theory U is axiomatized by the scheme $\mathcal{S}$ if $[\mathcal{S}]$ proves the same theorems as U.

§3. Theories of number(s). We formulate theories that can be both theories of *a number* and of *all numbers*. The theory TN_0 is intended to be possibly the theory of a number. It is given as follows. We have, apart from equality, the following symbols in the signature: a constant 0, two binary relation symbols S and $\leqslant$, and two ternary relations A and M. The theory is axiomatized as follows.

tn1 $\vdash (\mathsf{S}xy \wedge \mathsf{S}uv) \to ((x = u \wedge y = v) \vee (x \neq u \wedge y \neq v))$,
tn2 $\vdash \neg\, \mathsf{S}x0$,
tn3 $\vdash x = 0 \vee \exists y\, \mathsf{S}yx$,

[6] There is an extensive literature on interpretations. Some pointers are: [14], [10], [9], and [6].

tn4 $\vdash (\mathsf{A}xyu \wedge \mathsf{A}xyv) \to u = v$,
tn5 $\vdash \mathsf{A}x0x$,
tn6 $\vdash \exists u\,(\mathsf{S}yu \wedge \mathsf{A}xuv) \leftrightarrow \exists z\,(\mathsf{S}zv \wedge \mathsf{A}xyz)$,
tn7 $\vdash (\mathsf{M}xyu \wedge \mathsf{M}xyv) \to u = v$,
tn8 $\vdash \mathsf{M}x00 \wedge \mathsf{M}0x0$,
tn9 $\vdash x \neq 0 \to (\exists u\,(\mathsf{S}yu \wedge \mathsf{M}xuv) \leftrightarrow \exists z\,(\mathsf{A}zxv \wedge \mathsf{M}xyz))$,
tn10 $\vdash x \leqslant y \leftrightarrow \exists z\,\mathsf{A}zxy$.

The assumption of $x \neq 0$ in Axiom tn9 is essential. To see that, suppose we drop it. Since we have $\mathsf{A}000$ and $\mathsf{M}0y0$, it would follow that, for some u, $\mathsf{S}yu$. This frustrates our intention to formulate a theory that can also be the theory of *a number*. Note that if we would both drop the assumption of Axiom tn9 and the second conjunct of tn8, we still would be able to prove the undesirable:

$$\exists x_1, \ldots, x_n\,(S0x_1 \wedge \cdots \wedge Sx_{n-1}x_n),$$

for every $n > 0$.

Our system TN_0 is closely related to Hájek's variant of Grzegorczyk's weak version of Q. Modulo a few details, TN_0 is the Grzegorczyk–Hájek system where we replace the total successor function of this system by a partial successor function. See [13] for information about the Grzegorczyk–Hájek system.

We will use partial function notation. Partial terms will be eliminated via the algorithm $(\cdot)^\star$. We adopt the small scope convention and we employ weak evaluation for multiplication with 0. We first translate formulas of the form $t = x$, where x does not occur in t.

- $(y = x)^\star := (y = x)$,
- $(0 = x)^\star := (0 = x)$,
- $(\mathsf{S}t = x)^\star := \exists x_0\,((t = x_0)^\star \wedge \mathsf{S}x_0x)$,
- $(t_0 + t_1 = x)^\star := \exists x_0\,\exists z\,((t_0 = x_0)^\star \wedge (t_1 = x_1)^\star \wedge \mathsf{A}x_0x_1x)$,
- $(t_0 \times t_1 = x)^\star = \exists x_0\,((t_0 = x_0)^\star \wedge x_0 = 0 \wedge x = 0) \vee$
$\exists x_1\,((t_1 = x_1)^\star \wedge x_1 = 0 \wedge x = 0) \vee$
$\exists x_0\,\exists x_1\,((t_0 = x_0)^\star \wedge (t_1 = x_1)^\star \wedge \mathsf{M}x_0x_1x)$.

We now define the full translation for all formulas not of the form $t = x$, where x is not in t, as follows:

- $(t_0 = t_1)^\star := (\exists x_0\,\exists x_1\,((t_0 = x_0)^\star \wedge (t_1 = x_1)^\star \wedge x_0 = x_1))$,
- $(t_0 \leqslant t_1)^\star := (\exists x_0\,\exists x_1\,((t_0 = x_0)^\star \wedge (t_1 = x_1)^\star \wedge x_0 \leqslant x_1))$,
- $(\cdot)^\star$ commutes with the propositional connectives and quantifiers.

We use $t\downarrow$ for: $\exists x\,(t = x)^\star$; in words: t is defined. We write $t\uparrow$ for: t is not defined. We use $t \simeq u$ for: $\forall x\,(t = x \leftrightarrow u = x)^\star$; in words: either t and u are both defined and their values are equal, or both are undefined. We use $+$ for A and $\times$ or $\cdot$ for M. In these notations we have the following consequences of our axioms:

tnf1 $\vdash t = u \to t\downarrow \wedge\, u\downarrow$,
tnf2 $\vdash 0\downarrow$,
tnf3 $\vdash \mathsf{S}t\downarrow \to t\downarrow$,
tnf4 $\vdash \mathsf{S}x = \mathsf{S}u \to x = u$,
tnf5 $\vdash \mathsf{S}x \neq 0$,
tnf6 $\vdash x = 0 \vee \exists y\, \mathsf{S}y = x$,
tnf7 $\vdash t + u\downarrow \to (t\downarrow \wedge\, u\downarrow)$,
tnf8 $\vdash x + 0 = x$,
tnf9 $\vdash x + \mathsf{S}y \simeq \mathsf{S}(x + y)$,
tnf10 $\vdash t \times u\downarrow \to (t = 0 \vee u = 0 \vee (t\downarrow \wedge\, u\downarrow))$,
tnf11 $\vdash x \times 0 = 0 \wedge 0 \times x = 0$,
tnf12 $\vdash x \times \mathsf{S}y \simeq (x \times y) + x$,
tnf13 $\vdash x \leqslant y \leftrightarrow \exists z\, z + x = y$.

It is easy to see that these principles follow indeed from the axioms. Note that to prove tnf12, we need to split into the cases whether x is or is not 0 and whether y is or is not 0. The antecedent of tn9 disappears in tnf12, since our conventions do not demand the successor to be defined in that case. Note that, conversely, we can derive our original axioms from the new ones except tn8 and part of tn9.

The disappearance of the assumption of tn9 is the main point of the use of the 'weak convention'. It allows us to compute with undefined terms in the usual way, giving us, e.g., Lemma 3.1(4).

We write $\underline{n}$ for $\mathsf{S}^n 0$. The theory TN_n is $\mathsf{TN}_0 + \underline{n}\downarrow$. The theory TN is the union of the TN_n.

Consider any natural number n. We define 0, S, A, M, $\leqslant$ on $\{0, \ldots, n\}$ in the obvious way, e.g., $\mathsf{A}xyz$ means: $x + y = z$ and $x, y, z \leqslant n$. We call this model $\mathbb{N}_n$. We have $\mathbb{N}_n \models \mathsf{TN}_n$. Of course, we also have $\mathbb{N} \models \mathsf{TN}$.

Here is a lemma collecting some basic facts about the TN_n.

