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MODEL THEORY OF ANALYTIC FUNCTIONS: SOME HISTORICAL COMMENTS

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Abstract. Model theorists have been studying analytic functions since the late 1970s. Highlights include the seminal work of Denef and van den Dries on the theory of the p-adics with restricted analytic functions, Wilkie's proof of o-minimality of the theory of the reals with the exponential function, and the formulation of Zilber's conjecture for the complex exponential. My goal in this talk is to survey these main developments and to reflect on today's open problems, in particular for theories of valued fields.

When I was invited to give this talk for the ASL annual meeting 2011, I decided it would be a good opportunity to review the history and the development of ideas that has led to today's rich area of research into the study of analytic functions from a model theoretic point of view (and vice-versa, as interesting questions in model theory arise from the geometric understanding of analytic functions). As I started reading, and wondering what I had let myself in for, I very quickly had to make decisions about what I would not be talking about. One very major area I am not talking about is model theory and *analysis*, in the sense of the general study of topological spaces equipped with a metric. There is much that is going on in this area, from descriptive set theory to continuous logic, and it would make a very interesting talk to hear about the historical development of these ideas. However, it is not this talk. What I do want to talk about is the model-theoretic study of analytic functions which begins with Tarski.

In the early 1930s, Tarski proved the decidability of the real numbers as a field. This work was finally published in 1948 as a Rand volume, and reprinted with annotations in 1951 by the University of California press (this is the version which appears in the collected works) [36]. In a discussion of related decision problems, Tarski says

> ... the decision problem is open ... for the system obtained by introducing the operation of exponentiation.

and comments on its potential interest. What we will see, in the course of this talk, is the continuing role that the exponential function has played. Of

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course, this is partly because Tarski posed this problem, and much work has been devoted to attempting to solve it. But it is also because of the wonderful properties of the exponential function which make it so natural and yet so powerful.

I think it is useful at this point to be precise about the objects that we are talking about. Recall the following definition.

DEFINITION 1. Let K be a field with an associated metric on it (and hence a notion of differentiability) which is complete with respect to the metric. A function $f\colon K \to K$ (more generally $K^n \to K$) is *analytic* in a neighborhood U of a point $a \in K$ if it is infinitely differentiable at a and the Taylor series expansion converges to the values of the function on U: for all $x \in U$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n .$$

Of course

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

which converges for all $|x| < \infty$, so is the canonical example of an analytic function.

The problem of decidability of various theories was of central interest in the late 1940's and early 1950's. In 1951, in a paper addressing the decision problem for many different theories of rings, R. Robinson [33] proved the following theorem relevant to the present discussion.

THEOREM 2. *The ring of entire functions over* $\mathbb{C}$; *that is, analytic functions that are defined on all of* $\mathbb{C}$, *is undecidable.*

The fact that an analytic function is given intrinsically as a convergent power series, and that the collection of formal power series, or of *convergent* power series, over a field forms a ring, gives us a choice of approach as highlighted by the above result. On the one hand, we can consider the set of power series as the universe of a structure in the language of rings, possibly with further structure on it (e.g., a derivation or valuation). Model-theoretically, we can then study the algebraic properties of the collection of analytic functions, and derive interesting results, of which the above theorem by R. Robinson is one example. On the other hand, we can consider one (or many) analytic functions on a field as the interpretation of one (or many) function symbols in the language. From this point of view, we are led model-theoretically to study the properties of the sets definable from these functions. The latter point of view is the one I plan to address for most of the talk. However, many of the results in this direction depend on algebraic properties of the functions, so the former point of view is also relevant to the analysis. Indeed, in his survey for the Logic Colloquium in 1997, Wilkie [41] comments

> . . . I believe that the future might lie more in considerations of the former type . . .

It is also significant in the historical development, so let me say a little bit more about results along these lines in the 1960s and 1970s, and leave the interested reader to follow-up with Wilkie's survey for more recent results.

