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GENTZEN'S PROOF SYSTEMS: BYPRODUCTS IN A WORK OF GENIUS

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Abstract. Gentzen's systems of natural deduction and sequent calculus were byproducts in his program of proving the consistency of arithmetic and analysis. It is suggested that the central component in his results on logical calculi was the use of a tree form for derivations. It allows the composition of derivations and the permutation of the order of application of rules, with a full control over the structure of derivations as a result. Recently found documents shed new light on the discovery of these calculi. In particular, Gentzen set up five different forms of natural calculi and gave a detailed proof of normalization for intuitionistic natural deduction. An early handwritten manuscript of his thesis shows that a direct translation from natural deduction to the axiomatic logic of Hilbert and Ackermann was, in addition to the influence of Paul Hertz, the second component in the discovery of sequent calculus. A system intermediate between the sequent calculus *LI* and axiomatic logic, denoted *LIG* in unpublished sources, is implicit in Gentzen's published thesis of 1934–35. The calculus has half rules, half "groundsequents," and does not allow full cut elimination. Nevertheless, a translation from *LI* to *LIG* in the published thesis gives a subformula property for a complete class of derivations in *LIG*. After the thesis, Gentzen continued to work on variants of sequent calculi for ten more years, in the hope to find a consistency proof for arithmetic within an intuitionistic calculus.

1. Genius at work. Gerhard Gentzen launched in early 1932 a program the aim of which was to clear the problems of consistency of arithmetic and analysis, rendered enigmatic after Gödel's great discovery a year earlier. By September 1932, he had localized the natural rules of basic logical reasoning and by the next spring, modified them so that the structure of such reasoning by the newly discovered rules could be made completely transparent. Next he applied this understanding to the outstanding problem of the consistency of the formal axioms of the theory of natural numbers; After Gödel's incompleteness results of 1931, it was known that no simple approach to the consistency problem was possible. Despite great practical difficulties, including the Nazi takeover and ensuing destruction of his university department Gentzen solved also that problem towards the end of 1934.

Because I describe Gentzen's discovery as a work of genius, maybe some comparison could be in place. A fine example of a surprising, fast, and

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profound discovery is Albert Einstein's theory of Brownian motion. His 1905 paper on the topic is very instructive in the way it shows how the discovery was made: There was a starting point that was new, simple, and right. It is of course some kind of an arithmetic necessity that simple new things are few in number and therefore hard to find. Whenever there is such an idea, success follows if things are in the right order from the start. If one rushes at this point, one is bound to just grope around in the shadows. Thus, Einstein's predecessors were chasing the velocity of a Brownian particle, but from his basic insight[1] it followed that there simply is no such thing: The Brownian particle experiences continuously collisions by the surrounding molecules and there is no limit to the change of pace in time, i.e., no velocity as it is defined in kinematics. The theory dictates that the observable quantity is mean displacement in time. –New, simple, and right ideas get soon accepted, as was the case with Einstein's work on Brownian motion. It led soon to a final consolidation of the atomic structure of matter.

I find that Gentzen's discovery of the rules of logical reasoning in 1932–33 is easily comparable to the three or four discoveries Einstein published in his miraculous year 1905. In discussing Gentzen's logical work, I shall in part dwell on historical sources, in part record my own impressions, and at places also proceed through a systematic study, with some new insights and results here and there; history of logic with theorems, as it were.

My basic insight into Gentzen's achievement in logic is: His amazing discovery of natural deduction and sequent calculus in 1932–33, with a full control over the structure of derivations, followed from the use of a **tree form** for derivations. That was the new, simple, and right idea through which it could be seen what depends on what, to compose derivations, and to permute the order of application of rules. The last one is the central methodological point in Gentzen and soon led to results such as normalization and cut elimination. As with Brownian motion, the idea was absorbed so fast in proof theory that one is not even aware today that a decisive novelty was introduced here.

The dates:[2] Gentzen's first paper, on what are known as Hertz systems, was received by the ***Mathematische Annalen*** on 6 February 1932. He sent an offprint to his first university professor Hellmuth Kneser in Greifswald on 13 December 1932, also telling that he had quit working on its topic but is instead working on the foundations of mathematics, especially the consistency of arithmetic, since almost a year. Stenographic notes show that the system of natural deduction was in its final form on 23 September 1932. Gentzen's thesis with

[1]For the *cognoscenti*: The molecules of a liquid move fast and immersed particles of a visible size don't move at all. At what size would the random bombarding of a particle by the surrounding molecules produce a visible effect, and of what form? The size can be determined by what is called the equipartition of energy. Result: particle size of about one part in ten thousand of a millimeter, but with a nondifferentiable trajectory. For more details, *cf*. section 3.4 of my book ***Creating Modern Probability***.

[2]Instead of creating discontinuities in the text with distracting footnotes, I gather most of the data on sources and related items at the end of the sections.

the development of sequent calculus was finished in May 1933. A letter to Heyting of 25 February 1933 shows that Gentzen had postponed work on the consistency of arithmetic to 1934. A letter to Kneser of 5 December 1934 tells that he had finished the consistency proof.

A systematic presentation of all of Gentzen's work is found in my chapter on *Gentzen's logic* written in 2005 for the ***Handbook of the History of Logic***. Here the emphasis is on its earliest purely logical phase.

2. The pursuit of truth. What were Gentzen's predecessors chasing that he was not? I would say absolute mathematical truth, and logical truth as its generalization. Gentzen was just interested in how one establishes mathematical theorems, namely by proof, and saw logic as the general pattern behind all such proofs. With this attitude, Gödel's incompleteness results lost most of their dramatic character. Gentzen reasoned, in October 1932, as follows: The axioms of arithmetic are obviously correct, and the principles of proof obviously preserve correctness. Why can't one simply conclude consistency, i.e., what is the true meaning of the second incompleteness theorem, the one by which the consistency of arithmetic cannot be proved by arithmetic means? "*Where is the Gödel-point hiding?*"

The pursuit of truth in logic had begun with Gottlob Frege and was continued by Russell and the rest, so that axiomatic logic had become the norm by the 1920s. The axioms were supposed to express the most basic logical truths, and just two rules of proof the passage from instances of axioms to the theorems of logic. The latter were, supposedly, the less basic truths, but sometimes theorems were simpler than axioms. The rules of passage were detachment, from $A \supset B$ and A to conclude B, and the rule of universal generalization.

There was another aspect to axiomatic logic, namely a fundamental relativity in the choice of the basic notions. Russell's standard axiomatization of propositional logic, from the ***Principia Mathematica***, uses disjunction and negation as the primitive connectives. A slightly modernized notation is:

TABLE 1. Principia Mathematica style logical axioms.

1. $\neg(A \vee A) \vee A$,
2. $\neg A \vee (A \vee B)$,
3. $\neg(A \vee B) \vee (B \vee A)$,
4. $\neg(A \vee (B \vee C)) \vee ((A \vee B) \vee C)$,
5. $\neg(\neg A \vee B) \vee (\neg(C \vee A) \vee (C \vee B))$.

Implication is defined by $A \supset B \equiv \neg A \vee B$ and its use would make the above axioms look a little less bad. Say, the first axiom is $A \vee A \supset A$ and the last one $(A \supset B) \supset (C \vee A \supset C \vee B)$. The choice of axioms in Russell and Whitehead is motivated by the algebraic tradition of logic of Ernst Schröder, with such algebraic properties of an operation as idempotence (axiom 1), commutativity (3), and associativity (4).

In a paper of 1906, *The theory of implication*, Russell uses negation and implication as primitives, with a much better-looking axiomatization as a result:

TABLE 2. Russell's 1906 theory of implication.

1. $A \supset A$,
2. $A \supset (B \supset A)$,
3. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$,
4. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$,
5. $\neg\neg A \supset A$,
6. $(A \supset \neg A) \supset \neg A$,
7. $(A \supset \neg B) \supset (B \supset \neg A)$.

The axioms are nicely motivated by intuitive considerations, partly in reference to Frege, partly to Peano, in whose work they appear. Next to the rule of detachment, there is an explicit rule for taking instances of the axioms in formal derivations.

After the axioms (pp. 168–9) there follow more than twenty pages of formal derivations; not that these axioms would have made the provability of theorems of propositional logic any more apparent to Russell than those based on negation and disjunction. He comments wryly (p. 159):

> In the present article, certain propositions concerning implication will be stated as premisses, and it will be shown that they are sufficient for all common forms of inference. It will not be shown that they are all *necessary*, and it is probable that the number of them might be diminished.

The net effect of the study of axiomatic logic was a scandalously unmanageable logical machinery. Thus, Russell and Whitehead put years into the production of hundreds and hundreds of pages of formal logical proofs within the axiomatic system, but had little idea of its properties as a logical calculus. This aspect becomes obvious through the work of Paul Bernays of 1918. He was writing a habilitation thesis on the propositional calculus of the ***Principia Mathematica*** and wanted to prove the mutual independence of the axioms, exactly as one does in axiomatic geometry *à la Hilbert*. To this effect, he invented interpretations of the axioms of logic with more than two truth values. Then, certainly unexpectedly, the fourth axiom resisted attempts at proving its independence by such a semantical method. Bernays began suspecting that it could be a theorem and managed to find a derivation for it from the rest of the axioms.

Axiomatic logic had, in the hands of Russell, moved far away from its origin in Frege, namely from a formula language for arithmetic in which, as Frege wrote, "everything necessary for a correct inference is expressed in full, but what is not necessary is generally not indicated." Frege, in contrast to Russell, had a clear idea of the practical meaning of his axioms. I would bet, were it possible, that he could have proved any ordinary logical theorem of the propositional and predicate calculus from his axioms.

Sources: The "Gödel-point" story is from Gentzen's manuscript that bears the signum **INH**. Bernays' *Habilitationsschrift* has the title *Beiträge zur axiomatischen Behandlung des Logik-Kalküls* (Contributions to the axiomatic treatment of the calculus of logic). It is a 60-page manuscript kept in the Bernays collection at ETH Zurich, beautifully written in black ink with the finest *Stahlfeder* pen. Part of the results were published in Bernays (1926).

3. Axiomatic logic in the 1920s. In Bernays (1918), a step back from the relativism of the Russellian axiomatizations had been taken, in that all the standard connectives, conjunction, disjunction, implication, and negation, were present. Bernays hoped to pinpoint the role of negation in logical axiomatics, motivated by the possibility of using negation and any one of conjunction, disjunction, or implication as basic connectives. He asks (p. 19) whether "it is possible to set up a negation-free axiom system from which all negation-free formulas provable in our calculus, and only these, can be derived." A negation-free system is one in which conjunction and implication are "not abbreviations, but symbols for basic operations ... the question has a positive answer" (ibid.). Instead of giving the answer, Bernays sets out to derive the fourth of the Russell–Whitehead axioms and to show that the rest are mutually independent. In the latter part of the *Schrift*, he explores various possibilities of replacing axioms by rules, for example, one that is rather peculiar (p. 47): There is just one axiom, and among the rules there is one that allows to replace a subformula of the form $\alpha \vee \beta$ by $\beta \vee \alpha$.

The formal derivations are written in a linear form, with a stacking of the premisses of the rule of detachment before an inference line. Disjunction is just concatenation:

$$\frac{\begin{cases}\overline{\alpha}\,\beta \\ \alpha\end{cases}}{\beta} \qquad\qquad \frac{\begin{cases}\alpha \to \beta \\ \alpha\end{cases}}{\beta}$$

The notation at right is given as an alternative.

Formal derivations start appearing on p. 22 of Bernays' work. They have always the major premiss of the rule first, then the minor, and the conclusion below the inference line. Repeated applications of the rule are displayed so that this arrangement is kept. It of course happens soon that not all derivations can be arranged under the pattern, unless previously derived formulas are numbered and can be used as premisses in rule instances. A good example in his respect is the derivation of Russell and Whitehead's axiom 4 (p. 26).

A two-premiss rule of syllogism, the use of which is indicated by a double curly bracket, is shown to be derivable (p. 23):

$$\frac{\left\{\!\!\left\{\begin{array}{l}\alpha \to \beta \\ \beta \to \gamma\end{array}\right.\right.}{\alpha \to \gamma}$$

In the proof of derivability of this rule, the premisses appear as formulas assumed to be derivable, so it is in fact an admissible rule; a notion weaker than derivability. In the rest of the derivations, this rule replaces the rule of detachment.

The negationless axioms Bernays alludes to can be gathered from various sources. One place is the first volume of the ***Grundlagen der Mathematik*** of 1934 (p. 66), but earlier sources exist, such as Hilbert (1928, 1931), and the earliest I have found that had the axioms for conjunction and disjunction, namely the doctoral thesis of Hilbert's student Wilhelm Ackermann (1924). Zach (1999, p. 352) reports that the axioms stem from a lecture course Hilbert and Bernays taught jointly in 1922–23:

TABLE 3. The Hilbert–Bernays axioms.

1. $A \to (B \to A)$,
2. $(A \to (A \to B)) \to (A \to B)$,
3. $(A \to B) \to ((B \to C) \to (A \to C))$,
4. $A \,\&\, B \to A, \quad A \,\&\, B \to B$,
5. $(A \to B) \to ((A \to C) \to (A \to B \,\&\, C))$,
6. $A \to A \vee B, \quad B \to A \vee B$,
7. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$,
8. $(A \to B) \to (\neg B \to \neg A)$,
9. $A \to \neg\neg A$,
10. $\neg\neg A \to A$.

These axioms appear beautiful to anyone with experience in elementary logic. The three first are, in terms of sequent calculus, axiomatic equivalents to the rules of weakening, contraction, and cut. Then follow what look like axiomatic equivalents to the rules of natural deduction for conjunction and disjunction, then the axioms of negation. The implicational axioms go back to Frege. Sometimes an axiom that corresponds to a rule of exchange in sequent calculus is seen, as in Hilbert (1923).

Bernays wrote in 1918 that there is an axiom system for his negation-free fragment of classical propositional logic. If so, axioms 1–7 of Table 3 are not sufficient, for they axiomatize the negationless fragment of intuitionistic logic that falls short of the corresponding classical fragment, witness a purely classical theorem such as $(A \supset B) \vee (B \supset A)$.

There appeared in 1928 the book ***Grundzüge der theoretischen Logik***, with the names of Hilbert and Ackermann on it. The preface tells that the text was produced by Ackermann on the basis of previous versions prepared by Bernays from Hilbert's lectures. It is further told that Bernays is preparing another book on the foundations of mathematics ("Grundlagen der Mathematik" in the German original). From the way the birth of the logic book is described, it seems very odd that the name of Bernays should not appear on the cover. However, the really odd thing is the book's presentation of logic in the old axiomatic form of Russell and Whitehead that Hilbert and Bernays had used in 1918, as if Hilbert and Ackermann had not published several papers on a radically improved axiomatization in the intervening ten years.

Hilbert and Ackermann's book had a bad effect on logic, because its presentation precluded the possibility of understanding the difference between classical and intuitionistic logic. The exception are those who followed the research literature or undertook to study the big book that appeared in two parts, in 1934 and 1939. Intuitionistic logic took definitive shape right after the appearance of the Hilbert–Ackermann book, the main paper being Heyting (1930).

4. The early days of intuitionistic logic. Intuitionistic logic began with L. Brouwer's great insight that behind the purely indirect existence proofs of mathematics, there are the logical laws $A \vee \neg A$ and $\neg\neg A \supset A$. Further, the validity of these laws in infinite domains is an extrapolation from our experience in finite situations in which latter the laws are valid. Brouwer came to realize in the 1920s that the classical logical laws are not contradictory, by which he could argue for the correctness of both $\neg\neg(A \vee \neg A)$ and $\neg\neg(\neg\neg A \supset A)$. Another intuitionistically correct law was $\neg\neg\neg A \supset \neg A$ that Brouwer described, in his characteristic idiosyncratic terminology, as: "absurdity must follow from absurdity of absurdity of absurdity." By 1928, he had arrived at another profound insight: Namely, the classical form of excluded middle for a property is $\forall x(A(x) \vee \neg A(x))$, but its double negation $\neg\neg\forall x(A(x) \vee \neg A(x))$ fails intuitionistically. Brouwer (1928) showed this negative result by a counterexample from the intuitionistic theory of real numbers.

