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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Steve Awodey (Managing Editor), John Burgess, Mark Colyvan, Anuj Dawar, Noam Greenberg, Rahim Moosa, Ernest Schimmerling, Alex Simpson, Kai Wehmeier, and Matthias Wille. Authors and publishers are requested to send, for review, copies of books to *ASL, Box 742, Vassar College, 124 Raymond Avenue, Poughkeepsie, NY 12604, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

Games, scales, and Suslin cardinals. The Cabal Seminar, Volume I, edited by A. S. Kechris, B. Löwe, and J. R. Steel, Lecture Notes in Logic, vol. 31. Association for Symbolic Logic and Cambridge University Press, Cambridge, 2008, xi + 445 pp.—therein:

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Descriptive set theory is a diverse subject, with many different techniques and threads of research. The common underlying theme is definability over the reals. Here and below we shall adopt the logician's view of the real numbers as elements of any fixed uncountable Polish space such as the Baire space ${}^{\omega}\omega$, the Cantor space ${}^{\omega}2$, or the Euclidean spaces $\mathbb{R}^k$. *Polish* means separable and completely metrizable. It is customary to use the symbol $\mathbb{R}$ to denote any uncountable Polish space. The rationale for this blurring of vision is that all uncountable Polish spaces are Borel isomorphic, and many questions in descriptive set theory are invariant under Borel isomorphisms.

In a cartoonish description of the subject, two large sub-areas of descriptive set theory are singled out: the pure and the applied. Before moving on we should stress that this distinction is just a matter of expositional convenience, rather than a real divide in the field, since many results can be ascribed to either camp. The adjective *pure* only suggests that the results and techniques are closer to topics in logic such as recursion theory or inner model theory, while *applied* indicates connections with other areas of mathematics such as ergodic theory, operator algebra, etc. Most of the early work was motivated by problems in real analysis, and, for this reason, the applied side of the subject is also known as *classical* descriptive set theory. I am now going to focus on the pure, more abstract aspects of the subject, which henceforth will be simply called descriptive set theory.

In the wake of Cohen's breakthrough on the Continuum Hypothesis, all areas of set theory experienced a renaissance. In addition, methods from recursion theory and infinite games became prominent in descriptive set theory. By the late seventies, southern California had became the hotbed of the subject, and logicians from UCLA and Caltech started the tradition of Friday afternoon meetings where new results were exposed. These became known as the *Cabal Seminar*, and some of the mathematics presented there was collected in a series of papers published in the Springer Lecture Notes in Mathematics series: ***Cabal Seminar 76–77*** (vol. 689), ***Cabal Seminar 77–79*** (vol. 839), ***Cabal Seminar 79–81*** (vol. 1019), ***Cabal Seminar 81–85*** (vol. 1333). A fifth volume was planned but did not appear. The four books have been out of print for a while but their mathematical content is just as relevant now as it was when they were written, since they constitute the basis for current research in inner model theory and determinacy; see Steel's article *Games and scales* in the book under review. Therefore the ASL's decision to reprint the original papers, together with the material originally planned for the fifth volume, is excellent news for anyone interested in descriptive set theory and inner model theory. These new books are organized around specific topics, which makes them more useful than the original volumes, where papers appeared in chronological order.

The book under review is the first in a planned series of four, and it is divided into two parts: the first focused on *games and scales*, the second devoted to *Suslin cardinals, partition properties, and homogeneous trees*. Each part contains a new introductory essay, articles from the original Cabal volumes, and new research papers. Before looking at the content of the book, let me first recall some basic definitions and results.

Given sets X and $A \subseteq {}^{\omega}X$, consider the game in which two players **I** and **II** take turns choosing elements from X, with **I** moving first. Player **I** wins a run of the game if the resulting sequence belongs to A; otherwise **II** wins. The set A is said to be determined iff either player has a winning strategy. In the last four decades, set theorists have devoted an enormous amount of energy to investigating which sets of reals are determined. One reason for this (wonderful) obsession is that assumptions about determinacy imply intuitively correct answers to many natural questions in analysis. Let me digress a bit on this point. A (boldface) pointclass $\mathbf{\Gamma}$ is a nonempty collection of sets of reals closed under continuous preimages, its dual $\check{\mathbf{\Gamma}} = \{{}^{\omega}\omega \setminus A \mid A \in \mathbf{\Gamma}\}$ is the family of all complements, and $\exists^{\mathbb{R}}\mathbf{\Gamma}$ is the pointclass of all projections of sets in $\mathbf{\Gamma}$, i.e., sets of the form $\{x \in {}^{\omega}\omega \mid \exists y\,(x, y) \in A\}$ where $A \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ belongs to $\mathbf{\Gamma}$. (Here and below, I blatantly identify the Baire space with its own square.) Examples of pointclasses are the Borel sets $\mathbf{\Delta}^1_1$, $\mathbf{\Sigma}^1_1 = \exists^{\mathbb{R}}\mathbf{\Delta}^1_1$ (the analytic sets) and its dual $\mathbf{\Pi}^1_1$ (the coanalytic sets), and the projective sets $\bigcup_n \mathbf{\Sigma}^1_n$, where $\mathbf{\Sigma}^1_{n+1} = \exists^{\mathbb{R}}\mathbf{\Pi}^1_n$ and $\mathbf{\Pi}^1_n$ is the dual of $\mathbf{\Sigma}^1_n$. Going back to determinacy: assuming $\mathsf{Det}(\mathbf{\Gamma})$, that is, every set in $\mathbf{\Gamma}$ is determined, every set in $\exists^{\mathbb{R}}\mathbf{\Gamma}$ has useful analytic properties such as Lebesgue measurability, Baire measurability and the perfect set property.