In the 1960s, J. Ax and S. Kochen, and independently, Y. Ersov, looked at power series fields in the language of valued fields (about which more in a moment). Their major contribution is still repeatedly cited today in the model theory of valued fields (where one often refers to an AKE-type result), but they also proved the following theorem along the lines of R. Robinson's undecidablity (this is an amalgamation of several theorems in the literature) [2], [4], [10].

THEOREM 3. *Let K be a field, $K((t))$ the field of formal power series in one variable over K. If K is undecidable as a field, then $K((t))$ is undecidable as a valued field. For K a field of characteristic* 0, *if K is decidable as a field, then $K((t))$ is decidable as a valued field.*

Thus for K a field of characteristic 0, the field of power series over K is decidable as a field if and only if it is decidable as a valued field ([4]). This is not known for characteristic non-zero, and indeed, the following problem is still open.

Open problem. Is the theory of $\mathbb{F}_p((t))$ as a valued field decidable?

At about the same time as the Ax–Kochen–Ersov results, the following, prima facie orthogonal, conjecture was expressed in a book by Lang [19] where it is attributed to Schanuel.

Schanuel's conjecture. Let $\alpha_1, \dots, \alpha_n$ be complex numbers which are linearly independent over $\mathbb{Q}$. Then the field $\mathbb{Q}(\alpha_1, \dots, \alpha_n, \exp(\alpha_1), \dots, \exp(\alpha_n))$ has transcendence degree over $\mathbb{Q}$ at least n.

To see the breadth of this conjecture, consider the case when $n = 2$, $\alpha_1 = 1$ and $\alpha_2 = i\pi$. If Schanuel's Conjecture holds in this case, then

$$\operatorname{trdeg}(\mathbb{Q}(1, i\pi, e^1, e^{i\pi})/\mathbb{Q}) = \operatorname{trdeg}(\mathbb{Q}(i\pi, e)/\mathbb{Q}) \geq 2$$

and hence e and π are algebraically independent. The conjecture essentially sums up everything that is known or guessed about transcendental number theory. In 1971, Ax [3], proved the power series version of Schanuel's Conjecture using differential-algebraic methods.

THEOREM 4. *Let $\alpha_1, \dots, \alpha_n$ be complex power series in $t\mathbb{C}[[t]]$ (that is, with constant term* 0) *which are linearly independent over $\mathbb{Q}$. Then the transcendence degree over $\mathbb{C}(t)$ of the field $\mathbb{C}(t)(\alpha_1, \dots, \alpha_n, \exp(\alpha_1), \dots, \exp(\alpha_n))$ is at least n.*

A field of power series has an obvious way of defining a derivation on it. The algebraic study of the properties of a field equipped with a derivation was initiated by Ritt in the 1920's (see [30], for example). They were first studied model-theoretically by A. Robinson [31], who gave axioms for a differentially closed field. These were simplified by L. Blum (1968) who showed further that the theory of differentially closed fields is a model companion to the theory of fields with a derivation, and proved that this theory has good model-theoretic properties, for example:

THEOREM 5. *The theory of differentially closed fields has quantifier elimination and is ω-stable.*

An account of these results can be found in [7]. A differentially closed field is, in the first place, algebraically closed. In the situation of a real field, one has instead the notion of a Hardy field. Moving away from functions defined by power series, but remaining in a class which is closed under differentiation, A. Robinson [32] proved the following theorem.

THEOREM 6. *Let $\mathcal{H}$ be a set of germs at ∞ of infinitely differentiable, real-valued functions. Assume $\mathcal{H}$ is closed under derivation, and that it forms a field. Then $\mathcal{H}$ has a real closure which is also closed under derivation.*

It remains an open problem, currently being studied by Aschenbrenner, van den Dries, van der Hoeven, to understand the model theory of Hardy fields in general.

All this, although interesting, is not the main direction of what I want to talk about today. Rather, I want to look at the study of sets defined by analytic functions, and what model theory can say about them. The kind of field on which the functions are defined is obviously important, as also are differential-algebraic properties of the functions themselves. So let us return to Tarski's problem and sets defined on the real numbers by analytic functions.