Discussions about the logic of Brouwer's intuitionistic mathematics were conducted on the pages of the *Bulletin* of the Royal Belgian Academy during the last years of the 1920s. In the midst of these discussions, Heyting figured out in 1928 the proper axiomatization of intuitionistic propositional and predicate logic. There were three predecessor, all unknown to him. In chronological order, Skolem (1919) had found a structure that is today known as Heyting algebra, one that relates to intuitionistic logic in the same way as Boolean algebra to classical logic. The matter was known to Tarski and others when they in the latter part of the 1930s figured out the algebraic semantics of intuitionistic logic (see my 2007 for details). A second precursor to Heyting was Kolmogorov in his Russian paper of 1925. A third precursor was Bernays, who after a talk by Brouwer in Göttingen found out in 1925 that it is sufficient to leave out the law of double negation from a suitable axiomatization of classical logic, as the one in Table 3 above (see Troelstra 1990, pp. 6–7).

Brouwer was enthusiastic of Heyting's work, and concluded that he would not have to finish such a work himself (see van Dalen 2005, p. 676). There has been for some reason a general belief that Brouwer was somehow against the formalization of intuitionistic logic, but this is a clearly erroneous idea. Heyting's paper was scheduled to appear in the ***Mathematische Annalen***; However, after Hilbert had thrown Brouwer out of the editorial board, Heyting did not want to publish in that journal. The paper came out in the *Sitzungsberichte* of the Prussian Academy in 1930, in three parts. There is a charming letter of Bernays to Heyting, reproduced in Troelstra (1990), that shows how well intuitionistic logic was received in the "Hilbert school." Even von Neumann, counted to be among that school, met with Brouwer in April 1929 (cf. his letter to Brouwer, in van Dalen 2011), so one starts to think that perhaps the Göttingen antagonism towards Brouwer was Hilbert's private folly.

We can conclude that intuitionistic logic was, with the exception of Hilbert, well received in the first years of the 1930s. The central ideas of intuitionism, as in Brouwer's (1928) "four insights," were well understood. They were: first, that "the formalists" have to differentiate between the generation of theorems in formal systems and the contentful theory of these systems, and that the latter is based on "the intuitionistic theory of the set of natural numbers." The second insight was that classical logic is "the logic of finite domains." The third insight was that excluded middle equals the assumption of "the solvability of every mathematical problem." The fourth insight was: "The recognition that the justification in terms of content of formalistic mathematics by a proof of its consistency contains a vicious circle." By this Brouwer meant that if one assumes the principle of excluded middle contradictory, then proves this assumption false, excluded middle would follow only if the law of double negation is used, hence the circularity (as in Brouwer 1927, note 8).

A consistency proof of arithmetic was, of course, Gentzen's first objective, but before that, the purely logical basis had to be cleared. To that end, intuitionistic logic got in Gentzen's hands a formulation that soon led to a realization of some of the intuitionistic conditions on a formal level, such as the disjunction property and the underivability of excluded middle. A highlight, one that unfortunately appears only as an announced result in a letter of Gentzen's to Heyting, was a syntactic proof of the underivability of what is known as the "double-negation shift" $\forall x\neg\neg A \supset \neg\neg\forall x A(x)$ in intuitionistic predicate logic, in place of a counterexample in terms of constructive reals.

5. The discovery of natural deduction.

Wir wollen einen Formalismus aufstellen, der möglichst genau das wirkliche logische Schliessen bei mathematischen Beweisen wiedergibt.

The setting up of natural deduction in Gentzen's printed thesis begins with this one-sentence paragraph. *We want to set up a formalism that reproduces as precisely as possible the real logical reasoning in mathematical proofs.* Prior to 2005, knowledge of the discovery of natural deduction was limited to such remarks in Gentzen's published thesis. He goes there through three examples of theorems of logic, taken from Hilbert and Ackermann's axiomatic formulation, trying to "see their correctness in the most natural way possible." Then he states that the characteristic property of such logical reasoning is that it starts from assumptions, contrary to the "logistic" tradition. How he actually found the calculus, presented as a finished set of rules, is not detailed; it is only verified that the intuitive reasoning with the three examples can be reproduced in the calculus. A similar path is taken in his proof of the consistency of arithmetic, Gentzen (1936). It uses natural deduction with a notational variant taken over from sequent calculus. A lengthy example is used for determining the modes of actual inference in mathematical proofs.

One assumption about the origins of natural deduction is that Gentzen converted the handsome axioms of Hilbert and Bernays into rules. Yet another possibility is the explanation of the logical connectives by Heyting, what came to be known as the "BHK-explanation," but it is unknown if Gentzen knew about it. If not, the letter G could be added to BHK. It is instead clear that Gentzen had studied carefully Heyting's (1930) formalization of intuitionistic logic, the notation of which he mostly followed. Heyting produced an axiomatization with eleven axioms for the propositional part, not as much along the model of ***Principia Mathematica*** as has been often said, but along the Hilbert–Bernays axioms in which all connectives and quantifiers are present. Contrary to earlier practice, implication elimination was not the only propositional rule, but there was also an explicit rule of conjunction introduction. Heyting writes that it could be dispensed with, but that the proofs would then be "even more intricate" (*verwickelt*, a word with a negative connotation). Indeed, a little reconstruction shows that this is the case: Heyting's first theorem is $A \,\&\, B \supset A$, and three axiom instances and a step of implication elimination are given as a proof outline.[3] I made

[3]Note also that the final symbol in Heyting's proof should be a, not b.

a complete proof that has eleven lines, three of them conjunction introductions. Such steps can be replaced by a combination of axioms and just implication eliminations.

A reader who undertakes to turn Heyting's proof outlines into formal derivations will soon notice two things: There is so much work that one lets it be after a while, but the way to proceed instead becomes clear, because one starts reading Heyting's axioms and theorems in terms of their intuitive content. Say, the axiom $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$ is seen as a rule by which two previous axioms or theorems $A \supset B$ and $B \supset C$ combine into a new one, exactly as in the work of Bernays (1926) Heyting refers to. His first theorem $A \,\&\, B \supset A$ becomes: from $A \,\&\, B$ follows A. Axioms such as $A \supset A \vee B$ and $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$ turn into what are now the familiar natural deduction rules of disjunction. From this perspective, Gentzen's discovery of natural deduction was nothing but a systematic undoing of the intricate axiomatic formal proofs *à la Heyting* into the intuitive terms of what follows from what.

In February 2005, during a visit to Erlangen, I found among the articles and documents Christian Thiel had collected about Gentzen a photocopy of a handwritten early version of Gentzen's thesis that he had obtained through the good offices of Eckart Menzler-Trott in 1988. Remarkably, part of page two of the manuscript is reproduced photographically among the frontmatter of Gentzen's ***Collected Papers*** of 1969, with the indication that the manuscript comes from Bernays. Most of the manuscript is written in a formal hand ("deutsche Schrift") that is not so easy to read, as one can gather from the photograph. In January 2008, I recovered a missing last page from the original manuscript, checked some symbols that were "indicated in red" as Gentzen wrote but not clearly visible in a BW-photocopy, and so on. The piece is kept in the Bernays collection of the ETH.

Another piece I found in Erlangen back in 2005 were a few pages of Gentzen's in shorthand notation that I tried to decipher. A complete translation to German was soon prepared by Thiel who reads fluently Gentzen's "unified shorthand" (*Einheitskurzschrift*) that had replaced Gabelsberger and some other earlier systems. The piece is titled *Five forms of natural calculi*, dated 23 September 1932, and I saw that it made understandable the way in which Gentzen arrived at the specific form of natural deduction. The easy answer is that he tried out all possible ways consonant with his idea that actual mathematical reasoning proceeds by hypotheses or assumptions, rather than instances of axioms, and that, as indicated in my opening Section, he was a genius who got things right.

The handwritten thesis indicates a proof of the equivalence of natural deduction and axiomatic logic, but the axioms of the latter are those of Hilbert and Ackermann, not the elegant axioms of Hilbert and Bernays that also Heyting used in part. Direct translations are given from the calculus of Hilbert and Ackermann to natural deduction and back. In the printed version of the thesis, the logical axioms come from the Hilbert–Bernays formulation, and there is no direct translation from natural deduction to axiomatic logic. First a straightforward translation from derivations in axiomatic logic to natural deduction is given: It is sufficient to substitute the axioms by their derivations in natural deduction and to leave implication elimination intact. There follows, instead of a translation back, a translation from natural deduction to sequent calculus, and finally the hard task that closes the circle, the translation from sequent calculus to axiomatic logic. The part of the last one that shows how to convert to axiomatic form a step of derivation that moves from B concluded from the assumptions Γ and A into $A \supset B$ concluded from Γ, so in terms of sequent calculus, the step from $\Gamma, A \rightarrow B$ to $\Gamma \rightarrow A \supset B$, contains in fact a proof of the deduction theorem.

As said, what is left of the road to natural deduction are the few pages of *Five forms of natural calculi*. The rules appear in the beginning of that document, with a "boxed" date typical of the stenographic notes. The letters in the rules are direct adaptations of Gentzen's German nomenclature:

TABLE 4. The *Five forms* rules of natural deduction.

$\boxed{23.\text{IX}.32}$ *Inference schemes*:

$$
\begin{array}{cccccc}
AI & AE & OI & OE & FI & FE \\[6pt]
\dfrac{A \quad B}{A \,\&\, B} &
\dfrac{A \,\&\, B}{A} \quad \dfrac{A \,\&\, B}{B} &
\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B} &
\dfrac{\begin{array}{ccc} A & & B \\ C & A \vee B & C \end{array}}{C} &
\dfrac{\begin{array}{c} A \\ \vdots \\ B \end{array}}{A \rightarrow B} &
\dfrac{A \quad A \rightarrow B}{B}
\end{array}
$$

$$
\begin{array}{ccccccc}
UI & UE & EI & EE & RA & REND & CI \\[6pt]
\dfrac{Pa}{x\ Px} &
\dfrac{x\ Px}{Pa} &
\dfrac{Pa}{ExPx} &
\dfrac{ExPx}{Pa} &
\dfrac{\begin{array}{cc} A & \\ \vdots & \\ B & \neg B \end{array}}{\neg A} &
\dfrac{\neg\neg A}{A} &
\dfrac{\begin{array}{cc} & Pa \\ & \vdots \\ P1 & Pa' \end{array}}{Ph}
\end{array}
$$

$$
\begin{array}{ccccc}
NEE & W & D & RA2 & H \\[6pt]
\dfrac{\begin{array}{cc} A & \\ \vdots & \\ B & \neg B \end{array}}{\neg A} &
\dfrac{\qquad}{\neg . A \,\&\, \neg A} &
\dfrac{\qquad}{A \vee \neg A} &
\dfrac{\begin{array}{cc} \neg A & \\ \vdots & \\ B & \neg B \end{array}}{A} &
\dfrac{A \quad \neg A}{B}
\end{array}
$$

The basic rules are from AI (and-introduction) to EE (existence-elimination). Then follow rules for negation, the rule of complete induction, and so on. *REND* should be something like "reduction of negation doubled" (almost the same in German). Rules *W* (for *Widerspruch*, contradiction) and *D* (for *Dilemma*) have zero premisses. *H* stands for "Heyting's rule." The inference schemes for the connectives and quantifiers are the ones we are used to, with the letters that indicate the connective and whether it is an introduction or an elimination. The presence of a rule of induction stems from the main aim of the exercise: namely, to find a proof of the consistency of arithmetic. It is only by January or February 1933 that Gentzen drops, for the time being, arithmetic from his scheme and considers pure logic in isolation.

There follows a division into three calculi, beginning with the weakest calculus that is designated *C*, possibly for *consequentia*, the pure logic of consequence and called minimal logic after Johansson (1936):

Division of the inference schemes:

(C) *AI* to *EE*.

Intuitionistic (I): *AI* to *RA, H*.

Classical (K): *AI* to *REND*.

It goes exactly as it should, especially, falsity elimination, or rule *H*, is not needed in *K*. Gentzen writes *J* for *I*, but this is pure typography. The same goes for the names of his calculi *NI* and *LI*, as well as rule *CI* that is *VJ* in the original German.

There are two basic forms of proofs, tree proofs and net proofs:[4]

Formal shape of proof: each proposition conclusion of at most one inference, no circle in inference.

T *Tree form: each proposition except for the endproposition a premiss in exactly one inference.*

N *Net form: each proposition except for the endproposition premiss in at least one inference, eventually several.*

I have tried to find out who the first one was to put logical inferences in the form of a tree. It is the kind of question almost no one asks, but I found the paper Hertz (1923) as a best candidate, suggested by Schroeder–Heister (2002, p. 251). The notation in Hertz is somewhat awkward, with premisses partially stacked. Gentzen's paper on Hertz systems of 1932, as well as all of his later papers, take it as a matter of course that one does derivation trees. Since the papers of Hertz remained unread, our habit of drawing derivation trees stems from the work of Gentzen, and it would be of some interest to chart how that habit spread out. In *Five forms* we see that he also considered what are today called directed acyclic graphs ("dags," from the first letters) as an alternative.

Derivation trees have several extremely useful properties that the earlier linear derivations do not have. Assumptions are top formulas so that it

[4]Gentzen uses for the former the word *Stammbaum*, family tree.

can be seen what depends on what, and it is possible to prove that two derivations can be composed into one, i.e., that composition is an admissible rule. Further, the order of application of rules can be permuted, a thing noted and used already by Hertz.

Permutation of rules became the central single method in Gentzen's work on proof theory. The aim is to get a hold of the combinatorial possibilities offered by an axiom system, pure logic in the first place. One shows through permutability arguments that formal proofs by rules that correspond to an axiomatic system can be brought to a standard form, such as normal derivations in natural deduction or cut-free derivations in sequent calculus. Thus, we do not say that we can master the structure of an arbitrary proof, but instead that any proof transforms into one that has a transparent structure.

Logicians are struck by the purity of Gentzen's method and understand and learn to use it. A comparison to the work of Skolem is instructive in this respect: He gave a profound analysis of proofs by the axioms of lattice theory and projective geometry in a paper that, as much as any paper in logic, must have been studied by many, namely the 1920 paper that began with the Löwenheim–Skolem theorem. No one understood his arguments in the latter parts of the paper, and his results, such as a polynomial-time algorithm for what is known as the word problem for free lattices, had to be reinvented in 1988! His geometrical results were used once between 1920 and 1995.[5]

As to the tree form, Gentzen took it to be a deviation from actual reasoning that is linear. The thought seems to have been that proofs are texts written on lines after lines.[6]

Gentzen's *Five forms* continues with various ways of treating free and bound variables, of closing open assumptions, and so on. Each alternative has a number and he studies the combinations of the alternatives, as shown by the following passage:

> *Variable conditions* . . .
>
> Condition 10: There is a linear order of propositions in a proof, in which each conclusion comes after its premisses, and in which no common free variable appears before its *EE*.
>
> Condition 30: In a *UI*, there must not occur any common variable that depends on the all-variables.

[5]It took me ten years to understand in full Skolem's method; to have done the permutations of order of application of axioms in his head or on discarded paper, with no clue left to the reader; *cf.* my 2007 paper for a reconstruction of Skolem's results in lattice theory and plane projective geometry in terms of proof trees and permutation arguments.

[6]I don't doubt that he would have been delighted to see modern theories of sentence structure that analyze them into a two-dimensional syntax tree form and consider the linear writing a codification of a deeper tree structure. Such is the case even with linear derivations: The lines are numbered and the justifications of the steps, through reference to the line numbers of the premisses, codify a tree derivation.

> Condition 34: There is to each free variable an eigenvariable equal to it.

One remains flabbergasted in front of such numbers, a minimum of at least 34 different conditions on formal proofs!

Next Gentzen starts combining the various possibilities, rules for variables, and so on, to locate his *five forms of natural calculi*:

> *Important forms of proof*:
>
> *NNA*: natural net proof general form, i.e., net form with conditions 6.2; 11, 10, 30, 33. (sheet 78.3–79.)
>
> *N1*: Tree form, with conditions 6.2; 27, Bb4. (Bb 1, Bb3, Bb 4.) (sheet 66.3.)
>
> *N2*: Tree form, inference forms *AI, AE, UI, UE, RA* and *REND*, with condition 6.2; Bb 4 (only the special case of condition 32.) (sheet 80.2)
>
> *NN2*: Net form, inference forms *AI, AE, UI, UE, RA* and *REND*, with condition 6.2; 11. (sheet 84.1)
>
> *NO*: Tree form, with condition 6.2; 10, 30, 33. Corresponds to *NNA*, condition 11 and superfluous because of the tree form. For complete "generality" there is still missing the admission of several *EE*'s with the same variable ...