Martin proved in ZFC that all Borel sets are determined. The conclusion cannot be strengthened to the determinacy of larger pointclasses by a result of Harrington. (Analytic determinacy implies that ZFC is consistent and more.) However, with large cardinal assumptions, more determinacy follows. For $n < \omega$, Martin and Steel proved that $\mathsf{Det}(\mathbf{\Sigma}^1_{n+1})$ follows from the existence of n many Woodin cardinals with a measurable cardinal above them all. (The case $n = 0$ is due to Martin.) Woodin showed that if there are infinitely many Woodin cardinals and a measurable cardinal above them all, then every set of reals in $\mathsf{L}(\mathbb{R})$ is determined, where $\mathsf{L}(\mathbb{R})$ is the smallest proper class transitive model of ZF that contains $\mathbb{R}$. Put another way: large cardinals imply that $\mathsf{L}(\mathbb{R})$ is a model of the Axiom of Determinacy (AD), which says that every set of reals is determined. Note that AD implies that every set of reals is Lebesgue measurable, which contradicts the Axiom of Choice (AC). Descriptive set theorists have not proposed AD as an alternative to AC but they are interested in models of AD, so they keep tabs on when AC is used. Some choice is needed to get things off the ground, for example, to make sense of Lebesgue measurability. The Axiom of Dependent Choice (DC) suffices for this and holds in $\mathsf{L}(\mathbb{R})$. We always assume ZF and usually DC too.

Let κ be a cardinal. For $A \subseteq {}^{\omega}\omega$, define A to be κ-Suslin iff A is the projection of a closed subset of ${}^{\omega}\omega \times {}^{\omega}\kappa$. This means that there is a tree T on $\omega \times \kappa$ such that A is the projection in the first coordinate of $[T]$, the set of all branches through T. Let $\mathbf{S}(\kappa)$ be the pointclass of κ-Suslin sets. Then $\mathbf{S}(\omega) = \mathbf{\Sigma}^1_1$ and $\mathbf{\Sigma}^1_2 \subseteq \mathbf{S}(\omega_1)$ by the proof of Shoenfield's absoluteness theorem. If $A \neq \emptyset$ and (x, y) is the left-most branch of T, then x is a member of A that is definable from T. In one more dimension, if T is a tree on $\omega \times \omega \times \kappa$ and $A \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ is the projection of $[T]$, then there is a function f that is definable from T and uniformizes A in the sense that $f \subseteq A$ and $\mathrm{dom}(f) = \mathrm{dom}(A)$. What we really want are conditions on a pointclass $\mathbf{\Gamma}$ that imply every binary relation $A \in \mathbf{\Gamma}$ can be uniformized by a function f with graph in $\mathbf{\Gamma}$. But κ and T might not be definable in any meaningful way, so finding the right conditions takes additional insight. The motivation is the Kôndo-Novikov theorem, which says every $\mathbf{\Pi}^1_1$ relation can be uniformized by a $\mathbf{\Pi}^1_1$ function. Observe that this sort of uniformization is a kind of definable choice, certain to be useful in our context.

If $A \subseteq {}^{\omega}\omega$ and κ is a cardinal, then A admits a κ-scale iff there are maps $\varphi_n \colon A \to \kappa$ such that whenever $\lim_{i\to\infty} x_i = x$ with $x_i \in A$ and $\lim_{i\to\infty} \varphi_n(x_i) = \lambda_n$, then $x \in A$ and $\varphi_n(x) \leq \lambda_n$. It turns out that A is κ-Suslin iff A admits a κ-scale. We say that A admits a $\mathbf{\Gamma}$-scale iff there is an ordinal κ and a κ-scale $\langle \varphi_n \mid n < \omega \rangle$ on A such that for all y, if $y \in A$ then $\forall n\, \forall x[(x \in A \wedge \varphi_n(x) \leq \varphi_n(y)) \iff S_{\mathbf{\Gamma}}(n, x, y) \iff S_{\check{\mathbf{\Gamma}}}(n, x, y)]$, where $S_{\mathbf{\Gamma}} \in \mathbf{\Gamma}$ and $S_{\check{\mathbf{\Gamma}}} \in \check{\mathbf{\Gamma}}$. The pointclass $\mathbf{\Gamma}$ is scaled iff every member of $\mathbf{\Gamma}$ admits a $\mathbf{\Gamma}$-scale. It turns out that if $\mathbf{\Gamma}$ is scaled, $\mathbf{\Gamma} = \forall^{\mathbb{R}}\mathbf{\Gamma}$ and $\mathbf{\Gamma}$ satisfies an additional minor closure condition that we will not state, then every relation in $\mathbf{\Gamma}$ can be uniformized by a function in $\mathbf{\Gamma}$.

The pointclass $\mathbf{\Pi}^1_1$ is scaled by the proof of the Kôndo-Novikov theorem, and so is $\mathbf{\Sigma}^1_2$. But $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_2$ are not. These are theorems of $\mathsf{ZF} + \mathsf{DC}$. Moschovakis, who formulated the notion of scale, extended this pattern of results to higher pointclasses assuming AD. The analysis of which pointclasses have what sorts of scales has been a leitmotif of the Cabal Seminar. The existence of arbitrarily large scaled pointclasses (equivalently, the assertion that every binary relation on the reals can be uniformized) is not a consequence of AD. For example, $\boldsymbol{\Sigma}^2_1$ is the largest scaled pointclass of $\mathrm{L}(\mathbb{R})$. In fact it is the largest pointclass $\boldsymbol{\Gamma}$ such that every set in $\boldsymbol{\Gamma}$ can be uniformized by some function in $\mathrm{L}(\mathbb{R})$. To ensure that every subset of the plane can be uniformized, we use the stronger Axiom of Real Determinacy ($\mathsf{AD}_{\mathbb{R}}$), the analogue of AD for games with payoff $A \subseteq {}^{\omega}\omega$ in which **I** and **II** play reals $x_n \in {}^{\omega}\omega$ and **I** wins iff the real obtained by dovetailing the x_n's belongs to A.