First recall the notion of a *semi-algebraic set*. In the language of ordered fields, the quantifier-free definable (with parameters) subsets of $\mathbb{R}^n$ are boolean combinations of sets of the form

$$\{y \in \mathbb{R}^n \colon f(y) = 0 \wedge \bigwedge_{i=1}^{k} g_i(y) > 0\}$$

where f, g_i are polynomials with real coefficients. Tarski's decidability result goes via quantifier elimination: the collection of semi-algebraic sets is closed under projection. Analogously, we can define the notion of a *semi-analytic set*.

DEFINITION 7. A set $S \subset \mathbb{R}^n$ is *semi-analytic at $a \in \mathbb{R}^n$* if there is an open neighborhood U of a in $\mathbb{R}^n$ such that $S \cap U$ is a finite union of sets of the form

$$\{y \in U : f(y) = 0 \wedge \bigwedge_{i=1}^{k} g_i(y) > 0\}$$

where the functions f and g_i are analytic. The set S is *semi-analytic* if it is semi-analytic at every point in $\mathbb{R}^n$ and it is *globally semianalytic* if its image under the map $(x_1, \ldots, x_n) \to (x_1/\sqrt{1+x_1^2}, \ldots, x_n/\sqrt{1+x_n^2})$ is semianalytic. (This function is an analytic isomorphism from $\mathbb{R}^n$ to the open unit cube $(-1,1)^n$.)

Semi-analytic sets have reasonable geometric properties. They were studied in the 1960s by S. Łojasiewicz, and his school ([21], although this does not appear to be available on MathSciNet. A more complete study of both semianalytic and subanalytic sets can be found in [6]). However, the collection of semi-analytic sets is not closed under projection, or more generally, under images by analytic functions. Thus one is led to make a further definition.

DEFINITION 8. The set $S \subset \mathbb{R}^n$ is *subanalytic at* $a \in \mathbb{R}^n$ if there are an open neighborhood U of a, and a bounded semianalytic set $S' \subset \mathbb{R}^{n+m}$ such that

$$S \cap U = \{y \in U : \exists x \in \mathbb{R}^m((y,x) \in S')\}.$$

It is *subanalytic* if it is subanalytic at every point in $\mathbb{R}^n$ and it is *globally subanalytic* if the image of S under the above map to the unit cube is subanalytic.

The subanalytic sets were studied under various names, including semi-analytic shadows by Hardt [12], and $\mathcal{P}$-sets by Gabrielov [11], but the name given by Hironaka [14] is the one that stuck. In 1968 we see the first result with a really model-theoretic flavor. Gabrielov was an undergraduate when he was given this problem by his advisor. They were clearly informed by the model-theoretic point of view on the Tarski–Seidenberg theorem, but were interested in proving a geometric result.

THEOREM 9. *The complement of a subanalytic set is subanalytic.*

To see this as a model-theoretic result, we need to specify an appropriate language. We build the language $\mathcal{L}_{an}$ by adding a function symbol to the language of rings for every power series f (in n variables) over $\mathbb{R}$ which converges in a neighborhood of $[-1,1]^n$. For every new function symbol f, interpret f on $\mathbb{R}^n$ by the function $\tilde{f}$ defined by

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [-1,1]^n, \\ 0, & \text{if } x \notin [-1,1]^n. \end{cases}$$

(we call this a *restricted* analytic function). Write $\mathbb{R}_{an}$ for this expansion of the real field in the language $\mathcal{L}_{an}$. This seems like a fairly brutal operation to perform on an analytic function, since it immediately loses its analyticity on the boundary of the unit cube. However it serves to capture the local character of the semi-analytic and subanalytic sets. In fact, the globally semi-analytic subsets of $\mathbb{R}^n$ are precisely the quantifier-free definable sets in $\mathbb{R}_{an}$, and the globally subanalytic sets are those which are existentially definable. Thus Gabrielov's theorem says that the collection of subanalytic sets is closed under complement, and hence that the theory of $\mathbb{R}_{an}$ is model-complete.