Fragments of the full language are considered, as in *N2* and *NN2*, and there appear mysterious codes such as "Bb," (most probably for *Beweisbedingung*; proof condition). Numbers such as "sheet 84.1" refer to Gentzen's notes from a series he called **D**, possibly for "dissertation." Each sheet was a four-page legal size paper (*Kanzleiformat* in German), written in stenographic notation, with the sheet number followed by a dot followed by a page number from 1 to 4. Thus, 84 sheets, say, could amount to anything up to 500 and more printed pages! Moreover, **D** was not the only series. There is even a reference to "my book on propositional logic."

A certain attitude of starting from a scratch, as well as intelligence and concentrated work, produce results. This is the essential secret behind the discovery of natural deduction, if one has to guess from the few pages that have remained of the series **D**.

Sources: The contents of Gentzen's handwritten thesis manuscript as well as his stenographic manuscripts are detailed in my 2009 handbook chapter on Gentzen.

6. Normalization. The first, introductory chapter of Gentzen's thesis manuscript ends with a summary of his five central results. The fourth is:

> IV. Theorem:
>
> If a logical proposition is provable in the calculus *N1I*, there is a proof for it in which only subpropositions of it appear, eventually with different variables.

The word "conjectured" has been cancelled from the theorem. The calculus *NJI* of intuitionistic natural deduction is given in the first chapter. The rules are now, in comparison to the *Five forms*, stable and fixed, and notational improvements have been taken. Even if there is no date in the manuscript, the changes indicate that it is a clearly later product than the *Five forms* of 23 September:

TABLE 5. Gentzen's thesis ms. rules of natural deduction.

AI	AE	OI	OE
$\dfrac{A \quad B}{A \,\&\, B}$	$\dfrac{A \,\&\, B}{A} \quad \dfrac{A \,\&\, B}{B}$	$\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}$	$\dfrac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$
UI	UE	EI	EE
$\dfrac{Dy}{(x)Dx}$	$\dfrac{(x)Dx}{Da}$	$\dfrac{Da}{(Ex)Dx}$	$\dfrac{(Ex)Dx \quad \begin{matrix}[Dy]\\ C\end{matrix}}{C}$
FI	FE	V	
$\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}$	$\dfrac{A \quad A \supset B}{B}$	$\dfrac{\curlywedge}{A}$	

Rules *UI* and *EE* have the standard variable restrictions. Square brackets indicate any number $\geqslant 0$ of discharged assumption formulas. In actual derivations of which Gentzen gives some examples, instances of rules are indicated by writing the name of a rule next to the inference line, and assumptions are indicated as closed by numerical markers next to the rule symbol and above the closed assumptions.

Three rules in the table are cancelled: a rule T that concludes A from A, and rules R and V that are like rules RA and H of Table 4. Rule T is perhaps there to make the derivation of $A \supset A$ look less awkward. The derivations with and without the rule would be:

$$\dfrac{\dfrac{\overset{1}{A}}{A}\,T}{A \supset A}\,FI\,1 \qquad\qquad \dfrac{\overset{1}{A}}{A \supset A}\,FI\,1$$

It takes some mental effort to see the premiss A in the right derivation simultaneously as an assumption and a conclusion. As to the letter T, Gentzen notes later in the manuscript that the assumption of A in natural deduction is written as $A \to A$ in sequent calculus and calls it "the tautological sentence."

Rules R and V are two primitive rules for negation, and a transformation is given that reduces two successive instances of R into one such instance (page 9 of the manuscript):

$$\begin{array}{c}\dfrac{\begin{array}{c}\begin{bmatrix}2\\C\end{bmatrix}\\ \zeta\\ A\end{array}\qquad \dfrac{\begin{array}{c}\begin{bmatrix}1\\A\end{bmatrix},\begin{bmatrix}2\\C\end{bmatrix}\\ \gamma\\ B\end{array}\qquad \begin{array}{c}\begin{bmatrix}1\\A\end{bmatrix},\begin{bmatrix}2\\C\end{bmatrix}\\ \delta\\ \neg B\end{array}}{\neg A}{\scriptstyle R1}}{\neg C}{\scriptstyle R2}\\ \varepsilon\end{array} \quad \text{becomes:} \quad \begin{array}{c}\dfrac{\begin{array}{c}\begin{bmatrix}\begin{bmatrix}2\\C\end{bmatrix}\\ \zeta\\ A\end{bmatrix},\begin{bmatrix}3\\C\end{bmatrix}\\ \gamma\\ B\end{array}\qquad \begin{array}{c}\begin{bmatrix}\begin{bmatrix}2\\C\end{bmatrix}\\ \zeta\\ A\end{bmatrix},\begin{bmatrix}3\\C\end{bmatrix}\\ \delta\\ \neg B\end{array}}{\neg C}{\scriptstyle R3}\\ \varepsilon\end{array}$$

An instance of R followed by V reduces similarly into just an instance of V:

$$\begin{array}{c}\dfrac{\begin{array}{c}\alpha\\ A\end{array}\qquad \dfrac{\begin{array}{c}[A]\\ \gamma\\ B\end{array}\quad \begin{array}{c}[A]\\ \delta\\ \neg B\end{array}}{\neg A}{\scriptstyle R}}{C}{\scriptstyle V}\\ \varepsilon\end{array} \quad \text{becomes:} \quad \begin{array}{c}\dfrac{\begin{array}{c}\begin{bmatrix}\alpha\\A\end{bmatrix}\\ \gamma\\ B\end{array}\quad \begin{array}{c}\begin{bmatrix}\alpha\\A\end{bmatrix}\\ \delta\\ \neg B\end{array}}{C}{\scriptstyle V}\\ \varepsilon\end{array}$$

Gentzen now notes that one can define negation through the use of a "false proposition F": $\neg A$ becomes $A \to F$, and "the inference R can be expressed as follows with the help of FE and FI":

$$\text{In place of} \quad \dfrac{\begin{array}{c}\overset{1}{[A]}\\ \gamma\\ B\end{array}\quad \begin{array}{c}\overset{1}{[A]}\\ \delta\\ \neg B\end{array}}{\neg A}{\scriptstyle R1} \quad \text{enters:} \quad \dfrac{\dfrac{\begin{array}{c}\overset{1}{[A]}\\ \gamma\\ B\end{array}\quad \begin{array}{c}\overset{1}{[A]}\\ \delta\\ B \to F\end{array}}{F}{\scriptstyle FE}}{A \to F}{\scriptstyle FI\ 1}$$

Thus, the rules with primitive negation become derivable by the pairs $\supset E$, $\supset I$ and $\supset E$, $\bot E$, respectively. It is now seen clearly that the two successive instances *R, R* and *R, V* pinpoint precisely the possible convertibilities and the transformations Gentzen gives their normalization steps: Writing out with the defined notion of negation the above derivations with R followed by another R and R followed by V, respectively, we find a detour convertibility on implication in both. Writing out Gentzen's reduced derivations, we find that the detour conversions have been done. Thus, the defined notion of negation fits perfectly to Gentzen's scheme of introduction and elimination rules and their conversion into normal form, whereas the conversions with the primitive rules for negation do not follow any general pattern.[7]

[7]These observations answer questions posed about primitive negation by Ole Hjortland and Stephen Read during the Gentzen conference in St Andrews, November 2009.

In a chapter on normalization that has been added later, negation is defined through *falsum* and a modified rule V with conclusion $\curlywedge$ added next to the cancelled rules. The printed thesis is half-way between primitive and defined negation, with I- and E-rules for negation and rule V, all written with $\curlywedge$.

The implication symbol in the first chapter was an arrow but got changed into the horseshoe in the new chapter on normalization. The universal and existential quantifiers were still written as $(x), (Ex)$. One earlier chapter became the article on translation from classical to intuitionistic arithmetic and has the older notation, whereas the printed thesis instead takes one step further from the manuscript and uses the now standard quantifier symbols $\forall, \exists$, the former of which is original to Gentzen. It can be used at places to date his manuscripts. The translation part was submitted as an article in mid-March, but one cannot simply conclude that the normalization proof would have been produced that late. The translation uses axiomatic logic and the notation is that of Heyting's 1930 article.

In Gentzen's original presentation of natural deduction, finished formal derivations ended with the closing of all open assumptions. The same is in fact true of his printed thesis, although this aspect is always ignored. Thus, implication introductions build up the theorem to be proved. At some point, Gentzen must have realized that the rules of natural deduction move, by and large, in the *I*-rules to conclusions that contain the premisses as components, and the other way around for the *E*-rules. With the closing of all open assumptions in the end, the conclusion became the longest formula in a derivation that also contained all the formulas in the derivation as subformulas, if no detours had been taken. It is seen from his summary that he first conjectured the normalization theorem for derivations in intuitionistic natural deduction, then found a proof and cancelled the word "conjecture." So much is even stated in his printed thesis, even if not directly (introduction, paragraph 2):

> A closer inspection of the natural calculus led me in the end to a general theorem that I want to call the "Hauptsatz" in what follows.
>
> ⋮
>
> To be able to express and prove the Hauptsatz in a convenient form, I had to lay as a basis a logical calculus especially adapted to the purpose. The natural calculus proved not to be suitable. It is true that it has the essential properties needed for the validity of the Hauptsatz, but only in its intuitionistic form.

Gentzen is not exactly stating that he has a detailed proof. Today, knowing better, we can draw conclusions from his various remarks. Subsequently, the emphasis in research was on Gentzen's systems of sequent calculi until 1965, when Dag Prawitz gave a direct proof of normalization for a system of classical predicate logic without disjunction and existence. Then he extended it to the full language of intuitionistic predicate logic. Another, less known proof was given by Andres Raggio the same year 1965.

Gentzen's handwritten thesis manuscript contained as the greatest surprise a detailed proof of normalization for intuitionistic logic. It is clearly a later addition, because it contains the result conjectured in the summary. The proof forms a separate chapter, with Gentzen's name on it, but with no other title than *"Dissertation. Part III."* The writing is different form the formal hand of the rest, what could be described as a schoolboy's hand, rather clumsy, but eminently legible, and perhaps intended to be read by a printer. The text is perfect and finished, "ready for publication" as Gentzen might have said.[8] What happened to this proof and why Bernays ignored and forgot about it, is not known. As Gentzen at least intimated in the printed thesis, he was unable to extend the result to classical natural deduction, therefore invented classical sequent calculus and proved the *Hauptsatz* that applied to both a classical and an intuitionistic calculus. There was no need for the normalization proof and he could afford to never mention it again.

After having found the proof in 2005 and checked it for what it is, I phoned Dag Prawitz and told about the discovery, to which his cool reply was: "Of course he knew the result, for his remarks in the thesis are so suggestive." We also talked of why Bernays never mentioned that he was in the possession of a detailed proof of normalization, not even after the appearance of Prawitz' proof. To this Dag said that he is sure Bernays never took it up in their discussions, even if they met well after the normalization proof of 1965 was generally known.

Szabo's translation of Gentzen's thesis was published first in the ***American Philosophical Quarterly*** in 1964–65. It has a brief introductory comment by Bernays, but nothing new is revealed. There is also a letter at the ETH collections from Bernays to Szabo, dated 14 May 1968 (manuscript Hs. 975:4536, ETH Zurich), that gives a lengthy commentary to various issues, but the normalization proof is not mentioned. Commenting the latter's introduction to Gentzen's ***Collected Papers***, Bernays asks:

> How do you mean that Gentzen was led to his Hauptsatz through an investigation of the "natural calculus"? Consider that modus ponens is an essential elimination rule in the natural calculus.

Further, there is the somewhat disturbing set of lectures Bernays gave in Princeton in 1935–36, titled *Logical Calculus*. These lectures explain natural deduction, but in a messy and awkward style of notation for derivation trees. Only classical logic is covered and none of the aims that Gentzen had, normalization, subformula property, consistency, is mentioned. In the end of the part on natural deduction (p. 34), Bernays suggests that we might as well stick to truth tables.

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[8] In a couple of places in his stenographic notes, Gentzen describes a piece in such terms (*publikationsreif verfasst*).

Reflecting on Bernays, I see a clear lack of appreciation of Gentzen's discovery of natural deduction in his writings. One may ask what effect the failure to prove normalization for the classical calculus can have had on his judgment and later recollection, against the clear-cut situation in sequent calculus, with the *Hauptsatz* covering in one result both intuitionistic and classical logic. So, natural deduction seems to have been for Bernays just a part of the way to sequent calculus, and his question and comment to Szabo, quoted above, seem further to indicate that he thought that the subformula property fails in natural deduction because of the indispensability of rule $\supset E$. He must have been led to this thought by Herbrand's result in which modus ponens plays the role of cut.[9] Thus, he had not realized that Gentzen had proved a normalization theorem for intuitionistic natural deduction, even if he was in possession of the detailed proof for decades.[10] He was instead careful to mention in connection with natural deduction what he considered to be a simultaneous discovery and development of natural deduction by Jaskowski (1934), and suggested in the mentioned letter Szabo to do the same. If one reads Jaskowski's paper and compares it to Gentzen, it will be very difficult to see anything at all developed in a way that could be set on a par with Gentzen in the way Bernays did. The calculus is classical and the derivations fashioned in a linear form. The latter feature, especially, makes it practically impossible to obtain any profound insights into the structure of derivations. Consequently, there are no results in this work beyond what, more or less, Aristotle could have said about the hypothetical nature of proof.

From today's perspective, Gentzen's proof of normalization is exactly what it should be: First permutative convertibilities are eliminated, so that no *I–E*-pairs separated by an *E*-rule remain, next detour convertibilities are eliminated. This is the standard proof procedure of normalization. Gentzen's almost maniac attention to such details as the avoidance of clash of variables through the "purification of derivations" strikes the eye, for this was in the very beginning of proof theory as we know it, early in 1933.

Finally, a few words about the reception of Gentzen's logical calculi by the intuitionists: Gentzen had sent his double negation translation work to Heyting early in 1933, and mentioned some results from his thesis in later correspondence. A letter of January 1934 to Heyting mentions that he can prove the intuitionistic underivability of what today is called "double negation shift" in predicate logic: $\forall x \neg\neg A \supset \neg\neg\forall x A(x)$. Here is just another

[9]Let the premisses be A and $A \supset B$, so the conclusion of modus ponens is B. The premisses in sequent notation are $\rightarrow A$ and $\rightarrow A \supset B$. Rule $R\supset$ is invertible, so $A \rightarrow B$ is derivable. Then, a cut with cut formula A gives the conclusion $\rightarrow B$ and trace of A may get lost.

[10]The package of Gentzen's handwritten manuscript pages is folded and at the back of the normalization proof there is written obliquely across the blank page in Bernays' hand: "Konzepte von Herrn Gentzen."

indication of his careful reading of Heyting's 1930 paper: Its second part mentioned two classical theorems of predicate logic, the double negation of the universal form of excluded middle discussed above, and the double negation shift, that both fail intuitionistically, as shown by Brouwerian counterexamples. Gentzen was writing in the letter in response to a card of Heyting's in which the latter asked about the decidability of intuitionistic propositional logic, mentioned in a letter of Gentzen's of November 1933. Heyting was obviously struck by Gentzen's proof of the said decidability and by natural deduction in general; In the first of a series of Dutch papers that began in 1935, titled "Intuitionistische wiskunde" (Intuitionistic mathematics), he explains this new formulation of the logical principles. The naturalness of formalization after the model of Gentzen's calculus is one conclusion (p. 81):

> If one then introduces in the logical calculus the usual mathematical signs and writes the axioms of the mathematical theory that one wishes to develop, with the aid of the logical signs, it would be possible to write a mathematics book with only formulas, without a connecting text, because the reasonings that are usually included in the text, we can now also write with formulas.
>
> ⋮
>
> It is in principle remarkable that a proof can be assessed independently of its content in thought, exclusively on the basis of the intuitive properties of the collection of symbols that make up the proof.

It was a real pity that Gentzen did not present his normalization theorem for natural deduction in the published thesis or explain it to Heyting in correspondence. The result would have cut short the talk about the possible circularity of Heyting's explanation of implication, at least for first-order logic. Heyting's initial enthusiasm for Gentzen's natural deduction is not reflected in his later work; Heyting's book of 1956 looks as if it had gone through the hands of a censor who deleted from the ample references all those to Gentzen.

One can also ask why Brouwer remained silent in front of one who came from Hilbert's Göttingen but agreed with the intuitionists on all essential points, as in the "four intuitionistic insights on formalism" listed in Brouwer (1928) and referred to and endorsed in Gentzen (1936).