It has been expedient to discuss only boldface pointclasses above but many of the papers under review are about lightface pointclasses, which are the effective versions. So the results in these papers are finer.

Let us now look at the content of the book under review. Part I starts with the introductory paper *Games and scales* by Steel. This, together with the next paper *Notes on the theory of scales* by Kechris and Moschovakis from the first Cabal volume, are mandatory reading for anyone interested in descriptive set theory. Steel's paper is particularly useful because it serves as a guiding map to read (and put in perspective) all the other papers in Part I. Next is a new paper, *Propagation of scales by games* by Neeman, that, by collecting several results and techniques scattered in the literature, is a natural companion to the first two papers. The next four papers, *Scales on* Σ^1_1 *sets* by Steel, *Inductive scales on inductive sets* and *Scales on coinductive sets* by Moschovakis, and *The extent of scales in* $\mathrm{L}(\mathbb{R})$ by Martin and Steel, from the first and third Cabal volumes, study the existence and definability of scales in larger and larger pointclasses. Martin's paper, *The largest countable this, that, and the other* from the third Cabal volume, characterizes the largest countable Σ^1_{2k} set C_{2k} in terms of the complexity of its elements. More precisely $\mathrm{C}_{2k} = \bigcup_n \Game^{2k-1}\omega \cdot n\text{-}\Pi^1_1$ where $\Game$ is the game quantifier, and $\omega \cdot n\text{-}\Pi^1_1$ is the pointclass of all $\omega \cdot n$ differences of Π^1_1 sets. It turns out that C_{2k} is the set of reals in the canonical inner model with $2k-2$ Woodin cardinals. In particular $\mathrm{C}_2 = \mathbb{R} \cap \mathrm{L}$ if $\mathbb{R} \cap \mathrm{L}$ is countable. The next two papers by Steel—*Scales in* $\mathrm{L}(\mathbb{R})$ from the third Cabal volume, and the new *Scales in* $\mathrm{K}(\mathbb{R})$—investigate the level-by-level definability of scales in $\mathrm{L}(\mathbb{R})$, as well as in the *real core model* $\mathrm{K}(\mathbb{R})$. These are technical papers, combining ideas from different parts of set theory, such as Jensen's fine structure theory and higher recursion theory, but the techniques developed there have become increasingly important for further development of inner model theory. The next three papers: Martin's *The real game quantifier propagates scales* from the third Cabal volume, Steel's *Long games* from the fourth Cabal volume, and Steel's new *The length-*ω_1 *quantifier propagates scales*, provide increasingly sophisticated extensions of Moschovakis' game comparison technique to propagate scales.

Part II of the book is centered around the notion of Suslin cardinal. An infinite cardinal κ is Suslin if there is a set which is κ-Suslin but not λ-Suslin for any smaller λ. That is, $\mathbf{S}(\kappa) \neq \bigcup_{\lambda<\kappa} \mathbf{S}(\lambda)$. Many questions about arbitrary cardinals can be reduced to problems on the Suslin cardinals. The first two Suslin cardinals are ω and ω_1. Other examples of Suslin cardinals are the projective cardinals $\boldsymbol{\delta}^1_n$, defined as the supremum of the $\boldsymbol{\Delta}^1_n$ prewellorderings of $\mathbb{R}$. The first paper of Part II, *Suslin cardinals, partition properties, homogeneity* by Jackson, serves as survey and introduction to the topics discuss there. Kechris' paper, *Suslin cardinals, κ-Suslin sets, and the scale property in the hyperprojective hierarchy* from the second Cabal volume, gives a nice account of the theory of Suslin cardinals up to the late seventies.

An ordinal that plays a key role in descriptive set theory is Θ, the supremum of ordinals α for which there is a surjection from $\mathbb{R}$ to α. In ZFC, it is easy to see that $\Theta = (2^{\aleph_0})^+$, but we are not assuming AC, so $\mathbb{R}$ need not have a cardinality. Moschovakis' Coding Lemma, an important tool in descriptive set theory, implies that, under AD, if $\alpha < \Theta$, then $\mathcal{P}(\alpha)$ is also a surjective image of $\mathbb{R}$. In *A coding theorem for measures* from the fourth Cabal volume, Kechris extends this by proving that, under $\mathsf{AD}_{\mathbb{R}}$, if $\alpha < \Theta$, then $\boldsymbol{\beta}(\alpha)$, the collection of all ultrafilters on α, is also a surjective image of $\mathbb{R}$.

In *The axiom of determinacy, strong partition properties, and nonsingular measures* from the second Cabal volume, Kechris, Kleinberg, Moschovakis and Woodin show that $\mathsf{AD} + \mathsf{DC}$ implies there are arbitrarily large strong partition cardinals below Θ. A cardinal κ has the strong partition property iff $\kappa \to (\kappa)^{\kappa}_{2}$. In *The equivalence of partition properties and determinacy*, Kechris and Woodin prove that the converse is true in $\mathrm{L}(\mathbb{R})$, namely that if $\mathrm{V} = \mathrm{L}(\mathbb{R})$, then AD is equivalent to the existence of arbitrarily large strong partition cardinals below Θ. The other paper by Kechris and Woodin, *Generic codes for uncountable ordinals, partition properties, and elementary embeddings*, develops category-theoretic arguments for coding ordinals. The previous two papers were not in the original Cabal volumes, and are based on circulated handwritten notes from the early 80s.