To continue the model-theoretic development, we need to digress for a moment to recall valued fields. The field F is *valued* if there is a valuation function $v\colon F^* \to \Gamma$ from the field to an ordered group Γ such that $v(xy) = v(x) + v(y)$, $v(x+y) \geq \min\{v(x), v(y)\}$. For convenience, we define $v(0) = \infty$, where $\gamma < \infty$ for every $\gamma \in \Gamma$. The set $\{x \in F\colon v(x) \geq 0\}$ is called the *valuation ring*; it has a unique maximal ideal and the quotient of the valuation ring by the maximal ideal is called the *residue field*. For example, a field of formal Laurent series carries a valuation in a natural way: take $F = K((t))$; $v(\sum_{i=M}^{\infty} a_i t^i) = M$, where $a_M \neq 0$. Here the residue field is K and the value group is $\mathbb{Z}$.

A different example is seen in the construction of the p-adic numbers. Fix a prime p, take $F = \mathbb{Q}$, and define $v(\frac{p^r s}{t}) = r$, where $p \nmid st$, $r \in \mathbb{Z}$. Here the field has characteristic 0, and the residue field is $\mathbb{F}_p$, with characteristic p. The valuation naturally gives rise to a metric $|x|_p = p^{-v(x)}$, and the field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to this metric. That the theory of the p-adic numbers as a field is decidable was proved already by Ax and Kochen, and Ersov, in the papers referred to above, with a primitive recursive decision procedure given by Cohen in [8]. In 1976, Macintyre [22] proved the theory has quantifier elimination in a language with predicate symbols for the sets of nth powers, which are thought of as analogous to the ordering (equivalently, the set of squares) in the reals.

We can build the language $\mathcal{L}_{an}$ on top of the algebraic language for valued fields just as we built it on the field language for the reals. For the p-adics, the "restricted" in restricted analytic function becomes restriction to the valuation ring. Convergence is easier in the p-adics, because of the ultrametic property of the metric defined from the valuation — any power series with coefficients eventually decreasing in size will converge as a function on the valuation ring. With this setup, J. Denef and L. van den Dries [9] proved the following quantifer elimination results for the theory of the p-adics with analytic functions.

Theorem 10. *Let $\mathbb{Q}_{p,an}$ be the structure with domain $\mathbb{Q}_p$ in the language $\mathcal{L}_{an}$ with function symbols for the functions on the valuation ring defined by convergent power series. The theory of $\mathbb{Q}_{p,an}$ is model-complete. Add a binary* *function symbol D to the language, which is interpreted by a restricted division function. The theory of* $\mathbb{Q}_{p,an}$ *in the language* $\mathcal{L}_{an}^{D}$ *has quantifier elimination.*

In the same paper, they show the analogous result for the theory of $\mathbb{R}$ in a language with symbols for the restricted analytic functions. Thus they recapture Gabrielov's theorem, and also get a quantifier elimination result.

The new idea of Denef and van den Dries was to really exploit the fact that the functions are given as power series (at least in the standard model), and thus satisfy Weierstrass preparation. What this says is that, locally, possibly after a change of variables, a power series in n variables can be expressed as a product of an analytic function, which is a unit in the power series ring, and a polynomial in one of the variables with coefficients that are power series in the other variables. For algebraic conditions, such as being equal to zero or having a kth root, the unit is irrelevant. Thus one can apply the algebraic quantifier elimination results to the last variable, and proceed by induction on n. Although Weierstrass preparation is a local result, the fact that the valuation ring is compact means that only finitely many different products are needed. The change of variables to get into a position to apply Weierstrass preparation requires division, and this is why we only get model-completeness in the original language.