7. A persistent problem. Gentzen's coding of the logical principles behind mathematical proofs led to a system of introduction and elimination rules. He wrote in his thesis that the former were definitions of sorts of the connectives and quantifiers, in that they gave sufficient conditions for inferring a proposition of the corresponding form. The elimination rules were inverses of sorts of the introduction rules.

The system of introduction and elimination rules falls short of classical logic. A classical calculus of natural deduction, *NK* in Gentzen's nomenclature, was obtained by letting instances of the law of excluded middle appear as top formulas in derivations. If one applies disjunction elimination, the formulas in the instance may be lost so that the subformula property of normal derivations need not hold. Indeed, Gentzen says in the quoted passage of the previous Section that only intuitionistic logic "has the essential properties needed for the validity of the Hauptsatz," i.e., that normalization fails for classical logic.

Gentzen had obviously explored various ways of obtaining classical logic from intuitionistic logic. One such way was to add a rule of double negation elimination, from $\neg\neg A$ to conclude A. Even here, the symmetry of introductions and eliminations broke down with negation. For negation introduction, one could take $\neg A$ to be, as in intuitionistic natural deduction, the same as $A \supset \bot$, with $\bot$ the false formula. There was, however, no obvious way of getting rid of a negated formula, because double negation elimination did not fit into the pattern.

One might think *a priori* that double negation elimination works, if a trivial modification to the subformula property is added, namely that also negations and double negations of formulas known from the open assumptions and the conclusion can appear in a derivation. Such is, however, not the case. The matter is easiest seen if we formulate a rule of indirect proof:

TABLE 6. The rule of indirect proof.

$$\dfrac{\begin{matrix}\overset{1}{\neg A}\\ \vdots\\ \bot\end{matrix}}{A}\,{\scriptstyle DN,1}$$

The assumption $\neg A$ is closed by the inference.

The rule of indirect proof has the same deductive force as the rule of double negation elimination: In one direction, assume $\neg\neg A$ and $\neg A$. Now $\bot$ follows, so by *DN*, A can be concluded under the open assumption $\neg\neg A$. In the other direction, assume there to be a derivation of $\bot$ from $\neg A$. By implication introduction, we have $\neg A \supset \bot$, i.e., $\neg\neg A$, and by double negation elimination, A.

Looking at rule *DN*, we realize that nothing prevents its conclusion A from being the major premiss of an elimination rule. If A is such a major premiss, there would be no guarantee for the subformula property. This possibility is the likely reason for Gentzen's thought that normalization fails for classical natural deduction. The real novelty of Dag Prawitz' thesis *Natural Deduction: A Proof-Theoretical Study*, was not the normalization theorem for intuitionistic logic, but the corresponding result for classical natural deduction for the language of predicate logic that does not have disjunction or existence. His deft move was to limit indirect proof to atomic formulas, and to show that it is *admissible* for arbitrary formulas in the $\vee, \exists$-free fragment. Once the rule is limited to atoms, its conclusion cannot be a major premiss in an elimination, and the subformula property of normal derivations holds, as shown by Prawitz in 1965. The proviso is that with atomic subformulas in a derivation, also their single negations and the formula $\perp$ can appear.

There have been many attempts at extending Prawitz' result to the full language of predicate logic. In propositional logic, it is possible to generalize indirect proof into a "rule of excluded middle" by which the case of disjunction is cleared (cf. Negri and von Plato 2001, section 8.6). For predicate logic, restrictions have been put on disjunction and existence elimination, as in Statman (1974) and Stålmarck (1991). In Andou (1995), a global proof transformation leads to the normalization of derivations in the full language. After so many attempts, it came as quite a surprise when I realized that Prawitz' result contains the key to a general solution with no restrictions or *ad hoc* features. In an earlier work (von Plato 2001b), to be discussed in Section 15 below, I had changed the elimination rules for conjunction, implication, and universal quantification so that they concluded an arbitrary formula, in exact analogy to disjunction and existence elimination. The effect of the use of these "general elimination rules" was that the permutative conversions worked uniformly, not just for disjunction and existence elimination, and I could define a derivation in intuitionistic natural deduction to be normal whenever *all major premisses of elimination rules are assumptions.* Now this is a property that has precisely the effect of Prawitz' restriction on indirect proof, for if no major premiss of an elimination rule has been concluded, none has been concluded by indirect proof either. Contrary to Prawitz, no restrictions are needed, but the rule applies to arbitrary formulas. It turned out easy to prove normalization and the subformula property, the former by routine conversions in which rule *DN* is permuted below elimination rules, the latter as just a little addition to the subformula property for the intuitionistic calculus (cf. von Plato and Siders 2012).

By the correspondence between natural deduction with general elimination rules and standard natural deduction (as in von Plato 2011), the above solution can be carried over to give a normalizing system of classical natural deduction with Gentzen's intuitionistic rules and indirect proof, as detailed out in von Plato and Siders (2012).

The *admissibility* of a rule, as in Prawitz' solution, means in general that the conclusion of a rule is derivable whenever its premisses are. It is thus a weaker notion than the *derivability* of a rule in which a derivation of the conclusion is required from the premisses. Proofs of admissibility work on how the premisses have been derived and show in each case how such derivations can be transformed into a derivation of the conclusion.

The point in leaving out disjunction and existence in Prawitz' calculus is that with them, the application of a rule of indirect proof does not reduce to the components. It is contrariwise with conjunction, implication, and universal quantification.

Gentzen was not far from the above, in a remarkable one-page note from the beginning of 1933. It is titled *Decision in classical predicate calculus reducible to decision in intuitionistic calculus with only* $\supset$ *and* ( )*?* and contains a transformation of derivations in classical logic to what came to be called minimal logic. He writes that propositions of classical logic can be written with implication, a false formula $\mathcal{F}$, the universal quantifier, and with atomic formulas occurring only negated. Then:

> This proposition is classically and intuitionistically of the same value. (I.e., correct or incorrect in both systems.) By the subformula theorem it is therefore provable, if at all, already with the inferences for $\supset, (\;)$, and V. I would just like to have that one does not need the inference V. This should be possible. It is done more or less as follows: One puts before each proposition and subproposition always $\neg\neg$. Then one has to redo the inferences. This is still a bit unclear.

When working through the inferences, Gentzen saw that the rule of double negation elimination (DN) does no work here. He shows first that in the fragment, application of DN to compound formulas reduces to the components, quite analogously to Prawitz' transformations, until it applies to atomic formulas, to conclude A from $\neg\neg A$. When A is double negated at the outset, DN removes two out of four negations, and "it is obvious that the inference DN can be completely eliminated by these steps."

Sources: Gentzen's note with the translation of derivations is dated by him afterwards as "should be about the beginning of 1933." It is found in the series **PLF** that stands for *Forms of predicate logic*. Note that the subformula property, and therefore normalization, were done by its writing, at least for the fragment he discusses.

8. Two failures and guesswork about timing. Gentzen met, by early 1933, with two failures: He was unable to analyze classical logic by the means of natural deduction. Secondly, his plan to prove the consistency of arithmetic within a system of natural deduction faltered. To overcome the first failure, he invented the second main form of proof-theoretical calculi as we know them today, namely *sequent calculi*. The invention of sequent calculi was arguably the single most important event in the development of systems of proof.

Gentzen's way to sequent calculus is our next topic. Before the rediscovery of the handwritten manuscript version of his thesis, the only clues on the discovery of sequent calculus had been in his printed doctoral thesis, Gentzen (1934–35), a few remarks in later papers, and a forgotten paper by

Bernays from 1965 titled *Betrachtungen zum Sequenzenkalkül* (Considerations on the sequent calculus). The handwritten manuscript has the same title, *Untersuchungen über das logische Schliessen* (Investigations into logical reasoning) as the published version Gentzen (1934–35), but the order of things and contents are rather different. The manuscript begins with an introduction that presents Gentzen's overall plan for a consistency proof of arithmetic:

I. To put up the calculus of natural deduction.
II. To prove that it is equivalent to standard axiomatic calculi.
III. To prove that classical arithmetic reduces to intuitionistic arithmetic.
IV. To prove normalization.
V. To extend normalization and the subformula property to arithmetic.

Of these parts, the first was finished by late September 1932. We know this from the *Five forms of natural calculi*. There was just some wavering about the rules for negation, but the working through of the detailed proof of normalization dictated a final solution. Gentzen cancelled the various negation rules and jotted down in the margin of the introductory chapter of the thesis manuscript the falsity elimination rule.

The second part is briefly sketched in the manuscript and shows the origin of sequent calculus: the main single component is the translation of derivations in natural deduction into derivations in the axiomatic logical calculus of Hilbert and Ackermann's book. The third part is known today as the Gödel–Gentzen translation. The latter sent his manuscript *Über das Verhältnis zwischen intuitionistischer und klassischer Arithmetik* (On the relation between intuitionistic and classical arithmetic) in January or February 1933 to Heyting, then submitted it to the ***Mathematische Annalen***, the leading mathematics journal of the time. Gödel's publication of the result made him retrieve the submission that had already reached the stage of proofs. These were preserved and the paper got published, first in English translation in the Szabo edition, and then in the original in 1974 with an introduction by Bernays. An examination of the thesis manuscript shows that the text of the article was taken out directly from it. Namely, there is a list of corrections to the paper left in the manuscript, with all corrections made in the submitted typeset article. For the fourth part, the manuscript contains a detailed proof of normalization for intuitionistic natural deduction, a piece that was carefully written to be ready for publication. An English translation and introduction to the proof is found in von Plato (2008).

Now to the second of Gentzen's failures: Some time after having written a letter to Hellmuth Kneser, dated 13 December 1932 and containing the words that he "hoped to finish soon" his proof of consistency of arithmetic, Gentzen must have realized that the subformula property of normal derivations in arithmetic, the object of part V of the thesis plan, does not go through. Thus, the projected proof of consistency would not succeed. The specific reason is that it is not possible to keep track of possible induction formulas in his induction rule, written as follows:

TABLE 7. Gentzen's rule of induction in natural deduction.

$$\dfrac{A(1) \qquad \begin{matrix} \overset{1}{A(x)} \\ \vdots \\ A(x') \end{matrix}}{A(t)}\,{\scriptstyle CI,1}$$

The assumption $A(x)$ with the eigenvariable x is closed at the inference, as marked by the label 1. The base case $A(1)$ of induction need not be a subformula of the open assumptions it depends on. If the conclusion $A(t)$ of the induction rule is decomposed through an elimination rule, traces of the induction formula can disappear from the overall conclusion of the derivation. The consideration of these possibilities led to revisions in Gentzen's plan for a doctoral thesis on the consistency of arithmetic, including a submission of the translation from classical to intuitionistic arithmetic as a separate article and concentration on the part that dealt with pure logic. The latter led soon to the sequent calculi for intuitionistic and classical logic that became the main topic of the finished thesis.

How and when Bernays got the early handwritten version of the thesis is not clear. However, as noticed by Eckart Menzler-Trott, there is a marginal remark in Bernays' hand in the summary part of Gentzen's results. This remark is that the translation from classical to intuitionistic logic is itself intuitionistic, "even finitary in a strict sense." These self-same words are found in the correction sheet of Gentzen's original version, indicated by him as to be cancelled, and indeed they are absent in the preserved galley proofs. Thus, Bernays had read the original manuscript, then probably suggested to Gentzen that a separate article be made of the double negation translation part, for which Gentzen needed it back.[11]

A presumably typewritten thesis was handed in by 26 May 1933, with the same title *Untersuchungen über das logische Schliessen* as the extant manuscript and the printed version, but no copy has been preserved. At that time, Gentzen's teacher Bernays had already been expelled from the university on racial grounds and a thesis evaluation was signed instead by Hermann Weyl on 9 June. Only minor changes can have been made to the version that was printed, because it was submitted on 21 July, after a final oral exam for the doctoral degree on 12 July. It was the same with Gentzen's *Habilitationsschrift*, finished in 1939, which was published in 1943 with only

[11]There is in addition just one small correction in Bernays' hand, Gentzen's "gäbe" corrected twice into "gabe" in the introduction, and nothing in the other parts.

minor modifications as his last article. – Incidentally, even that script has no original left in Göttingen. There exists by chance a stencil copy of all but the first four pages of the 32-page typewritten manuscript with changes made in ink. Gentzen must have gotten the manuscript back in 1943 from the printer's together with the galley proofs. Short of paper, he started writing his notes with the title *Beweistheorie der intuitionistischen Zahlentheorie*, the series **BTJZ**, at its back, ripping off sheets one after the other from a stapled set. The result is that, compared to the stenographic notes for **BTJZ**, the pages of the *Habilitationsschrift* are in reverse order and upside down!

Here is my best guess as to what happened: Gentzen had given the thesis manuscript to Bernays, not including the detailed proof of normalization of chapter IV nor chapter V. This may have been for the turn of the year 1932–33, and anyway before writing the one-page note of the beginning of 1933 discussed in the previous Section. A planned chapter V was to contain the main result, namely a proof of the consistency of arithmetic. This much is clearly indicated in the summary of the introductory chapter. In the meanwhile, perhaps in January, Gentzen came to the conclusion that the proof would not go through satisfactorily, so no chapter V seems to have been written. Chapter III became a separate article, obviously appreciated by Bernays, for he mentioned repeatedly the result in his work and even wrote 35 years later that he was "surprised by Gentzen's discovery as he explained it to me." (Letter to Szabo, 14 May 1968.)

Of the original plan there remained most of I (introduction to natural deduction), II (translation to axiomatic logic), IV (normalization, now numbered III), and the short sketches of sequent calculi. The translation chapter was then moved last in the pile that formed the manuscript, as in the printed thesis. The detailed text of III is a later product. The consistency problem of arithmetic abandoned for the time being by February 1933, it is a small miracle that Gentzen was able to turn the pieces left into a detailed treatment of natural deduction and sequent calculi by the late spring.

The remaining package must have been given in turn back to Bernays, with marginal indications on how the new thesis be organized. In the meantime, the latter had met with other problems, for we are perhaps in February or March 1933. He was also very busy with the preparation of the first volume of the ***Grundlagen der Mathematik*** that must have been his main occupation from the late 1920s on. Thus, Bernays seems not to have studied the newly added normalization proof. The presence of the sheet of corrections to the old chapter III remains unexplained. Perhaps there was a typewritten version given to Bernays and the corrections gave a final version, but who knows.

Gentzen mentions in the printed thesis what I listed as his first failure, namely to prove normalization and the subformula property for classical natural deduction. He states that he had to invent another calculus, namely classical sequent calculus, to this purpose. How this happened will now be detailed out.

9. The first steps towards a sequent calculus. A narrow path led Gentzen to the discovery of sequent calculus. The first component was a translation from a classical natural calculus to the axiomatic system of logic of Hilbert and Ackermann. Secondly, there was Gentzen's experience with the sequents of Hertz systems, which made him see a similarity of structure in the translation and in sequent notation.

A four-page text titled *The N2-formalism* ends the handwritten thesis manuscript. It has been shifted to the end from before other parts, the reason clearly being a decision to present the translations between the different logical calculi at the end of the thesis. The numbering of the calculi indicates that it has been written before the part that has the title *Calculus NL3, "natural-logistic" parallel calculus*. *N2* is the classical ¬-&-∀-fragment of natural deduction. The rules are:

TABLE 8. The classical natural calculus *N2*.

$$\begin{array}{c} AI \\ \dfrac{A \quad B}{A \,\&\, B} \end{array} \qquad \begin{array}{c} AE \\ \dfrac{A \,\&\, B}{A} \quad \dfrac{A \,\&\, B}{B} \end{array} \qquad \begin{array}{c} UI \\ \dfrac{Pa}{(x)Px} \end{array} \qquad \begin{array}{c} UE \\ \dfrac{(x)Px}{Ph} \end{array}$$

$$\begin{array}{c} DN \\ \dfrac{\begin{array}{cc} A & A \\ \vdots & \vdots \\ B & \neg B \end{array}}{\neg A} \end{array} \qquad \begin{array}{c} TND \\ \dfrac{\neg\neg A}{A} \end{array} \qquad \begin{array}{c} CI \\ \dfrac{\begin{array}{cc} & Pa \\ & \vdots \\ P1 & Pa' \end{array}}{Ph} \end{array}$$

Schematic letters $A, B, C, \ldots$ are used in the rules, except when free and bound variables appear and the writing becomes Pa, Ph, and so on. The variable condition is that a is an eigenvariable in rules *UI* and *CI*. The latter, as well as *UE*, have an arbitrary term h in the conclusion. Furthermore, an eigenvariable must not occur in formula A in rule *DN*.