LEMMA 3.1. (1) *The relation $\simeq$ is TN_0-provably an equivalence relation between terms.*
(2) *Suppose t and t' are substitutable for x in Ax, then*: $\mathsf{TN}_0 \vdash t \simeq t' \to (At \leftrightarrow At')$.
(3) *Suppose $t\vec{m} = n$. Then,* $\mathsf{TN}_0 \vdash t\underline{\vec{m}} \simeq \underline{n}$.
(4) *Suppose $t\vec{m} = t'\vec{m}$. Then,* $\mathsf{TN}_0 \vdash t\underline{m} \simeq t'\underline{\vec{m}}$.
(5) *Suppose $t\vec{m} \leqslant t'\vec{m}$. Then,* $\mathsf{TN}_0 \vdash t'\underline{\vec{m}}\downarrow \to t\underline{m} \leqslant t'\underline{\vec{m}}$.
(6) *If $t\vec{m} \neq t'\vec{m}$, then* $\mathsf{TN}_0 \vdash t\underline{\vec{m}} \neq t'\underline{\vec{m}}$.
(7) *If $t'\vec{m} < t\vec{m}$, then* $\mathsf{TN}_0 \vdash \neg\, t\underline{\vec{m}} \leqslant t'\underline{\vec{m}}$.
(8) *Suppose $t\vec{m} = t'\vec{m} \leqslant n$. Then,* $\mathsf{TN}_n \vdash t\underline{\vec{m}} = t'\underline{\vec{m}}$.
(9) *Suppose $t\vec{m} \leqslant t'\vec{m} \leqslant n$. Then* $\mathsf{TN}_n \vdash t\vec{m} \leqslant t'\vec{m}$.
(10) $\mathsf{TN}_0 \vdash \bigvee_{m<n} x = \underline{m} \vee \underline{n} \leqslant x$.
(11) $\mathsf{TN}_0 \vdash (\underline{n}\uparrow \vee\, x \leqslant \underline{n}) \leftrightarrow \bigvee_{m\leqslant n} x = \underline{m}$.
(12) $\mathsf{TN}_n \vdash x \leqslant \underline{n} \vee \underline{n} \leqslant x$.
(13) $\mathsf{TN}_0 \vdash (x \leqslant \underline{n} \wedge \underline{n} \leqslant x) \to x = \underline{n}$.

(14) $\mathsf{TN}_0 \vdash (\underline{n} \leqslant x \wedge x \leqslant y) \to \underline{n} \leqslant y$.
(15) $\mathsf{TN}_0 \vdash (x \leqslant \underline{n} \wedge \underline{n} \leqslant y) \to x \leqslant y$.
(16) $\mathsf{TN}_0 \vdash (x \leqslant y \wedge y \leqslant \underline{n}) \to x \leqslant \underline{n}$.

PROOF. The verifications of the consequences are routine. We just provide a typical example. We prove (10) by induction on n. The case that $n = 0$ is easy. Suppose $n = k + 1$ and we have $\mathsf{TN}_0 \vdash \bigvee_{m<k} x = \underline{m} \vee \underline{k} \leqslant x$. Reason in TN_0. If we have $\bigvee_{m<k} x = \underline{m}$, then we also have $\bigvee_{m<n} x = \underline{m}$ and we are done. Suppose we have $\underline{k} \leqslant x$. So, for some u, $u + \underline{k} = x$. In case $u = 0$, we find $0 + \underline{k} = x$ and hence, by an easy argument, $\underline{k} = x$ and, thus, $\bigvee_{m<n} x = \underline{m}$. In case $u = \mathsf{S}v$, we find $\mathsf{S}v + \underline{k} = x$. By an easy argument this gives $v + \underline{n} = x$, so $\underline{n} \leqslant x$. ⊣

LEMMA 3.2. *Suppose $A\vec{x}$ is a Δ_0-formula (in the sense of our logic with partial terms). There is a term $t_A\vec{x}$, which majorizes every x in $\vec{x}$, such that, for every n and for every $\vec{m}$ with $t_A\vec{m} \leqslant n$, we have:*

(1) *If $A\vec{m}$, then $\mathsf{TN}_n \vdash A\underline{\vec{m}}$.*
(2) *If $\neg A\vec{m}$, then $\mathsf{TN}_n \vdash \neg A\underline{\vec{m}}$.*

PROOF. The proof is by induction, simultaneous in both cases, on the complexity of A. Suppose A is $t\vec{x} = t'\vec{x}$. Then, we can take $t_A\vec{x} := t\vec{x} + t'\vec{x}$. The cases of the Boolean connectives are obvious.

Suppose $A\vec{x}$ is $\forall y{\leqslant}t\vec{x}\, B\vec{x}y$. We take $t_A\vec{x} := t\vec{x} + t_B(\vec{x}, t\vec{x})$. Suppose $t_A\vec{m} \leqslant n$. Let $t\vec{m} = k$. Since, $t\vec{m} \leqslant n$, we find, by Lemma 3.1(8), $\mathsf{TN}_n \vdash t\underline{\vec{m}} = \underline{k}$. By Lemma 3.1(11), we find:

$$\mathsf{TN}_n \vdash A\vec{m} \leftrightarrow \bigwedge_{p \leqslant k} B(\underline{\vec{m}}, \underline{p}).$$

Since, $t_B(\vec{m}, p) \leqslant t_B(\vec{m}, k) \leqslant t_A(\vec{m}) \leqslant n$, the desired result is immediate from the Induction Hypothesis. The case of the bounded existential quantifier is similar. ⊣

The next definition will play a fundamental role in this paper.

DEFINITION 3.1. Suppose $S\vec{x}$ is a Σ_1-formula. Say, $S\vec{x} = \exists\vec{y}\, S_0\vec{x}\vec{y}$, where $S_0\vec{x}\vec{y}$ is Δ_0. We define $S^z\vec{x}$, by $\exists\vec{y}{\leqslant}z\, (t_{S_0}\vec{x}\vec{y} \leqslant z \wedge S_0\vec{x}\vec{y})$. Here t_{S_0} is the term provided by Lemma 3.2.

We have the following lemma.

LEMMA 3.3. *Let $S\vec{x}$ be a Σ_1-formula.*

(1) *We have: $S\vec{m}$ iff, for some n, $S^n\vec{m}$.*
(2) *Suppose $S^n\vec{m}$. Then, $\mathsf{TN}_n \vdash S^{\underline{n}}\underline{\vec{m}}$.*
(3) *Suppose $\neg S^n\vec{m}$. Then, $\mathsf{TN}_0 \vdash \neg S^{\underline{n}}\underline{\vec{m}}$.*

PROOF. We treat (2). Suppose $S^n\vec{m}$. So, for some $\vec{k}$, we have $t_{S_0}\vec{m}\vec{k} \leqslant n$ and $S_0\vec{m}\vec{k}$. By Lemma 3.1(9), $\mathsf{TN}_n \vdash t_{S_0}(\underline{\vec{m}}, \underline{\vec{k}}) \leqslant \underline{n}$. By Lemma 3.2, we have $\mathsf{TN}_n \vdash S_0(\underline{\vec{m}}, \underline{\vec{k}})$. Hence, $\mathsf{TN}_n \vdash S^n\underline{\vec{m}}$.

We treat (3). Suppose $\neg S^n\vec{m}$. Let $U := \mathsf{TN}_0 + S^{\underline{n}}\underline{\vec{m}}$. Note that U extends TN_n. By Lemma 3.2, we find, for $\vec{k}$ such that $t_{S_0}\vec{m}\vec{k} \leqslant n$, that $U \vdash \neg S_0(\underline{\vec{m}}, \underline{\vec{k}})$. Also, by Lemma 3.1(7), if $n < t_{S_0}\vec{m}\vec{k} \leqslant n$, then $U \vdash \neg t_{S_0}(\underline{\vec{m}}, \underline{\vec{k}}) \leqslant \underline{n}$. By Lemma 3.1(11), we find

$$U \vdash \bigvee_{\vec{k}<n} (t_{S_0}(\underline{\vec{m}}, \underline{\vec{k}}) \leqslant \underline{n} \wedge S_0(\underline{\vec{m}}, \underline{\vec{k}})).$$

Combining, we find $U \vdash \bot$. So, we may conclude that $\mathsf{TN}_0 \vdash \neg S^n\underline{\vec{m}}$. ⊣

LEMMA 3.4. *Suppose $S(\vec{x}, y)$ is a Σ_1-formula representing a function in $\mathbb{N}$. Then, there is a Σ_1-formula $S^\circ(\vec{x}, y)$, such that, for all $\vec{m}$, if $S(\vec{m}, n)$, then, for some k,*
$\mathsf{TN}_k \vdash S^\circ(\underline{\vec{m}}, \underline{n}) \wedge \forall y\, (S^\circ(\underline{\vec{m}}, y) \to \underline{n} = y)$.