L. Lipshitz (1993) [20] applied the same method to an algebraically closed valued field. Here the valuation ring is not compact, and so one needs to treat the convergent power series with more care. Lipshitz uses power series with two sorts of variables — one for the valuation ring and one for the maximal ideal. Power series with coefficients of eventually convergent norm will converge for any inputs, but power series over the valuation ring need greater control over the convergence of the coefficients in order for the functions to converge uniformly. Lipshitz defined a collection called the *separated* power series for which he then proves model-completeness and quantifier elimination results.

In the meantime, people continued to think about Tarski's problem in different ways. One observation that Tarski had made is that it follows from his quantifier elimination that definable subsets of $\mathbb{R}$ are finite unions of points and intervals. L. van den Dries, in his lecture at the Logic Colloquium 1982 [37], showed that many nice properties of definable sets in $\mathbb{R}^n$, including uniform finiteness of fibres of sets in definable families and that definable functions are piecewise continuous, follow just from this simple observation. J. Knight, A. Pillay and C. Steinhorn [17] then observed that van den Dries's results depend only on this property of the one-variable definable sets in an ordered structure, and not on particular properties of the real numbers (nor on the complete quantifier elimination). They called such structures *order minimal*, by analogy with the property of a structure being minimal (definable sets are finite or cofinite).

DEFINITION 11. A structure in a language with a linear ordering is *o-minimal* if every definable set in one variable is quantifier-free definable just using the ordering.

It was very soon observed by van den Dries [38] that $\mathbb{R}_{an}$ is o-minimal, and thus formed the first non-trivial expansion of the algebraic structure on the reals which is o-minimal. In particular, since the exponential function is analytic, it follows that the theory of $\mathbb{R}$ with a function symbol for the *restricted* exponential function is also o-minimal (the property of being o-minimal is closed under reducts.)

The intuition about restricted analytic functions is that, because of Weierstrass preparation, they are not really more complicated than polynomials. The challenge then was to try to expand the reals by some analytic function which is defined on all of the real numbers, yet does not have the periodicity which would mean that it, in return, defines the integers. The big breakthrough was achieved by A. Wilkie [40] (a preprint was circulating already in 1991).

THEOREM 12. *The theory of* $\mathbb{R}_{\exp}$ *is model complete and o-minimal.*

The proof uses the o-minimality of $\mathbb{R}_{\exp|[0,1]}$, as well as a detailed understanding of the nature of solutions of exponential polynomial equations that comes from work of Khovanskii, as well as earlier work of Wilkie.

This result does not solve Tarski's original problem. Five years later, Macintyre and Wilkie essentially settled it in the following unexpected way [23].

THEOREM 13. *If Schanuel's conjecture is true then the theory of* $\mathbb{R}_{\exp}$ *is decidable. On the other hand, the decidability implies a weak form of Schanuel's conjecture.*

Thus the solution to Tarski's problem depends on this far-reaching conjecture which is not expected to be resolved any time soon.

Tarski's problem aside, Wilkie's proof of the o-minimality of the real field with exponentiation opened the doors to a wealth of new problems. One significant feature of his proof is to use differential-algebraic properties of the exponential function. Some years later, he developed these methods to show that the expansion of the reals by a large class of functions, the Pfaffian functions, is o-minimal [42].

DEFINITION 14. A sequence $f_1, \ldots, f_n$ of differentiable functions is a *Pfaffian chain* if there are polynomials $p_1(t_1, t_2), \ldots, p_n(t_1, \ldots, t_{n+1})$ such that $f_i'(x) = p_i(x, f_1, \ldots, f_i)$ for each i.

THEOREM 15. *Expansions of the real field by Pfaffian chains of analytic functions are o-minimal.*

The same year, Speissegger [35] constructed an o-minimal expansion of the reals in a different way, by adding relation symbols to the structure for an analytically defined object.