Gentzen now takes into use an axiomatic formulation of logic that he calls the "*H-A*-formalism," from Hilbert and Ackermann. The axioms are (Hilbert and Ackermann 1928, p. 22 and 53):

TABLE 9. The H-A axioms of Gentzen's thesis ms.

(a) $A \vee A \rightarrow A$,
(b) $A \rightarrow A \vee B$,
(c) $A \vee B \rightarrow B \vee A$,
(d) $(A \rightarrow B) \rightarrow (C \vee A \rightarrow C \vee B)$,
(e) $(x)Px \rightarrow Ph$,
(f) $Ph \rightarrow (Ex)Px$.

The rules of inference are *modus ponens*, or *FE* in Gentzen's notation, and for the quantifiers the following two (*ibid.*, p. 54):

> Let $B(x)$ be an arbitrary logical expression that depends on x, and A one that doesn't. If now $A \to B(x)$ is a correct formula, also $A \to (x)B(x)$ is. One obtains similarly from a correct formula $B(x) \to A$ the new $(Ex)B(x) \to A$.

To prove that derivations in the axiomatic calculus of *H-A* can be reproduced in *N2*, Gentzen makes the

DEFINITION. *$A \vee B$ is $\neg.\neg A \,\&\, \neg B$, $(Ex)Px$ is $\neg(x)\neg Px$, $A \to B$ is $\neg.A \,\&\, \neg B$.*

The axioms of *H-A* now read:

(a) $\neg...\neg.\neg A \,\&\, \neg A.. \,\&\, .\neg A$,
(b) $\neg..A \,\&\, \neg\neg.\neg A \,\&\, \neg B$,
(c) $\neg...\neg.\neg A \,\&\, \neg B.. \,\&\, ..\neg\neg.\neg B \,\&\, \neg A$,
(d) $\neg\{\neg.A \,\&\, \neg B.. \,\&\, \neg\neg...\neg.\neg C \,\&\, \neg A.. \,\&\, ..\neg\neg.\neg C \,\&\, \neg B\}$,
(e) $\neg.(x)Px \,\&\, \neg Ph$,
(f) $\neg.Ph \,\&\, \neg\neg(x)\neg Px$.

These are easily proved in *N2*. Gentzen gives as an example the derivation of axiom *b*:

$$\dfrac{\dfrac{\overset{1}{A \,\&\, \neg\neg.\neg A \,\&\, \neg B}}{A}\,{\scriptstyle AE} \qquad \dfrac{\dfrac{\dfrac{\overset{1}{A \,\&\, \neg\neg.\neg A \,\&\, \neg B}}{\neg\neg.\neg A \,\&\, \neg B}\,{\scriptstyle AE}}{\neg A \,\&\, \neg B}\,{\scriptstyle TND}}{\neg A}\,{\scriptstyle AE}}{\text{Axiom } b}\,{\scriptstyle DN\ 1}$$

The inference schemes of *H-A* can be reproduced in *N2*:

The inference scheme $\dfrac{A \quad \neg.A \,\&\, \neg B}{B}$ becomes:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \quad \overset{1}{\neg B}}{A \,\&\, \neg B} \qquad \neg.A \,\&\, \neg B}{\neg\neg B}\,{\scriptstyle DN\ 1}}{B}\,{\scriptstyle TND}}{}$$

The generalization scheme $\dfrac{\neg.A \,\&\, \neg(x)Px}{\neg.A \,\&\, \neg Pa}$ becomes:

$$\dfrac{\dfrac{\overset{1}{A \,\&\, \neg(x)Px}}{\neg(x)Px}{\scriptstyle AE} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{A \,\&\, \neg(x)Px}}{A}{\scriptstyle AE} \quad \overset{2}{\neg Pa}}{A \,\&\, \neg Pa}{\scriptstyle AI} \qquad \neg.A \,\&\, \neg PA}{\neg\neg Pa}{\scriptstyle DN\ 2}}{Pa}{\scriptstyle TND}}{(x)Px}{\scriptstyle UI}}{\neg.A \,\&\, \neg(x)Px}{\scriptstyle DN\ 1}$$

UI fulfils the variable condition, because a does not occur in $\neg.A \,\&\, \neg(x)Px$.

The scheme of existential instantiation is treated analogously. The induction axiom is derived by

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{P1 \,\&\, .(x)\neg.Px \,\&\, \neg Px' .. \,\&\, \neg Ph}}{P1 \,\&\, .(x)\neg.Px \,\&\, \neg Px'}{\scriptstyle AE}}{P1}{\scriptstyle AE} \qquad \dfrac{\dfrac{\dfrac{\overset{3}{Pa} \quad \overset{2}{\neg Pa'}}{Pa \,\&\, \neg Pa'}{\scriptstyle AE} \qquad \dfrac{\dfrac{\dfrac{\overset{1}{P1 \,\&\, .(x)\neg.Px \,\&\, \neg Px' .. \,\&\, \neg Ph}}{P1 \,\&(x)\neg.Px \,\&\, \neg Px'}{\scriptstyle AE}}{(x)\neg.Px \,\&\, \neg Px'}{\scriptstyle AE}}{\neg.Pa \,\&\, \neg Pa'}{\scriptstyle UE}}{\dfrac{\neg\neg Pa'}{Pa'}{\scriptstyle TND}}{\scriptstyle DN2}}{Ph}{\scriptstyle CI\ 3} \qquad \dfrac{\overset{1}{P1 \,\&\, .(x)\neg.Px \,\&\, \neg Px' .. \,\&\, \neg Ph}}{\neg Ph}{\scriptstyle AE}}{\neg ... P1 \,\&\, .(x)\neg.Px \,\&\, \neg Px' .. \,\&\, \neg Ph}{\scriptstyle DN\ 1}$$

By the above, the calculus *N2* is at least as strong as the standard axiomatic calculus of Hilbert and Ackermann. In the other direction, derivations in *N2* are reproduced in axiomatic logic as follows:

> One finds for each proposition A of the *N2*-proof all those assumptions above it the respective *DN*'s (*CI*'s) of which still stand under A. These are $B_1, \dots B_\varphi$. Then one substitutes A by $B_1 \,\&\, \dots \,\&\, B_\varphi \to A$.

If A is an assumption, $A \to A$ takes its place. The steps of inference of natural deduction become:

TABLE 10. Gentzen's translation from *N2* to axiomatic logic.

AI	*AE*	*UE*	*TND*
$\dfrac{D \to A \quad E \to B}{D \,\&\, E \to A \,\&\, B}$	$\dfrac{C \to A \,\&\, B}{\begin{array}{c} C \to A \\ \phantom{C \to {}} B \end{array}}$	$\dfrac{C \to (x)Px}{C \to Ph}$	$\dfrac{C \to \neg\neg A}{C \to A}$

TABLE 10. *continued.*

$$\frac{C \to Pa}{C \to (x)Px}\,UI \qquad \frac{D \,\&\, A \to B \quad E \,\&\, A \to \neg B}{D \,\&\, E \to \neg A}\,DN \qquad \frac{D \to P1 \quad E \,\&\, Pa \to Pa'}{D \,\&\, E \to Ph}\,CI$$

Now it remains to show that these steps can be reproduced in axiomatic logic. With AI, one proves first $(D \to A) \to ((E \to B) \to (D \,\&\, E \to A \,\&\, B))$ to be a theorem of axiomatic logic. Then the translation of AI in Table 10 is replaced by two implication eliminations:

$$\frac{\dfrac{(D \to A) \to ((E \to B) \to (D \,\&\, E \to A \,\&\, B)) \quad D \to A}{(E \to B) \to (D \,\&\, E \to A \,\&\, B)} \qquad E \to B}{D \,\&\, E \to A \,\&\, B}$$

The pattern is the same for the rest of the translations.

The quoted passage shows how the derivability of a formula A from the open assumptions $B_1, \dots, B_\varphi$ in natural deduction becomes expressed as an implication $B_1 \,\&\, \dots \,\&\, B_\varphi \to A$ in axiomatic logic. The next step is to read the implication arrow as the derivability relation instead of a connective. In this step, Gentzen was greatly helped by his work from the summer of 1931 on what are known as Hertz-systems. They originate with Paul Hertz, also known for his early work on the foundations of statistical mechanics.[12]

Hertz put up in the 1920s a general theory of "systems of sentences" (Hertz 1923, 1929).[13] A *sentence* is an expression of the form $a_1, \dots, a_n \longrightarrow b$. It has several interpretations: the circumstances given by $a_1, \dots, a_n$ bring about b, objects with the properties $a_1, \dots, a_n$ have also the property b, etc. The logical interpretation is that from the propositions $a_1, \dots, a_n$, proposition b follows. The arrow thus represents a notion of logical consequence.

Hertz systems have two forms of rules of inference, the first with zero premisses:

TABLE 11. The rules of Hertz systems.

$$a_1, \dots, a_n \longrightarrow a_i \quad \text{for } 1 \leqslant i \leqslant n,$$

$$\frac{\left\{\begin{array}{l} a_{1_1}, \dots, a_{1_n} \longrightarrow b_1 \\ \quad\vdots \\ a_{m_1}, \dots, a_{m_k} \longrightarrow b_m \end{array}\right. \qquad b_1, \dots, b_m, c_1, \dots, c_l \longrightarrow c}{a_{1_1}, \dots, a_{1_n}, \dots, a_{m_1}, \dots, a_{m_k}, c_1, \dots, c_l \longrightarrow c}$$

[12]He discovered, simultaneously with Einstein, what is known as the canonical ensemble, for which see Einstein 1911.

[13]Several of his papers bear the same title, *Über Axiomensysteme für beliebige Satzsysteme.*

Hertz called the former kind of rules "immediate inferences" and the latter "syllogisms." What are called the "upper premisses" in syllogisms are stacked and lumped together with a big curly bracket and the "lower premiss" written at right in isolation.

If a collection of sentences S is closed with respect to these rules, it forms a *sentence system*. A collection of sentences T is an *axiom system* for a sentence system S if each sentence in S is derivable from the ones in T.

One of the principal questions for Hertz was the existence of independent axiom systems, with independence defined in the obvious way, as underivability by the rules. Bernays had followed and clearly also sustained the work of Hertz and he suggested that Gentzen work with the problem. The result was Gentzen's first paper, titled, in English translation, "On the existence of independent axiom systems for infinitary sentence systems." He showed by a counterexample that there is an infinite sentence system that has no axiom system with independent axioms. In connection with the preparation of Gentzen's ***Collected Papers***, Bernays suggested in the 14 May 1968 letter to the editor Manfred Szabo that the work of Hertz be explained, and then added: "It would be worth mentioning that Gentzen showed here his great mathematical ability, bringing to a complete resolution a difficult problem that had remained open in the theory of sentence systems."

Gentzen noticed in the course of his work, conducted in the summer of 1931, that the second form of Hertz' rules can be replaced by one in which there is just one term b as a middle term of syllogism and none of the ugly stacking:

TABLE 12. Gentzen's cut rule in Hertz systems.

$$\frac{a_1, \dots, a_m, \rightarrow b \quad b, c_1, \dots, c_n \rightarrow c}{a_1, \dots, a_m, c_1, \dots, c_n \rightarrow c}$$

He called this form of rule *cut* (Schnitt), and the first form of Hertz' rules *thinning* (Verdünnung). The "sentences" of the calculus of Hertz became the "sequents" of Gentzen's later work. Gentzen's results in the paper use the method of permuting cuts suitably, which became the crucial point in the proof of his *Hauptsatz*, the equivalent of normalization for sequent calculus. The details of the 1932 paper can be found explained in Schroeder–Heister (2002). Among others, he notes that Gentzen's main results contain a normal form for derivations that amounts, in modern terms, "to the completeness of propositional SLD-resolution in logic programming" (ibid., p. 246).

The calculus of Hertz was one of the two main sources of Gentzen's sequent calculus. It suggested the transition from an implication to a something that lies one level higher, so to speak, namely a formal derivability relation. Similarly, in place of a connective, the commas of this higher level represent a collection of assumptions that can be used conjunctively. Further discussion of the background of sequent calculus in the work of Hertz and in Gentzen's first paper can be found in Bernays (1965).

The conceptual ground was now sufficient to support the idea of a formal calculus for the derivability relation. Before I go to that, a final remark about *N2*: It appears from the summary of results of the handwritten thesis that Gentzen had first obtained the normalization theorem for what he called calculus *N2I*. There are no permutative conversions in the absence of disjunction and existence and assumptions are closed only in a rule for negation. The addition of a rule of double negation elimination gives *N2* and makes the fragment complete for classical logic. The definitions of disjunction, implication, and existential quantification and the realization that double negation elimination can be restricted to negative formulas, an intuitionistically justified step, led to the reduction of classical to intuitionistic arithmetic.

10. From natural rules to sequent calculus. In the order in which it was left, the proof of normalization in the thesis manuscript is followed by the presentation of a "natural-logistic parallel" calculus denoted *NL3*. The parallelism comes from the way Gentzen arrived at the sequent rules. He writes:

> The construction of a proof is not, as so far, undertaken from top to downwards, i.e., from the conditions to the final result, but from *inside* out. New propositions are added partly to below, partly to above, and such parts of proof are also "hung one beside the other."

This vivid language may leave something to be desired; Gentzen continues with a detailed exposition:

> I explain the concept: "Proof of a proposition B from the propositions $A_1, \dots A_r$."[14]
>
> I shall use the following symbolic writing when one such is at hand: $A_1, \dots A_r \supset B$ to be read as: B is provable from $A_1, \dots A_r$. This is a "sentence" in the sense of P. Hertz, I shall use the word "sentence" only in this sense.[15]
>
> I designate by $\Gamma, \Delta, \Theta, \Lambda$ complexes of propositions, they can be empty.

One sees that the arrow still stands for implication, so the inverted C of Peano is used for derivability. The latter symbol is soon used for implication, namely in the chapter on normalization. Formal derivations in the two systems run parallel and are listed by Gentzen in two columns, the left for

[14]A marginal addition reads: The order and [changed into: though not the] multiplicity of the A is irrelevant. The A could be empty; B not.

[15]The German is *Satz*, that for proposition is *Aussage*.

a "natural writing" of a step of proof, the right for a "symbolic writing for its formation." The parallel begins with: "A is a proof of A from A." It is rendered as A in the natural writing and as $A \supset A$ in the symbolic writing. Rule *UE* of natural deduction becomes the sentence $(x)Px \supset Ph$. Next the rest of the rules of natural deduction are turned into the symbolic writing:

$$\mathit{EI})\ \frac{Ph}{(Ex)Px} \qquad\qquad Ph \supset (Ex)Px$$

$$\mathit{AE})\ \frac{A \,\&\, B}{A} \qquad \frac{A \,\&\, B}{B} \qquad\qquad A \,\&\, B \supset A \qquad A \,\&\, B \supset B$$

$$\mathit{OI})\ \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad A \supset A \vee B \qquad B \supset A \vee B$$

$$\mathit{FE})\ \frac{A \qquad A \to B}{B} \qquad\qquad A, A \to B \supset B$$

The above were the single-step natural rules and their corresponding symbolic writings. The latter received the name **groundsequents** some time later. For the rest of the rules, Gentzen explains first rule *UI*:

> A proof of Pa from $A_1 \dots A_\nu$ gives (through the joining of $(x)Px$, to below) rise to a proof of $(x)Px$ from $A_1 \dots A_\nu$. Here a must not occur in $A_1 \dots A_\nu, (x)Px$.

The two parallel writings are, with $\Gamma = A_1 \dots A_\nu$:

$$\begin{array}{c} \underline{A_1 \dots A_\nu} \\ A_1 \dots A_\nu \\ \vdots \\ \underline{Pa} \\ (x)Px \end{array} \qquad\qquad \frac{\Gamma \supset Pa}{\Gamma \supset (x)Px}$$

The variable condition is that a must not occur in $A_1 \dots A_\nu, (x)Px$.[16] Rule *EE* has the parallel writing:

$$\begin{array}{c} \underline{\Gamma\ (Ex)Px} \\ \Gamma\ [Pa] \\ \vdots \\ \underline{B} \\ B \end{array} \qquad\qquad \frac{\Gamma\ [Pa] \supset B}{\Gamma\ (Ex)Px \supset B}$$

[16]A circle is drawn in the left column around the subderivation from $A_1 \dots A_\nu$ to Pa. It is further explained that "[A] means $A \dots A$, i.e., A occurs in an arbitrary number, even "not at all" is allowed."