PROOF. Suppose $S(\vec{x}, y)$, is $\exists \vec{z}\, S_0\vec{x}y\vec{z}$, where S_0 is in Δ_0. Let:

- $S_0^\circ(\vec{x}, y, z) :\leftrightarrow y \leqslant z \wedge \exists \vec{u} \leqslant z\, S_0\vec{x}y\vec{u} \wedge \forall \vec{v}, w \leqslant z\, (S_0\vec{x}w\vec{v} \to w = y)$.
- $S^\circ(\vec{x}, y) :\leftrightarrow \exists z\, S_0\vec{x}yz$.

Suppose $S\vec{m}n$. By the functionality of S, we can clearly find a q such that $S_0^\circ\vec{m}nq$. Let $k \geqslant t_{S_0^\circ}\vec{m}nq$. By Lemma 3.2, we have $\mathsf{TN}_k \vdash S_0^\circ(\underline{\vec{m}}, \underline{n}, \underline{q})$.

We reason in TN_k. Suppose $S_0^\circ(\underline{\vec{m}}, y, z)$. By Lemma 3.1(12), either $z \leqslant \underline{q}$ or $\underline{q} \leqslant z$.

Suppose $z \leqslant \underline{q}$. Since $S_0^\circ(\vec{m}, n, \underline{q})$, for every $\vec{v}, w \leqslant \underline{q}$, if $S_0(\underline{\vec{m}}, w, \vec{v})$, then $w = \underline{n}$. On the other hand, by $S_0^\circ(\underline{\vec{m}}, y, z)$, we have $y \leqslant z$ and, for some $\vec{u} \leqslant z$, $S_0(\underline{\vec{m}}, y, \vec{u})$. By Lemma 3.1(16), we find $\vec{u}, y \leqslant \underline{q}$. So, it follows that $y = \underline{n}$.

Suppose $\underline{q} \leqslant z$. Since $S_0^\circ(\underline{\vec{m}}, y, z)$, for every $\vec{v}, w \leqslant z$, if $S_0(\underline{\vec{m}}, w, \vec{v})$, then $w = y$. On the other hand, by $S_0^\circ(\underline{\vec{m}}, \underline{n}, \underline{q})$, we have $\underline{n} \leqslant \underline{q}$ and, for some $\vec{u} \leqslant \underline{q}$, $S_0(\underline{\vec{m}}, n, \vec{u})$. By Lemma 3.1(15), we find $\vec{u}, \underline{n} \leqslant z$. So, $\underline{n} = y$. ⊣

LEMMA 3.5 (Gödel Fixed Point Lemma). *Suppose that K is an interpretation of TN_0 in U. Let $Ax\vec{y}$ be a formula in the language of U. Suppose that*:

$$U \vdash Ax\vec{y} \to \delta_K(x) \text{ and } U \vdash x_0 =_K x_1 \to (Ax_0\vec{y} \leftrightarrow Ax_1\vec{y}).$$

Then, there is a formula $B\vec{v}$ and an n, such that

$$U + (\underline{n}\downarrow)^K \vdash B\vec{z} \leftrightarrow \exists x\, ((x = \ulcorner B\vec{v} \urcorner)^K \wedge Ax\vec{z}).$$

PROOF. Suppose sub is a Σ_1-representation of the Gödel substitution function or the language of U, such that

$$\mathsf{sub}(n, \ulcorner Cx\vec{y} \urcorner, \ulcorner \exists x\, (\delta_K(x) \wedge (x = \underline{n})^K \wedge Cx\vec{y} \urcorner)),$$

where x is a designated variable. Let:

- $A^*y\vec{v} :\leftrightarrow \exists x\, (\delta_K(x) \wedge (\mathsf{sub}^\circ(y, y, x))^K \wedge Ax\vec{v})$,
- $m := \ulcorner A^*u\vec{v} \urcorner$,
- $B\vec{v} :\leftrightarrow \exists y\, (\delta_K(y) \wedge (\underline{m} = x)^K \wedge A^*y\vec{v})$.
- $k := \ulcorner B\vec{v} \urcorner$.

By Lemma 3.4, we can find n such that:

$$\mathsf{TN}_n \vdash \mathsf{sub}^\circ(\underline{m}, \underline{m}, \underline{k}) \wedge \forall y\,(\mathsf{sub}^\circ(\underline{m}, \underline{m}, y) \to \underline{k} = y).$$

Thus, we have, in $U + (\underline{n}\downarrow)^K$:

$$\begin{aligned} B\vec{z} &\leftrightarrow \exists y\,(\delta_K(y) \wedge (\underline{m} = y)^K \wedge \exists x\,(\delta_K(x) \wedge (\mathsf{sub}^\circ(y, y, x))^K \wedge Ax\vec{z})) \\ &\leftrightarrow \exists x\,\exists y\,(\delta_K(x) \wedge \delta_K(y) \wedge (\underline{m} = y \wedge \mathsf{sub}^\circ(y, y, x))^K \wedge Ax\vec{z}) \\ &\leftrightarrow \exists x\,(\delta_K(x) \wedge (\underline{k} = x)^K \wedge Ax\vec{z}) \end{aligned}$$ ⊣

§4. Sets. We define a theory of sets and numbers TS_0 as follows. Our signature consists of two unary symbols N, Z, five binary symbols S, $\leqslant$, E, $\in$, η, and two ternary symbols A and M. The intended interpretation of η is the relation between a set and its cardinality. However, our axioms do not go far in constraining η to be so.

We let $a, b, c, \ldots$ range over N. This means that if, e.g., 'a' occurs freely in a formula, then we assume that $\mathsf{N}(a)$ is present in the context. The formula $\exists a\, A$ means $\exists a\,(\mathsf{N}a \wedge A)$, and similarly for the universal quantifier. We use $x, y, z, w, \ldots$ to range over all objects.[7]

We note that axioms ts1–ts4 are 'administrative' axioms about the proper behavior of Z, E, etc., and that ts5–ts14 are just translations of the axioms of TN_0 into the new language. Thus, the really new axioms are ts15 and ts16.

- ts1 Axioms stating that Z, E, S, $\leqslant$, A, M are relations on N. E.g., $\vdash \mathsf{A}xyz \to (\mathsf{N}x \wedge \mathsf{N}y \wedge \mathsf{N}z)$.
  An axiom stating that η is a relation on N in the second component, i.e., $\vdash x\;\eta\;y \to \mathsf{N}y$,
- ts2 Axioms stating that E is an equivalence relation of N which is a congruence w.r.t. Z, S, $\leqslant$, A, M. An axiom stating that E is a congruence w.r.t η in the second component, i.e., $\vdash (x\;\eta\;a \wedge a\;\mathsf{E}\;b) \to x\;\eta\;b$,
- ts3 $\vdash \exists a\,\mathsf{Z}a$,
- ts4 $\vdash (\mathsf{Z}a \wedge \mathsf{Z}b) \to a\;\mathsf{E}\;b$,
- ts5 $\vdash (\mathsf{S}ab \wedge \mathsf{S}cd) \to ((a\;\mathsf{E}\;c \wedge b\;\mathsf{E}\;d) \vee (\neg\, a\;\mathsf{E}\;c \wedge \neg\, b\;\mathsf{E}\;d))$,
- ts6 $\vdash \neg\,(\mathsf{S}ab \wedge \mathsf{Z}b)$,
- ts7 $\vdash \mathsf{Z}a \vee \exists b\,\mathsf{S}ba$
- ts8 $\vdash (\mathsf{A}abc \wedge \mathsf{A}abd) \to c\;\mathsf{E}\;d$,
- ts9 $\vdash \mathsf{Z}b \to \mathsf{A}aba$,
- ts10 $\vdash \exists d\,(\mathsf{S}bd \wedge \mathsf{A}ade) \leftrightarrow \exists c\,(\mathsf{S}ce \wedge \mathsf{A}abc)$,
- ts11 $\vdash (\mathsf{M}abc \wedge \mathsf{M}abd) \to c\;\mathsf{E}\;d$,
- ts12 $\vdash \mathsf{Z}b \to (\mathsf{M}abb \wedge \mathsf{M}bab)$,
- ts13 $\vdash \neg\,\mathsf{Z}a \to (\exists d\,(\mathsf{S}bd \wedge \mathsf{M}ade) \leftrightarrow \exists c\,(\mathsf{A}cae \wedge \mathsf{M}abc))$,
- ts14 $\vdash a \leqslant b \leftrightarrow \exists c\,\mathsf{A}cab$.