There is a strengthening of the notion of κ-Suslin that requires the witnessing tree T to be equipped with a homogeneity system, which is a certain kind of system of countably additive measures that we will not define here. Should A have this property for some κ, then A is said to be homogeneously Suslin, and, as a consequence, A is determined. Martin, Steel and Woodin take this route in the proofs of the determinacy theorems mentioned earlier. The projection of a homogeneously Suslin subset is called weakly homogeneously Suslin. Such sets are Lebesgue measurable, Baire measurable and have the perfect set property, but they might not be determined. The complement of a weakly homogeneously Suslin set can be seen to be Suslin. Martin and Steel, in *The tree of a Moschovakis scale is homogeneous*, show that, under $\mathsf{AD} + \mathsf{DC}$, if A and $\mathbb{R} \setminus A$ are Suslin, then they are both homogeneously Suslin. Martin and Woodin, in *Weakly homogeneous trees*, show that $\mathsf{AD}_{\mathbb{R}}$ implies that every tree is weakly homogeneous.

To summarize: this book contains some very interesting mathematics that will continue to influence the development of set theory for many years to come.

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Rodney G. Downey and Denis R. Hirschfeldt. ***Algorithmic randomness and complexity.*** Theory and Applications of Computability. Springer, 2010, xxviii + 855 pp.

The book a whole community has been waiting for! Rod Downey and Denis Hirschfeldt's ***Algorithmic randomness and complexity*** was already cited dozens of times before it was published, as the authors maintained an online draft during most of the writing process. This online draft has now become a 850-page book, the first published by Springer in its new book series *Theory and Applications of Computability*.

The subject of this monograph is algorithmic randomness (a.k.a. effective randomness). The main purpose of this area of research is to answer the philosophical question: "What does it mean for a single object to be random?". Classical probability theory allows us to talk about the probability of an event to occur, but it says nothing about the outcome of an experiment, even when some outcomes look much less random than others (for example, the outcome 44444 … 4 [1000 times 4] for the experiment consisting of throwing a regular dice 1000 times looks "non-random" despite having the same probability of occurrence as any other outcome). The first attempts to formalize this intuition go back to von Mises in the 1910s, but it was only in the 1960s that this was done in a rigorous manner by Solomonoff, Chaitin and Kolmogorov for finite objects and by Martin-Löf for infinite ones. Both approaches involved computability theory. More progress was made in the 1970s by researchers such as Levin, Solovay and Schnorr (it is worth mentioning that Solovay never published his work in this area, and only left a manuscript containing numerous deep theorems; Downey and Hirschfeldt got ahold of this manuscript and present its central results, which nobody had done before them). The field decreased somewhat in popularity in the 1980s and 1990s (some remarkable results were still proven at the time, e.g., by Kučera and Gács), but regained the interest of computability theorists in the early 2000s, and has been growing rapidly since then. It is currently one of the most central and active topics in computability theory.

Algorithmic randomness and complexity consists of four parts. The first hundred pages are a crash course in computability theory in an attempt to make the book as self-contained as possible. Then the central notion of algorithmic randomness for finite strings, Kolmogorov complexity, is introduced and the main results of this theory are presented. The first part also includes a chapter on effective real numbers (mostly lower semicomputable ones) which are of utmost importance in the rest of the book.

The second part deals with the theory of effective randomness for infinite sequences, and how one can define a *random* sequence. The "big three" paradigms of algorithmic randomness are presented:

- *typicality*: a random sequence should have all properties of measure 1 that can be tested algorithmically
- *incompressibility*: a random sequence should have no simple description; in other words, it should be hard to compress (this is the paradigm that relates randomness for finite strings and randomness for infinite ones)
- *unpredictability*: a random sequence should be hard to guess; there is no effective strategy which can guess its bits with better-than-average accuracy.

Of course, these paradigms are rather informal, and different interpretations lead to different randomness notions. More than twenty different randomness notions have emerged from these paradigms, and most of them are introduced in this second part. A large part of the exposition, however, focuses on four or five of them, with particular attention paid to the notion that is considered to be the best of all: Martin-Löf randomness. The delicate interactions between the three above paradigms and the subsequent randomness notions are illustrated by a great number of results. These interactions are so deep that most of the interesting randomness notions can be expressed via all three of them. This part of the book also illustrates the deep interplay between effective randomness and computability theory, which holds in both directions: computability theory is essential to define and study effective randomness, but the objects and tools of this theory give a fresh point of view on some computability questions, yielding new results. Readers who are familiar with computability theory but not with algorithmic randomness might be amazed to see in Chapter 8 ("Algorithmic randomness and Turing reducibility") that high degrees, diagonally non-computable degrees, PA degrees, GL_1 degrees, hyperimmune-free degrees, jump inversion theorems, to name only a few, all have something to do with algorithmic randomness.

This interplay between randomness and computability is perhaps even more present in the third part of the book, which addresses the natural question: when can we say that an infinite sequence is *more* random than another one? Again, there are several ways to approach this question, and the authors discuss many of them: comparison of initial segment complexity, randomness relativized to an oracle, Solovay reducibility for lower semicomputable random reals, etc. This study of relative randomness naturally leads one to look at the other extreme: sequences that are far from random. A particular class, the class of K-trivial sequences, plays a fundamental role in this line of study. Originally defined by Chaitin as sequences of minimal prefix-free Kolmogorov complexity, the K-trivial sequences have been proven by Nies and several co-authors to coincide with two classes of computationally weak sequences introduced independently by Zambella (the class of oracles that do not affect Martin-Löf randomness) and Muchnik (the class of oracles that do not affect prefix-free complexity). The declinations of this idea ("far from random equals computationally weak") has become one of the central topics in algorithmic randomness and is discussed at length by Downey and Hirschfeldt, in particular in Chapter 11 and Chapter 12. Another gem of algorithmic randomness is presented in Chapter 13: effective Hausdorff dimension. The work of Lutz, Mayordomo, Staiger and others show that the most natural constructive version of Hausdorff dimension (in the set of infinite binary sequences with its usual distance) has a very natural characterization in terms of Kolmogorov complexity. It is quite amazing that these two seemingly unrelated notions are in fact so tightly connected, and the study of effective Hausdorff dimension has lead to fascinating developments, such as the work of Downey, Greenberg and Miller on the links between this notion and diagonally non-computable Turing degrees.