The condition is that a must not occur free in Γ, B. This is the first rule in which the principal formula appears in the antecedent part of a sequent.

The rest of the parallel writings produce the following:

$$\frac{\Gamma \supset A \quad \Delta \supset B}{\Gamma\,\Delta \supset A \,\&\, B} \qquad \frac{[A]\Gamma \supset C \quad [B]\Delta \supset C}{(A \vee B)\Gamma\,\Delta \supset C} \qquad \frac{[A]\Gamma \supset C}{\Gamma \supset A \to B} \qquad \frac{\Gamma\,[Pa] \supset Pa'}{\Gamma\, P1 \supset Ph}$$

The structural rules come last. They consist in the addition of a condition and in the "hanging together" of two proofs. The parallel writing for the rule of weakening is:

$$\begin{array}{ccccc} & \Gamma & & \Gamma\, A & \\ & \vdots & & \vdots & \\ \text{from} & B & \text{becomes} & B & \qquad \dfrac{\Gamma \supset B}{\Gamma\, A \supset B} \end{array}$$

It is not told what (if any) the sense of the left part is. The word *Verdünnung* (thinning) is added to the right part. The second rule is cut (*Schnitt*):

$$\begin{array}{cccccccc} & & & & & & \Gamma & \\ & & & & & & \vdots & \\ & \Gamma & & \Delta\, A & & \Delta & A & \\ & \vdots & & \vdots & & & \vdots & \\ \text{from} & A & \text{and} & B & \text{becomes[17]} & & B & \qquad \dfrac{\Gamma \supset A \quad \Delta\, A \supset B}{\Gamma\,\Delta \supset B} \end{array}$$

The upshot of Gentzen's parallel, really a translation from natural deduction, is a specific sequent calculus, one that he used actively in his work for the next ten years or so, but on which he never published anything separate. He called it with the name *LIG*, an intuitionistic **sequent calculus with groundsequents** (*LJG* in the German original, a calculus with "Grundsequenzen"). There was, as one can expect, a corresponding classical calculus denoted *LKG*.

A careful reading of Gentzen's thesis is again revealing, when one knows what to look for. The printed thesis has the two sequent calculi *LI* and *LK* that have become standard. Now look at the translation from *LI* to axiomatic logic (chapter 5, §5), and you find there the list of the groundsequents. The translation is in two stages: First derivations in *LI* are translated into ones in *LIG*, and in the next stage from *LIG* to axiomatic logic. The first stage of the translation begins with rules $L\,\&$ and $L\neg$ of *LI*:

Table 13. Gentzen's translation from *LI* to *LIG*.

$$\frac{A, \Gamma \to \Theta}{A \,\&\, B, \Gamma \to \Theta}\,L\& \quad \rightsquigarrow \quad \frac{\overset{Gs1}{A \,\&\, B \to A} \quad A, \Gamma \to \Theta}{A \,\&\, B, \Gamma \to \Theta}\,Cut$$

[17]The derivation of A from Γ after composition has an extra smaller A between Γ and A.

TABLE 13. *continued.*

$$\dfrac{\Gamma \to A}{\neg A, \Gamma \to}\,L\neg \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\Gamma \to A \quad \dfrac{\overset{Gs7}{\neg A, A \to}}{A, \neg A \to}\,Ex}{\Gamma, \neg A \to}\,Cut}{\neg A, \Gamma \to}\,Ex^*$$

Here Θ is either one formula or empty, and $Gs1$, $Gs7$ stand for the first and seventh of the groundsequents in Gentzen's list, respectively. Ex^* (my notation) stands for possibly repeated instances of the rule of exchange of order in a list. The rest of the translations are similar.

LIG with its equal share of rules and groundsequents is a calculus half-way between sequent calculus and an axiomatic calculus.

The discussion of sequent calculi in §2 of chapter II of the printed thesis, right after the rules have been given, makes two remarks. The first one states that the rules are not independent, but if some were left out, the *Hauptsatz* would be lost.[18] The second remark is that "if one put no value on the *Hauptsatz*," one could simplify the calculus by replacing some rules by groundsequents. Had Gentzen just given a name to the ensuing calculus defined on top of p. 194 of his thesis, it would have been common property long since.

11. "A nice problem for learning". Gentzen's thesis manuscript contains a brief part with the title

Calculus LDK. (Logistic-dualistic.) (classical.)

Without further explanation, sequents with a multiple succedent appear:

$$A_1, \dots, A_\mu \supset B_1, \dots B_\nu \qquad \text{in short: } \Gamma \supset \Delta$$

This classical calculus goes in the other direction, as compared to the development of the printed thesis the calculi of which have just rules: Here there are in the propositional part no logical rules but just logical groundsequents and structural rules. Gentzen writes that the order of formulas in the antecedents and succedents, as well as their multiplicity, is not counted. Later he made exchange and contraction into explicit rules. We shall do so with the latter but ignore the order of formulas. The rules of *LDK*, modified so that contraction is explicit, and with an arrow instead of the horseshoe, are as follows:

TABLE 14. The calculus LDK, propositional part.

1. *Initial sequents*: $A \to A$.

[18] Difficult to understand: One suggestion is in my 2009, p. 688.

TABLE 14. *continued.*

2. *The structural rules of weakening, contraction, and cut*:

$$\frac{\Gamma \to \Delta}{\Theta, \Gamma \to \Delta, \Lambda}\,Wk \qquad \frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta}\,CL \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}\,CR$$

$$\frac{\Gamma \to \Delta, A \quad A, \Theta \to \Lambda}{\Gamma, \Theta \to \Delta, \Lambda}\,Cut$$

3. *Logical groundsequents*:

$$\begin{array}{ll} A \,\&\, B \to A & A \to A \vee B \\ A \,\&\, B \to B & B \to A \vee B \\ A, B \to A \,\&\, B & A \vee B \to A, B \\ A, \neg A \to & \to A, \neg A \end{array}$$

Implication is taken as a defined connective, but conjunction is included contrary to Hilbert and Ackermann's axiomatization. Gentzen's choice matches exactly the needs for writing formulas in conjunctive normal form, a method used heavily in Hilbert and Ackermann.

A multisuccedent calculus for classical predicate logic is obtained by adding to the above the following:

TABLE 15. The calculus LDK, predicate logic.

4. *Logical groundsequents*:

$$\forall x A(x) \to A(t) \qquad\qquad A(t) \to \exists x A(x)$$

5. *Logical rules of inference*:

$$\frac{\Gamma \to \Delta, A(y)}{\Gamma \to \Delta, \forall x A(x)}\,R\forall \qquad\qquad \frac{A(y), \Gamma \to \Delta}{\exists x A(x), \Gamma \to \Delta}\,L\exists$$

The eigenvariable y must not occur in the conclusion. In Gentzen's writing of these two rules, the contexts on the side with the active and principal formulas are missing, which I take to be an oversight.

The following appears after the propositional part (with, as above, an arrow in place of the original horseshoe):

> Remarks: $\to A$ means intuitively: A is correct,
> $A \to$ means: A is wrong. ($A \to$ is equivalent to $\to \neg A$.)

> The sentence: $\rightarrow$ with empty antecedent and succedent means "the contradiction." (It is equivalent to $\rightarrow A \,\&\, \neg A$.)

We can verify these remarks by Gentzen:

EQUIVALENCE OF $A \rightarrow$ AND $\rightarrow \neg A$.

$$\dfrac{\rightarrow A, \neg A \quad A \rightarrow}{\rightarrow \neg A}\,Cut \qquad \dfrac{\rightarrow \neg A \quad A, \neg A \rightarrow}{A \rightarrow}\,Cut$$

The equivalence is suggestive of the right negation rule. The left negation rule is similarly close at hand.

EQUIVALENCE OF $\rightarrow$ AND $\rightarrow A \,\&\, \neg A$.

One direction is immediate:

$$\dfrac{\rightarrow}{\rightarrow A \,\&\, \neg A}\,Wk$$

The other direction requires a number of cuts:

$$\dfrac{\dfrac{\rightarrow A \,\&\, \neg A \quad A \,\&\, \neg A \rightarrow \neg A}{\rightarrow \neg A}\,Cut \qquad \dfrac{\dfrac{\rightarrow A \,\&\, \neg A \quad A \,\&\, \neg A \rightarrow A}{\rightarrow A}\,Cut \quad A, \neg A \rightarrow}{\neg A \rightarrow}\,Cut}{\rightarrow}\,Cut$$

The passage on *LDK* in the handwritten manuscript ends with a stenographic remark that must be a later addition, deciphered by Thiel. It begins with:

> *I have proved that weakening and cut form a complete system of inference in the propositional calculus.*

There is in Gentzen's last known letter, written to Paul Lorenzen on 12 November 1944 and by chance in my possession in original, the following advice:

> Allow me to give you a well-meant advice: Don't begin with so difficult problems, but with, say, the propositional calculus, as I have done; It is especially suited for learning to think by oneself and for becoming used to the logical concepts. The road to the predicate calculus is a lot stonier! A couple of nice problems are, for example: a proof of the completeness of my "structural rules of inference," or: to give a handy decision method for the intuitionistic propositional logic.

For the latter problem, there is an attempt at a semantical decision procedure in the Gentzen papers, but I never understood the former before the above stenographic remark was decoded.

The proof of completeness of Gentzen's classical propositional calculus is not difficult. I shall now make the self-study suggested by Gentzen, which indeed was a nice exercise.

Consider a tautology C in conjunctive normal form $C_1 \,\&\, \dots \,\&\, C_k$. Each C_i is of the form

$$\neg P_1 \vee \cdots \vee \neg P_m \vee Q_1 \vee \cdots \vee Q_n$$

Here the P's and Q's are atomic formulas. In general, a sequent of the form $\to \Delta, A \,\&\, B$ is derivable if and only if both $\to \Delta, A$ and $\to \Delta, B$ are derivable, as is seen by cutting twice with $A, B \to A \,\&\, B$ and contracting, and similarly in the other direction. Therefore the sequent $\to C$ is derivable if and only if each of the $\to C_i$ are derivable. For the latter,

$$\to \neg P_1 \vee \cdots \vee \neg P_m \vee Q_1 \vee \cdots \vee Q_n$$

has to be derivable. In general, $\to \Delta, A \vee B$ is derivable if and only if $\to \Delta, A, B$ is derivable, as is easily shown by cuts. Similarly, $\Gamma \to \Delta, \neg A$ is derivable if and only if $A, \Gamma \to \Delta$ is derivable, by adding contexts to Gentzen's remark. Therefore, $\to C$ is derivable if and only if each of the

$$P_1, \dots, P_m \to Q_1, \dots, Q_n$$

is derivable. The latter is derivable if and only if, for some $i, j, P_i \equiv Q_j$. This, precisely, is the case if C is a tautology.

The above argument comes rather close to the proof of completeness of Ketonen's invertible classical propositional sequent calculus, a result even explained in the Bernays review (1945).

12. The first sequent calculus *LIG*. The rules of sequent calculus in Gentzen's first sketch have what are today called **independent contexts**: The side formulas in the "complexes of propositions" Γ and Δ in the rules are added up in the conclusion, exactly as in natural deduction. In the rules of the printed thesis, the contexts are **shared**, i.e., the same in all two-premiss rules except implication at left. The printed thesis notes that the proof of cut elimination is simplified somewhat if the side formulas are the same. I shall now change the system with groundsequents a bit so that it is in harmony with the structural features of the rules in the printed thesis. The aim is to have a better understanding of Gentzen's way towards the cut elimination theorem. I also start using the notation of the printed thesis that has become the standard one in proof theory. Gentzen himself indicated a change of the roles of the arrow and the horseshoe in a marginal note and carried it through in the third chapter of the handwritten manuscript, the one on the normalization proof.

Sequents have the general form $A_1, \dots, A_n \to C$, with the possibility that the antecedent is empty, and the same for the succedent. The antecedent formulas form a **list** or **sequence**, which latter is the origin of the name sequent calculus. There was some variation in how the formulas in the antecedent were treated: At first Gentzen stated that the order and multiplicity of the A_i does not matter, then changed it so that multiplicity but not order matters.

In the printed thesis both order and multiplicity were handled formally by rules of **exchange** and **contraction**. Today, one tends to work with multisets in which multiplicity but not order is counted, the middle one of the three possibilities that Gentzen considered.

Gentzen became by 1934 aware of the crucial role of the rule of contraction when attempting to prove the consistency of arithmetic. It can be shown, in fact, that if multiplicities of formulas could be dismissed with, or, equivalently, a rule of contraction left out, then the consistency of arithmetic would have a proof by ordinary arithmetic induction. I plan to discuss this matter in detail in a later paper on Gentzen's proofs of the consistency of arithmetic.

In the rules of the calculus *LIG*, L and R stand for left and right (A and S in Gentzen):

TABLE 16. Logical rules and groundsequents of the calculus *LIG*.

$$A \,\&\, B \to A, \quad A \,\&\, B \to B, \qquad \frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \,\&\, B}\,R\&$$

$$\frac{A, \Gamma \to C \quad B, \Gamma \to C}{A \vee B, \Gamma \to C}\,L\vee \qquad A \to A \vee B, \quad B \to A \vee B$$

$$\frac{A, \Gamma \to B}{\Gamma \to A \supset B}\,R\supset \qquad A \supset B, A \to B$$

$$A, \neg A \to$$

$$\forall x A(x) \to A(t), \qquad \frac{\Gamma \to A(y)}{\Gamma \to \forall x A(x)}\,R\forall$$

$$\frac{A(y), \Gamma \to C}{\exists x A(x), \Gamma \to C}\,L\exists \qquad A(t) \to \exists x A(x)$$

The sequents have at most one formula at right, in the succedent part. Thus, falsity elimination of natural deduction is replaced by the groundsequent $A, \neg A \to$. The quantifier rules have the usual variable restriction. The quantifier groundsequents have an arbitrary term t.

Next we come to the **structural rules** of *LIG*. In them, W stands for weakening, C for contraction, and E for exchange. Rule *Cut* makes possible the **composition** of two derivations, one with the conclusion A, the other with the assumption A.

To the structural rules can be added what Gentzen sometimes called axioms, sometimes initial sequents: $A \to A$. These and groundsequents start derivation branches.

TABLE 17. The structural rules of the sequent calculus *LIG*.

$$\frac{\Gamma \to C}{A, \Gamma \to C}\,LW \qquad \frac{\Gamma \to}{\Gamma \to C}\,RW$$

$$\frac{A, A, \Gamma \to C}{A, \Gamma \to C}\,LC \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, B, A, \Delta \to C}\,LE$$

$$\frac{\Gamma \to A \quad A, \Delta \to C}{\Gamma, \Delta \to C}\,Cut$$

Looking at the translation from *LI* to *LIG* in Table 13, one might think that *LIG* is just a trivial variant of *LI*: To recover a derivation in the latter, it is sufficient to substitute each groundsequent by its derivation in *LI*. Somewhat of a mess is instead produced, for the calculus with groundsequents is not complete if the rule of cut is left out. Cuts, on the other hand, do not in general maintain the subformula property. The indispensability of cuts in *LIG* is seen from the derivation

$$\frac{A \to A \vee B \quad A \vee B \to (A \vee B) \vee C}{A \to (A \vee B) \vee C}\,Cut$$

It is clear that there is no cut-free derivation of the conclusion of this cut. However, cuts can be so permuted that the cut formula is a subformula of the conclusion:

CUT ELIMINATION FOR *LIG*. *A derivation in LIG can be so transformed that it contains only cuts in which the cut formula is a subformula of the conclusion of the cut.*

The most interesting aspect of *LIG* and cut elimination concerns the rule of contraction. In the printed thesis, the calculus is *LI*, and *LK* for classical logic, and a **mix rule** is introduced in place of cut. It allows to cut more than one occurrence of the cut formula. The problem arose as follows: Assume that the right premiss of cut has been derived by contraction. If one tries to permute cut up by cutting twice the contracted formula, there is no reduction in the cut elimination parameters.