[7] It would be more natural to give TS_0 a two-sorted formulation. However, that option is blocked by our implicit design decision to have only one-sorted schemes for a one-sorted theory.

ts15 $\vdash \mathsf{Z}a \to \exists x\,(x\ \eta\ a \wedge \forall y\ y \notin x)$,
ts16 $\vdash (x\ \eta\ a \wedge \mathsf{S}ab) \to$
$$\forall u\,(u \notin x \to \exists y\,(y\ \eta\ b \wedge \forall v\,(v \in y \leftrightarrow (v \in x \vee v = u)))).$$

Let us define:

- $\mathsf{empty}(x) :\leftrightarrow \forall y\ y \notin x$,
- $\mathsf{adj}_u(x, y) :\leftrightarrow \forall v\,(v \in y \leftrightarrow (v \in x \vee v = u))$,

With these definitions, our new axioms read:

- $\vdash \mathsf{Z}a \to \exists x\,(x\ \eta\ a \wedge \mathsf{empty}(x))$,
- $\vdash (x\ \eta\ a \wedge \mathsf{S}ab) \to \forall u\,(u \notin x \to \exists y\,(y\ \eta\ b \wedge \mathsf{adj}_u(x, y)))$.

Par abus de langage, we suppress the standard interpretation of TN_0 in TS_0. We define $\mathsf{TS}_n := \mathsf{TS}_0 + \underline{n}\!\downarrow$

THEOREM 4.1. *The theory* TS_n *is directly interpretable in* VS_2 *via a one dimensional interpretation without parameters. If follows that* TS, *the union of the* TS_n, *is locally directly interpretable in* VS_2.

PROOF. Let k be the minimal number such that $2^k \geqslant n$. We define:

- $x \in^0 y :\leftrightarrow x = y$, $x \in^{j+1} y :\leftrightarrow \exists z\,(x \in z \wedge z \in^j y)$.
- $\mathsf{set}(y) :\leftrightarrow \forall u_0, \dots, u_n\,(\bigwedge_{i \leqslant n} u_i \in^k y \to \bigvee_{i<j\leqslant n} u_i = u_j)$.
- For $j \leqslant n$,
  $j(x) :\leftrightarrow \mathsf{set}(x) \wedge \exists u_0, \dots u_{j-1}\,(\bigwedge_{i<j} u_i \in^k x \wedge$
  $$\bigwedge_{s<i<j} u_s \neq u_i \wedge \forall v{\in^k}x\ \bigvee_{i<j} v = u_i).$$

We note that, under the obvious conventions, $0(x)$ is equivalent to $\forall v\ v \notin x$. We define a direct translation τ_n as follows.

- $\mathsf{N}_{\tau_n}(x) :\leftrightarrow x = x$,
- $\mathsf{Z}_{\tau_n}x :\leftrightarrow 0(x)$,
- $\mathsf{S}_{\tau_n}xy :\leftrightarrow \bigvee_{j<n}(j(x) \wedge (j+1)(y))$,
- $\mathsf{A}_{\tau_n}xyz :\leftrightarrow \bigvee_{s+i=j\ \&\ j\leqslant n}(s(x) \wedge i(y) \wedge j(z))$,
- $\mathsf{M}_{\tau_n}xyz :\leftrightarrow \bigvee_{s\times i=j\ \&\ s,i,j\leqslant n}(s(x) \wedge i(y) \wedge j(z))$,
- $x \leqslant_{\tau_n} y :\leftrightarrow \bigvee_{i\leqslant j\leqslant n}(i(x) \wedge j(y))$,
- $x\ \mathsf{E}_{\tau_n}\ y :\leftrightarrow \bigvee_{i\leqslant n}(i(x) \wedge i(y))$,
- $x \in_{\tau_n} y :\leftrightarrow x \in^k y \wedge \mathsf{set}(y)$,
- $x\ \eta_{\tau_n}\ y :\leftrightarrow x\ \mathsf{E}_{\tau_n}\ y$.

Note that in VS_2 we can prove that there are infinitely many objects, by considering iterated singletons on empty sets. So we can show in VS_2 that n (as interpreted by τ_n) exists. E.g., in case $n = 7$, we produce an empty set a_0 and singletons $a_1, \dots, a_6$, where $a_i \in a_{i+1}$. We may show that these are pairwise distinct. If we had extensionality, a representative of the cardinal 7 would be:

$$\{\{\{a_0, a_1\}, \{a_2, a_3\}\}, \{\{a_4, a_5\}, \{a_6\}\}\}.$$

Without extensionality, a representative looks as follows. We find $b_0, \ldots, b_3$, where b_0, b_1, b_2 are unordered pairs, b_3 is a singleton and $a_0, a_1 \in b_0$, $a_2, a_3 \in b_1$, $a_4, a_5 \in b_2$ and $a_6 \in b_3$. Note that the b_i are pairwise distinct. Next we find unordered pairs c_0 and c_1 such that $b_0, b_1 \in c_0$ and $b_2, b_3 \in c_1$. Note that $c_0 \neq c_1$. Finally, we find an unordered pair d with $c_0, c_1 \in d$. The object d will be a set with 7 elements under τ_7. Note that $k = 3$.

Using these ideas we may easily show that τ_n yields an interpretation of TS_n in VS_2. ⊣

REMARK 4.2. The interpretation constructed in the preceding proof is highly inefficient in terms of the size of the interpreting formulas as a function of n. Employing the usual definitions of set theory we can interpret TS_n in far more efficient ways. Also, we can do much better in defining $\in^k$. Such efficiency could be necessary if we want to verify our theorem in S^1_2. Also, it would be quite interesting to try to adapt our methods to reprove the lower bound result of Ferrante and Rackoff (see [5]). This would also ask for more efficient interpretations. However, for our present purposes such efficiency is not needed.

§5. Satisfaction.

Consider any theory U. We extend the language of U with the vocabulary of TS_0 plus a new binary predicate sat. We define the formula SAT that says roughly that 'below c' the predicate sat behaves as a satisfaction predicate with the first argument in the role of assignment and the second argument in the role of formula. To keep things readable we assume that the language of U has a unary predicate P, a binary predicate Q and that the only propositional connectives are the 0-ary connectives $\top$ and $\bot$, negation and conjunction and that the only quantifier is universal quantification.

We may find Σ_1-formulas defining some useful predicates:

- $\mathsf{form}(a)$: a codes a formula.
- $\mathsf{var}(b)$: b codes a variable.
- $\mathsf{fv}(b, a)$: a codes a formula and b codes a free variable occurring in a.
- $[\![P]\!](b, a)$: b codes a variable 'v' and a codes 'Pv'. Similarly for $[\![Q]\!](b, a)$.
- $[\![\top]\!](a)$: a codes '$\top$'. Similarly for $\bot$.
- $[\![\neg]\!](b, a)$: b codes a formula B and a codes $\neg B$.
- The other cases are as expected.