Perhaps a bit artificially separated from the rest of the book, the last part discusses three particular topics: strongly jump traceable sequences (a subclass of the K-trivials), the relativizations of Chaitin's Omega numbers (which are exactly the lower semicomputable, Martin-Löf random reals) and the initial segment complexity of computably enumerable sets.

Overall, this book is an absolutely remarkable piece of work, both for its breadth and its depth. The authors have managed to keep an eye on most of the interesting developments in the field over the last ten years (with a few notable exceptions, such as randomness for spaces other than the space of infinite binary sequences, or the interactions between Kolmogorov complexity and combinatorics), and have included the most important results in their book. ***Encyclopedia of algorithmic randomness*** would have been a suitable alternative title! Of course, the downside of the encyclopedic nature of Downey and Hirschfeldt's book is that it is rather difficult to "navigate". Many chapters present a series of theorems, some of truly fundamental importance, some only of interest to experts, and the distinction between the two is not always made clear. This unfortunately means that this book cannot really serve as a textbook for a first course on effective randomness. However, it is nicely structured and very well written, and therefore, with some guidance, it will make a valuable introduction to algorithmic randomness for graduate students and researchers outside the field. Downey and Hirschfeldt's ***Algorithmic randomness and complexity*** is a most impressive book, playing at least in the same league as modern classics such as Soare's or Odifreddi's monographs on computability theory. It definitely sets a very high standard for the *Theory and Applications of Computability* book series it initiates.

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Foundational theories of classical and constructive mathematics, edited by Giovanni Sommaruga, Springer, 2011. xi + 314 pp.

Foundational theories of classical and constructive mathematics is an anthology that, as the editor notes in the preface, "grew out of the Swiss Society for Logic and Philosophy of Science (SSLPS) annual meeting in Fribourg (Switzerland) on October 11/12 2006" (p. v). Nevertheless, the contributions contained in the volume are not all new—a portion of Penelope Maddy's ***Naturalism in mathematics***, Clarendon Press, 1997, is reprinted here, as is a paper by Sol Feferman originally published in 2006.

The volume is divided thematically into four sections: *Part I Senses of Foundations of Mathematics*, *Part II Foundations of Classical Mathematics*, *Part III Between Foundations of Classical and Foundations of Constructive Mathematics*, and *Part IV Foundations of Constructive Mathematics*.

The first section of the volume is the most philosophical. Each of the four chapters in this section emphasizes one or more roles that a foundation of mathematics should play, and then evaluates one or more proposed accounts of the foundations of mathematics in terms of the highlighted role or roles. Geoffrey Hellman's chapter ("Foundational Frameworks") emphasizes those aspects of foundations tied up with expressiveness, explanation, proof, pluralism, and open-endedness, and he argues that his modal structuralism fares better in these regards than either set theory or category theory. Bob Hale ("The Problem of Mathematical Objects") asks how we can be justified in believing that there are infinitely many mathematical objects, and he argues that Fregean (or Scottish) logicism (or, more generally, what he calls a property-based approach) fares better with regard to answering this question than accounts based on intuition or on some other type of direct access to potentially or actually infinite sequences of objects. Maddy ("Set Theory as a Foundation") identifies six different roles that a foundation might play, and argues that although set theory cannot play some of these roles, it can nevertheless provide us with what she calls a 'methodological foundation' insofar as it provides us with set-theoretic surrogates for those structures studied in most if not all areas of modern mathematics. Finally, Stewart Shapiro ("Foundations: Structures, Sets, and Categories") distinguishes between ontological, epistemological, and organizational roles that a foundation might play, and evaluates set theory, category theory, and Scottish logicism in terms of each of these three roles. Unlike the previous papers, Shapiro does not advocate one approach over another, instead claiming only to have "sufficiently illuminated the terrain" (p. 110), pointing out strengths and weaknesses in each case along the way.

The remaining three sections of the text involve (for the most part) a sharp shift from primarily philosophical concerns to a decidedly more formal approach. This is not to say, of course, that there is not much of philosophical interest in these papers, and most of the authors take pains to outline the philosophical underpinnings motivating the theorem-proving. Nevertheless, these later chapters (with one exception—McCarty's paper, discussed in detail at the end of this review) involve moving from a focus on asking philosophical questions about mathematical frameworks to a focus on answering philosophical questions via work within these mathematical frameworks (or via drawing connections between two or more such frameworks, etc.).

Before examining the content of these sections, it should also be noted that, in moving from the first section to sections two through four, it is not only the emphasis on philosophical investigation that disappears. In addition, all mention of Scottish logicism disappears as well. This is, on one level, perhaps understandable—after all, Scottish logicism has not benefited from the sort of sustained technical development that has, over the past century, transformed set theory, category theory, and type theory from philosophical ideas into mature mathematical theories—K. Fine, ***The limits of abstraction***, Oxford University Press, 2002, notwithstanding. Nevertheless, the reader might feel that an opportunity was lost here, since a volume of this sort would have been the perfect venue in which to develop the beginnings of an examination of the mathematics of Scottish logicism. Given the fact that logicism features prominently in two out of four of the more philosophical papers in the first section of the volume, some examination of the mathematics of this view in the later sections would have been both appropriate and welcome.