Gentzen remarks in passing in his printed thesis, when he begins the proof of cut elimination, that he uses the mix rule "to make the proof easier." Such a remark is a sure sign that he had also a proof with the standard cut rule. Intrigued by the remark, I set out to produce a proof of the *Hauptsatz* without multicut (von Plato 2001a). The procedure involves tracing up in the right premiss the origin of the cut formula. In the critical case, it is found that cut can be permuted up in one sweep to a place where the contraction formula is principal in a left rule, with the consequence that contraction can be applied to shorter formulas. With *LIG*, a cut formula is never principal in both premisses of cut. Therefore, if the right premiss has been derived by contraction, either cut can be permuted up at left until the left premiss is a groundsequent and the cut harmlessly repeated twice, or else the duplication in the right premiss can be deleted because it comes from weakening.

Gentzen did not leave *LIG* aside but used it in, for example, attempts at doing consistency proofs for an intuitionistic sequent calculus for arithmetic in 1938, with the rules written out very prominently exactly as in Table 16. The calculus keeps appearing in his notes until 1944.

As mentioned, there is a longish paper by Bernays on sequent calculus published in 1965 that, in my opinion, shows that Gentzen had all the properties of *LIG* under control. Bernays discusses first at length the work of Hertz, how Gentzen contributed to it, and how all this was one of the two components in the discovery of sequent calculus. The presentation is careful; Bernays shows, for example, the following result (p. 8): In a single-succedent calculus, weakening and context-sharing cut render the rest of the structural rules derivable, including the rule of exchange.

The second component in the way to sequent calculus is "the translation of Gentzen's rules of natural deduction into sequent calculus" (ibid., p. 9). The result is, as given by Bernays:

TABLE 18. Variant of *LIG* from Bernays 1965.

$$A, B \to A \,\&\, B \qquad\qquad A \,\&\, B \to A \qquad\qquad A \,\&\, B \to B$$

$$A \to A \vee B \qquad\qquad B \to A \vee B \qquad\qquad \frac{A, B \to D \quad A, C \to D}{A, B \vee C \to D}$$

$$\frac{A, B \to C}{A \to B \supset C} \qquad\qquad A, A \supset B \to B$$

Only the propositional rules are discussed in Bernays. The calculus has a groundsequent in place of $R\&$, a possibility mentioned in Gentzen's manuscript. Further, the printed thesis has this groundsequent (pp. 193–94). Bernays discusses also the possibility of a calculus with inversions of rules, similar to Gentzen's discussion in the handwritten thesis manuscript.

Section 3 of Bernays' paper is devoted to a discussion of cut elimination. First Bernays recalls the calculus of Table 16, obtained through a translation from natural deduction, then, as he writes (p. 20):

> Beside the system thus obtained, we put up a second one, all rules of which are passages from one sequent to another such that the passages are pairwise inverses to each other.

> The system of intuitionistic sequent calculus used by Gentzen himself in his "Investigations into logical reasoning" is different from both of the mentioned forms of sequent calculus. The setting up of this calculus is directed toward a specific goal.

The goal was, naturally, a cut-free calculus that has the subformula property. The reader may remain somewhat puzzled by the "we" in Bernays: Is it an editorial we? Should the reader not have been told that both of the alternative earlier calculi mentioned are a work of Gentzen's? The impression possibly given by the above passage gets corrected later, on p. 32, where classical calculi are discussed. There, as we shall see, one obtains a particularly nice calculus with just groundsequents and weakening and cut (Gentzen's calculus *LDK*, Table 14 above). Bernays writes (p. 32) that "if we consider the dual-symmetric sequent calculus without a view towards cut elimination, we can, as already emphasized by Gentzen, simplify remarkably the rules for the connectives." It is clear from Bernays' paper that he had studied the original package for a thesis given to him by Gentzen, or that the latter had explained its contents and results carefully, except for the part on normalization. It is moreover clear that Bernays had himself worked quite a bit with both calculi, the variant of *LIG* and the invertible calculus.

Bernays' paper reproduces with such precision Gentzen's so far unknown calculi from the handwritten thesis manuscript that I take the further development in Bernays (1965) as part of what Gentzen himself had accomplished. First of all, Bernays shows by an example that *LIG* is not closed with respect to cuts.[19] Secondly, Bernays notes (p. 25):

> Even if the eliminability of cuts fails for this system, there remains for *suitable* derivations a subformula property.

A brief sketch of how one arrives at this result is given: Starting from a derivation in Gentzen's cut-free calculus *LI*, instances of rules are substituted by instances of groundsequents of *LIG*:

> The substitutions come out almost all simply through cuts. Only the rules for negation ... require a careful discussion.

Note that these substitutions are precisely Gentzen's steps of translation from *LI* to *LIG*, as in Table 13 above. By the completeness of cut-free *LI*, a complete class of derivations in *LIG* with the subformula property follows. This indirect argument for the subformula property in suitable *LIG*-derivations, through the calculus *LI* of the printed thesis and the translation, may well be a reverse of the historical order of things.

Sources and timing: Gentzen's papers from the time of his thesis work have mostly been lost, possibly some in 1964 together with the destruction of the papers of Arnold Schmidt, possibly later when in the 1970s papers from the attic of the Göttingen mathematics building

[19] Incidentally, the example (p. 21) is the same I gave, modulo associativity, originally suggested to me by Sara Negri in 2005.

were, it seems, hauled to the garbage dump. With some luck, more sources than the two slim folders of stenographic notes in Thiel's possession and the thesis manuscript in Zurich are found, and more can be said.

As to the timing of the discovery of sequent calculus, I believe the earliest notes, as in the handwritten thesis manuscript, to be from late 1932. A letter to Heyting from the end of February 1933 would have mentioned the decidability of intuitionistic propositional logic, one of the corollaries to cut elimination in *LI*, had he been that far by then. So sequent calculus proper seems to have been developed between March and May 1933.

13. Variants of sequent calculi. Natural deduction, as fixed in the handwritten thesis after the proof of normalization, remained stable until the 1980s. In sequent calculus, changes were made early on. Foremost among these was the work of Oiva Ketonen, my old philosophy professor in Helsinki. He had studied with Gentzen in Göttingen in 1938–39, or at least so he tried. He arrived there on the day of the *Kristallnacht*, 9 November 1938, the night of destruction of synagoues, homes, and shops, and arrestation and deportation to concentration camps of tens of thousands of Jews. It appears from Gentzen's manuscripts that he was working in those very same days with the extension of his (1938) "new consistency proof" of arithmetic into analysis, a thing he had hoped in that paper to be soon ready. About a month of attempts was enough this time, and he turned into the preparation of his *Habilitationsschrift*. Ketonen, in a frustrated letter to his mother, writes in January 1939 that finally Gentzen had given him a problem to work with. His thesis, finished in 1944 and titled *Untersuchungen zum Prädikatenkalkül*, introduced what is today called the *G3*-sequent calculus for classical logic. The propositional part has the remarkable property that all of its rules are invertible, which gives a strikingly elegant proof of completeness. Another result was a sharpening of the midsequent theorem, by which the number of quantifier inferences with eigenvariables could be minimized. The existence of a weakest possible midsequent followed, in the sense that if any midsequent is derivable, the weakest one is. Turning this into a contrapositive, Ketonen had a purely syntactic method for proofs of underivability that he applied to projective and affine geometry. To my regret, I never asked Ketonen what specific suggestions Gentzen had made to him. Little was found after his death in 2000. One letter to Ketonen in Göttingen from the spring of 1939, by a student friend of his in Helsinki, contains the enigmatic sentence: "Gentzen's proof is in the typewriting office." Judging by their titles, some notes by Gentzen from June 1939, from the series **BTJZ**, may contain results that relate to those of Ketonen, but they have not been transcribed so far.

Ketonen's results became known mainly through the long review Bernays wrote of his thesis to ***The Journal of Symbolic Logic*** in 1945. His calculus was taken up by Kleene and Beth and Curry, among others. (The last-mentioned wrote to Ketonen that he would be interested in any other work on logic as well, “even in Finnish.”) After Ketonen, a whole list of different sequent calculi were presented by Kleene in his book ***Introduction to Metamathematics***, with explicit acknowledgement to Ketonen through the Bernays review, and by Beth in his tableau systems of 1955, with failed acknowledgement of what he had learned from Ketonen, a work he referred to in the late 1940s.

Sequent calculi seem to leave room for variants, as compared to natural deduction. Gentzen himself begins it in the short section on *LIG* in his thesis manuscript:

> There is to each logical sign an axiom and an inference scheme. Another division is into an introduction (I) and an elimination (E) of a sign, which lies close at hand in the natural form of proof.
>
> Inference AI can as well be conceived as an axiom, $A, B \supset A \,\&\, B$. It was not done because of symmetry.
>
> All logical-sign-axioms can be replaced by inference schemes, namely, for example:

$$\frac{\Gamma \supset (x)Px}{\Gamma \supset Ph} \quad \frac{\Gamma\ (Ex)Px \supset C}{\Gamma\ Ph \supset C} \quad \frac{\Gamma \supset A \,\&\, B}{\begin{array}{l}\Gamma \supset A \\ \qquad\ B\end{array}} \quad \frac{A \vee B\ \Gamma \supset C}{\begin{array}{l}A\ \Gamma \supset C \\ B\end{array}} \quad \frac{\Gamma \supset A \rightarrow B}{A\ \Gamma \supset B}$$

> I shall show as an example the equivalence with the axiom in the case of the all-inference: The inference follows from the axioms like this:

$$\frac{\Gamma \supset (x)Px \qquad \overset{\text{Axiom}}{(x)Px \supset Ph}}{\Gamma \supset Ph}\ \text{Cut}$$

> The axiom follows from the inference like this:

$$\frac{\overset{\text{Str. Axiom}}{(x)Px \supset (x)Px}}{(x)Px \supset Ph}$$

> These inference schemes are to a large extent analogous (inversions) of those in 2).[20] I use the axioms instead of the inference schemes because they are more convenient in the search of peaks.

Here is a third system of rules of sequent calculus to consider, still different from the ones of the published thesis, and discussed in some detail in Bernays (1965). In some work from the late 1930s, Gentzen refers to it as “the U-calculus,” where U comes from the German word for inversion (*Umkehrung*). I comment briefly on what the “peaks” are in the last Section.

One of the last known writings from Gentzen’s hand is an eight-page, extremely neatly written stenographic manuscript that contains an overview of

[20]Those of Table 14 above.

sequent calculi. It bears the date VIII.44 and signum **WKRd**. Here the letter d stands for *Durchsicht*, overview, W naturally for *Widerspruchsfreiheit* or possibly the longer jaw-breaking *Widerspruchsfreiheitsbeweis*, consistency or consistency proof, respectively, and K and R for *Kalküle* and *Reduktionen*. In all the title amounts to *Consistency proof, overview of calculi and reductions*. Three calculi are considered: the sequent calculus *LI* of the doctoral thesis, the calculus *LIG* as detailed above, and the invertible calculus that is designated by *LIU*.

What was the point of such detailed work on variants of sequent calculi? Here we would have to go into the details of Gentzen's various proofs of the consistency of arithmetic, which would make for a long story. Anyway, when he had come into formulating inconsistency in terms of sequent calculus, it was expressed as the derivability of an **empty sequent** $\rightarrow$ in a sequent calculus for arithmetic. If there is such a derivation, it must have an **endpiece** in which only the structural rules and the rule of induction are met. Thus, the endpiece is limited from above by initial sequents, or possibly groundsequents, and by instances of logical rules. Cuts in the endpiece typically lead to the multiplication of formulas, as in

$$\dfrac{\begin{matrix}\vdots\\ A, \Gamma \rightarrow D\end{matrix} \quad \begin{matrix}\vdots\\ D, A, \Delta \rightarrow C\end{matrix}}{A, A, \Gamma, \Delta \rightarrow C}\,\mathit{Cut}$$

Now there could be a contraction on A, followed by a cut, so we have altogether

$$\dfrac{\begin{matrix}\vdots\\ \Theta \rightarrow A\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ A, \Gamma \rightarrow D\end{matrix} \quad \begin{matrix}\vdots\\ D, A, \Delta \rightarrow C\end{matrix}}{A, A, \Gamma, \Delta \rightarrow C}\,\mathit{Cut}}{A, \Gamma, \Delta \rightarrow C}\,\mathit{Ctr}}{\Gamma, \Delta, \Theta \rightarrow C}\,\mathit{Cut}$$

The central problem in the proof of consistency along these lines is to transform the endpiece so that a "principal cut" is created, with the left premiss concluded by a right logical rule and the right premiss by a left logical rule. Say, we try the following transformation in which we assume that $\Theta \rightarrow A$ and $A, \Gamma \rightarrow D$ are uppermost sequents in the endpiece, with *Ctr** denoting repeated contractions:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \Theta \rightarrow A\end{matrix} \quad \begin{matrix}\vdots\\ A, \Gamma \rightarrow D\end{matrix}}{\Gamma, \Theta \rightarrow D}\,\mathit{Cut} \qquad \dfrac{\begin{matrix}\vdots\\ \Theta \rightarrow A\end{matrix} \quad \begin{matrix}\vdots\\ D, A, \Delta \rightarrow C\end{matrix}}{D, \Delta, \Theta \rightarrow C}\,\mathit{Cut}}{\Gamma, \Delta, \Theta, \Theta \rightarrow C}\,\mathit{Cut}}{\Gamma, \Delta, \Theta \rightarrow C}\,\mathit{Ctr}^*$$

If A is the principal formula in logical rules with the conclusions $\Theta \to A$ and $A, \Gamma \to D$, respectively, the situation with the left part is good: Both premisses of cut are concluded by logical rules in a way that makes possible a standard step of cut elimination. On the other hand, we have duplicated the sequent $\Theta \to A$, which is problematic. How to give a weight to the endpieces of derivations of the empty sequent such that the weight diminishes in the presence of such duplications? Gentzen tried all sorts of solutions, for example, **WKRd** contains the explicit use of multisets to get rid of the rule of exchange, and it contains what is very close to a "context-sharing" cut, namely a cut of the following kind:

$$\frac{\Gamma, \Theta \to D \quad D, \Delta, \Theta \to C}{\Gamma, \Delta, \Theta \to C}$$

Now any duplications, the formulas in Θ, are eliminated in the cut. However, an easy check shows that two such context-sharing cuts cannot in general be permuted without messing up the endpiece.[21]

These were the problems Gentzen was wrestling with: first, trying to improve his 1938 consistency proof for arithmetic that used the classical sequent calculus *LK*. It contains a construction of a "height-line" that may be seen as a desperate move to save the proof, to be replaced by something better. Secondly, Gentzen wanted to give a proof that is based on an intuitionistic sequent calculus. He clearly believed that a suitable formulation of the rules of sequent calculus could lead to an assignment of weights for the endpieces of derivations that solves his two problems. None of his attempts led to any success, and it seems that solutions have to be sought from a different direction.[22]

14. Linear derivations, or logical calculus for the common reader. Gentzen's famous proof of the consistency of elementary arithmetic is found in a lengthy, 71-page paper published in 1936 in the ***Mathematische Annalen***. Its first part is a broad discussion of foundational research and the nature of consistency proofs, meant to introduce the topic to newcomers. Bernays in a letter apparently suggested its deletion, but Gentzen persisted. Even

[21] Reading Gentzen's overview, I was reminded of my own attempt about ten years ago at a proof of consistency that used a rule of multicut in which repeated occurrences of a cut formula in the right premiss were deleted, with no contractions at all needed in the endpiece. The move led to an ordinal assignment that was below Gentzen's ε_0, so what went wrong? Nobody saw the mistake, until Michael Rathjen pointed out that two cuts permute, two multicuts instead don't. The experience serves as a warning to those who think they can do a "sequent calculus with sets," i.e., to ignore the multiplicity of formulas in sequents.

[22] Inspired by Gentzen's attempts, my student Annika Siders (formerly Kanckos) found a consistency proof that is based on intuitionistic natural deduction (Kanckos 2010), another one based on intuitionistic sequent calculus (Siders 2012a), and finally one (Siders 2012b) that avoids the height-line construction.

the logical calculus is presented with a similar aim: It is a natural calculus, but two novelties are avoided: the, at the time, perhaps esoteric use of an intuitionistic calculus, and the use of derivation trees. I take both aspects to have purely expository motivations.

The presentation of the calculus begins with an introduction of the sequent notation $A_1, A_2, \ldots, A_\mu \rightarrow B$, after which follows the definition of a derivation (p. 513):

> A derivation consists of a number of sequents in succession, such that each of these is either a "groundsequent" or results from some previous sequents through a "structural modification" or a "rule of inference."