In the definitions given below, we will employ the '$S^z\vec{x}$' notation of Definition 3.1.

- $x \subseteq y :\leftrightarrow \forall z\, (z \in x \to z \in y)$.
- $\mathsf{pair}(x, y, z) :\leftrightarrow \exists u, v\, \forall w\, ((w \in z \leftrightarrow (w = u \vee w = v)) \wedge$
  $(w \in u \leftrightarrow w = x) \wedge (w \in v \leftrightarrow (w = x \vee w = y)))$.
- $\mathsf{rel}(x) :\leftrightarrow \forall y {\in} x\, \exists u, v\, \mathsf{pair}(u, v, y)$.

$$
\begin{aligned}
&\mathsf{SAT}(c,\mathsf{sat}) :\leftrightarrow \forall x\,\forall a{\leqslant}c\,[(\mathsf{form}^c(a) \wedge \mathsf{ass}(c,x,a)) \to \\
&\quad \{\mathsf{sat}(x,a) \leftrightarrow \langle \exists b{<}c\,(\,[\![P]\!]^c(b,a) \wedge P(x(b))) \vee \\
&\qquad \exists b_0,b_1{<}c\,(\,[\![Q]\!]^c(b_0,b_1,a) \wedge Q(x(b_0),x(b_1))) \vee \\
&\qquad [\![\top]\!]^c(a) \vee \\
&\qquad \exists b{<}c\,(\,[\![\neg]\!]^c(b,a) \wedge \neg\,\mathsf{sat}(x,b)) \vee \\
&\qquad \exists b_0,b_1{<}c\,\exists x_0,x_1\,(x_0 \subseteq x \wedge x_1 \subseteq x \wedge \\
&\qquad\qquad \mathsf{ass}(c,x_0,b_0) \wedge \mathsf{ass}(c,x_1,b_1) \wedge \\
&\qquad\qquad [\![\wedge]\!]^c(b_0,b_1,a) \wedge \\
&\qquad\qquad \mathsf{sat}(x_0,b_0) \wedge \mathsf{sat}(x_1,b_1)) \vee \\
&\qquad \exists b_0,b_1{<}a\,(\,[\![\forall]\!]^c(b_0,b_1,a) \wedge \\
&\qquad\qquad \forall w\,(x[b_0]w \to \mathsf{sat}(w,b_1)))\rangle\}]
\end{aligned}
$$

FIGURE 1. Definition of “sat is a satisfaction predicate below c”.

- $\mathsf{fun}(x) :\leftrightarrow \forall y,z{\in}x\,\forall u,v,w\,((\mathsf{pair}(u,v,y) \wedge \mathsf{pair}(u,w,z)) \to y = z)$.[8]
- $x(y) = z :\leftrightarrow \mathsf{fun}(x) \wedge \exists u{\in}x\,\mathsf{pair}(y,z,u)$.
- $\mathsf{dom}(y,x) :\leftrightarrow \exists z\,x(y) = z$.
- $\mathsf{funn}(x) :\leftrightarrow \mathsf{fun}(x) \wedge \forall y\,(\mathsf{dom}(y,x) \to \mathsf{N}(y)) \wedge$
  $\forall a,b,u,v\,((a \mathrel{\mathsf{E}} b \wedge x(a) = u \wedge x(b) = v)$
  $\to (a = b \wedge u = v))$.
- $x \star a = y :\leftrightarrow \mathsf{funn}(x) \wedge \exists b\,(a \mathrel{\mathsf{E}} b \wedge x(b) = y)$.
- $\mathsf{ass}(c,x,a) :\leftrightarrow \mathsf{funn}(x) \wedge \forall b\,(\mathsf{dom}(b,x) \leftrightarrow \mathsf{var}^c(b))$.
- $x[a]y :\leftrightarrow \mathsf{funn}(x) \wedge \mathsf{funn}(y) \wedge \forall b\,(\neg\, a \mathrel{\mathsf{E}} b \to x \star b \simeq y \star b) \wedge y \star a \downarrow$.

The definition of the ‘this is a satisfaction predicate below c’-predicate SAT is given in Figure 5.

In the next theorem, we will consider a theory U in which TS_0 is interpretable via a direct interpretation K, i.o.w., $K\colon U \rhd_{\mathsf{dir}} \mathsf{TS}_0$. There is the obvious interpretation, say N, of TN_0 in TS_0. So, $M := K \circ N$ interprets TN_0 in U. In our development we will suppress both K and M. So, when we speak, e.g., about $\ulcorner A \urcorner$, this is an object determined modulo E_K.

THEOREM 5.1. *Suppose we have an interpretation $K\colon U \rhd_{\mathsf{dir}} \mathsf{TS}_0$. Let A be $A(v_0,\dots v_{k-1})$. Suppose we have $\mathsf{form}^n(\ulcorner A \urcorner)$ in $\mathbb{N}$.*[9] *We assume that $n \geqslant 2$. We have*:

$$
U + \underline{n} \leqslant b + \mathsf{SAT}(b,B) + \mathsf{ass}(b,x,\ulcorner A \urcorner) \vdash A(x \star \ulcorner v_0 \urcorner,\dots,x \star \ulcorner v_i \urcorner) \leftrightarrow B(x,\ulcorner A \urcorner).
$$

[8]We have ‘$y = z$’ rather than ‘$v = w$’ in the consequent, because our pairing is non-functional. To have lots of copies of the same pair in the representation of a function seems a bit wasteful.

[9]We employ the superscript notation introduced in Definition 3.1.

PROOF. We proceed by induction on A. We treat the case that $A = \forall v\, C$, where C is $C(v, v_0, \ldots, v_{k-1})$.

We have, in $\mathbb{N}$, $\mathsf{form}^n(\ulcorner A \urcorner)$ and, hence, $\mathsf{form}^n(\ulcorner C \urcorner)$, $[\![\forall]\!]^n(\ulcorner v \urcorner, \ulcorner C \urcorner, \ulcorner A \urcorner)$ and $\mathsf{fv}^n(\ulcorner x \urcorner)$. Let V_b be the theory $U + \underline{n} \leqslant b + \mathsf{SAT}(b, B) + \mathsf{ass}(b, x, \ulcorner A \urcorner)$. Note that '$b$' occurs as a free parameter in V_b. In V_b, we have $\mathsf{form}^b(\ulcorner A \urcorner)$, $\mathsf{form}^b(\ulcorner C \urcorner)$, $[\![\forall]\!]^b(\ulcorner v \urcorner, \ulcorner C \urcorner, \ulcorner A \urcorner)$ and $\mathsf{fv}^b(\ulcorner v \urcorner)$. The induction hypothesis gives us:

$$\begin{aligned} V_b \vdash\ & \mathsf{ass}(b, w, \ulcorner C \urcorner) \\ & \to (B(b, w, \ulcorner C \urcorner) \leftrightarrow C(w \star \ulcorner v \urcorner, w \star \ulcorner v_0 \urcorner, \ldots, w \star \ulcorner v_{k-1} \urcorner)). \quad (\dagger) \end{aligned}$$

We work in V_b. Suppose $B(b, x, \ulcorner A \urcorner)$. It follows that

$$\forall w\, x[\ulcorner v \urcorner]w\, B(b, w, \ulcorner C \urcorner).$$

Consider any z. Since $n > 2$, we have that $\underline{2}$ exists. So we have unordered pairing. Hence, we can construct an ordered pair p such that $\mathsf{pair}(\ulcorner v \urcorner z, p)$. Since the number of free variables in C is smaller than $\ulcorner A \urcorner$ and $\ulcorner A \urcorner$ exists, we can, using repeated adjunction, find a y such that on $\ulcorner v_i \urcorner$, for $i < k$, we have $y \star \ulcorner v_i \urcorner = x \star \ulcorner v_i \urcorner$, and $y \star \ulcorner v \urcorner = z$. It is easy to see that $\mathsf{ass}(b, y, \ulcorner C \urcorner)$ and $x[\ulcorner v \urcorner]y$. By $(\dagger)$, we find $C(y \star \ulcorner v \urcorner, y \star \ulcorner v_0 \urcorner, \ldots, y \star \ulcorner v_{k-1} \urcorner)$. We may conclude that

$$C(z, x \star \ulcorner v_0 \urcorner, \ldots, x \star \ulcorner v_{k-1} \urcorner).$$

Since z was arbitrary, it follows that $\forall z\, C(z, x \star \ulcorner v_0 \urcorner, \ldots, x \star \ulcorner v_{k-1} \urcorner)$, in other words, $A(x \star \ulcorner v_0 \urcorner, \ldots, x \star \ulcorner v_{k-1} \urcorner)$.