Setting aside this complaint, the content of the latter three sections of the book is excellent, and even the reader who is well-versed in the mathematics involved will likely find much that is new and of interest here. Steve Awodey begins the second section ("From Sets to Types, to Categories, to Sets") with a novel argument for category theory: after examining mappings from set theory to type theory, from type theory to category theory, and from category theory back to set theory, he argues that categories fare better as a foundation since category theory, unlike set theory or type theory, retains its full content under such transformations. Solomon Feferman's contribution ("Enriched Stratified Systems for the Foundations of Category Theory") continues his ongoing examination into the set-theoretical foundations of category theory, examining whether and how such a foundation can be provided in a version of Quine's *New Foundations*. Colin McLarty ("Recent Debate over Categorical Foundations") defends category theory as a foundation of mathematics from criticisms found in earlier work by Hellman and Feferman (amongst others), arguing in addition that the defenses of category theory that he outlines were already present in the early development of category theory by William Lawvere and Saunders Mac Lane. The only complaint one could reasonably level at the papers in this section is that, jointly, they display a preoccupation with category theory that seems odd in a section of the text whose title (*Foundations of Classical Mathematics*) suggests a more balanced overview of various foundational approaches.

The third section of the volume begins with John Bell's ("The Axiom of Choice in the Foundations of Mathematics") examination of the role that the axiom of choice has played in the development of foundations, and includes an insightful discussion of Razvan Diaconescu's proof that, in extensional contexts, the axiom of choice intuitionistically implies the law of excluded middle. Joachim Lambek and Philip J. Scott ("Reflections on the Categorical Foundations of Mathematics") then examine Gödel's theorem and its connection to various ways of understanding mathematical truth within the context of intuitionistic type theory, concluding that:

> Gödel's theorem does not assert that provability fails to capture the notion of absolute truth in the Platonic universe. Rather, it asserts that other models of set theory are required than those which resemble the alleged Platonic universe. (p. 184).

The fourth, and final, section of the volume is, as noted above, titled *Foundations of Constructive Mathematics*, but a warning is in order: The reader should not enter into this section expecting five essays on constructive mathematics in the narrow sense—that is, on the mathematical systems developed by L. E. J. Brouwer, Everett Bishop, and their followers. Instead, this section covers a much broader grab-bag of topics, unified by the fact that each involves some sort of restriction or rejection of central classical principles or assumptions. The section begins with Peter Aczel's ("Local Constructive Set Theory and Inductive Definitions") examination of local constructive set theory—essentially, a many-sorted version of constructive set theory—which focuses on recapturing various theorems regarding inductive definitions in the local context, where there is no direct quantification over the entire universe. John Mayberry ("Euclidean Arithmetic: The Finitary Theory of Finite Sets") then continues his work on Euclidean arithmetic—a version of intensional, finitistic set theory inspired by ancient Greek understandings of number—focusing on various counterintuitive consequences of this alternative conception of number and set, including the fact that there are natural number systems incomparable in length. Richard Tieszen's contribution ("Intentionality, Intuition, and Proof in Mathematics") argues that we should understand proofs as satisfying mathematical meaning-intensions (where this intentionality is understood in terms of the 'aboutness' of our mathematical thinking), comparing and contrasting classical and constructive notions of proof within this intensional framework. Finally, Paul Taylor ("Foundations for Computable Topology") rejects all of the foundational approaches addressed so far, instead arguing that foundations for particular mathematical disciplines should be formulated in terms of adjunctions, and he develops a version of computable topology based on this approach.

The reader who is counting carefully will have noticed that I skipped one paper from the fourth part of the book in the quick survey given above—Charles McCarty's "Proofs and Constructions". The reason is simple: This paper is (at least, in this reviewer's opinion) by far the most important contribution to the volume, and to our understanding of the foundations of mathematics more generally. While the other papers tread relatively familiar ground, or at least unfamiliar ground that is located within a short stroll of familiar ground, McCarty's paper argues, in effect, for a bold and nearly incoherent sounding—but quite correct—conclusion: That Brouwer, Heyting, and Dummett (and as a result, most of their acolytes, nevermind their critics!) misunderstood, and continue to misunderstand, central tenets of intuitionism. McCarty examines a number of claims that are central to what we might call the 'folk' understanding of intuitionism (where here the folk are professional mathematicians and philosophers, including intuitionists themselves), arguing that these claims are mistaken, if not outright incoherent. In particular, McCarty soundly refutes the common idea that intuitionist logicians and classical logicians mean different things by negation (or by any other connective or logical operator). In response to the idea that intuitionists mean to be understood as proposing a novel reading of the connectives, he writes:

> But [this idea] is pure moonshine. As an intuitionist, whether chatting about the theorems of the day with a fellow true believer, or at home with the missus, the word "not" in my mouth means "not". Neither I nor Brouwer can change the meanings of logical signs by fiat, even when driving under the influence of some 200-proof philosophy. Besides, it is a simple theorem of intuitionistic mathematics that there is only one operation of negation. This is as provable a fact as anything in mathematics, logic, or life can be. (p. 213)

Moonshine indeed! This insight in turn refutes the idea that the classical logician can coherently describe the intuitionist rejection of excluded middle along the the following lines:

> The intuitionist refuses to assert $P \vee \neg P$ because they cannot prove P and they cannot prove $\neg P$.

where the negation expressed by "$\neg$" is intuitionistic (or *internal*), and the negation expressed by the English expression "cannot" is classical (or *external*). As a result, the intuitionistic versus classical logic debate must be understood as a disagreement regarding whether excluded middle (and other, similarly contested classical principles) are or are not logical truths, where these principles are understood by both parties in the debate in exactly the same way. McCarty extends these lessons to fatally skewer both the coherence of Brouwerian weak counterexamples and the idea that intuitionists are mobilizing a different notion of truth than that mobilized by their classical counterparts. The resulting insights provide a refreshing (and immensely important) reconceptualization of exactly what is, and is not, at issue in the debate between intuitionists and classical logicians and, more generally, what is at issue between classical approaches to foundations and non-classical alternatives (whether constructive or not).