THEOREM 16. *The expansion of an o-minimal structure on* $\mathbb{R}$ *by any* Rolle leaf (*solution of a continuous, definable* 1*-form with the Rolle property*) *is still o-minimal.*

Work on o-minimal expansions of the reals continues to grow and prosper. One longterm goal is to tackle what remains of Hilbert's 16th problem; that is, to find a bound on the number of limit cycles of polynomially defined plane vector fields. Researchers continue to extend the range of o-minimality beyond the analytic category, as is seen, for example, in the 2003 paper of Rolin, Speissegger and Wilkie, where they develop an axiomatic method for showing that an expansion of the reals by quasi-analytic functions is o-minimal [34]. Some proofs of o-minimality now do not go via model-completeness, so then it is also a goal to understand the quantifier complexity of the definable sets.

But where is complex analysis in all of this? In 1993, B. Zilber gave a talk at the 10th Berlin Easter conference in Model Theory. A paper is published in the proceedings volume [44], and available from Zilber's website. Instead of adding function symbols to the language, he takes the domain to be a compact complex manifold, and adds relation symbols to the language for all of the complex analytic subsets.

THEOREM 17. *This structure has quantifier elimination and is* ω*-stable with finite Morley rank.*

A. Pillay took up the model-theoretic study of compact complex varieties (where a complex variety is a reduced irreducible complex analytic space). With his student R. Moosa, they extended Zilber's analysis to the multi-sorted structure, whose sorts are all compact complex varieties, each sort having all complex analytic subsets as basic relations. One thing they prove, as noted in [27] is:

THEOREM 18. *The multi-sorted structure above eliminates imaginaries.*

Subsequent work (Hrushovski, Kowalski, Moosa, Pillay, Scanlon) relates sophisticated model-theoretic ideas (for instance, internality and Zariski geometries) to geometric properties of complex manifolds. And vice-versa, the expression of geometric concepts in a purely model-theoretical way has brought interesting developments. The two surveys by Moosa [27, 28] give a very good account of these.

A very different approach to the study of complex analytic functions is taken by Peterzil and Starchenko ([29] and subsequent work). They look at what we might call an 'o-minimal complex field' as the algebraic closure of an o-minimal real field, and show that many results of classical complex analysis are true for these more general algebraically closed fields.

In both of the above approaches, analytic functions are not added to the language, and with good reason. As D. Marker noted in his 2006 JSL article [25],

> When studying the model theory of $\mathbb{C}_{\exp}$ the first observation is that the integers can be defined ... . Since $\mathbb{C}_{\exp}$ is subject to all of Gödel's phenomena, this is also often the last observation.

As Marker points out, $\mathbb{C}_{\exp}$ is undecidable, and is also not model-complete. Nevertheless, one can still try to understand it model-theoretically. In the worst case it would turn out that $\mathbb{R}$ is definable in $\mathbb{C}_{\exp}$. The best case would be that $\mathbb{C}$ is *quasi-minimal*; that is, every definable set is countable or co-countable. Zilber has outlined a program to try to prove this latter statement. In a paper in 2005 [45], he proposed axioms for an algebraically closed field K in a language with a surjective homomorphism E from the additive structure to the multiplicative structure of K. These axioms include the complex Schanuel conjecture, an existential closure axiom which is a strong converse to Schanuel's conjecture, and an axiom called the countable closure property (which limits the number of existential generic points). Of course these cannot be first-order axioms; rather they are given in $\mathcal{L}_{\omega_1,\omega}(Q)$. Zilber then proves a categoricity theorem for this theory.

THEOREM 19. *For every uncountable cardinal κ, there is a unique model of the above theory of cardinality κ and this model is quasi-minimal.*

In particular, if $\mathbb{C}_{\exp}$ is a model of the axioms, then it is the unique model of cardinality the continuum, and hence is quasi-minimal. There are partial results in the direction of this program. In the original paper, Zilber proved that $\mathbb{C}_{\exp}$ satisfies the countable closure property (incidentally, using the theorem of Ax on the power series version of the Schanuel conjecture). Marker proved that $\mathbb{C}_{\exp}$ satisfies a first case of the existential closure property. Of course, the completion of this program as currently outlined would require proving the full complex Schanuel's Conjecture, and this is not expected to be done any time soon. Nevertheless, to a model-theorist, the categoricity result is perhaps the strongest reason for believing in the truth of Schanuel's Conjecture (as remarked to me by Rahim Moosa).