Conversely, suppose $A(x \star \ulcorner v_0 \urcorner, \ldots, x \star \ulcorner v_{k-1} \urcorner)$. This means:

$$\forall z\, C(z, x \star \ulcorner v_0 \urcorner, \ldots, x \star \ulcorner v_{k-1} \urcorner).$$

Consider a y with $x[\ulcorner v \urcorner]y$. We find: $C(y \star \ulcorner v \urcorner, y \star \ulcorner v_0 \urcorner, \ldots, y \star \ulcorner v_{k-1} \urcorner)$. Hence, by $(\dagger)$, $B(b, y, \ulcorner C \urcorner)$. Thus, we find that, for all y with $x[\ulcorner v \urcorner]y$, we have $B(b, y, \ulcorner C \urcorner)$. It follows that $B(b, x, \ulcorner A \urcorner)$. ⊣

THEOREM 5.2. *Let K be any direct interpretation of TS_0 in U, possibly with free parameters $z_0, \ldots, z_{p-1}$. Fix any U-formula $A(b, x, a, \vec{z})$. Then, for some n, we have*: $U + \mathsf{SAT}(b, A) \vdash \bigvee_{i \leqslant n} b = \underline{i}$.

PROOF. By Lemma 3.5, for some n, and for some formula $B\vec{v}$ with Gödel number m:

$$U + \underline{n}\downarrow \vdash B\vec{y} \leftrightarrow \exists a, s\, (\mathsf{ass}(b, s, a) \wedge \underline{m} = a \wedge \bigwedge_{i<p} s \star \ulcorner z_i \urcorner = z_i \wedge \neg A(s, a, \vec{z})).$$

We take n so big that also $U + \underline{n}\downarrow \vdash \mathsf{form}^{\underline{n}}(\underline{m})$.

Reason in U. Suppose $\mathsf{SAT}(b, A(x, a, \vec{z}))$. Suppose $\underline{n} \leqslant b$. Then we have, by Theorem 5.1,

$$B\vec{z} \leftrightarrow \exists a, s\, ((\underline{m} = a)^M \wedge \bigwedge_{i<p} s(\ulcorner z_i \urcorner) = z_i \wedge \neg A(s, a, \vec{z}))$$
$$\leftrightarrow \neg B\vec{z}.$$

We may conclude, by Lemma 3.1(10), that $U + \mathsf{SAT}(b, A) \vdash \bigvee_{i<n} b = \underline{i}$. ⊣

§6. **Proof of the main result.** Consider any theory U. Let α be a Σ_1-formula defining the set of Gödel numbers of the axioms of U. We assume that $\alpha^n(m)$ implies $\mathsf{form}^n(m)$. We define the following scheme (for the signature of U):

- $\mathcal{S}_\alpha := \bigwedge \mathsf{TS}_0 \to \forall a, b\, ((\alpha^b(a) \wedge \mathsf{SAT}(b, \mathsf{sat})) \to \exists s\, (\mathsf{empty}(s) \wedge \mathsf{sat}(b, s, a)))$.

The scheme $\mathcal{S}_\alpha$ will be the main ingredient of the scheme we will give to axiomatize a given RE pair theory. Note that our scheme both contains schematic predicates and predicates from the signature of U. We prove that U satisfies $\mathcal{S}_\alpha$.

THEOREM 6.1. *Suppose the axiom set of U is given by the Σ_1-formula α. Then, U validates $\mathcal{S}_\alpha$.*

PROOF. Consider any direct interpretation K (possibly with free parameters $\vec{z}$) for the language of TS_0. In the following we will suppress K. Let $A := A(b, s, a, \vec{z})$ be any formula. Let $V_{a,b} := U + \mathsf{TS}_0 + \alpha^b(a) + \mathsf{SAT}(b, A)$. By Theorem 5.2, we find, for some n, $V_{a,b} \vdash \bigvee_{j<n} b = \underline{j}$. It follows that $V_{a,b} \vdash \bigvee_{i<n} a = \underline{i}$. Fix $i, j < n$. If we have $\neg \alpha^j(i)$, then, by Lemma 3.3, we find $V_{a,b} \vdash \neg \alpha^{\underline{j}}(\underline{i})$. So, it follows that $V_{a,b} \vdash \neg(a = \underline{i} \wedge b = \underline{j})$ and a fortiori,

$$V_{a,b} + a = \underline{i} + b = \underline{j} \vdash \exists s\, (\mathsf{empty}(s) \wedge \mathsf{A}(b, s, a))).$$

If we have $\alpha^i(j)$, then we find, by Lemma 3.3, $V_{a,b} \vdash \alpha^{\underline{j}}(\underline{i})$. Suppose i is the Gödel number of C. Since, $U \vdash C$, we find, by Theorem 5.1,

$$V_{a,b} + a = \underline{i} + b = \underline{j} \vdash \exists s\, (\mathsf{empty}(s) \wedge A(b, s, a))).$$

So, we obtain $V_{a,b} \vdash \exists s\, (\mathsf{empty}(s) \wedge A(b, s, a))$. ⊣

Note the generality of the previous theorem: for every RE theory U we produced a scheme of a certain form only depending on the formula α and the signature of U, such that U validates that scheme.

THEOREM 6.2. *Consider a theory U whose axiom set is given by the Σ_1-formula α. We suppose that we have a direct interpretation M, of VS_2 in U. Then, $\mathsf{VS}_2^M + [\mathcal{S}_\alpha]$ proves the same theorems as U.*[10] *Thus, U is schematically axiomatized by the scheme $\bigwedge \mathsf{VS}_2^M \wedge \mathcal{S}_\alpha$.*

[10] Remember that we write $[\mathcal{S}_\alpha]$ for the set of instances of $\mathcal{S}_\alpha$ in the signature of U.

The sentence $\bigwedge \mathsf{VS}_2^M$ is in the *non-schematic* vocabulary and $\mathcal{S}_\alpha$ consists of a mixture of *non-schematic* vocabulary and *schematic* vocabulary. Thus, the instances of the scheme $\bigwedge \mathsf{VS}_2^M \wedge \mathcal{S}_\alpha$ are of the form $\bigwedge \mathsf{VS}_2^M \wedge A$, where A is in $[\mathcal{S}_\alpha]$.

PROOF. The fact that U proves all principles of $V := \mathsf{VS}_2^M + [\mathcal{S}_\alpha]$ is immediate by Theorem 6.1. We prove the converse.

Consider an axiom A of U and suppose $\alpha^n(\ulcorner A \urcorner)$. Theorem 4.1 provides a direct interpretation K of TS_n in VS_2. So, $L := K \circ M$ is a direct interpretation of TS_n in U. We define a satisfaction predicate $B(x, c)$, for all formulas C, such that $\mathsf{form}^n(\ulcorner C \urcorner)$, just by making it the disjunction of all formulas:

- $(c = \ulcorner C \urcorner) \wedge \mathsf{ass}(\underline{n}, x, \ulcorner C \urcorner) \wedge C(x \star \ulcorner v_0 \urcorner, \ldots, x \star \ulcorner v_{k-1} \urcorner)$, where $C = C(v_0, \ldots, v_{k-1})$ and $\mathsf{form}^n(\ulcorner C \urcorner)$.[11]

We find in V that, under interpretation L, we have $\mathsf{SAT}(\underline{n}, \ulcorner B \urcorner)$. Let s be any empty set. By $\mathcal{S}_\alpha$, instantiated via L for the signature of VS_n and by substituting B for sat, we find: $V \vdash B(\underline{n}, s, \ulcorner A \urcorner)$. Hence, $V \vdash A$. $\dashv$

We note that the result works both for the case that we consider closed schemes or schemes with parameters.