The volume concludes with a short three-page essay by Sommaruga and Bell, providing an overview of the current state of play as reflected in the articles discussed above and outlining directions for future research. This closing essay suffers from the same (minor) misplaced emphases—for example, Scottish logicism is not even mentioned here, despite the fact that the formal development of this promising alternative to set theory, category theory, and type theory is one of the most pressing unexplored (or, at the very least, embarrassingly underexplored) technical areas in foundations of mathematics. Nevertheless, as is highlighted by this very nice (albeit overly concise) concluding essay, the volume provides many informative, exciting, and important contributions that will be of great interest to both philosophers and mathematicians working in foundations and related areas.

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S. S. Goncharov. *Autostability and computable families of constructivizations*. ***Algebra and Logic***, vol. 14 (1975), no. 6, pp. 392–409.

S. S. Goncharov. *The quantity of nonautoequivalent constructivizations*. ***Algebra and Logic***, vol. 16 (1977), no. 3, pp. 169–185.

S. S. Goncharov and V. D. Dzgoev. *Autostability of models*. ***Algebra and Logic***, vol. 19 (1980), no. 1, pp. 28–37.

J. B. Remmel. *Recursively categorical linear orderings*. ***Proceedings of the American Mathematical Society***, vol. 83 (1981), no. 2, pp. 387–391.

TERRENCE MILLAR. *Recursive categoricity and persistence*. ***The Journal of Symbolic Logic***, vol. 51 (1986), no. 2, pp. 430–434.

PETER CHOLAK, SEGEY GONCHAROV, BAKHADYR KHOUSSAINOV AND RICHARD A. SHORE. *Computably categorical structures and expansions by constants*. ***The Journal of Symbolic Logic***, vol. 64 (1999), no. 1, pp. 13–137.

PETER CHOLAK, RICHARD A. SHORE AND REED SOLOMON. *A computably stable structure with no Scott family of finitary formulas*. ***Archive for Mathematical Logic***, vol. 45 (2006), no. 5, pp. 519–538.

CHRIS ASH, JULIA KNIGHT, MARK MANASSE AND THEODORE SLAMAN. *Generic copies of countable structures*. ***Annals of Pure and Applied Logic***, vol. 42 (1989), no. 3, pp. 195–205.

JOHN CHISHOLM. *Effective model theory vs. recursive model theory*. ***The Journal of Symbolic Logic***, vol. 55 (1990), no. 3, pp. 1168–1191.

One property which is well studied in model theory is countable categoricity; a theory is *countably categorical* if any two countable models of the theory are isomorphic. In computable model theory, it is natural to restrict attention to computable structures and computable isomorphisms; in this case, we can arrive at a more complicated notion by considering isomorphism types rather than theories. A computable structure is called *computably categorical* if any two computable copies are computably isomorphic.

Goncharov began a program of study of which structures are computably categorical with *Autostability and computable families of constructivizations* [***Algebra and Logic***, vol. 14 (1975), pp. 392–409]. He introduced as a sufficient condition what we now refer to as having a Σ_1^0 Scott family. A Σ_1^0 *Scott family* for a structure $\mathcal{A}$ is a computably enumerable set X of Σ_1 formulas $\phi(\overline{c},\overline{x})$ (in the language of $\mathcal{A}$) with $\overline{c}$ a fixed tuple of parameters from $\mathcal{A}$, satisfying: (1) for every tuple $\overline{a} \in \mathcal{A}$, there is some formula $\phi \in X$ with $\mathcal{A} \models \phi(\overline{c},\overline{a})$; and (2) if $\phi \in X$ and $\overline{a},\overline{b} \in \mathcal{A}$ with $\mathcal{A} \models \phi(\overline{c},\overline{a}) \wedge \phi(\overline{c},\overline{b})$, then there is an automorphism of $\mathcal{A}$ taking $\overline{a}$ to $\overline{b}$.

Goncharov observed that if a structure has a Σ_1^0 Scott family, then one can build a computable isomorphism between any two computable copies using a back-and-forth argument. He also showed that with the additional hypothesis of 2-decidability, every computably categorical structure has a Σ_1^0 Scott family.

However, building on work of Selivanov in *On enumerations of families of general recursive functions* [***Algebra i Logika***, vol. 15 (1976), pp. 205–226], Goncharov showed in *The quantity of nonautoequivalent constructivizations* [***Algebra and Logic***, vol. 16 (1977), pp. 169–185] that there is a computably categorical structure with no Σ_1^0 Scott family. Selivanov had previously constructed a computable family of functions which is discrete (as a subset of Baire space) but not effectively discrete, and has only a single computable enumeration up to computable permutation. Goncharov created his structure by coding this family: because the family has only a single enumeration, the structure is computably categorical; because the family is not effectively discrete, the structure has no Σ_1^0 Scott family.

Within many natural theories, however, every computably categorical structure has a Σ_1^0 Scott family. For example, in *Autostability of models* [***Algebra and Logic***, vol. 19 (1980), pp. 28–37], Goncharov and Dzgoev classified the computably categorical linear orders as those with only finitely many adjacencies, and the computably categorical relatively complemented distributive lattices as those with only finitely many atoms. Remmel independently classified the computably categorical linear orders in *Recursively categorical linear orderings* [***Proceedings of the American Mathematical Society***, vol. 83 (1981), pp. 387–391].

Such structures can be seen to have Σ_1^0 Scott families: for example, for a linear order with only finitely many adjacencies, let the parameters $\overline{c}$ be the points in these adjacencies, and let the family be the set of all formulas $\phi(\overline{c},\overline{x})$ which define a total order on $\overline{c} \cup \overline{x}$.