Indeed, if Schanuel's Conjecture fails, one wonders what might be the mysterious homomorphism which gives the unique model of Zilber's theory of cardinality $2^{\aleph_0}$. It follows from work of Wilkie [43] and Koiran [18] on Liouville functions that there is some entire function on $\mathbb{C}$ which satisfies Schanuel's conjecture and a variant of the existential closure property (see also Aschenbrenner's review [1] of Zilber's paper). However, given the construction as a power series whose coefficients converge very rapidly, such a function will not be a homomorphism from the multiplicative group of $\mathbb{C}$ to its additive group, so does not give insight into this question.

We turn now to non-archimedean analysis. A valued field naturally has an associated notion of distance on it. Indeed, the definition of a valuation can be expressed as the existence of a function $|\cdot|\colon K^* \to \Gamma$, where Γ is an ordered multiplicative group such that $|xy| = |x||y|$ and $|x + y| \leq \max\{|x|, |y|\}$. This latter property is expressed by saying the function is an *ultrametric*. The ultrametric makes convergence of infinite series very easy, but unfortunately the field is totally disconnected in the associated topology. There is thus no good notion of analytic continuation. For the quantifier elimination results, the method of working locally gets around this, but we can see the associated difficulties when comparing the p-adics versus an algebraically closed valued field. In the former case, the valuation ring is compact, and the arguments for the p-adics are very similar to those for the reals (indeed, the p-adic analytic quantifier elimination came before the result for the reals, as we have already seen). The difficulties in the algebraically closed case can be attributed to the fact that the functions do not have a natural analytic continuation beyond the (non-compact) valuation ring.

V. Berkovich [5] proposed a way to find a better topology on a valued field. This can be described, very roughly, as considering the space of ultrametrics defined on the ring of polynomials over a valued field K. This is analogous to the operation in algebraic geometry of considering the spectrum of the ring of polynomials over K. As in the algebraic case, the field K embeds naturally into the space of ultrametrics. There is a reasonable way to define a topology on this space which is compact and Hausdorff, and hence is amenable to defining analytic functions on it. As yet, the model theory of Berkovich space has not been tackled. However, Berkovich space has been studied using model-theoretic tools. In a recent preprint, E. Hrushovski and F. Loeser [15] study the space of *stably dominated types* on the definable subsets of an algebraically closed valued field. The Berkovich space of an algebraic variety is such a space of stably dominated types, where the definable set is the given variety and the value group of the field is a subset of $\mathbb{R}$.

To finish the talk, I would like to mention a very recent result about analytic functions on a valued field, which illustrates again the fundamental role of the exponential function. In 2006 [13], Hrushovski, Macpherson and I proved that the theory of algebraically closed valued fields has elimination of imaginaries in a sorted language $\mathcal{L}_{\mathcal{G}}$ which includes sorts for an infinite collection $\mathcal{G}$ of definable modules and their torsors. This was swiftly followed by proofs of the analogous result for real closed valued fields [26] and p-adically closed fields [16]. Given the elimination of quantifiers results for all of these theories in an appropriate language with restricted analytic functions, we wondered if the same sorted language would suffice to eliminate the further imaginaries which can be defined with these additional functions. In fact, we have shown that properties of the exponential function mean that this is not true.

THEOREM 20. *Add function symbols to the language to be interpreted by restricted analytic functions on the field. Assume the collections of functions is rich enough so that the resulting structure eliminates quantifiers, and also assume that the (restricted) exponential and logarithm functions are included. Let h be the function from a translate of the valuation ring to a translate of the maximal ideal defined by* $h(a + x) = b \exp(x)$. *The imaginary which is the graph of h cannot be eliminated in the above language.*

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