If M and N are both direct interpretations of VS_2 in U, then, by Theorem 6.2, we have: $[\mathcal{S}_\alpha] \vdash \bigwedge \mathsf{VS}_2^M \leftrightarrow \bigwedge \mathsf{VS}_2^N$. So, in a sense, the choice of the direct interpretation of VS_2 is immaterial.

Where is the main result, Theorem 6.2, verifiable? We freely used big disjunctions (of the form $\bigvee_{i<n} C_i$) and big conjunctions (of the form $\bigwedge_{i<n} C_i$), so, at least prima facie, we need exponentiation. Inspection of our proofs shows that modulo one subtlety the proof works in EA, aka $\mathrm{I}\Delta_0 + \mathsf{exp}$. The subtlety is as follows. In Theorem 6.1, we verified that each *instance of the scheme* $\mathcal{S}_\alpha$ is validated in U. The proof of Theorem 6.2 showed that every *axiom* of U is verifiable in $\mathsf{VS}_2^M + [\mathcal{S}_\alpha]$. The silent inference that *ipso facto* the theories U as given by α and $\mathsf{VS}_2^M + [\mathcal{S}_\alpha]$ as given by the scheme have the same *theorems* uses Σ_1-collection aka $\mathsf{B}\Sigma_1$. So, we have:

$$\mathsf{EA} + \mathsf{B}\Sigma_1 \vdash \forall a\, (\mathsf{prov}_\alpha(a) \leftrightarrow \mathsf{prov}_{\mathsf{VS}_2^M + [\mathcal{S}_\alpha]}(a)).$$

Ergo,

$$\mathsf{EA} + \mathsf{B}\Sigma_1 \vdash \mathsf{con}(\alpha) \leftrightarrow \mathsf{con}(\mathsf{VS}_2^M + [\mathcal{S}_\alpha]).$$

Moreover, if α is in $\Delta_1(\mathsf{EA})$, we may drop $\mathsf{B}\Sigma_1$ from our formulation.

§7. Comprehension principles. Schemes can be condensed into single sentences in the presence of suitable comprehension principles. This is illustrated by the well known cases of $\mathsf{PA}/\mathsf{ACA}_0$ and ZF/GB. These examples

[11] This is a very inefficient way, in terms of size, to make such a predicate. One can clearly do much better.

also illustrate the usefulness of this construction and the fact that, in terms of interpretability strength, the comprehension principles may add power.

We briefly consider what happens in two cases: First Order Comprehension or FOC and Predicative Comprehension or PC.

The phenomenon considered here is a special case of finite axiomatizability via extension of the language. See e.g., [4].

7.1. First order comprehension. Let us start with a pair theory U which is axiomatized by a scheme $\mathcal{S}_0$. We first show that $\mathcal{S}_0$ can be replaced by a unary scheme $\mathcal{S}$, i.e., a scheme in which only unary schematic predicates occur. We note that in VS_2 we can define Wiener–Kuratowski pairing. Let $\mathsf{pair}(x, y, z)$ be:

$$\exists u, v\,(\forall w\,(w \in z \leftrightarrow (w = u \vee w = v)) \wedge \\ \forall w'\,(w' \in u \leftrightarrow w' = x) \wedge \forall w''\,(w'' \in v \leftrightarrow (w'' = x \vee w'' = y))).$$

Using pair we can define n-tuple for standard n. Also we easily define the associated projection functions. The desired properties are verifiable in VS_2. We assign to every schematic predicate P a unary schematic predicate X_P. Consider an occurrence of a schematic predicate $P(x_0, \dots, x_{n-1})$ in $\mathcal{S}_0$. We replace this occurrence by $\exists s\,(n\text{-}\mathsf{tuple}(s) \wedge (s)_0 = x_0 \wedge \dots \wedge (s)_{n-1} = x_{n-1} \wedge s \in X_P)$. Similarly, for all other occurrences of schematic predicates in $\mathcal{S}_0$.[12] Suppose in the original scheme we would substitute $A\vec{x}$ for $P\vec{x}$. Clearly, we will obtain the same effect by substituting $(n\text{-}\mathsf{tuple}(s) \wedge \exists x_0, \dots, x_{n-1}\,((s)_0 = x_0 \wedge \dots \wedge (s)_{n-1} = x_{n-1} \wedge A\vec{x}))$ for X_P.

Based on the above considerations, we suppose that U is axiomatized by a unary scheme $\mathcal{S}$. We extend U with a new sort for classes $\mathfrak{c}$, identity on classes and a binary relation $\in$ of type $\mathfrak{oc}$. Here $\mathfrak{o}$ is the sort of the original objects of U. We use the variables $X, Y, Z, \dots$ to range over classes. The theory $\mathcal{S}^{\mathsf{foc}}$ in the extended signature is defined as follows.

$\mathcal{S}^{\mathsf{foc}}$1. $\vdash \forall \vec{y}\,\exists X\,\forall x\,(x \in X \leftrightarrow Ax\vec{y})$,
where A is a formula from the original language of U.

$\mathcal{S}^{\mathsf{foc}}$2. $\vdash \forall X, Y\,(X = Y \leftrightarrow \forall z\,(z \in X \leftrightarrow z \in Y))$.

$\mathcal{S}^{\mathsf{foc}}$3. $\vdash \forall \vec{X}\,\mathcal{S}^{\star}\vec{X}$,
where $\mathcal{S}^{\star}$ is the result of replacing all occurrences of schematic predicates in $\mathcal{S}$ by class variables in the obvious way, e.g., Xx by $x \in X$.

The next theorem gives us the obvious connection between U and $\mathcal{S}^{\mathsf{foc}}$.

Theorem 7.1. *Suppose U is a pair theory, axiomatized by a unary scheme $\mathcal{S}$. (Clearly, $\mathcal{S}^{\mathsf{foc}}$ is a conservative extension of U.) We have: U locally $\mathfrak{o}$-directly interprets $\mathcal{S}^{\mathsf{foc}}$.*

Proof. We work in U. Suppose we want to have comprehension for the formulas $A_i x\vec{y}_i$, for $i = 0, \dots, k-1$, where the length of $\vec{y}_i$ is n_i. We define:

[12] If a schematic predicate in $\mathcal{S}_0$ is 0-ary, we introduce a dummy variable.

- $\mathbf{i}(v) :\leftrightarrow \exists v_0, \ldots, v_{i-1}\,(\forall w\, w \notin v_0 \wedge \forall w\,(w \in v_1 \leftrightarrow w = v_0) \wedge \cdots \wedge \forall w\,(w \in v \leftrightarrow w = v_{i-1}))$.
- $\mathsf{class}_i(v) :\leftrightarrow (n_i + 1)\text{-}\mathsf{tuple}(v) \wedge \mathbf{i}((v)_0)$.
- $\mathsf{class}(v) :\leftrightarrow \bigvee_{i<k} \mathsf{class}_i(v)$.
- $x \in v :\leftrightarrow \mathsf{class}(v) \wedge \bigvee_{i<k}(\mathbf{i}((v)_0) \wedge A_i(x, (v)_1, \ldots, (v)_{n_i}))$.
- $v =_{\mathfrak{c}} w :\leftrightarrow \mathsf{class}(v) \wedge \mathsf{class}(w) \wedge \forall x\,(x \in v \leftrightarrow x \in w)$.