Although the Goncharov–Dzgoev proofs are indirect, both of those proofs and Remmel's proof all follow a similar approach: fixing a computable structure with infinitely many adjacencies (or atoms), a computable structure is constructed which is isomorphic to the original via a Δ^0_2-computable isomorphism. This structure is made not computably isomorphic to the original by arranging that every potential isomorphism maps some adjacency (or atom) of the original structure to a non-adjacency (resp. non-atom) of the constructed structure.

Returning again to model theory, if a structure $\mathcal{A}$ has a countably categorical theory, then when the signature of $\mathcal{A}$ is expanded by finitely many constants, the theory of the resulting model is also countably categorical. A natural question is whether the effective analog of this holds, namely is it true that every computably categorical structure remains computably categorical when its signature is expanded by finitely many constants. It is easily seen that the converse is true: a computably categorical structure remains computably categorical when finitely many constants are removed from its signature. It is also straightforward to see that the proposition is true if $\mathcal{A}$ has a Σ^0_1 Scott family (and thus is true if $\mathcal{A}$ is 2-decidable).

In *Recursive categoricity and persistence* [***The Journal of Symbolic Logic***, vol. 51 (1986), pp. 430–434], Millar showed that this remains true even if $\mathcal{A}$ is only 1-decidable. Millar began by observing that by induction, it suffices to show the result for a single constant. Then, given 1-decidable structures $(\mathcal{A}, a)$ and $(\mathcal{B}, b)$ with $\mathcal{A} \cong \mathcal{B}$, he constructed $\mathcal{C} \cong \mathcal{B}$ and a sequence of functions f_i such that if g_i is an isomorphism mapping $\mathcal{A}$ to $\mathcal{C}$, then f_i is an isomorphism mapping $\mathcal{C}$ to $\mathcal{B}$ such that $f_i \circ g_i(a) = b$.

Cholak, Goncharov, Khoussainov and Shore showed in *Computably categorical structures and expansions by constants* [***The Journal of Symbolic Logic***, vol. 64 (1999), pp. 13–137] that the assumption of 1-decidability in Millar's result is necessary; they constructed a computably categorical structure which is not computably categorical when expanded by *any* constant. They began by constructing a computably enumerable family X of pairs of sets with the following properties: if $(A, B) \in X$, then $(B, A) \in X$; there are precisely two enumerations of the family, up to computable permutation; and, if f is an enumeration with $f(i) = (A_i, B_i)$, the reverse enumeration $\overline{f}(i) = (B_i, A_i)$ is not a computable permutation of f. They then coded this family into the structure in such a way that every copy codes an enumeration f of X, but also codes the reverse enumeration $\overline{f}$, and there is no natural way to choose between the two. Fixing two computable copies $\mathcal{A}$ and $\mathcal{B}$, one of the enumerations coded by $\mathcal{A}$ will be a computable permutation of one of the enumerations coded by $\mathcal{B}$, and this permutation will give a computable isomorphism between the two structures. However, expanding the structure by a constant causes there to be a natural choice of one of the coded enumerations. If the natural enumeration coded by $(\mathcal{A}, a)$ is not a computable permutation of the natural enumeration coded by $(\mathcal{B}, b)$, then there can be no isomorphism between $(\mathcal{A}, a)$ and $(\mathcal{B}, b)$.

Since Goncharov had shown that having a Σ^0_1 Scott family was a sufficient but not necessary condition for a structure being computably categorical, Cholak, Shore and Solomon wondered if perhaps a weaker condition was equivalent. In *A computably stable structure with no Scott family of finitary formulas* [***Archive for Mathematical Logic***, vol. 45 (2006), pp. 519–538], they removed the conditions that the set of formulas be c.e. and that the formulas be Σ_1, and instead considered Scott families consisting of arbitrary sets of arbitrarily complex finitary formulas. They showed that even this is not a necessary condition, by constructing a computably categorical structure with no such generalized Scott family. Once again, their structure is created by coding a family with certain properties. In this case, it is a computably enumerable family of sets with precisely one enumeration up to computable permutation, such that infinitely many of the sets in the family infinitely often appear to be subsets of other sets in the family. Unlike the previous two coding arguments, the desired properties of the structure do not follow immediately from the properties of the family; instead, the authors first needed to show that the structure satisfied a sort of quantifier elimination.

Returning to Σ^0_1 Scott families, Goncharov had shown that having a Σ^0_1 Scott family was a property strictly stronger than being computably categorical. Ash, Knight, Manasse and Slaman in *Generic copies of countable structures* [***Annals of Pure and Applied Logic***, vol. 42 (1989), pp. 195–205], and independently Chisholm in *Effective model theory vs. recursive* *model theory* [***The Journal of Symbolic Logic***, vol. 55 (1990), pp. 1168–1191], introduced the notion of relatively computably categorical, and showed that this is equivalent to having a Σ^0_1 Scott family. A structure $\mathcal{A}$ is *relatively computably categorical* if for every structure $\mathcal{B}$ with $\mathcal{B} \cong \mathcal{A}$, there is a $\mathcal{B}$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$ (this differs from computably categorical in that we do not assume $\mathcal{B}$ is computable). The same back-and-forth argument that showed having a Σ^0_1 Scott family implies computable categoricity also serves to show that having a Σ^0_1 Scott family implies relative computable categoricity.

Both papers prove the other direction through a forcing argument. The notion of forcing is finite partial injective functions, and the resulting generic permutation g is used to induce a generic structure $\mathcal{B} \cong \mathcal{A}$, for $\mathcal{A}$ some relatively computable structure. Then an index for a $\mathcal{B}$-computable isomorphism f is fixed, and a Scott family is created by taking, for each tuple $\bar{b} \in \mathcal{B}$, the formula "there exists a finite partial function σ with $\bar{b} \subseteq \text{range}(\sigma)$ and $\sigma \Vdash [f \circ g^{-1} \restriction_{\bar{b}}$ is a partial embedding]".

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