The groundsquents are either assumptions, written $D \rightarrow D$, or of the form $\rightarrow A$ with A an arithmetic axiom. The structural modifications are, in today's terms, exchange, contraction, weakening, and α-conversion (change of a bound variable). Gentzen writes (pp. 513–4) that these rules are "purely formal in nature and inconsequential in their content; they have to be mentioned explicitly because of the peculiarities of the formalism." Indeed, as we shall see, these structural rules are implicit in proper natural deduction.

The origin of the sequent notation for natural deduction is in the translation from natural deduction to sequent calculus in Gentzen's thesis. In a first stage, Gentzen simply took a derivation tree in natural deduction and gathered together, for each formula in the tree, all the open assumptions it depends on. The calculus is now called "natural deduction in sequent calculus style." Gentzen denoted the calculus in his thesis manuscript by *NLI* (for "natural-logistic intuitionistic"), and *NLK* for the classical version. The essential difference to sequent calculus proper is the absence of left rules.

Contrary to these earlier calculi that used the novelty of derivation trees, Gentzen prefers in his consistency proof of 1936 a linear representation of formal derivations. The sequent notation is a crucial component, because otherwise one would not see what depends on what in a linear derivation. Indeed, as Gentzen writes (p. 512), a sequent is "a formal expression for the meaning of a proposition in a derivation in relation to its dependence on possible assumptions."

We come last to the logical rules. There are no surprises here: Conjunction introduction is the rule by which "the sequents $\Gamma \rightarrow A$ and $\Delta \rightarrow B$ give the sequent $\Gamma, \Delta \rightarrow A \ \& \ B$," and similarly for the rest. Disjunction elimination becomes: "$\Gamma \rightarrow A \vee B$ and $A, \Delta \rightarrow C$ and $B, \Theta \rightarrow C$ give $\Gamma, \Delta, \Theta \rightarrow C$." The formal aspects of this linear calculus are never at the forefront in the presentation, but Gentzen sticks to the form he has adopted; Not a single inference line is found in the whole paper.

In contrast to many subsequent presentations of natural deduction in a linear form, what are sometimes called Fitch-style calculi, the one of

Gentzen does not encounter problems such as how to prove the composition of derivations admissible, how to define a normalization procedure, etc.

15. Relations between Gentzen's logical calculi. As stated, the handwritten thesis contains direct translations between natural deduction and the Hilbert–Ackermann axiomatization. The printed thesis has instead a translation from natural deduction to sequent calculus. It was of course important to show the equivalence of the two calculi (or two plus two, if one considers the intuitionistic and classical variants of both), but on the whole, the matter has been considered somewhat of an obligatory business without real intrinsic interest. I shall try to argue that the translation is instead highly interesting in that it tells us something about the relations of these two forms of logical calculi.

One aspect of the difference between natural deduction and sequent calculus is notational: The latter displays all the open assumptions on a line, to the left of the sequent arrow. It is not specific to sequent calculus, and as mentioned, Gentzen used the same notation in his first published proof of the consistency of arithmetic in 1936. Natural deduction in sequent calculus style is the first stage in the passage from derivations in natural deduction to ones in sequent calculus. The second step consists in the replacement of E-rules by L-rules:

In the first stage, each formula C in a natural derivation is replaced by a sequent $\Gamma \to C$ in which Γ lists the open assumptions C depends on. If there was an I-rule in the natural derivation, we have, say,

$$\dfrac{\begin{matrix}\Gamma & \Delta\\ \vdots & \vdots\\ A & B\end{matrix}}{A \,\&\, B}\,\&I \qquad \rightsquigarrow \qquad \dfrac{\begin{matrix}\vdots & \vdots\\ \Gamma \to A & \Delta \to B\end{matrix}}{\Gamma, \Delta \to A \,\&\, B}$$

Gentzen's translation gives the step $R\,\&$, modulo a second phase of weakenings in the premisses to get a rule with the shared context Γ, Δ in the premisses. The other I-rules translate similarly, and there is a direct correspondence between introductions in natural deduction and right rules in sequent calculus.

With elimination rules, the translation is different. For example, $\supset\! E$ is translated in the first phase as

$$\dfrac{\begin{matrix}\Gamma & \Delta\\ \vdots & \vdots\\ A \supset B & A\end{matrix}}{B}\supset\! E \qquad \rightsquigarrow \qquad \dfrac{\begin{matrix}\vdots & \vdots\\ \Gamma \to A \supset B & \Delta \to A\end{matrix}}{\Gamma, \Delta \to B}$$

Observe that, as stated, we still have a step in natural deduction, with a rule that operates on the right side of the arrow. In the next phase, an instance of a left rule of sequent calculus and a cut is inserted as follows:

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \to A \supset B\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \Delta \to A\end{matrix} \quad B \to B}{A \supset B, \Delta \to B}\,{\scriptstyle L\supset}}{\Gamma, \Delta \to B}\,{\scriptstyle Cut}$$

I find it strange that this step of Gentzen's has received no comment in the proof-theoretical literature before my paper on natural deduction with general elimination rules in 2001: The derivation in natural deduction could very well have been normal, a notion absolutely clear to Gentzen as we know after the rediscovery of the normalization proof, yet the result of translation is a sequent calculus derivation with a cut. Gentzen keeps absolute silence here. My belief is that he was in a terrible hurry around the time of March to May 1933, to finish his thesis. That was the case with many people, with their professors being expelled by the Nazis and the department about to go in ruins. So, I think that the correspondence between natural deduction and sequent calculus remained unclear in Gentzen's thesis mainly because of lack of time. Later, Gentzen's last section in the thesis was perhaps considered an obligatory piece of "logical bureaucracy," as Jean-Yves Girard might put it, without intrinsic interest.

The treatment of disjunction and existence elimination in the logical calculus of the 1936 consistency proof, as in the previous Section, would have easily given the right correspondence between a natural and a sequent rule: By rule $\vee E$, $\Gamma \to A \vee B$ and $A, \Delta \to C$ and $B, \Theta \to C$ give $\Gamma, \Delta, \Theta \to C$, so if the first premiss is a logical groundsequent, we get that $A \vee B \to A \vee B$ and $A, \Delta \to C$ and $B, \Theta \to C$ give $A \vee B, \Delta, \Theta \to C$. Dropping the first premiss as superfluous, we have the sequent rule.

There is another translation to consider that relates to the role of the structural rules of sequent calculus: If in rule $\supset I$ several, say two, or no copies of formula A were discharged, Gentzen's translation gives, respectively,

$$\dfrac{\begin{matrix}\overset{1}{A}, \overset{1}{A}, \Gamma\\ \vdots\\ B\end{matrix}}{A \supset B}\,{\scriptstyle \supset I,1} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\vdots\\ A, A, \Gamma \to B\end{matrix}}{A, \Gamma \to B}\,{\scriptstyle LC}}{\Gamma \to A \supset B}\,{\scriptstyle R\supset} \qquad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ B\end{matrix}}{A \supset B}\,{\scriptstyle \supset I} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \to B\end{matrix}}{A, \Gamma \to B}\,{\scriptstyle LW}}{\Gamma \to A \supset B}\,{\scriptstyle R\supset}$$

Thus, the left contraction stems from a **multiple** discharge of an assumption in natural deduction, and the left weakening from a **vacuous** discharge. The same holds for the translation of rules $\vee E$ and $\exists E$.

The problem, now, is to figure out the relation of structural rules in sequent calculus (SC) to whatever it is that corresponds to them in natural deduction (ND):

PROBLEM. *What is the correspondence between derivations in ND and SC?*

As I see it, there are two main solutions:

SOLUTION A. *Change rules $\&E$, $\supset E$, $\forall E$ of NI to have a calculus with normal derivations isomorphic to cut-free derivations in Gentzen's original LI.*

SOLUTION B. *Restrict rules $L\&$, $L\supset$, $L\forall$ of LI suitably to have derivations isomorphic to normal derivations in Gentzen's original NI.*

Solution **A** is what I have called natural deduction with general elimination rules. It is obtained by writing all the E-rules in the style of $\vee E$, $\exists E$:

$$\dfrac{A \,\&\, B \qquad \begin{array}{c}\overset{1}{A}, \overset{1}{B}\\ \vdots \\ C\end{array}}{C}{\scriptstyle\,\&E,1} \qquad \dfrac{A \supset B \quad A \quad \begin{array}{c}\overset{1}{B}\\ \vdots \\ C\end{array}}{C}{\scriptstyle\,\supset E,1} \qquad \dfrac{\forall x A \qquad \begin{array}{c}\overset{1}{A(t/x)}\\ \vdots \\ C\end{array}}{C}{\scriptstyle\,\forall E,1}$$

Next we define a root-first translation, perfectly exemplified by:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \quad \begin{array}{c}\Delta\\ \vdots\\ B\end{array}}{A \,\&\, B}{\scriptstyle\,\&I} \qquad \begin{array}{c}\overset{1}{A}, \overset{1}{B}, \Theta\\ \vdots \\ C\end{array}}{C}{\scriptstyle\,\&E,1} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \quad \begin{array}{c}\Delta\\ \vdots\\ B\end{array}}{\Gamma, \Delta \to A \,\&\, B}{\scriptstyle\,R\&} \qquad \dfrac{\begin{array}{c}A, B, \Theta\\ \vdots\\ C\end{array}}{A \,\&\, B, \Theta \to C}{\scriptstyle\,L\&}}{\Gamma, \Delta, \Theta \to C}{\scriptstyle\,Cut}$$

The effect of the translation is that derived major premisses of E-rules give cuts. If, instead, they are assumptions, we get:

$$\dfrac{A \,\&\, B \qquad \begin{array}{c}\overset{1}{A}, \overset{1}{B}, \Theta\\ \vdots \\ C\end{array}}{C}{\scriptstyle\,\&E,1} \quad \rightsquigarrow \quad \dfrac{A \,\&\, B \to A \,\&\, B \qquad \dfrac{\begin{array}{c}A, B, \Theta\\ \vdots\\ C\end{array}}{A \,\&\, B, \Theta \to C}{\scriptstyle\,L\&}}{A \,\&\, B, \Theta \to C}{\scriptstyle\,Cut}$$

In this case, the translation gives a *Cut* the left premiss of which is an initial sequent so that the conclusion of the cut is identical to its right premiss. Such cuts can be deleted, so we can put:

DEFINITION. *A derivation in natural deduction with general elimination rules is* **normal** *if all major premisses of E-rules are assumptions.*

OVERALL RESULT. *The order of logical rules is the same in normal derivations in ND with general elimination rules and in cut-free derivations in SC.*

The correspondence that Gentzen's work gives between weakening and vacuous discharge and contraction and multiple discharge goes only in one direction, from natural deduction to sequent calculus. The correspondence becomes exact through the general elimination rules:

INTERPRETATION OF WEAKENING AND CONTRACTION IN ND. *If a formula in an SC derivation is active in a logical rule and has been obtained through a weakening, translation into ND with general elimination rules produces a vacuous discharge of that formula in the logical rule. Similarly, if the formula has been contracted, translation to ND produces a multiple discharge.*

The cuts produced by Gentzen's translation of rules $\& E$, $\supset E$, and $\forall E$ find an explanation: They are produced by the hidden non-normalities of his elimination rules, those cases in which a major premiss of an E-rule has been derived by Gentzen's $\& E$, $\supset E$, or $\forall E$. Furthermore, cuts in which the cut formula is principal in a logical rule in the right premiss of cut, so in a left logical rule, can be interpreted as non-normal instances of the corresponding elimination rule of natural deduction.

Next consider solution B, the change of some sequent rules. To do this in a transparent way, we look at Gentzen's NI-rules as special cases of the general rules: Just make the auxiliary derivations in the general rules degenerate, and the result is:

$$\frac{A \,\&\, B \quad \overset{1}{A}}{A}\,{\scriptstyle \& E,1} \qquad \frac{A \,\&\, B \quad \overset{1}{B}}{B}\,{\scriptstyle \& E,1} \qquad \frac{A \supset B \quad A \quad \overset{1}{B}}{B}\,{\scriptstyle \supset E,1} \qquad \frac{\forall x A \quad \overset{1}{A(t/x)}}{A(t/x)}\,{\scriptstyle \forall E,1}$$

The application of isomorphic translation to Gentzen's rules gives:

$$\frac{\begin{matrix}\Gamma\\ \vdots\\ A \,\&\, B\end{matrix} \quad \overset{1}{A}}{A}\,{\scriptstyle \& E,1} \quad \rightsquigarrow \quad \frac{\begin{matrix}\vdots\\ \Gamma \to A \,\&\, B\end{matrix} \quad \dfrac{A \to A}{A \,\&\, B \to A}\,{\scriptstyle L\&}}{\Gamma \to A}\,{\scriptstyle Cut}$$

$$\frac{\begin{matrix}\Gamma\\ \vdots\\ A \supset B\end{matrix} \quad \begin{matrix}\Delta\\ \vdots\\ A\end{matrix} \quad \overset{1}{B}}{B}\,{\scriptstyle \supset E,1} \quad \rightsquigarrow \quad \frac{\begin{matrix}\vdots\\ \Gamma \to A \supset B\end{matrix} \quad \dfrac{\begin{matrix}\vdots\\ \Delta \to A\end{matrix} \quad B \to B}{A \supset B, \Delta \to B}\,{\scriptstyle L\supset}}{\Gamma, \Delta \to B}\,{\scriptstyle Cut}$$

$$\frac{\begin{matrix}\Gamma\\ \vdots\\ \forall x A\end{matrix} \quad \overset{1}{A(t/x)}}{A(t/x)}\,{\scriptstyle \forall E,1} \quad \rightsquigarrow \quad \frac{\begin{matrix}\vdots\\ \Gamma \to \forall x A\end{matrix} \quad \dfrac{A(t/x) \to A(t/x)}{\forall x A \to A(t/x)}\,{\scriptstyle L\forall}}{\Gamma \to A(t/x)}\,{\scriptstyle Cut}$$

The results are pretty obvious:

RESULT 1. *Derivations in Gentzen's NI are isomorphic to derivations in sequent calculus when the following restrictions apply*:

(i) *Cut formulas are principal in right premisses of cuts.*

(ii) *Premisses of rules $L\,\&$ and $L\forall$, and the second premiss of rule $L\supset$, are initial sequents and the principal formulas cut formulas in the next rule.*

RESULT 2. *A derivation in Gentzen's NI is normal if and only if the following holds for its translation into sequent calculus in which $L\&$, $L\supset$, and $L\forall$ are "restricted":*

(i) *Cut formulas are not principal in left premisses of cuts.*
(ii) *Left premisses of cuts are not derived by $L\vee$ or $L\exists$.*

There remains the question of the relation of normalization to cut elimination. There are cuts in *SC* that don't have any interpretation in the above terms (*cf.* the remarks after the interpretation of weakening and contraction). Therefore there is still room for a final word on the relation of *ND* and *SC*.

Sources: The idea of changing some of Gentzen's elimination rules occurred perhaps first in Schroeder–Heister (1984) who changed the conjunction elimination rule. The full system of general E-rules is found in the rare publication Dyckhoff (1988), then in work of Tennant (1992) and Lopez-Escobar (1999). Solution A, the isomorphic translation between cut-free derivations in sequent calculus and normal derivations in natural deduction with general elimination rules, was defined in von Plato (2001b).

Solution B, the dual idea of restricting Gentzen's sequent calculus so as to obtain isomorphism with his original system of natural deduction, is carried through in von Plato (2011).

16. Closing remarks about the consistency of arithmetic. Gentzen's first attempts in the foundations of mathematics were directed towards resolving the question of the consistency of elementary arithmetic. Natural deduction and sequent calculi, the two main types of proof systems today, as well as the profound understanding of their structure, were a byproduct of his enterprise.

The "peak search" mentioned in the end of the quotation in Section 11 was a component of Gentzen's strategy in trying to prove the consistency of arithmetic and the passage that follows in the manuscript an early clue on how he thought he would obtain such a proof within sequent calculus. These peaks appear in the endpiece of a derivation, very prominent in the published 1938 consistency proof of arithmetic within the classical sequent calculus *LK*. When sending that later work to Bernays, Gentzen also wrote that the proof is now in a form in which he originally found the consistency proof, namely, similarly to the elimination of cuts in the purely logical calculus. We have seen to some extent, towards the end of Section 13, how the peaks can be permuted up in the endpiece to be eliminated, but the whole story of Gentzen's successes and failures with consistency proofs has to await another occasion.

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