We take class as the domain for the sort of classes, $\in$ in the obvious role and $=_{\mathfrak{c}}$ as identity of classes. It is easy to check that this interpretation has the desired properties. ⊣

An interpretation is $\mathfrak{o}$-direct if it interprets the sort $\mathfrak{o}$ as the sort $\mathfrak{o}$ and if it is unrelativized for $\mathfrak{o}$ and sends $\mathfrak{o}$-identity to itself. We have an interesting connection between $\mathsf{VS}_2^{\mathsf{foc}}$ and VS.

THEOREM 7.2. *We have* $\mathsf{VS}_2^{\mathsf{foc}} \rhd_{\mathfrak{o}\text{-}\mathsf{dir}} \mathsf{VS}$.

PROOF. We work in $\mathsf{VS}_2^{\mathsf{foc}}$. We first define Wiener–Kuratowski pairing. We work with ordered pairs. To give the heuristic, let's ignore for a moment the fact that our ordered pairing is not necessarily functional. The basic idea is to code, e.g., the set consisting of a, b, c as $\langle\langle\langle 0, a\rangle, b\rangle, c\rangle$, where 0 is a non-pair. Now forget about functionality again. We define:

- $\mathsf{dc}(Y) :\leftrightarrow \forall u, v, p\,((\mathsf{pair}(u, v, p) \wedge p \in Y) \to u \in Y)$,
  (We will also write $Y{:}\mathsf{dc}$ for $\mathsf{dc}(Y)$.)
- $x \in^\star y :\leftrightarrow \forall Y{:}\mathsf{dc}\,(y \in Y \to \exists w, q\,(\mathsf{pair}(w, x, q) \wedge q \in Y))$.

Consider any non-pair z. We clearly have $\{z\}{:}\mathsf{dc}$. If we would have $x \in z$, then, for some pair q, we would have that q is in the class $\{z\}$, quod non. So z is an empty $\star$-set. This gives us VS_0.

We have the following insight (†): Consider any x and y. Pick any z with $\mathsf{pair}(y, x, z)$. Clearly, $x \in^\star z$. Moreover, we find $y \subseteq^\star z$. For, suppose $u \in^\star y$, $\mathsf{dc}(Y)$ and $z \in Y$. We find $y \in Y$, and, hence, for some w and q, $\mathsf{pair}(w, u, q)$ and $q \in Y$. So $u \in z$.

We now show VS_n be induction on n. Let $x_0, \ldots, x_{n-1}$ be given. We construct a sequence of $\star$-sets $z_0, \ldots, z_n$ as follows. Let z_0 be any non-pair. We take z_{i+1} to be an object such that $\mathsf{pair}(z_i, x_{i-1}, z_{i+1})$. We note, by (†), that $x_j \in^\star z_i$, for all $j < i$.

Let Y_i be the class $\{z_0, \ldots, z_i\}$. We easily verify Y_i is downwards closed. Clearly for any u not among $x_0, \ldots x_{i-1}$, we find that no v such that $\mathsf{pair}(w, u, v)$, for some w, is in Y_i. On the other hand z_i is in Y_i. It follows that $u \notin^\star z_i$. Ergo, for all u, $u \in^\star z_i$ iff $u = x_0$ or ... or $u = x_{i-1}$. ⊣

Theorem 7.2 suggests the following questions.

OPEN QUESTION 7.3. Suppose U is a pair theory axiomatized by a scheme $\mathcal{S}$. Do we have, for any W with $W \rhd U \rhd_{\mathsf{loc}} W$, that $\mathcal{S}^{\mathsf{foc}} \rhd W$?[13]

OPEN QUESTION 7.4. Do we have $\mathsf{VS} \rhd \mathsf{VS}_2^{\mathsf{foc}}$?

[13] We know that this holds when U is a *sequential theory*.

7.2. Predicative comprehension. In our paper [20], we elaborate the equation "consistency = predicative comprehension". In that context, Vaught's result provides one way of articulating what predicative comprehension for infinitely axiomatized RE pair theories means.

Consider a pair theory U that is axiomatized by a scheme $\mathcal{S}$. As before we may assume that $\mathcal{S}$ only contains unary predicate variables. We extend the language of U with class variables. The theory $\mathcal{S}^{\mathsf{pc}}$ in the extended signature is defined as follows.

$\mathcal{S}^{\mathsf{pc}}1$. $\vdash \forall \vec{y}\, \exists X\, \forall x\, (x \in X \leftrightarrow Ax\vec{y}\,\vec{Y})$, where A is a formula that contains no bound class variables and that does not contain X.

$\mathcal{S}^{\mathsf{pc}}2$. $\vdash \forall X, Y\, (X = Y \leftrightarrow \forall z\, (z \in X \leftrightarrow z \in Y))$.

$\mathcal{S}^{\mathsf{pc}}3$. $\vdash \forall \vec{X}\, \mathcal{S}^{\star}\vec{X}$, where $\mathcal{S}^{\star}$ is the result of replacing all occurrences of schematic predicates in $\mathcal{S}$ by class variables.

Clearly, $\mathcal{S}^{\mathsf{pc}}$ is a conservative extension of U. One can show that $\mathcal{S}^{\mathsf{pc}}$ is finitely axiomatizable. In case that U is sequential one can prove that

$$(\mathsf{Q} + \mathsf{con}([\mathcal{S}])) \equiv \mathcal{S}^{\mathsf{pc}}.[14]$$

The details of this result are explained in [19] and [20]. So, we note that, by the Second Incompleteness Theorem, $\mathcal{S}^{\mathsf{pc}}$ is really stronger than U.

Now consider any consistent sequential theory U, axiomatized by a $\Delta_1(\mathsf{EA})$-formula α. Let M be any direct interpretation of VS_2 in U. We have, as we noted earlier,

$$\mathsf{EA} \vdash \mathsf{con}(\alpha) \leftrightarrow \mathsf{con}(\mathsf{VS}_2^M + [\mathcal{S}_\alpha]).$$

Since on both sides of the equivalence we have, modulo EA-provable equivalence, a Π_1-sentence of the form $\forall\Delta_0$, we may apply a meta-theorem from [21], which gives us:

$$(\mathsf{Q} + \mathsf{con}(\alpha)) \equiv (\mathsf{Q} + \mathsf{con}(\mathsf{VS}_2^M + [\mathcal{S}_\alpha])). \qquad (\ddagger)$$

Hence:

$$(\mathsf{Q} + \mathsf{con}(\alpha)) \equiv (\mathsf{VS}_2^M + [\mathcal{S}_\alpha])^{\mathsf{pc}}.$$

It is easy to see that, for any true Π_1-sentence P, we can find a $\Delta_1(\mathsf{EA})$-axiomatization β of U such that $(\mathsf{Q} + \mathsf{con}(\beta)) \rhd (\mathsf{Q} + P)$. Thus, $\mathcal{S}^{\mathsf{pc}}$ can be made arbitrarily strong, when $\mathcal{S}$ ranges over schemes for a theory U. This illustrates that a scheme is 'intensional' from the point of view of consistency strength.

It seems to me that, with a bit more work, we can also get equation $(\ddagger)$, when the theory U is given by a Σ_1-predicate α. Note that in this context, e.g., $\mathsf{con}(\alpha)$ is *not* Π_1—i.e., of the form $\forall\Delta_0$—modulo EA-provable equivalence. We will not pursue this improvement here.

[14] We remind the reader that $\equiv$ means *mutual interpretability*: $U \equiv V$ iff $U \rhd V$ and $V \rhd U$.

We end with a question.

OPEN QUESTION 7.5. Do we have $(\mathsf{Q} + \mathsf{con}([\mathcal{S}])) \equiv \mathcal{S}^{\mathsf{pc}}$, for pair theories U axiomatized by $\mathcal{S}$?

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